

Topological Mixed States: Phases of Matter from Axiomatic Approaches

Tai-Hsuan Yang,¹ Bowen Shi^{1,2}, and Jong Yeon Lee^{1,3}

¹*Department of Physics, Illinois Quantum Information Science and Technology Center,
and Institute of Condensed Matter Theory, The Grainger College of Engineering,
University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA*

²*Department of Computer Science, University of California, Davis, California 95616, USA*

³*Korea Institute for Advanced Study, Seoul 02455, South Korea*

 (Received 8 July 2025; revised 4 April 2026; accepted 20 July 2026; published 2 October 2026)

For closed quantum systems, topological orders are understood through the equivalence classes of ground states of gapped local Hamiltonians. The generalization of this conceptual paradigm to open quantum systems, however, remains elusive, often relying on operational definitions without fundamental principles. Here, we fill this gap by proposing an approach based on three axioms: (i) local recoverability, (ii) absence of long-range correlations, and (iii) spatial uniformity. States that satisfy these axioms are fixed points; requiring the axioms only after coarse-graining promotes each fixed point to an equivalence class, i.e., a phase, presenting the first step toward the axiomatic classification of mixed-state phases of matter: the *mixed-state bootstrap program*. From these axioms, a rich set of topological data naturally emerges; importantly, these data are *robust* under relaxation of axioms. For example, each topological mixed state supports locally indistinguishable classical and/or quantum logical memories with distinct responses to topological operations. These data label distinct mixed-state phases, allowing one to distinguish them. We further uncover a hierarchy of secret-sharing constraints: In non-Abelian phases, reliable recovery—even of information that looks purely classical—demands a specific coordination among spatial subregions, a requirement different across non-Abelian classes. This originates from non-Abelian fusion rules that can stay robust under decoherence. Finally, we performed large-scale numerical simulations to corroborate stability: Weakly decohered fixed points respect the axioms once coarse-grained. Surprisingly, the axiom violations at different length scales exhibit an exact crossing at the critical point, enabling its identification without any rescaling or data-collapse ambiguities. These results lay the foundation for a systematic classification of topological states in open quantum systems.

DOI: [10.1103/3bpt-f9pd](https://doi.org/10.1103/3bpt-f9pd)

Subject Areas: Condensed Matter Physics,
Quantum Physics, Quantum Information

I. INTRODUCTION

Entanglement is one of the fundamental principles underpinning the classification of phases of matter. Building on this concept, it has been proposed that two pure states belong to the same phase if they can be continuously deformed into one another via finite-depth local unitary operations (FDLUs) [1–4]. Extending this idea to mixed states, however, is nontrivial. A straightforward generalization—substituting finite-depth local unitary operations with finite-depth local quantum channels (FDLCs)—overlooks a crucial requirement: the smooth deformation of the underlying entanglement structure. While finite-depth local unitary operations modify

entanglement only gradually, local quantum channels can induce abrupt changes, potentially destroying the topological correlation across the system. This subtle oversight has contributed to some ambiguity in the study of mixed-state topological phases. This paper aims to clarify this point and provide a more robust and consistent framework.

The classification of phases of matter has been a central theme in quantum many-body physics, traditionally focused on identifying distinct ground states of local gapped Hamiltonians. Over the past few decades, entanglement has emerged as a pivotal tool in this endeavor, offering a perspective that goes beyond Landau's spontaneous symmetry breaking. In particular, quantum phases can be broadly categorized by their entanglement structure into short-range-entangled or long-range-entangled states [2,5], which can be further enriched under the presence of symmetries. Long-range-entangled states that are robust against local perturbations are called *topological*, and, in two dimensions, these states are described by

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

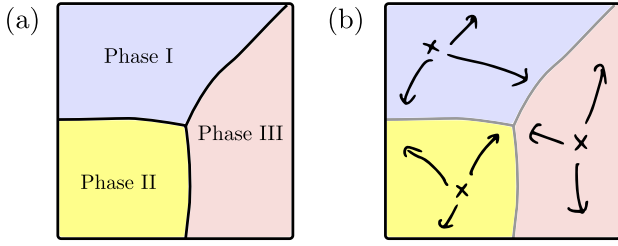


FIG. 1. Phases of matter defined through (a) equivalence relations (adiabatic connection without closing gaps of a local Hamiltonian), which specify boundaries as well, and (b) fixed points and allowed deformations, which are finite-depth (quasi) local operations.

anyon theories [6–8], which characterize the emergent quasiparticles and their braiding statistics.

A foundational principle in defining and classifying gapped phases is the existence of an energy gap in the parent local Hamiltonian [1,6,7,9]. Traversing from one phase to another typically necessitates gap closing that marks the phase boundary as in Fig. 1(a). The entanglement bootstrap program [10–17] offers a complementary perspective. It begins with a set of axioms designed to strip away all short-range structures, retaining only long-range entanglement. Two fixed-point wave functions that satisfy these axioms but cannot be connected by an FDLU circuit define distinct topological phases; see Fig. 1(b). Conversely, relaxing the axioms around a chosen fixed point and allowing FDLU deformations generates the entire phase anchored to that point. This construction yields a robust definition of topological order, independent of explicit Hamiltonian constructions [18]. Despite these advances in the pure-state regime, much less is known about how to classify mixed states in a similarly rigorous manner, particularly when the guiding notion of a spectral gap is unavailable.

From the information-theoretic perspective, understanding which aspects of topological order survive after decoherence is both conceptually intriguing and practically important. Recent studies have shown that decohered topological states—intrinsic or symmetry protected—can still encode nontrivial quantum or classical logical information across various regimes [19–27]. These findings support the growing consensus that one can meaningfully define mixed-state generalizations of topological phases [19–50]. Substantial progress has been made along these lines. The doubled space formalism offers a qualitative organizing framework for decoherence-driven transitions [19,22], while generalizations of strange correlators [19,36], information-theoretic quantities [20], and separability conditions [37] have emerged as promising order parameters. In parallel, the divergence of the Markov length has been proposed as a hallmark of transitions between phases defined via local Lindbladian reversibility [26].

However, a comprehensive framework is still missing. The main hurdle is to specify principled constraints on

mixed states that allow a meaningful classification. Equally uncertain is which properties of pure-state topological order would survive under decoherence. Do anyonic braiding statistics persist, and can generalized fusion rules—or viable analogs of the modular S and T matrices in two dimensions—still be defined? Finally, a complete theory must explain how information storage capacity—whether classical, quantum, or hybrid—naturally emerges across different mixed-state phases.

In this work, we propose a novel systematic framework for topological mixed states. By generalizing the ideas from the entanglement bootstrap [10], we state a set of axioms that identify fixed points of topological mixed states and explore invariants to label and classify them. Furthermore, by relaxing the axioms, we define mixed-state phases of matter where the equivalence class is defined by axioms under coarse-graining. This new approach naturally provides the notion of topological channel connectivity, which resolves issues in previous attempts to define the equivalence class using two-way connectivity. Our framework not only delineates the fundamental constraints that topologically ordered mixed states must satisfy, but also clarifies how such states can encode and preserve information—whether quantum, classical, or both. We thus offer the first coherent step toward a classification of topological mixed states.

The paper is organized as follows. Section II formulates the mixed-state bootstrap, including the axioms and definition of fixed point. We introduce the information convex set and its connection to memory capacity, relating its bound to topological entropy. Section III presents explicit fixed-point examples and analyzes their classical and quantum memories. For a class of classical fixed points, we discuss the method to prepare them through a local quantum channel. Section IV defines mixed-state topological phases through coarse-graining. We provide extensive numerical results demonstrating that axioms hold away from fixed points under coarse-graining. We also introduce a refinement of a previous approach based on two-way channel connectivity. Section V refines this phase equivalence via coarse-graining by demonstrating that topological invariants are robust under approximate axioms. Section VI discusses a multipartite secret-sharing property unique to certain non-Abelian mixed-state fixed points, exposing a hierarchy of topological *classical* memories. Section VII extends the framework to higher dimensions, and Sec. VIII summarizes with future directions.

II. MIXED-STATE BOOTSTRAP

A. Axioms

We study bosonic lattice systems whose total Hilbert space factorizes into a tensor product of finite-dimensional local Hilbert spaces. To isolate intrinsically topological features, we assume the absence of any global symmetry.

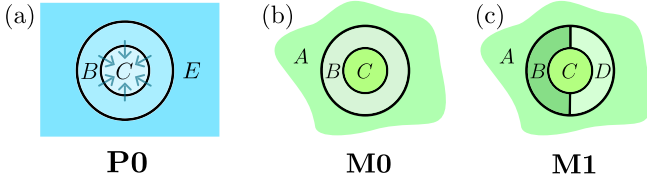


FIG. 2. Bootstrap axioms. Given a mixed state σ , we impose axioms (a) **P0**, (b) **M0**, and (c) **M1** on it. Axiom **P0** is imposed on the entire system, while BC remains local. The axioms **M0** and **M1** are stated on local patches indicated by green regions.

Our goal is to formulate mixed-state bootstrap axioms for the fixed-point density matrix of topological mixed states, as an analog of the *reference state* in the pure-state bootstrap framework [10]. The relevant information-theoretic measures are the mutual information $I(A:C)$ and conditional mutual information $I(A:C|B)$ defined for nonoverlapping regions A , B , and C as

$$I(A:C)_\rho = S(\rho_A) + S(\rho_C) - S(\rho_{AC}), \quad (1)$$

$$I(A:C|B)_\rho = I(A:BC)_\rho - I(A:B)_\rho, \quad (2)$$

where $S(\rho) := -\text{Tr}(\rho \log \rho)$ is the von Neumann entropy of density matrix ρ . These quantities capture elementary forms of multipartite correlations [51]. Given a physical system, consider a large ball partitioned in a way described in Fig. 2. A density matrix σ is called a fixed point of topological mixed states if it satisfies the following axioms.

Axiom 1 (P0: local recoverability)— $I(E:C|B)_\sigma = 0$ for Fig. 2(a), where E is the complement of BC . The condition is the saturation of strong subadditivity [52], which is the condition for the existence of a map that acts on B to recover C only when C is corrupted or removed [53–55], i.e., $\rho_{BCE} = \mathcal{E}_{B \rightarrow BC}(\rho_{BE})$ for some quantum channel $\mathcal{E}_{B \rightarrow BC}$. The condition enforces the state to have a zero Markov length mixed state.

P0 implies an important aspect of a topological mixed state: The nontopological (local) structure of the underlying mixed state can always be recovered locally, and, thus, local behavior should not contribute toward the properties we care about. This requirement is consistent with the fact that $I(E:C|B)$ diverges at a critical point separating two mixed-state phases [26]. By imposing the axiom, we focus our attention on fixed points located far away from the phase boundary.

Furthermore, this condition enforces the absence of the novel type of spontaneous symmetry breaking that can occur in the mixed-state systems [21,56], which was recently dubbed as strong to weak spontaneous symmetry breaking (SWSSB) [57–61]. Analogous to an ordinary spontaneous symmetry breaking, in SWSSB a quantum channel that breaks a strong symmetry to a weak one [62,63]—whether applied either infinitesimally across the entire system or finitely on a local patch—thermalizes the

conserved-charge sector and induces a global change in the density matrix state, behavior incompatible with **P0**. More details for the motivation of this axiom are explained in Appendix A 5.

Formally, if we consider a channel $\mathcal{E}_C$ acting on a local patch C , the condition together with the monotonicity of the conditional mutual information implies that $0 = I(E:C|B)_\rho \geq I(E:C|B)_{\mathcal{E}_C[\rho]} = 0$. This, in turn, implies that the difference between the original and perturbed density matrices of the entire system, captured by the relative entropy [53], is given as

$$D(\rho_{EBC} \| \mathcal{E}_C[\rho_{EBC}]) = D(\rho_{BC} \| \mathcal{E}_C[\rho_{BC}]), \quad (3)$$

which is only a function of the compact region BC . Accordingly, with **P0**, the local channel acting on C cannot induce a change that is intrinsically global in the density matrix. More details can be found in Appendix A 6.

The next two axioms can be stated on local patches of the density matrix.

Axiom 2 (M0: no long-range order)— $I(A:C)_\sigma = 0$ for Fig. 2(b), where A is the complement of BC on the local patch. This condition implies that there is neither classical nor quantum correlation between separated regions.

M0 is physically natural: It eliminates states that exhibit long-range order, such as those with spontaneous symmetry breaking. Furthermore, the condition precludes some pathological situations that arise in a mixed-state setup, such as the classical mixture $\rho = \sum_i p_i \rho_i$ with $p_i > 0$, where each ρ_i realizes a different topological phase. Therefore, through **P0** and **M0**, we can exclude pathological density matrices.

Axiom 3 (M1: uniformity and smoothness)— $I(A:C|B)_\sigma = 0$ for Fig. 2(c), where A is the complement of BCD on the local patch. The condition excludes domain walls between two different phases, which can satisfy both **P0** and **M0**. Thus, it should be included if we wish to say the given state is uniform enough to represent a single mixed-state phase. Without it, one risks including pathological cases where two different phases are simply juxtaposed.

M1 is quite subtle, yet extremely important. For pure states, it implies the absence of defects in the wave function, which allows one to discuss the physics from the topology of the manifold rather than local details [64]. In other words, the axiom requires a *smoothness* of the fabric of many-body correlation or entanglement, the quintessential premise when it comes to the discussion of topological states. On the other hand, a finite-depth local channel can violate **M1** since channels can break the entanglement. As we see, the violation of **M1** also implies that the topological entropy (TE) can change abruptly.

One potential concern is that highly fine-tuned FDLUs can generate *spurious* entanglement that violates **M1** on specially chosen partitions [70–74]. Two points address

this. (i) A generic FDLU does not violate **M1** on sufficiently large local patches. (ii) Even when such violations occur, they do not obstruct phase identification: We pass to a *clean* representative obtained by a shallow-depth local unitary that removes the axiom violations and then evaluate phase diagnostics on the clean state. A soft-invariant theorem [75] ensures that, within a given phase, all clean representative states related by FDLUs yield identical diagnostic values. In particular, the Levin-Wen topological entropy γ_{LW} of any clean representative is a global minimum over its FDLU orbit [76]. Hence, for any two states ρ_0 and ρ_1 in the same phase, their clean counterparts ρ'_0 and ρ'_1 have the same topological entropy $\gamma_{\text{LW}}(\rho'_0) = \gamma_{\text{LW}}(\rho'_1)$; for relevant discussions, see Sec. V. Therefore, spurious contributions do not impede a well-defined notion of mixed-state phases.

Definition 1 (topological mixed-state fixed point). A density matrix satisfying three axioms **P0**, **M0**, and **M1** for any partition above length scale l is called a *fixed point* of a topological mixed state. In particular, if these axioms hold for all shapes of partitions at length scale l_0 , they hold for length scale $l \geq l_0$ [10].

We emphasize that the novelty of this work does not lie in the individual entropy conditions **P0**, **M0**, or **M1**; as we elaborate above, these are well-understood physical conditions one expects from fixed points of topological mixed states. Rather, the novelty is to show that the three conditions taken together furnish a coherent and self-contained axiomatic framework for mixed-state topological phases. In particular, this framework underlies our definitions of fixed-point states (Definition 1) and phase equivalence (Definition 6) and supports a consistent classification theory for mixed states.

It is worth comparing these mixed-state axioms with those of the entanglement bootstrap program for gapped ground states [10], where we have two axioms **A0**: $S(BC) + S(C) - S(B) = 0$ and **A1**: $S(BC) + S(CD) - S(B) - S(D) = 0$, for the partition of a ball into BC or BCD as shown in Figs. 2(b) and 2(c), respectively. The first two axioms **P0** and **M0** follow from **A0** [77], and **M1** follows from **A1**. However, these mixed-state conditions are strictly weaker because they may hold on volume-law states, whereas the entanglement bootstrap axioms for gapped ground state imply an area law. The axiom **A0** is known as a local purity condition—this is because it implies that the state in C can be purified with a subspace of B ; see Appendix A 2. From now on, we refer to entanglement bootstrap with **A0** and **A1** as “pure-state bootstrap”.

Furthermore, these three axioms are independent: (i) **P0** and **M0** together do not imply **M1**, e.g., a state with non-Abelian anyons located at C . This is because adding an anyon a to a disk will increase the von Neumann entropy of the disk by $\log d_a$ [78], where d_a is the quantum dimension of a . Suppose we create an anyon pair with a unitary string

operator, one anyon in A and another in C . Then, the violation of **M1** will be $2 \log d_a$, because S_{AB} and S_{BC} are increased by $\log d_a$ each without affecting S_B and $S_{ABC} = S_D$. Note that only non-Abelian anyons give a violation, because $d_a = 1$ for Abelian anyons. (ii) **P0** and **M1** together do not imply **M0** by the example of an incoherent mixture of distinct SSB states, such as $\rho = \frac{1}{2}(|\bar{0}\rangle\langle\bar{0}| + |\bar{1}\rangle\langle\bar{1}|)$, where $|\bar{a}\rangle = |a\rangle^{\otimes N}$. (iii) **M0** and **M1** together do not imply **P0** by the example of the state $(1 + \prod_{i=1}^N X_i)/2^N$. This state reduces to the maximal mixed state after tracing out one site.

These mixed-state axioms are naturally satisfied by pure states satisfying pure-state axioms, and, thus, pure-state fixed points are naturally incorporated into our framework. Furthermore, volume-law-entangled decohered states can satisfy mixed-state axioms, as we see in the example section (Sec. III). More physically, we allow small errors to the axioms as long as the errors decay as we zoom out to larger length scales. For a more careful discussion, see Sec. IV.

B. Information convex set

Given two density matrices ρ_1 and ρ_2 satisfying mixed-state axioms, how can we distinguish them? The first step is to characterize their information-theoretic properties, which can be achieved by investigating the structure of an information convex set, a set of mixed states locally equivalent to each density matrix satisfying the axioms. Information convex sets can be defined on the entire system as well as on pieces [10,11,79] (including embedded and locally embedded regions [80]). For pure states, the information convex set of a closed manifold simply corresponds to the ground state manifold for the corresponding parent Hamiltonian [81–83]. For mixed states, the information convex set corresponds to the set of mixed states that can be prepared in a similar manner yet encode different topological information (see Fig. 3).

Definition 2 (information convex set). Given a fixed-point state σ under Definition 1 defined on a physical

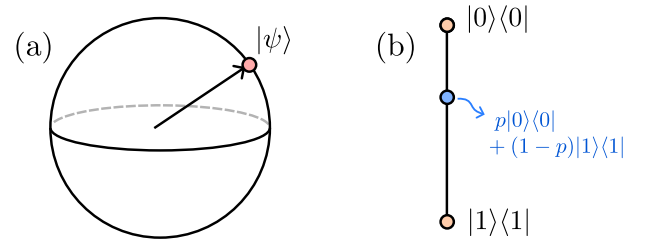


FIG. 3. Information convex set. For topological mixed states encoding (a) a single qubit (quantum memory) and (b) a single bit (classical memory). While quantum memories reside on the surface of the Bloch sphere, classical memories reside in the linear combination of extreme points. Mathematically, the convex set in (a) is the convex hull $\text{conv}(\{|\psi\rangle\langle\psi| \mid |\psi\rangle \in \mathcal{H}\})$, and the convex set in (b) is $\text{conv}(\{|0\rangle\langle 0|, |1\rangle\langle 1|\})$.

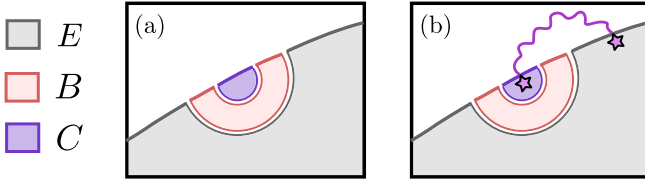


FIG. 4. Manifold with boundary. (a) Partitioning near the boundary for the definition of information convex set (Definition 2). (b) For a subsystem with a boundary, a nontrivial entanglement between boundary degrees of freedom can exist. Imposing $I(E:C|B) = 0$ precisely excludes this complication.

system N , the information convex set on a manifold $\Omega \subseteq N$, possibly with boundary, is defined as

$$\Sigma(\Omega|\sigma) = \{\rho_\Omega|(1), (2)\}, \quad \text{where} \quad (4)$$

- (i) *local indistinguishability*—for any bounded radius ball $b \subset \Omega$, we require $\rho_b = \sigma_b$;
- (ii) *local recoverability*—**P0** is satisfied for ρ_Ω . When Ω has a boundary, we require $I(E:C|B)_\rho = 0$ for configurations in Fig. 4. This allows the partition to be located near the boundary.

These two conditions imply that distinct elements of the information convex set cannot be converted into each other through local operations. Furthermore, $\rho \in \Sigma$ must satisfy **M0** and **M1** because the state is identical to the fixed-point state locally. This indicates the topological nature of the information that we can encode using states in the information convex set.

When Ω is a closed manifold, the proof that $\Sigma(\Omega|\sigma)$ is a compact convex set can be found in Appendix A 4. The role of **P0** in the definition of information convex set is explained in Appendix A 5.

When Ω is a manifold with boundary embedded in a larger manifold N , Σ is still a compact convex set; see Appendix B. More importantly, given a fixed-point state σ on N , one can show that $\Sigma(\Omega|\sigma) \cong \Sigma(\Omega'|\sigma)$ whenever Ω can be smoothly deformed into Ω' on the background N . This isomorphism theorem, further elaborated in Eq. (B2), is central to the advanced application of our framework.

C. Channel

In the study of mixed states, the generalization of local perturbations or unitary transformations is a local quantum channel.

Definition 3 (finite-depth local channel). A quantum channel $\mathcal{E}$ acting on a lattice system is called a finite-depth local channel if it can be written as a finite composition of layers of local quantum channels, each consisting of commuting, spatially disjoint operations. More precisely, $\mathcal{E}$ is said to be of depth D if

$$\mathcal{E} = \mathcal{E}^{(D)} \circ \mathcal{E}^{(D-1)} \circ \dots \circ \mathcal{E}^{(1)}, \quad (5)$$

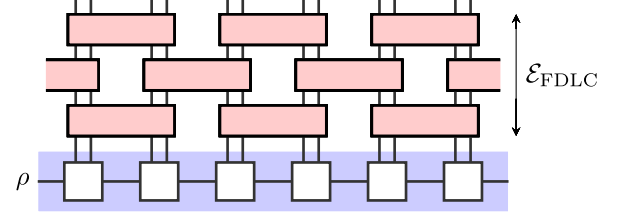


FIG. 5. FDLc. For each site, we have two physical indices corresponding to bra and ket. The red block represents a local quantum channel acting on two sites. The entire channel has a finite depth of three.

where each $\mathcal{E}^{(j)} = \bigotimes_{x \in \Lambda_j} \mathcal{E}_x^{(j)}$ is a layer of local channels supported on disjoint regions (e.g., finite balls around lattice sites) and Λ_j is a subset of lattice sites such that the supports of $\mathcal{E}_x^{(j)}$ and $\mathcal{E}_{x'}^{(j)}$ are disjoint for $x \neq x'$. See Fig. 5 for illustration.

FDLCs provide a physically motivated model for local quantum processes, such as interactions with an environment, local noise, and decoherence protocols. Since local channels can *break* entanglement [84], the action of an FDLc can, though need not, drive a state toward classicality. From the perspective of state preparation, FDLcs correspond to a restricted set of operations: They allow only local operations and local classical communication. This is more stringent than the standard local operations and classical communication framework, which permits unrestricted classical communication and plays a central role in entanglement theory. In this regard, we introduce the following notion.

Definition 4 (classically preparable). Consider a state σ such that $\mathcal{E}(\rho_0) = \sigma$, where $\mathcal{E}$ is a FDLc. If ρ_0 is a classical mixture of product states, i.e.,

$$\rho_0 = \sum_a p_a (\bigotimes_i \rho_i^a), \quad (6)$$

then σ is classically preparable. Here, the probability $\{p_a\}$ mediates classical correlations.

Any product state $\rho = \bigotimes_i \rho_i$ is a classically preparable state, where the channel $\mathcal{E}$ is given as the tensor product of an on-site channel $\mathcal{E}_i$ that resets to a specific local density matrix ρ_i . On the other hand, the toric-code ground state is not classically preparable, because starting from a product state, one has to perform local measurements and nonlocal communication for error corrections.

D. Memory capacity and its bound by the topological entropy

How much classical or quantum information can be stored in a closed manifold M ? In the context of pure-state topological order, this is measured by the dimension of the degenerate ground state. To quantify the ability to store (classical or quantum) topological information of the

system M , we define the *memory capacity* of a mixed state σ as

$$\mathcal{Q}(M|\sigma) \equiv \max_{\rho, \lambda \in \Sigma(M|\sigma)} S(\rho) - S(\lambda). \quad (7)$$

Since two density matrices from the same information convex set are locally indistinguishable, the entropy difference can arise only from nonlocal topological information. The memory capacity of the sphere S^2 is always zero, as we show in Appendix A 7. Thus, we take the torus $M = \mathbb{T}^2$, as the minimal example.

To better connect this idea with a well-known quantity, we consider the topological entropy defined by

$$\gamma_{\text{LW}} \equiv \frac{1}{2} I(A:C|B)_\sigma \quad \text{for} \quad \text{[Diagram: A disk with an annulus ABC inside. The annulus is divided into three regions: A (green), B (blue), and C (green).]} \quad (8)$$

where the annulus ABC is embedded in a disk. Since this topological contribution can be nonzero even for entirely separable mixed states, we call this more general quantity *topological entropy* following Ref. [85]. For pure topological states, γ_{LW} is exactly the Levin-Wen topological entanglement entropy [86,87].

While we mainly talk about case $M = \mathbb{T}^2$, we make an important remark on the annulus. The memory capacity of the information convex set of a contractable annulus X (i.e., annulus contained in a disk) equals twice the topological entropy. Namely,

$$\mathcal{Q}(X|\sigma) = 2\gamma_{\text{LW}}, \quad (9)$$

for $X = ABC$ being the Levin-Wen partition. We explain this exact property in Appendix B 1, Proposition 8.

Proposition 1 (torus capacity). The capacity of a torus $M = \mathbb{T}^2$ is upper bounded by 4 times the Levin-Wen topological entropy as

$$\mathcal{Q}(\mathbb{T}^2) \leq 4\gamma_{\text{LW}}. \quad (10)$$

The proof of this (and a more powerful one on any closed manifold) is given in Appendix D. We see examples of mixed-state fixed points saturating this bound in Sec. III. The bound (10) should be contrasted with a bound by Kim [88] for pure-state topological order, which says $\mathcal{Q}(M) \leq 2\gamma_{\text{LW}}$. The violation of the pure-state bound for some mixed-state fixed points originates due to the absence of the nondegeneracy braiding statistics [13], which is a quantum mechanical effect.

E. Comparison to pure-state bootstrap

The bootstrap axioms developed in this section encompass topological mixed and pure states. While three “mixed-state axioms” **P0**, **M0**, and **M1** are strictly weaker

than “pure-state axioms” **A0** and **A1** introduced in Ref. [10], many of the intuitions and machinery carry over.

- (i) *Fixed-point interpretation*—These axioms, defined on (or centered at) local balls, imply that the same conditions hold true on larger length scales by strong subadditivity [52,53]. Think about coarse-graining as the renormalization group (RG) flowing to large length scales. This intuition agrees with the fixed-point interpretation of such states. This leads to the formal definition of fixed point in Definition 1.
- (ii) *Uniformity*—The axiom **M1** (which is an analog of **A1**) implies an emergent sense of uniformity, which is in the deep level related to the isomorphism property of the information convex set under a smooth deformation of region, as we review in Appendix B.
- (iii) *Topological entropy*—For both cases, the Levin-Wen topological entropy can be shown to be invariant under the deformation of regions. We show that this is true for states with decaying errors of the axioms; see Sec. V.
- (iv) *From mixed to pure*—A mixed-state fixed point which also satisfies the local purity condition **A0** will be a pure-state fixed point (see Appendix A 2):

$$\mathbf{A0}, \mathbf{M1} \Rightarrow \mathbf{A0}, \mathbf{A1}. \quad (11)$$

However, there are several distinctions.

- (i) The pure-state axioms *guarantee* the information convex set on a closed manifold to host pure quantum states on some Hilbert space; in other words, such states are fully coherent and can host topological quantum information. On the other hand, the mixed-state axioms do not; in particular, the mixed-state axioms allow the information convex set to host classical states as illustrated in Fig. 3(b), implying that there exists *classical topological information*. See Proposition 2 for a concrete example.
- (ii) When the pure-state axioms are satisfied, both the Levin-Wen and Kitaev-Preskill topological entanglement entropy can be shown to be quantized [10] according to the fusion rules. In contrast, we do not yet know if Levin-Wen TE is quantized following the mixed-state axioms. Furthermore, the Kitaev-Preskill topological entropy is no longer a topological invariant for mixed-state fixed points; see Appendix E.
- (iii) The information convex set of the annulus can be rigorously shown to be a simplex with orthogonal extreme points, under the pure-state axioms [10]. It is an open question if this “simplex theorem” is still true for an arbitrary mixed-state fixed point.
- (iv) The braiding nondegeneracy [89] of anyon systems can be derived under the pure-state axioms [13,90].

However, when the mixed-state axioms are satisfied, braiding may become degenerate, as happens in examples; see Ref. [35]. This is related to a special property of these mixed-state fixed-point wave functions detected by an immersed punctured torus; see Appendix G 1.

- (v) A modular commutator [91] is not an invariant anymore in the mixed state. This is because there exists a fixed-point density matrix with a nonuniform value of modular commutator. In particular, a nonzero modular commutator can appear in a state in the same phase as the trivial phase; see Appendix E.
- (vi) Finally, classification of phases is not only about finding fixed points, but also about their equivalence; the phases are fixed points modulo the equivalence condition. Because of the distinction of the axioms, the meaning of equivalence will change accordingly. See Definition 6 for our sense of equivalence.

III. EXAMPLES OF FIXED POINTS

In this section, we identify various solvable fixed points by Definition 1 and calculate the corresponding information convex set Σ , which demonstrates their classical and quantum memories.

A useful way to show whether the state satisfies axioms is by showing the existence of recovery channel $\mathcal{R}: B(\mathcal{H}_B) \mapsto B(\mathcal{H}_{BC})$ such that $(\text{Id}_A \otimes \mathcal{R})[\rho_{AB}] = \rho_{ABC}$, which is equivalent to the condition $I(A:C|B) = 0$ [53]. Here, $B(\mathcal{H})$ is a set of bounded operators acting on $\mathcal{H}$.

In general, showing the existence of a Petz recovery map can be cumbersome. But for stabilizer states, we have a simple argument. Consider a stabilizer state $\rho_B = \text{Tr}_C \rho_{BC}$ on some region B :

$$\rho_B = \prod_{S_i \in \mathcal{S}_B} \frac{1 + S_i}{2}, \quad (12)$$

where $\mathcal{S}_B = \{S_i\}$ is a set of stabilizer generators supported on B . Assume ρ_{BC} supports additional stabilizer generators $\{\tilde{S}_j\}$ that commute with $\{S_i\}$. We can have a recovery map as follows. For simplicity, assume we have a single extra stabilizer $\tilde{S}$ on BC , where its eigenvalue is 1. Let $\mathcal{P}_{\tilde{S}}^{\pm}$ be the projector onto the eigenspace of $\tilde{S}$ with eigenvalue ± 1 . Then, the recovery channel $\mathcal{R}$ is given as

$$\begin{aligned} \mathcal{R}[\rho_B] &= \mathcal{P}_{\tilde{S}}^+ \left(\rho_B \otimes \frac{\text{Id}_C}{d_C} \right) \mathcal{P}_{\tilde{S}}^+ + U \mathcal{P}_{\tilde{S}}^- \left(\rho_B \otimes \frac{\text{Id}_C}{d_C} \right) \mathcal{P}_{\tilde{S}}^- U^\dagger \\ &= (1 + \tilde{S}) \left(\rho_B \otimes \frac{\text{Id}_C}{d_C} \right) = \rho_{BC}, \end{aligned} \quad (13)$$

where U is some unitary acting on extra qubits such that $U\tilde{S}U^\dagger = -\tilde{S}$ and d_C is the Hilbert space dimension associated with region C . It is straightforward that its

Kraus representation satisfies the normalization condition. Even if we have multiple commuting stabilizers on BC , it should always be possible to apply a certain unitary that makes their signs correct since they are independent. Finally, if ρ_{ABC} does not contain any stabilizer that extends across A and C , then this recovery map $\mathcal{R}$ would recover ρ_{ABC} from ρ_{BC} .

Example 1 (product state)—Consider a generic product state $\rho = \otimes_i \rho_i$; here, ρ_i can be either a pure or mixed local state. In this case, the subadditivity is saturated and we have $S(\rho_{AB}) = S(\rho_A) + S(\rho_B)$. Accordingly, all mutual information vanishes and the three axioms are satisfied. Maximally mixed states with volume-law entropy belong to this scenario. Because product states have zero topological entropy $\gamma_{\text{LW}} = 0$, the bound in Eq. (10) is saturated, and the memory capacity $\mathcal{Q}(M|\rho) = 0$ for any closed manifold M . This further implies that the information convex set $\Sigma(M|\rho) = \{\rho\}$, i.e., no information can be stored.

Example 2 (ground states of topological order)—It is well known that the ground state of nonchiral topological orders has fixed points that exactly satisfy the axioms. Such representative wave functions can be the string-net wave functions [7,86]. In particular, on the torus, $\Sigma(\mathbb{T}^2|\sigma)$ is the state space of a finite-dimensional Hilbert space, where the Hilbert space dimension equals the number of anyons of the system:

$$\Sigma(\mathbb{T}^2|\sigma) = \text{conv} \left\{ |\psi\rangle\langle\psi| \mid |\psi\rangle = \sum_a c_a |\psi_a\rangle \right\}, \quad (14)$$

where $|\psi_a\rangle$ are minimal entangled states [92] on the torus, in one-to-one correspondence with anyon types, and $\sum_a |c_a|^2 = 1$. One such model is the ground state of the toric code (15), for which $\Sigma(\mathbb{T}^2|\sigma)$ is identical to the four-dimensional Hilbert space. Ground state of chiral topological orders can be treated as fixed points in the same manner, if we are allowed to neglect finite-size errors of the area law that decays with subsystem sizes [15,17,93].

Example 3 (dephased toric code)—The discussion here immediately generalizes to $\mathbb{Z}_n$ toric code. For the toric code, the density matrix for maximally mixed logical states is given as

$$A_v^1 := \left(1 + \prod_{e \ni v} X_e \right) / 2, \quad (15)$$

$$B_p^1 := \left(1 + \prod_{e \in p} Z_e \right) / 2, \quad (16)$$

$$\rho^{\text{tc}} = \prod_v A_v^1 \prod_p B_p^1. \quad (17)$$

Consider a mixed state dephased under maximal X error (note that the other option is mathematically equivalent

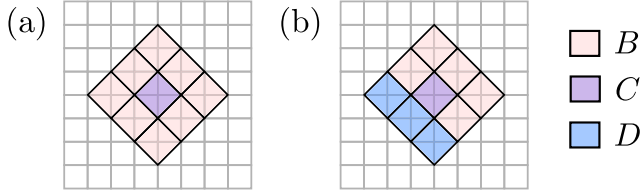


FIG. 6. Cases to check for (a) **M0** and **P0** and (b) **M1**. In both cases, a local region C is separated from the rest by an annulus (B or BD). Here, we show only one of the several configurations to check **M1**. See Appendix A 3 for a complete list.

due to $e \leftrightarrow m$ exchange duality):

$$\rho^e = \prod_v A_v^1. \quad (18)$$

This model has been considered in the literature of mixed-state phases [20–23,34,37]. We show that the state satisfies **P0**, **M0**, and **M1** and, thus, it is a valid fixed point under the bootstrap axioms. The regions A , B , and C (and D) of interest are recalled in Fig. 6.

- (i) *Proof of M0*—Consider tripartite region ABC such that B separates A and C . Here, a region B can be denoted using a collection of vertices and qubits supported on the edges connecting within this collection. Then, we have

$$\rho_{AC} = \prod_{\{v|N(v) \not\subseteq B\}} A_v^1 \left(\frac{1 + X_{\partial B_A}}{2} \right) \left(\frac{1 + X_{\partial B_C}}{2} \right), \quad (19)$$

where $N(v) = \{e|e \ni v\}$ and ∂B denotes a set of edge qubits whose one end belongs to B but not the other end; note that ∂B decomposes into two disjoint sets ∂B_A and ∂B_C , each of which is supported on A and C , separately. Note that the last factor on $X_{\partial B_C}$ is redundant as it can be absorbed into $\prod_v A_v^1$. ρ_{AC} decomposes as $\rho_{AC} = \rho_A \otimes \rho_C$, implying $I(A:C) = 0$.

- (ii) *Proof of P0*—Note that if the region C in Fig. 6(a) is traced out, we lose stabilizers that share links with C but keep only the product of them which is entirely supported on B . Now, using the aforementioned strategy in Eq. (13), we can construct local channels just acting on B , which recover individual stabilizers. More precisely, we prepare ancillas, measure lost stabilizers, and perform error correction on outcomes. Because $\prod_{e \in \partial A} X_e = 1$, we obtain only the desired outcomes.
- (iii) *Proof of M1*—If we trace out the region CD , we get

$$\rho_{AB} = \prod_{\{v|N(v) \not\subseteq B\}} A_v^1 \left(\frac{1 + X_{\partial(CD)}}{2} \right). \quad (20)$$

By using the protocol explained in **P0**, we can recover ρ_{ABC} by applying a channel that acts only on the support of B , which proves $I(A:C|B) = 0$.

Example 4 (fermionic decohered toric code)—Consider maximally $Z_r X_{r+\delta}$ -decohered toric code [23] in the square lattice, where $\delta = (1/2, 1/2)$ is a shift vector. It is written as

$$\rho_f \propto \prod_p (1 + W_p), \quad W_p := \prod_{e \ni (p-\delta)} X_e \prod_{e \ni p} Z_e. \quad (21)$$

The proof of **M0** is the same as the toric-code case; we can always choose a large enough B so that there is no stabilizer supported on both A and C , which leads to $I(A:C) = 0$. The proof of **P0** is also similar. After tracing out C , we have an additional stabilizer of ZX string at the boundary of C , which enforces the measurement outcome of W_p stabilizer supported on C to obey the desired condition $\prod_{p \in C} W_p = 1$. Accordingly, we can perform error correction on BC to recover the original ρ_{ABC} always. The proof of **M1** is also analogous.

Example 5 (stacked layers of fixed points)—We can get more fixed-point examples by stacking layers of known fixed-point examples. One can immediately check that stacking does not violate any of the axioms. In particular, stacking a decohered classical state with a quantum ground state is allowed.

A. Abelian quantum double

Next, we examine some canonical examples of decohered Abelian quantum double models, which generalize the toric-code example above. To proceed, we use the notation from Ref. [65], where A_v^q and B_p^f are commuting projectors on a gauge charge q on v and flux f on p , respectively [94]. For example, in the case of a $\mathbb{Z}_n$ toric-code state,

$$A_v^q := \frac{1}{n} \sum_{m=0}^{n-1} q^m \prod_{e \ni v} X_e^{m \epsilon_{e,v}} = \mathcal{P}_{q, \prod_{e \ni v} X_e^{\epsilon_{e,v}}},$$

$$B_p^f := \frac{1}{n} \sum_{m=0}^{n-1} f^m \prod_{e \in p} Z_e^{m \epsilon_{e,p}} = \mathcal{P}_{f, \prod_{e \in p} Z_e^{\epsilon_{e,p}}}, \quad (22)$$

where we use $\mathbb{Z}_n$ generalization of Pauli operators and $q, f \in \{\omega^m | m \in \mathbb{Z}_n\}$ label local gauge charge and flux, where $\omega = e^{2\pi i/n}$. The result immediately follows from the discrete Fourier transformation. The exponent $\epsilon_{e,v}$ and $\epsilon_{e,p}$ are defined in such a way that $\partial e = \sum_{v \in e} \epsilon_{e,v} v$ and $\partial p = \sum_{e \in p} \epsilon_{e,p} e$. With these notations, the ground state of an Abelian quantum double model is written as $\prod_v A_v^1 \prod_p B_p^1$. For mixed states, we define the following.

Definition 5 (canonically decohered Abelian quantum double). We introduce the canonically decohered states of

the $\mathbb{Z}_n$ quantum double by applying the maximal Pauli- X (flux) or Pauli- Z (charge) decoherence channel:

$$\rho^e = \prod_v A_v^1, \quad \rho^m = \prod_p B_p^1. \quad (23)$$

Their forms follow from the observation that, under the maximal Z or X decoherence, any stabilizer that fails to commute with error operators is removed from the density matrix. Because of $e \leftrightarrow m$ duality in $\mathbb{Z}_n$ quantum double, the studies of ρ^e and ρ^m are equivalent. Furthermore, since any finite Abelian group decomposes into a direct product of cyclic groups [95], it is sufficient to study this specific example to understand the generic behavior of the Abelian quantum double. As we see in Sec. IV, ρ^e and ρ^m belong to the same phase as well.

Following the arguments similar to Example 3, it is straightforward to show that $\rho^{e/m}$ satisfies mixed-state axioms. With this, the next step is to understand the information convex set $\Sigma(\mathbb{T}^2|\rho^e)$.

Proposition 2. The information convex set $\Sigma(\mathbb{T}^2|\sigma)$ for the canonically decohered $\mathbb{Z}_n$ quantum double ρ^e on the torus is a simplex with n^2 mutually orthogonal extreme points with the same entanglement entropy.

Proof. Consider a closed manifold $M = \mathbb{T}^2$. First, we consider a fixed-point state σ reduced to a ball $\bullet \subset M$:

$$\begin{aligned} X_e \sigma_\bullet &= \sigma_\bullet X_e, \\ A_v^1 \sigma_\bullet &= \sigma_\bullet A_v^1 = \sigma_\bullet, \\ B_p^f \sigma_\bullet &= \sigma_\bullet B_p^f. \end{aligned} \quad (24)$$

For any $\rho \in \Sigma(M|\sigma)$, its reduced density matrix is indistinguishable from the fixed-point state on any ball $\bullet \subset M$, i.e., $\rho_\bullet = \sigma_\bullet$. Thus, we can replace $\sigma_\bullet$ with $\rho_\bullet$ in Eq. (24). Does the same hold for any *unreduced* global state ρ ? Without loss of generality, we partition $\bullet$ into BC (as in **P0**) such that the operator in question (either A_v^1 , X_e or B_p^f) is supported on C , and $A = M \setminus BC$. For this partition, $I(A:C|B)_\rho = 0$ by condition **P0** on ρ . This implies the existence of a quantum channel $\Phi_{B \rightarrow AB}$ such that

$$\rho = \Phi_{B \rightarrow AB}(\rho_\bullet) = \Phi_{B \rightarrow AB}(\sigma_\bullet). \quad (25)$$

Applying the channel $\Phi_{B \rightarrow AB}$ to both sides of each equation in Eq. (24), we show that the global density matrix ρ commutes with B_p^f and X_e while absorbing A_v^1 .

The rest of the proof is done by expanding ρ in the generalized Pauli basis. Write

$$\rho = \sum_{P \in \mathcal{P}_n^{\otimes N}} c_P P, \quad (26)$$

where $\mathcal{P}_n^{\otimes N}$ is a set of n -generalized Pauli operators on N sites. The condition $[X_e, \rho_M] = 0$ and $[B_p^f, \rho_M] = 0$ for any i

and p implies that ρ commutes with an operator algebra generated by $\{X_e, B_p^f\}$. Therefore, $c_P = 0$ for Pauli strings involving Y and Z , and even for Pauli strings in terms of I and X , they should commute with B_p^f 's—this implies that they must be *closed* X strings on the dual lattice. The condition $A_v^1 \rho = \rho$ implies that ρ can be written as a product of A_v^1 and additional terms. Since any contractible closed X strings can be absorbed into $\prod_v A_v^1$, only remaining degrees of freedom are noncontractible X strings. For $\mathcal{M} = \mathbb{T}^2$ with two noncontractible strings, we get

$$\rho = \left[\sum_{a,b=0}^{n-1} c_{ab} \bar{X}_1^a \bar{X}_2^b \right] \cdot \prod_v A_v^1, \quad (27)$$

where $\bar{X}_{1,2}$ are generators of logical X operators along the x and y direction. Since ρ is diagonal in the joint eigenspaces of $\bar{X}_1$ and $\bar{X}_2$, it can be written as a convex combination of the corresponding joint-eigenspace states. That is, $\rho = \sum_{a,b} p_{ab} \rho_{ab}^e$, where $p_{ab} \geq 0$ and $\sum_{a,b} p_{ab} = 1$. The following n^2 density matrices are mutually orthogonal extreme points:

$$\rho_{ab}^e = \prod_v A_v^1 \times \left[\frac{1}{n} \prod_{j=0}^{n-1} (\omega^{-a} \bar{X}_1)^j \right] \left[\frac{1}{n} \prod_{k=0}^{n-1} (\omega^{-b} \bar{X}_2)^k \right], \quad (28)$$

where factors next to $\rho^e = \prod_v A_v^1$ project ρ^e onto an eigenspace of $\bar{X}_{1,2}$ with eigenvalues ω^a, ω^b . Therefore, $\Sigma(\mathbb{T}^2|\sigma)$ is forming a simplex with n^2 orthogonal extreme points. ■

Corollary 2.1 (memory capacity). The memory capacity $\mathcal{Q}(\mathbb{T}^2)$ of the state ρ^e (ρ^m) is

$$\mathcal{Q}(M) = 2 \log n. \quad (29)$$

This follows immediately from the counting of the extreme points, the orthogonality, and the fact that all the extreme points have identical entropies. It is also easy to compute that $\gamma_{\text{LW}} = \frac{1}{2} \log n$. Thus, the class of Abelian models saturates the bound (10).

Lemma 1. The extreme points of $\Sigma(\mathbb{T}^2|\rho^{e,m})$ can be prepared by a finite-depth local channel from the product state.

Proof. Without loss of generality, we consider ρ^e only. The extreme points of $\Sigma(\mathbb{T}^2|\rho^e)$ are labeled by two integers $a, b \in \mathbb{Z}_n$. Define the product state $\lambda_{a,b}$ as

$$\lambda_{ab} := L_{a,b}^Z (|+\rangle\langle+|)^{\otimes N} (L_{a,b}^Z)^\dagger, \quad (30)$$

where $L_{a,b}^Z = \prod_{e \in C_1} Z_e^a \prod_{e \in C_2} Z_e^b$ is a logical Z string along noncontractible loops C_1 (horizontal) and C_2 (vertical) of the torus as shown in Fig. 7. By performing a positive-operator-valued measure (POVM) using Kraus

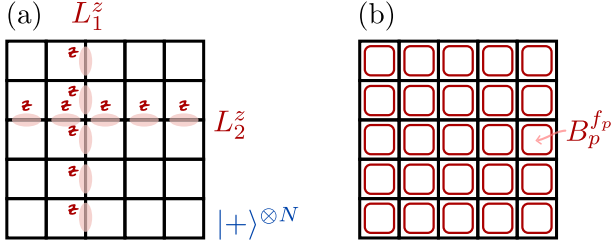


FIG. 7. Preparation of dephased toric code. Consider a square lattice on a torus. (a) First, we prepare a product state and apply logical-Z operators $L_{a,b}^Z$. (b) Second, we perform Z-stabilizer measurements by POVM $\{B_p^{f_p}\}$ and throw away measurement outcomes. The final state has an eigenvalue (a, b) under X -logical operator $(\bar{X}_1, \bar{X}_2)$.

operators $\{B_p^f\}$ (which commute with L^Z), we obtain

$$\rho_{ab}^e := \sum_{\{f_p\}} \prod_p B_p^{f_p} \lambda_{ab} B_p^{f_p}, \quad (31)$$

which is the extreme point labeled by (a, b) . To see this, note that the Pauli expansion of the density matrix λ_{ab} contains all possible dual X -string operators $X_{l^\perp}$ with coefficients given by $L_{a,b}^Z X_{l^\perp} (L_{a,b}^Z)^\dagger X_{l^\perp}^\dagger$. Next, under the POVM by B_p^f , the only surviving terms are dual *closed* strings. Defining $\omega = e^{2\pi i/n}$, the final density matrix can be rewritten in the form in Eq. (28). ■

Let us comment on the separable nature of the states. While we formulated as applying a channel with Kraus operators being projective measurement, this is equivalent to applying randomly unitary operators $B_p^m = \prod_{e \in p} Z_e^{m_{e,p}}$ obtained by inverse Fourier transformation [cf. Eq. (22)]. Thus, the state in Eq. (31) can be rewritten as

$$\rho_{ab}^e = \sum_{m_p} \prod_p B_p^{m_p} \lambda_{ab} B_p^{-m_p}, \quad (32)$$

which makes it manifest that ρ_{ab}^e is a classical separable state. The fact that such a state has $\gamma_{\text{LW}} = \frac{1}{2} \log n$ means the source of topological entropy is *not* from the entanglement. This is why we call γ_{LW} the topological entropy rather than topological entanglement entropy in the mixed states.

IV. MIXED-STATE PHASES OF MATTER

Thus far, we have proposed an axiomatic starting point for understanding topological mixed states. By focusing on axioms that isolate topological features, we characterized nontrivial behaviors of fixed points without being distracted by nonuniversal details. However, a fundamental question remains: How does this framework contribute to the broader understanding of mixed-state phases of matter? To define the equivalence of two topological mixed states satisfying the bootstrap axioms, one natural approach is to

propose that two mixed states belong to the same phase if they are connected by FDLCs in both directions [31,32]; also see Ref. [26]. However, our axiomatic analysis reveals a tension: For example, the trivial product state and the maximally dephased toric code represent distinct fixed points, with different classical topological memory capacities $\mathcal{Q}$. Yet, under the FDLC criterion, they appear to belong to the same phase. This discrepancy highlights the need for a more refined phase definition in the mixed-state setting.

In this section, we show that our approach provides a systematic framework to define mixed-state topological phases. First, we define the equivalence relation between two fixed-point states satisfying the axioms (Definition 6). Through this definition, we propose a new way to define an equivalence relation between generic topological mixed states. Second, we discuss how to relax axioms to explore *phases* away from the fixed points and show numerical results to justify our relaxation conditions. Finally, we investigate novel physics that arise at the domain wall between distinct topological mixed states.

A. Mixed-state phase equivalence

A reference state satisfying our axioms is a fixed point of topological mixed state (Definition 1). Two reference states σ^α and σ^β that can be sewn together along a boundary satisfying those axioms necessarily belong to the same phase. This is the consequence of the isomorphism theorem (see Appendix B), which implies that, whenever a topological invariant [e.g., γ_{LW} in Eq. (59)] is evaluated on any sufficiently large local patch, its value should be the same everywhere. This indicates that the topological correlations can be smoothly extended from a local patch to further neighborhood. Therefore, one can use the absence of an interpolating boundary satisfying axioms as a probe to distinguish two topological mixed states. This result is the conceptual foundation for the bootstrap approach, even for the pure-state setup [10].

To extend this approach to explore the mixed-state *phase diagram*, we may relax the axioms to hold only under *coarse-graining*. In practice, this means that a given axiom may be violated on short length scales, provided that such violations vanish under successive rescaling of the region and partition size. These considerations motivate the following formal definition.

Definition 6 (mixed-state phase equivalence). Two representative mixed states σ^α and σ^β are equivalent (i.e., in the same phase) if there exists a state containing a bubble of β in the background of α , such that **P0**, **M0**, and **M1** are preserved everywhere including the bubble domain wall, up to an error that decays under coarse-graining. See Fig. 8 for an illustration.

In other words, the domain wall becomes *transparent* under coarse-graining. The transparent domain wall can be formally defined as follows.

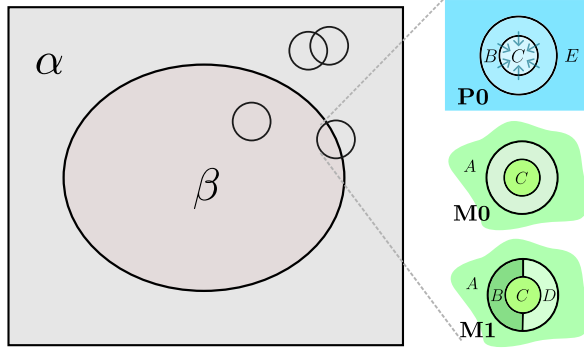


FIG. 8. Mixed-state phase equivalence. A bubble of phase β in the background of α . We say the two phases are equivalent, i.e., $\alpha \sim \beta$ if the axioms are preserved everywhere, including the domain wall. The associated domain wall is called a transparent domain wall.

Definition 7 (transparent domain wall). If a domain wall of a many-body system cannot be detected by checking any **P0**, **M0**, and **M1** centered in its neighborhood (under coarse-graining), we call it a transparent domain wall. See Fig. 8 for an illustration.

1. Application of phase equivalence

The definition of mixed phase equivalence (Definition 6) is a crucial ingredient of our theoretical framework. This is because a mixed-state phase should be classified as fixed point modulo equivalence. We first provide a few basic applications.

Example 6 (trivial mixed-state phase)—We say a fixed-point wave function is in the trivial phase if it is equivalent to the product state $\otimes_j |0\rangle_j$ in the sense of Definition 6. It is easy to see that the tensor product of arbitrary state $\otimes_j \rho_j$ can be joined with $\otimes_j |0\rangle_j$ by a transparent domain wall. We denote this as $\otimes_j \rho_j \sim \otimes_j |0\rangle_j$. Thus, $\otimes_j \rho_j$ is in the trivial mixed-state phase.

Example 7—The equivalence Definition 6 implies that the phases described by ρ^e and ρ^m —two canonically

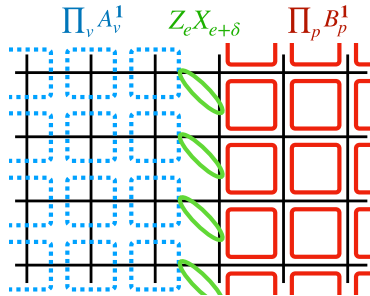


FIG. 9. Domain walls. Smooth sewing of ρ^e and ρ^m , which satisfy all axioms. Red and blue squares represent the Z and X stabilizers $\prod_{e \in p} Z_e$ and $\prod_{e \ni v} X_e$, respectively. Green diagonal circles correspond to the stabilizer $Z_r X_{r+\delta}$, where $\delta = (1/2, -1/2)$.

decohered toric code ($\mathbb{Z}_2$ quantum doubles)—are in the same phase. While a naive juxtaposition of the two results in a boundary that violates axiom **M1**, one can construct a domain wall that preserves all axioms, as illustrated in Fig. 9. We denote this result as

$$\rho^e \sim \rho^m. \quad (33)$$

Example 8 (gapped boundary of the toric code)—The toric code and the vacuum state represent different mixed-state phases, namely, $\rho^{\text{TC}} \not\sim \otimes_j |0\rangle_j$. For our definition to be consistent with this, the gapped domain wall of the toric code cannot be transparent. We explain why this is the case for the known gapped boundary of the toric code; see Fig. 10. Axioms **P0** and **M0** are satisfied on the gapped boundary. A version of **M1** is violated by $\log 2$ [67,68]. We find more kinds of nontransparent domain wall in Sec. IV C, especially the dry and wet boundary natural in mixed-state setups.

How can we tell if two fixed points σ^α and σ^β are in different phases? To show σ^α and σ^β are in different phases, it is *not* enough to identify a domain wall that violates **M1**. As studied in the pure-state case [76,96,97], the violations may be due to spurious contributions. It is necessary to show that it is impossible to construct another domain wall of the bubble, which preserves **M1**. The task may sound challenging. Luckily, there are simpler sufficient conditions.

First, two phases are distinct if two states have different topological entropy:

$$\gamma_{\text{LW}}(\alpha) \neq \gamma_{\text{LW}}(\beta). \quad (34)$$

Indeed, if two states can be connected by a transparent domain wall, then $\gamma_{\text{LW}}(\alpha) = \gamma_{\text{LW}}(\beta)$, because we can

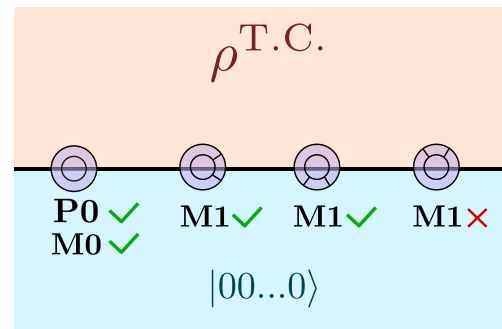


FIG. 10. Domain walls II. Verification of the preservation and violation of the axioms on a gapped domain wall between the toric-code ground state ρ^{TC} and the trivial product state. Axioms **P0** and **M0** are preserved on the wall. The violation happens to only axiom **M1** in the rightmost partition, whether we choose B to be the upper or lower disk contained in the purple annulus that thickens the local ball. The amount of the violation is $\log 2$. This makes the wall nontransparent.

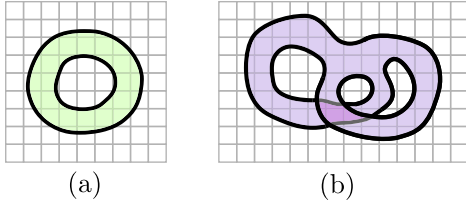


FIG. 11. Manifolds for information convex sets. (a) An annulus embedded in the physical system. (b) A punctured torus that is locally embedded but not globally embedded, known as the immersed punctured torus. This figure is adopted from Ref. [13].

smoothly deform the Levin-Wen partition through the domain wall. We demonstrate how this works in Sec. V, including the context that the axioms has errors decaying with the subsystem sizes. Since $\gamma_{\text{LW}}(\rho^{e,m}) = \log \sqrt{2}$ while $\gamma_{\text{LW}}(\otimes_i \rho_i) = 0$, it immediately follows that the product state and $\rho^{e,m}$ are not in the same phase according to our definition.

Second, our axiomatic approach implies that two mixed states belong to different phases if they possess different information convex sets. This approach is most useful when the regions are topologically interesting and embedded in the physical system, allowing entanglement boundaries such as the annulus; see Fig. 11(a). Further interesting and subtly important examples are regions that are only *locally* embedded, such as the immersed punctured torus in Fig. 11(b). This is because of an *isomorphism theorem* we explain in Appendix B, which guarantees that the information convex sets are isomorphic under any topological deformation as long as axioms are satisfied; one example is moving it across the transparent domain wall.

Example 9—Using the second approach, one *can* distinguish the ground state of chiral semion topological order (ρ^s) and ρ^e , namely,

$$\rho^s \not\sim \rho^e. \quad (35)$$

Note that they *cannot* be distinguished by their topological entropy, since $\gamma_{\text{LW}}(\rho^s) = \gamma_{\text{LW}}(\rho^e) = \log \sqrt{2}$. The difference can be seen from the information convex set of the immersed punctured torus, as in Fig. 11. Such an information convex set is a Bloch ball for the chiral semion but is a simplex with four mutually orthogonal extreme points for ρ^e . See Appendix G for details.

We discuss the robustness of the isomorphism of information convex sets under decaying errors in Appendix C. Although having identical information convex sets is a necessary condition for two fixed-point wave functions to belong to the same phase, it is not sufficient: For example, the toric code and doubled semion topological phases share the same information convex sets, yet they cannot be transformed into one another even by FDLCS [98]. In Appendix G, we outline an alternative approach

that extracts additional insight by exploiting nontrivial automorphisms of the information convex sets.

2. Space versus time

We have chosen our definition of phase equivalence in terms of transparent domain wall separating spatial patches (Definition 6). This is aligned with the entanglement bootstrap philosophy that the property of topological phases can be extracted from a ball-like region that is larger than the correlation (and Markov) length. Such an approach also naturally allows us to describe spatial boundaries of the system.

Alternatively, one may seek a time-based picture of the phase transition. For the traditional Hamiltonian phase transition of ground states, a common wisdom is that one must close the gap and the correlation length diverge. For mixed-state topological phases, the nice observation of Ref. [26] is that one needs to look into the Markov length associated with concentric partitions of a ball, as the correlation length fails to detect such a transition. By a Markov length, we mean a length ξ_{P0} that appears in the violation of axiom **P0** as [99]

$$I(A : C|B) \sim \text{poly}(A, C) \exp\left(-\frac{\text{dist}(A, C)}{\xi_{\text{P0}}}\right), \quad (36)$$

where $\text{dist}(A, C)$ is the minimum distance between A and C . This approach is a time-based approach in the sense we explain in Fig. 12; namely, one can interpret the divergence of the Markov length at some critical tuning parameter (error rate or Lindbladian evolution time) as the *temporal* domain wall between two distinct mixed-state phases across time.

However, a Markov length alone is not enough within our framework, as it is only one of the three pillars. What we propose precisely is that there are three correlations in question, which corresponds to the decaying of the three conditions **P0**, **M0**, and **M1**; we call these lengths ξ_{M1} , ξ_{M0} , and ξ_{P0} , respectively. Two of them are Markov lengths

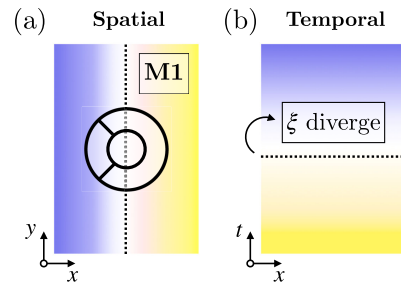


FIG. 12. Spatial versus temporal domain wall. (a) Spatial domain wall and the violation of **M1**. (b) Temporal domain wall and the divergence of Markov length or correlation length at a certain time slice.

with different partitions, and one of them is the correlation length.

Conjecture 1 (phase equivalence with time)—Consider a one-parameter family of states $\rho(t)$ for $t \in [0, 1]$. States $\rho(0)$ and $\rho(1)$ are in the same mixed-state phases if the three lengths $\xi_{\mathbf{M1}}$, $\xi_{\mathbf{M0}}$, $\xi_{\mathbf{P0}}$ are upper bounded by a finite constant for all t .

Importantly, we require *all three* length scales to be finite, and the statement does not involve the concept of channels, circuit depth, or Lindbladian. One may wonder whether we can drop the finiteness of one of the lengths, in the conjecture. Naive ways of relaxing these requirements lead to statements disproved by examples.

- (i) The traditional wisdom of phase transition suggests that one should require the finiteness of $\xi_{\mathbf{M0}}$. However, we cannot require only the finiteness of $\xi_{\mathbf{M0}}$; this observation, together with the observation that one should look at $\xi_{\mathbf{P0}}$, is the remarkable insight of Ref. [26].
- (ii) Suppose we require *only* the finiteness of $\xi_{\mathbf{P0}}$, but not $\xi_{\mathbf{M0}}$; we encounter the counterexample of classical 3D $\mathbb{Z}_2$ gauge theory transition at finite temperature. The transition is not detected by the Markov length $\xi_{\mathbf{P0}}$, as it is exactly zero for a thermal state of any stabilizer Hamiltonian [100]. Thus, the requirement of finite $\xi_{\mathbf{M0}}$ cannot be omitted.
- (iii) Suppose we require both $\xi_{\mathbf{P0}}$ and $\xi_{\mathbf{M0}}$ to be finite but put no requirement on $\xi_{\mathbf{M1}}$. Then, one can design a sequential circuit that grows a toric-code patch such that $\rho(0)$ is a trivial product state while $\rho(1)$ is a toric-code state. While both states have $\xi_{\mathbf{M1}} = 0$, in the intermediate time we get a state with a domain wall with diverging $\xi_{\mathbf{M1}}$. Such a one-parameter family of states all have finite $\xi_{\mathbf{P0}}$ and $\xi_{\mathbf{M0}}$. However, those in the intermediate time steps do not represent a uniform representative of mixed-state phases.

In summary, we made a conjecture of a time-based mixed-state phases equivalence. This statement is largely in the spirit of Ref. [26]. Nonetheless, we required three length scales to be finite, instead of just the Markov length for the concentric regions, and have given the physical reason for doing so. Especially, we highlight the need for the special uniformity: **M1** should be satisfied upon coarse-graining. Otherwise, the representative wave function can stop being a uniform representative of a phase, and we run into counterexamples. In fact, even for the ground state of topologically ordered systems, the common wisdom of “correlation length remains finite implies no phase transition” suffers from the same counterexample if one discards space uniformity.

B. Coarse-graining

For a generic topological mixed state, we do not expect the three axioms to hold exactly but only approximately. Given any local patch and associated partition considered

in the axioms, one can rescale all the regions with respect to a reference point by a multiplicative factor r .

In the case of critical states described by conformal field theory, it is known that the conditional mutual information for the partition in **P0** is invariant under such scaling [101]. In contrast, for gapped phases of matter, this condition is expected to decay as a function of r . Motivated by this behavior, we propose the following operational definition.

Definition 8 (coarse-graining). If the violations of the three axioms decay toward zero with the multiplicative factor r (by which we rescale all the regions) throughout the state, then we say the state *coarse-grains* into a particular topological mixed-state fixed point.

In what follows, we perform numerical simulations to study coarse-graining behavior of the axioms away from the fixed-point states.

1. Homogeneous system

Consider decoherence with a uniform error rate p that maps ρ^e [Eq. (18)] to product state at $p = 0.5$, where the error channel is

$$\mathcal{E} := \prod_i \mathcal{E}_i, \quad \mathcal{E}_i: \rho \mapsto (1-p)\rho + pZ_i\rho Z_i. \quad (37)$$

When p is 0, we have ρ^e as the initial state, and as we increase the error rate to 0.5, we get a maximally mixed state. For each $p \in [0, 0.5]$, we get a state that can be treated as a homogeneous system.

We investigate the violations of the three axioms for different values of p . We focus on **P0** and **M1** since the violation of **M0** is always zero for any p because the errors are on site and independent. The partition is given in Fig. 13(a), where we scale all regions proportionally. As we see in Figs. 13(b) and 13(c), the violations of both axioms **P0** and **M1** are exactly 0 for $p = 0$ and $p = 0.5$ as expected; at p away from the critical value, both axioms have their violations coarse-grain toward 0. Up to finite-size corrections to the violations, the errors decay exponentially with the separation r for both $p < p_c$ and $p > p_c$ as in Fig. 13(d), establishing the stability of our approach.

Moreover, in Figs. 13(b) and 13(c), we observe a clear crossing point near $p_c \simeq 0.109$, at which the curves for different length scales r intersect. This provides a simple empirical method for determining p_c with high precision. More broadly, this crossing-point method carries a practical advantage and may be applicable in a wider range of settings.

This is in contrast to the approach of Ref. [26], which relies on the Markov length: There, the size of region C is held fixed while the separation between A and C is rescaled rather than uniformly rescaling all regions as in our construction. Consequently, identifying p_c in Ref. [26] requires an additional curve-by-curve rescaling, because the peak height decays with r , making the analysis more complicated.

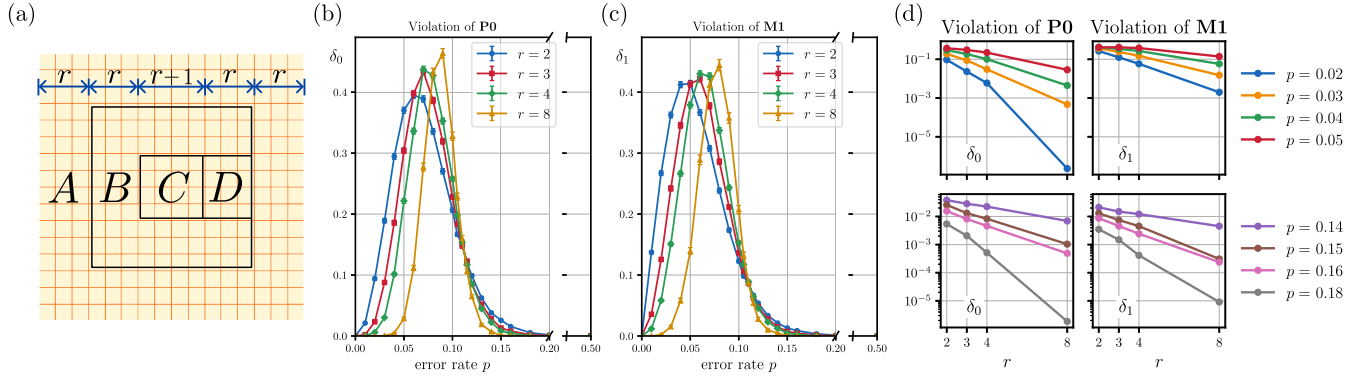


FIG. 13. Violation of $\mathbf{P0}$ and $\mathbf{M1}$ in the bulk. (a) We partition a bulk disk into A , B , C , and D as shown. (b) The (local) violation of $\mathbf{P0}$, $\delta_0 := S_{ABD} + S_{BCD} - S_{BD} - S_{ABCD}$, as a function of error rate p and subsystem sizes r . (c) The violation of $\mathbf{M1}$, $\delta_1 := S_{AB} + S_{BC} - S_B - S_{ABC}$, as a function of error rate p and subsystem sizes r . (d) At $p = [0.01, 0.04]$ and $p = [0.13, 0.16]$ the error of $\mathbf{M1}$ and $\mathbf{P0}$ decays exponentially with r . All data are shown in units of $\log 2$.

Our way of coarse-graining (Definition 8) by rescaling *all* regions by the same multiplicative factor raises another natural question. Is it true that the height of the peak in the violation of the axioms approaches a constant at large length scales at p_c ? This appears to be a viable conjecture, even though Ref. [26] sees a power law decaying of the violation of $\mathbf{P0}$ due to the fixed size of C . Indeed, if the critical point is described by a scale-invariant (or even conformal) effective theory, it is anticipated that the height of δ_0 and δ_1 at p_c should be a constant that does not depend on the length r , consistent with Figs. 13(b) and 13(c). It seems natural to look for connections to a disordered conformal field theory [102–104], which opens avenues for further investigations.

2. Spatial boundary: Inhomogeneous decoherence

Now suppose that we decohere a fixed-point state on a half-plane using local channels. This will create a domain wall (i.e., a spatial boundary) between two homogeneous states. We are interested in the properties of the domain

wall, e.g., whether it is transparent (Definition 7). We consider the following two examples. The first is identical to the previous example in the bulk.

From ρ^e to product state—Now consider the case that only lower half-plane of ρ^e is decohered under on-site Z errors with a tunable error rate p as in Eq. (37); see Fig. 14(a). At $p = 1/2$, the lower half-plane becomes a maximally mixed product state, which has a topological entropy γ_{LW} different from the upper half-plane. Again, we present only the violation of $\mathbf{P0}$ and $\mathbf{M1}$, since the violation of $\mathbf{M0}$ is always zero for every p . One can analytically find the violation of $\mathbf{M1}$ for the partition $I(A:C|B)$ in Fig. 14(a) to be $\log 2$, while the violation of $\mathbf{P0}$ is 0 at $p = 0.5$; see Appendix F.

For $p < p_c$, we expect the violation of both $\mathbf{P0}$ and $\mathbf{M1}$ to coarse-grain to zero. On the other hand, for $p > p_c$, the violation of $\mathbf{M1}$ shall coarse-grain to $\log 2$ while the violation of $\mathbf{P0}$ to zero. This is what we observe in Fig. 14(d) with both violations decay exponentially to the fixed-point values.

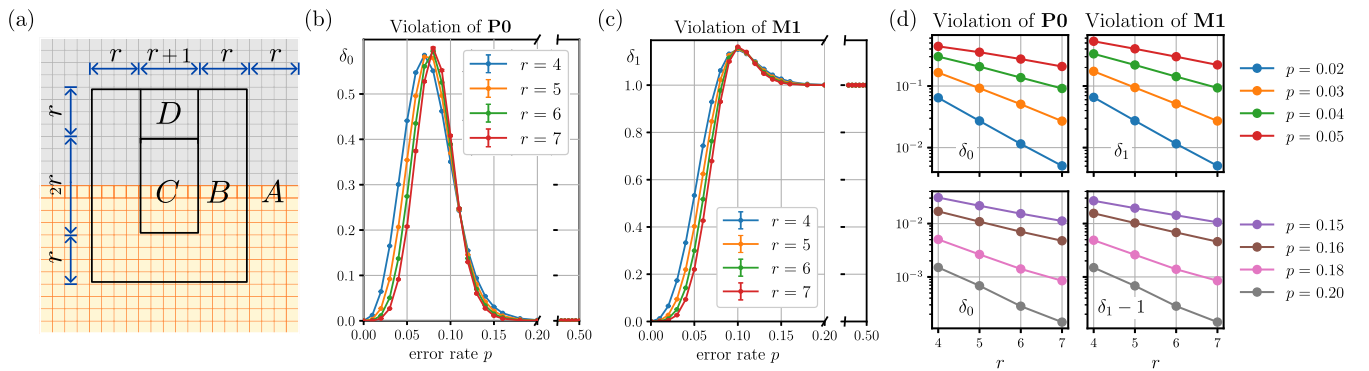


FIG. 14. Violation of $\mathbf{P0}$ and $\mathbf{M1}$ at the boundary. (a) The partitions near a domain wall. The decohered links are colored orange, located in the lower half-plane. r is the length of the subsystems. A is an annulus; BD is the annulus that surrounds the disk C . (b) The (local) violation of $\mathbf{P0}$. (c) The violation of $\mathbf{M1}$. (d) At $p = [0.02, 0.05]$ and $p = [0.15, 0.20]$, the error of $\mathbf{M1}$ and $\mathbf{P0}$ decays exponentially with r . All data shown are shown in units of $\log 2$.

At the critical point p_c , violations of both **P0** and **M1** in Figs. 14(b) and 14(c) exhibit scale invariance with respect to the region size r similar to the homogeneous case. Again, we emphasize that, in computing the violation of **P0** in both bulk and domain-wall boundary, we scale *all* regions A , BD , and C (and A , B , and C for **M1**).

As in the bulk data, we can locate the critical point directly from the crossings of curves at different r . We observe that the crossing occurs exactly at the known critical point p_c approximately equal to 0.109 [20,21,24,25]. We detail the numerical calculation in Appendix F 2.

From product state to ρ^e —We can also consider the opposite channel, which prepares ρ^e starting from the product state. This is the tunable version of the preparation protocol described in Fig. 7, where we perform the decoherence using B_p^f with error rate p . Applying this channel on the product state $|+\rangle^{\otimes n}$ gives the following density matrix:

$$\rho(p) = \frac{1}{2^n} \sum_{l_\perp \in \mathcal{S}_d} (1-2p)^{|l_\perp|} X^{l_\perp}, \quad X^{l_\perp} := \prod_{e \in l_\perp} X_e, \quad (38)$$

where $\mathcal{S}_d$ is the set of all possible strings on the dual lattice and $|l_\perp|$ is the number of boundary points of $l_\perp$.

To proceed, we start with $|+\rangle$ state every site and apply this protocol with error rate p on the upper half-plane of a given system. For the fixed-point wave function at $p = 1/2$, we have $I(A:C|B) = 0$ but $I(A:C|D) = \log 2$, near the domain wall in Fig. 14(a), which indicates the violation of **M1**. However, unlike the previous case, this FDLC is *fragile*, which means that the violation of axioms coarse-grains into zero at $p < 1/2$ and two half-planes belong to the same trivial phase; similar fragile behaviors in the bulk are observed. We show the numerical and theoretical details in Appendix F 3.

We can infer that the state in Eq. (38) with $p < 1/2$ exhibits vanishing topological entropy for sufficiently large partitions. At an intuitive level, the contribution of independent closed-loop operators to the density matrix becomes negligible under coarse-graining. This is because a string operator $X_{l_\perp} = \prod_{e \in l_\perp} X_e$ maintains a finite expectation value $(1-2p)^2$ regardless of loop length $|l_\perp|$ —behavior that cannot persist in a genuinely topological mixed state. From the statistical mechanics perspective, the physics maps to that of the two-dimensional $\mathbb{Z}_2$ gauge theory. In particular, Rényi-2 entropy would map to the free energy of the 2D Ising gauge theory at the inverse temperature $\beta = -2\ln(1-2p)$. Since 2D Ising gauge theory is disordered at any finite β , the (Rényi-2) topological entropy vanishes for $p < 1/2$. See Appendix F 3 for numerical data. Following numerous observations [20,21,24,25,27], we expect the von Neumann topological entropy would vanish, i.e., coarse-grain to zero for $p < 1/2$

since its behavior would follow random-plaquette $\mathbb{Z}_2$ gauge theory, which has no finite-temperature threshold.

C. Channel and boundary

We have proposed an equivalence of mixed-state fixed points based on their density matrices (Definition 6). How is this approach different from the existing approach based on the FDLC [31,32] or Markov length [26]? In this section, we discuss the connection and distinction between these approaches and ours. In particular, we explain why our definition naturally leads to a refinement of the FDLC approach.

The previous FDLC-based approach to which we compare our framework can be phrased as follows [105].

Definition 9 (channel connectivity). Two mixed-state fixed points σ^α and σ^β are channel connected if there exists a pair of FDLCs ($\Phi_{\alpha \rightarrow \beta}$ and $\Phi_{\beta \rightarrow \alpha}$), which can convert between σ^α and σ^β . Namely,

$$\Phi_{\alpha \rightarrow \beta}(\sigma^\alpha) = \sigma^\beta, \quad \Phi_{\beta \rightarrow \alpha}(\sigma^\beta) = \sigma^\alpha. \quad (39)$$

Suppose α and β are two fixed points that are channel connected as in Definition 9. The finite-depth quantum channels $\Phi_{\alpha \rightarrow \beta}$ and $\Phi_{\beta \rightarrow \alpha}$ can be truncated onto any (sufficiently large) local region. Such channels can create bubbles of $\alpha(\beta)$ over the background of $\beta(\alpha)$. Our idea of refining Definition 9 is that it is possible to find out more by looking at the correlation near the domain wall of the bubble. Based on this, we propose the *topological channel connectivity* below.

Definition 10 (topological channel connectivity). Two representative mixed states σ^α and σ^β are *topologically channel connected* if there exists a FDLC that can create a disklike bubble of β in the background of α (and vice versa), such that for every layer of the spatially truncated FDLC, **P0**, **M0**, and **M1** are preserved everywhere including along the domain wall of the bubble, up to decaying error.

It is clear that Definition 10 is a refinement of channel connectivity by requiring that the wall of the bubble can be made transparent (Definition 7). Let us make a few remarks.

- (1) Both channel connectivity (Definition 9) and topological channel connectivity (Definition 10) are equivalence relations. The latter is strictly finer, as we see in examples below.
- (2) The idea of having extra properties near the spatial boundary of the FDLC may look unfamiliar (or even strange). Nonetheless, we emphasize that similar considerations already occurred in the study of symmetry-protected topological phases (SPT) [2–4]. Suppose we want to deform an SPT state to another by finite-depth local unitaries; then, only deformations that respect the symmetry *locally* are permitted. For instance, in the 1D cluster state,

although the global application of controlled-Z gates (a depth-2 circuit) respects the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry as a whole, a local collection of these gates on local regions breaks the symmetry and is, therefore, disallowed. This makes the 1D cluster state a non-trivial SPT, which enjoys anomalous behavior at its spatial boundary. Analogously, when dealing with topological mixed states *protected* by long-range topological correlations or entanglements, we must restrict to deformations that preserve the underlying topological structure.

The topological channel connectivity (Definition 10) is seemingly stronger than our definition of equivalent fixed point (Definition 6). However, we do not know any pair of fixed-point density matrices that are identical by Definition 6, which do not have topological channel connectivity between them. This leads to the following conjecture.

Conjecture 2 (phase equivalence with topological channel)—Two mixed-state fixed points represent the same phase if and only if these two states are topologically channel connected.

To make the contrast between the channel connectivity and topological channel connectivity clearer, we consider a fixed point α and a trivial product state $\beta = \mathbb{1}$. Two types of boundaries are particularly natural to consider.

Definition 11 (dry and wet boundaries). Consider a fixed point α and a trivial product state $\beta = \mathbb{1}$. We call the domain wall of a bubble of $\mathbb{1}$ created on top of α as the dry boundary (created by truncated channel $\Phi_{\alpha \rightarrow \mathbb{1}}$). Suppose there exist a local channel $\Phi_{\mathbb{1} \rightarrow \alpha}$ that converts $\mathbb{1}$ to α ; we call domain wall of bubble of α created on top of $\mathbb{1}$ as a wet boundary (created by truncated channel $\Phi_{\mathbb{1} \rightarrow \alpha}$).

Note that any mixed-state fixed point α allows a dry boundary with the trivial product state $\mathbb{1}$. This is because we can always find the local reset or maximal depolarization channel that converts α to the trivial state $\mathbb{1}$. On the other hand, the existence of the wet boundary is possible only through channel connectivity (i.e., when $\Phi_{\mathbb{1} \rightarrow \alpha}$ exists and can be truncated to create a local bubble), which may or may not satisfy the stronger topological channel connectivity.

To be more explicit, let $\alpha = \rho^e = \prod_v A_v^1$ be the dephased toric-code fixed point (18) and $\mathbb{1} = |+\rangle^{\otimes N}$ be the product state. As we have discussed in the previous subsection, α and $\mathbb{1}$ are channel connected. The two channels are the following.

- (i) $\Phi_{\alpha \rightarrow \mathbb{1}}$ is the product of an on-site channel that resets each site to $|+\rangle$. The reset channel in this process is constructed by swapping a system qubit with an ancilla qubit $|+\rangle$ and tracing out the ancilla.
- (ii) $\Phi_{\mathbb{1} \rightarrow \alpha}$ is the measurement of $\prod_{e \in p} Z_e$ and throw away the outcome on each plaquette.

By restricting these channels to part of the plane, we create bubbles of one phase on the background of another phase. We illustrate the “dry boundary” and the “wet boundary” of

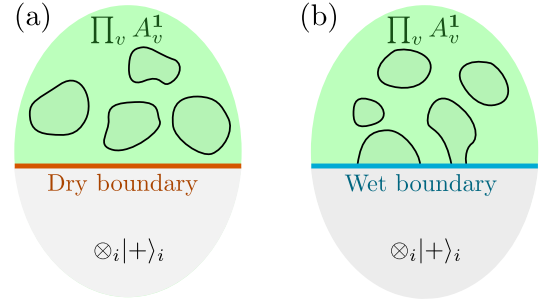


FIG. 15. Dry and wet boundaries (Definition 11): two types of boundary between $\alpha = \prod_v A_v^1$ and $\mathbb{1} = \prod_i |+\rangle^{\otimes N}$. (a) The “dry” boundary obtained by applying $\Phi_{\alpha \rightarrow \mathbb{1}}$ on the lower half-plane of α . (b) The “wet” boundary obtained by applying $\Phi_{\mathbb{1} \rightarrow \alpha}$ on the upper half-plane of $\mathbb{1}$. These two boundary types differ from each other on whether there exist Pauli-X string operators $\mathcal{O}_S$ terminating on the boundary, such that $\mathcal{O}_S \sigma = \sigma$. Such a distinction has physical implications as we explain in Fig. 17.

the dephased toric code in Fig. 15. Before doing any computation, we know that **M1** must be violated in the neighborhood of (any part of) the boundary. This is because $\gamma_{\text{LW}}(\alpha) = \log \sqrt{2}$, whereas $\gamma_{\text{LW}}(\beta) = 0$. If **M1** were satisfied on any portion of the boundary, these two values would be the same. [See the discussion around Eq. (34)].

Explicit computation tells us which of the axioms **P0**, **M0**, and **M1** are preserved or violated on these boundaries; see Appendix F. We state the result and comment on the physical interpretation.

Preserved conditions—Both the dry and wet boundaries have a set of **P0**, **M0**, and **M1**-like conditions preserved:

- (1) **P0** and **M0** are satisfied on the boundary, namely, $I(A:C|B) = 0$ and $I(A:C) = 0$ for Fig. 16(a).
- (2) **M1** is preserved on the boundary for some orientations. $I(A:C|B) = I(A:C|D) = 0$ for Fig. 16(b).

These conditions guarantee that the boundary is uniform in a sense that generalizes Ref. [11]. There is no obstruction to transport information along the boundary. If, instead, the boundary consists of segments joined at pointlike defects, we should expect the violation of some of these conditions.

Violated conditions—The dry boundary and the wet boundary both violate a certain type of **M1**. The relevant partitions are shown in Fig. 17(a). In fact, the violations give a pair of nontrivial topological entropies, and the two boundaries can be distinguished. We first define the pair of boundary topological entropies

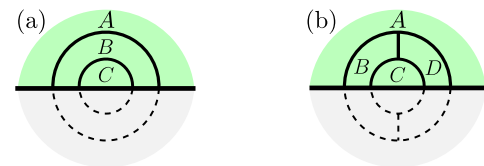


FIG. 16. Axioms are preserved for these classes of configurations (a) for **P0** and **M0** and (b) for certain classes of **M1**.

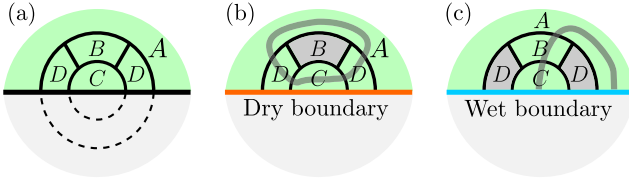


FIG. 17. (a) Configurations to check the violation of the **M1**. The resulting values are topological entropies γ_{dry} and γ_{wet} . These regions can be restricted to the upper half-plane, as the state is a product in the lower half-plane. (b) The intuition why the dry boundary has $I(A:C|D) = \log 2$ and (c) the wet boundary has nontrivial $I(A:C|B) = \log 2$. The gray closed (open) string is the support of a nontrivial string operator $\mathcal{O}_S$ for the dry (wet) boundary, such that its action on the state with the given boundary condition is $\mathcal{O}_S \sigma = \sigma$.

$$\gamma_{\text{dry}} := \frac{1}{2} I(A:C|D), \quad \text{for Fig. 17(a),} \quad (40)$$

$$\gamma_{\text{wet}} := \frac{1}{2} I(A:C|B), \quad \text{for Fig. 17(a).} \quad (41)$$

The computation result summarized in Table I justifies the name. Namely, γ_{dry} detects a dry boundary and γ_{wet} detects a wet boundary. Note that γ_{dry} and γ_{wet} are the only forms of violation of **M1** near the boundaries, as β is trivial.

Let us comment on the intuition why the dry boundary gives $I(A:C|D) = \log 2$. There exists a $\mathbb{Z}_2$ -valued nontrivial operator $\mathcal{O}_S = \prod_{i \in S} X_i$ supported on a closed string $S \subset ACD$ that stabilizes the state, i.e., $\mathcal{O}_S \sigma = \sigma$. However, its eigenvalue cannot be determined by either marginal state σ_{CD} or σ_{AD} . Therefore, conditioning on D leaves a single $\mathbb{Z}_2$ variable shared between A and C , contributing exactly one bit to the conditional mutual information; see Fig. 17(b).

Similarly, for the wet boundary, there exists a nontrivial operator $\mathcal{O}_S$ supported on an open string $S \subset ABC$ that stabilizes the state. However, its eigenvalue cannot be determined by the marginal states σ_{AB} and σ_{BC} , as shown in Fig. 17(c). Again, conditioning on B leaves a relevant degrees of freedom shared between A and C . See Refs. [106–108] for closely related ideas.

It is amusing to contrast the pure-state context. For a gapped boundary between a pure-state topological order and the vacuum, we always have $\gamma_{\text{dry}} = \gamma_{\text{wet}} = \frac{1}{2} \gamma_{\text{LW}}$. Thus, we cannot distinguish any pairs of pure-state gapped

TABLE I. Here, γ_{LW} is that for the nontrivial phase α being the $\prod_v A_v^1$ fixed point of the dephased toric code. Here, $\gamma_{\text{LW}} = \log \sqrt{2}$ is the Levin-Wen topological entropy of α .

Boundary type	γ_{dry}	γ_{wet}
Dry boundary	γ_{LW}	0
Wet boundary	0	γ_{LW}

boundaries with such topological entropies. In pure states, more interesting topological entropies exist only for domain walls between a pair of nontrivial phases [68,109]. For mixed-state phases, we have richer physics.

The distinction between the wet and dry boundaries in Table I encodes a genuine memory effect: Each boundary retains information about how it was created, reminiscent of hysteresis in a magnetic system. At a deeper level, one may ask how difficult it is to convert one boundary type into the other using FDLs acting only near the interface. This question closely parallels the stability analysis of the FDL that prepares the bulk phase, discussed in Sec. IV B 2, where it is discussed that a slightly imperfect channel ($p \neq 1/2$) fails to prepare the intended phase. More generally, if the preparation channel is noisy, does the resulting state remain in the same phase, or does it instead realize a different phase outside the bubble? In both cases, one encounters an apparent asymmetry. The relationship between these two cases is subtle and will be investigated in future work.

V. APPROXIMATE AXIOMS AND STABILITY OF TOPOLOGICAL INVARIANTS

Our axioms provide a rigorous foundation for discussing fixed points of nonequilibrium phases of matter under decoherence. To compare states beyond these fixed points, we permit controlled violations of the axioms that decay exponentially with the size of the partition and define phase equivalence accordingly (Definition 6); see Fig. 1(b). However, for this definition of phase equivalence to be meaningful, the relaxation of axioms should preserve the topological character of invariants—including the information convex set, topological entropy, and related quantities elaborated in Appendix B.

In this section, we establish the *stability* of these topological invariants under such relaxed conditions: If the axioms hold up to exponentially decaying corrections, the invariants are unchanged up to likewise corrections exponentially small in partition size. Concretely, we assume the violations of **P0**, **M0**, and **M1** obey the common bound

$$\delta(r) \leq \mu e^{-r/\xi} \quad (42)$$

for length scales $r \gg \xi$, where $\xi := \max\{\xi_{\text{corr}}, \xi_{\text{Markov}}\}$ is the larger of the correlation length and the Markov lengths and μ is an $\mathcal{O}(1)$ constant. This scaling is observed in our data for realistic decoherence (Fig. 14; see also Appendix F).

We consider the regime that the length scales $r \gg \xi$, and describe the system by a coarse-grained honeycomb lattice; see Fig. 18(c). The Levin-Wen partition is applied to a single coarse-grained unit cell. We then ask how the associated topological entropy varies with a length scale r . Concretely, we compare two adjacent placements of the

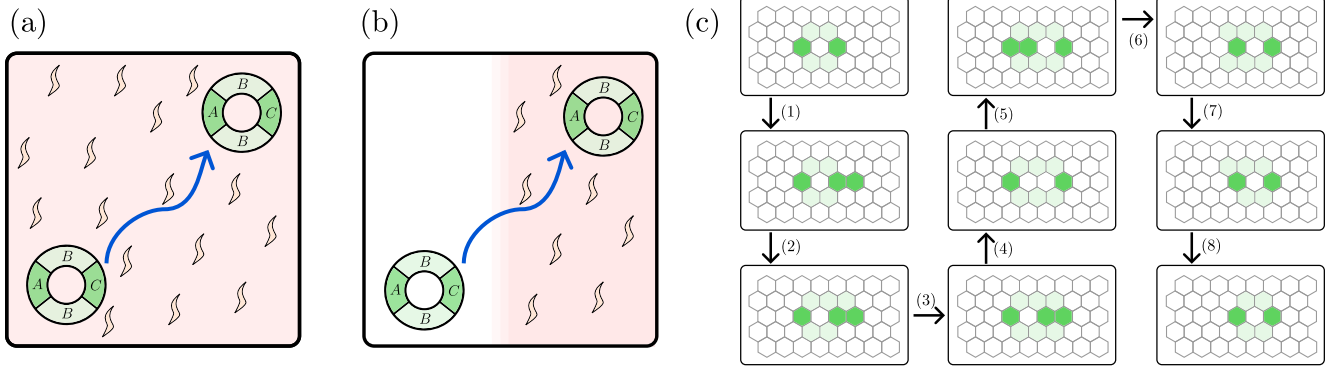


FIG. 18. Stability of topological invariant. (a) Decohered state. When the three axioms are approximately satisfied due to being away from the fixed point, still topological entropy remains invariant up to an exponentially small correction, no matter where we evaluate. (b) Connection to fixed points. A similar logic can be applied if we have a smooth interpolation of a fixed-point density matrix satisfying three axioms and a decohered density matrix with approximate axioms. Two states must show the same topological entropy up to small corrections that are exponentially suppressed under coarse-graining. (c) Sequence of moves. On a honeycomb coarse-grained lattice at length scale r , we consider two Levin-Wen partitions, relatively shifted by one lattice, showing in the first and the last figure. Their associated values of topological entropies are $\gamma_{\text{LW}}^{(\text{left})}$ and $\gamma_{\text{LW}}^{(\text{right})}$. The eight steps represent an explicit sequence of deformation of the partition.

partition that differ by a one-cell translation on the coarse-grained lattice (denoted “left” and “right”) and seek an r -dependent bound $|\gamma_{\text{LW}}^{(\text{left})} - \gamma_{\text{LW}}^{(\text{right})}| \leq f(r)$, where $f(r) \rightarrow 0$ as $r \rightarrow \infty$.

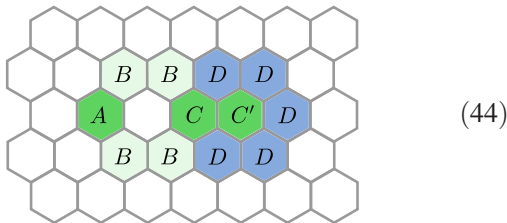
Proposition 3. The difference between the Levin-Wen TE on the left $\gamma_{\text{LW}}^{(\text{left})}$ and the partition on the right $\gamma_{\text{LW}}^{(\text{right})}$ is bounded as

$$|\gamma_{\text{LW}}^{(\text{left})} - \gamma_{\text{LW}}^{(\text{right})}| \leq 8\mu \times e^{-r/\xi}. \quad (43)$$

In particular, the error becomes exponentially smaller at larger length scales even though the two partitions are separated by a larger distance.

Proof. We need to estimate the entropy difference of each pair of nearby configurations; there are eight such pairs, as in Fig. 18(c).

Now let us show that the deformation in the first step gives a change at most $O(1)e^{-r/\xi}$. The difference of Levin-Wen topological entropy for the first step is the difference for the partition A, B, C and A, B, CC' in



which is computed as

$$\begin{aligned} |\delta\gamma_{\text{LW}}^{\text{step-1}}| &:= I(A:CC'|B) - I(A:C|B) \\ &= (S_{AB} + S_{BCC'} - S_B - S_{ABCC'}) \\ &\quad - (S_{AB} + S_{BC} - S_B - S_{ABC}) \\ &= S_{ABC} + S_{BCC'} - S_{BC} - S_{ABCC'} \\ &= I(A:C'|BC) \\ &\leq I(AB:C'|C) \\ &\leq \delta(r). \end{aligned} \quad (45)$$

The last two lines use the strong subadditivity, and the last line uses the error of **M1** is given by $\delta(r)$. The same analysis applies to steps 2–8, with $|\delta\gamma_{\text{LW}}^{\text{step-k}}| \leq \delta(r)$ for all $k = 2, \dots, 8$. While we omit the details, we list the equalities that are used:

$$I(A:CC'|B) - I(A:C|B) = I(A:C'|BC) \quad (46)$$

and

$$\begin{aligned} I(A:C|B) - I(A:C|BB') \\ = I(A:B'|B) - I(A:B'|BC). \end{aligned} \quad (47)$$

With these, we conclude the proof. $\blacksquare$

Corollary 3.1 (stability of topological entropy). Consider a fixed-point density matrix ρ_{fix} that exactly satisfies three axioms. Assume ρ belongs to the same phase in that one can construct a smooth spatial interpolation between ρ and ρ_{fix} where the violation of axioms is exponentially suppressed with scale r as in Eq. (42); see Fig. 18(b). Then, the topological entropies of ρ and ρ_{fix} , computed on two

Levin-Wen partitions separated by a constant number of deformation steps, are identical up to an exponentially small error at most $\mathcal{O}(1)e^{-r/\xi}$.

Therefore, our axioms indeed play the role of gap—as long as two states are connected while maintaining these axioms either exactly or approximately, topological entropies remain invariant up to exponentially small corrections. In fact, all other topological invariants described in the isomorphism theorem can be shown to exhibit only exponentially small corrections. We provide a physical sketch in Appendix C; the details will appear in a future dedicated work.

VI. SECRET SHARING: HIERARCHY OF TOPOLOGICAL MEMORY

In the broad family of topological mixed states, both classical and quantum logical information can be encoded in a topological manner [85,110]. This encoding implies that the logical information cannot be retrieved through local operations; rather, one must access a topologically nontrivial subregion of the system to recover it. A familiar example is the toric code [110], where logical operators are supported on nontrivial cycles of the torus. Put differently, if we partition the system as in Fig. 20(a) among four parties, recovery of the logical information requires collaboration between at least two parties.

Remarkably, even at the level of classical information, different topological mixed states exhibit distinct capacities for secret protection under this partition of the torus. For instance, in the canonically decohered Abelian quantum double defined in Definition 5, access to any two pairs (e.g., AB and AC , AD and BC , etc.) suffices to fully reconstruct the logical information stored along the two nontrivial cycles of the torus. However, as we see, for decohered non-Abelian topological mixed states, this level of access is insufficient. Even more intriguingly, a hierarchy of secret-sharing capacities emerges among different non-Abelian topological mixed states. We illustrate this phenomenon through a detailed analysis of the S_3 and A_4 quantum doubles.

A. Non-Abelian quantum double

Before going into details, we define the decohered states of non-Abelian quantum doubles [65,111]. For a non-Abelian group G , one can define the following projectors in a way analogous to the Abelian case in Eq. (22):

- (i) A_v^R on the charge R (irreducible rep.) at vertex v ,
 - (ii) B_p^C on the flux C (conjugacy class) at plaquette p .
- The notation of quantum double models is reviewed in Appendix H. For both the trivial irreducible representation and the conjugacy class of identity element, which correspond to trivial charge and flux, respectively, we use the label **1**. The quantum double ground state with maximally mixed logical space is then given as $\prod_v A_v^1 \prod_p B_p^1$.

In the Abelian case, we can start from the quantum double state and decohere it maximally in charge or flux, respectively, to obtain different topological mixed states (Definition 5). However, it turns out that the situation is different in non-Abelian cases.

Decohering into ρ^m —We apply the channel described by Kraus operators $\{T_e^g | g \in G\}$, where T_e^g is a projector onto $|g\rangle$ on edge e ; therefore, this channel corresponds to decohering non-Abelian charge. As $[B_p^1, T_e^g] = 0 \ \forall \ h, g \in G$, B_p^1 survives under this channel, while only the identity part survives in the expansion of $\prod_v A_v^1$. Thus, we eliminate $\prod_v A_v^1$ from the pure state and obtain $\rho^m = \prod_p B_p^1$.

As one might expect, this charge-decohered non-Abelian state ρ^m satisfies three axioms **P0**, **M0**, and **M1**. The argument to prove the axioms is similar to that of the Abelian case. The only subtlety is that the stabilizer measurements can give non-Abelian flux excitations when we prove **P0** and **M1**. However, partial trace always gives the constraint [cf. Eqs. (19) and (20)] that these fluxes must fuse into a trivial total flux as in the Abelian case. Therefore, the original stabilizers can always be recovered.

The topological mixed state ρ^m represents a classical mixture of loops subject to non-Abelian conservation laws, specifically the zero-flux condition:

$$\rho^m = \prod_p B_p^1 \propto \sum_c |c\rangle\langle c|, \quad (48)$$

where $|c\rangle$ is a product state in the group element basis satisfying $B_p^1|c\rangle = |c\rangle$. Since the state $|1\rangle^{\otimes N}$ ($1 \in G$) satisfies this condition, all other loop configurations can be generated by applying products of unitary operators A_v^g . For a manifold like $\mathbb{T}^2$, we can further introduce two fluxes around noncontractible cycles, which are trivial for loops generated in this way. To create a loop configuration with nontrivial flux through noncontractible cycles, a special operator should be applied along the cycle, as shown in Fig. 19(a). It is important to note that fluxes are defined only up to conjugacy class, meaning that two fluxes, g and $h^{-1}gh$, are equivalent.

We are ready to study the information convex set. On a lattice of the torus, define $\lambda_{g,h}$ to be the zero-flux configuration shown in Fig. 19(b), for $gh = hg$. We further define

$$C_{g,h} := \{(rgr^{-1}, rhr^{-1}) | r \in G\}, \quad (49)$$

which is the set of simultaneous conjugacy classes of pairs (g, h) with $g, h \in G$.

Proposition 4. The information convex set $\Sigma(\mathbb{T}^2 | \rho^m)$ on the torus $\mathbb{T}^2$ is a simplex with extreme points in one-to-one correspondence with $C_{g,h}$ for $gh = hg$. These extreme points are mutually orthogonal and have the explicit form

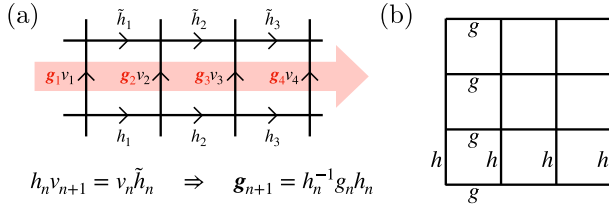


FIG. 19. Decohered Non-Abelian state. (a) The operator along the *dual* horizontal noncontractible cycle L , $\prod_{e \in L} L_e^{g_e}$, where $g_1 = g$ and $g_{n+1} = h_n^{-1} g_n h_n$. Applied on a loop state with trivial flux along the *vertical* cycle, this generates a conserved loop state with nontrivial flux C_g through the vertical noncontractible loop. (b) Initial product state $\lambda_{g,h}$ to prepare an extreme point of the information convex set $\Sigma(\mathbb{T}^2|\rho^m)$. The square lattice is on a torus with a periodic boundary condition. Unlabeled links are in the identity element.

$$\rho^{C_{g,h}} := \frac{1}{|G|^{N_v}} \sum_{\{g_v\}} \prod_v A_v^{g_v} \lambda_{g,h} A_v^{g_v^\dagger}. \quad (50)$$

Furthermore, these extreme points can be prepared by a finite-depth local circuit from the product state.

Proof. The proof is analogous to the one in Proposition 2. Following similar logic, we can show that, for any $\rho \in \Sigma(M|\rho^m)$, it satisfies

$$\begin{aligned} T_e^g \rho &= \rho T_e^g, \\ B_p^1 \rho &= \rho B_p^1, \\ A_v^g \rho &= \rho A_v^g. \end{aligned} \quad (51)$$

For any $\alpha \in \Sigma$, the first condition implies that $\alpha = \sum_c \lambda_c |c\rangle\langle c|$, where $|c\rangle$ is a configuration of group elements on each link. The second condition implies that every $|c\rangle$ must have zero flux locally, that is, $B_p^1 |c\rangle = |c\rangle$. The third condition implies that $\alpha = \mathcal{A}(\alpha)$, where

$$\mathcal{A}(\mathcal{O}) := \frac{1}{|G|^{N_v}} \sum_{\{g_v\}} \prod_v A_v^{g_v} \mathcal{O} A_v^{g_v^\dagger}. \quad (52)$$

Therefore, for any $\alpha \in \Sigma$, we can write

$$\alpha = \sum_c \lambda_c \mathcal{A}(|c\rangle\langle c|). \quad (53)$$

Next, $\mathcal{A}(|c\rangle\langle c|) = \mathcal{A}(|c'\rangle\langle c'|)$ if $|c\rangle$ and $|c'\rangle$ can be related by some “gauge transformation” $\prod_v A_v^{g_v}$, which we denote as $|c\rangle \sim |c'\rangle$. For any configuration $|c\rangle$ with zero local flux, one can always find a gauge transformation into $\lambda_{g,h}$ [the product state of the form in Fig. 19(b)], for some commuting pair $gh = hg$; this is a fact we explain in Appendix H 2. Thus, any element in $\Sigma(M|\rho^m)$ is given as a convex combination of states of the following extreme points:

$$\rho^{C_{g,h}} := \mathcal{A}(\lambda_{g,h}). \quad (54)$$

Since the conjugation by A_v^r generates all possible configuration in the class $C_{g,h}$ in Fig. 19(b), the above state is indeed labeled by the class $C_{g,h}$ in Eq. (49). Thus, Eq. (50) describes every extreme point of $\Sigma(\mathbb{T}^2|\rho^m)$. The extreme points $\rho^{C_{g,h}}$ are a classical mixture of configurations, and they are mutually orthogonal. Furthermore, since $\lambda_{g,h}$ is a product state and the above operation is equivalent to applying a mixed unitary channel [112] at every vertex, this provides a classical preparation protocol. ■

From the formula (50), we see that extreme points are classical separable states. If the group G is Abelian, then $C_{g,h} = \{(g, h)\}$, and the number of extreme points would be $|G|^2$. If the group G is non-Abelian, the number of $C_{g,h}$ equals the number of anyons in the quantum double models, which is less than $|G|^2$ [65]. It reflects the fact that non-Abelian fluxes along two noncontractible cycles cannot be fully independent. The entropies of the extreme points are

$$S(\rho^{C_{g,h}}) = S(\rho^{C_{1,1}}) + \log |C_{g,h}|. \quad (55)$$

This formula is obtained by computing the sizes of equivalent classes of the configurations, as detailed in Appendix H 2. While the entropy of each extreme point follows a volume law, the entropy difference $S(\rho^{C_{g,h}}) - S(\rho^{C_{1,1}}) = \log |C_{g,h}|$ has a topological origin.

This further implies that the memory capacity

$$\mathcal{Q}(\mathbb{T}^2|\rho^m) = \log |G| + \log |G_{\text{cj}}|, \quad (56)$$

where $|G_{\text{cj}}|$ is the number of conjugacy classes of G . A quick intuitive way to see this is noticing that the maximum entropy is achieved by mixing all extreme points with proper weights. To maximize $S(\rho)$, we can do it by uniformly mixing all the valid elements in the extreme points. To minimize $S(\lambda)$, we pick the extreme point corresponds to $C_{1,1}$, where $C_{1,1} = \{(1, 1)\}$ contains only one element. Therefore, we arrive at $\mathcal{Q}(M|\sigma) = \log N$, where

$$N = \{ \#(g, h) | gh = hg, g, h \in G \}. \quad (57)$$

For every element c in a conjugacy class $\mathcal{C}$, the number of elements in group G that commutes with c is $|G|/|\mathcal{C}|$. And the number of conjugacy class is $|G_{\text{cj}}|$; therefore,

$$N = \sum_{\mathcal{C} \in G_{\text{cj}}} \frac{|G|}{|\mathcal{C}|} |\mathcal{C}| = |G| |G_{\text{cj}}|. \quad (58)$$

For more systematic derivations, see Appendix H 3.

The Levin-Wen topological entropy for this model is

$$\gamma_{\text{LW}} = \log \sqrt{|G|}. \quad (59)$$

The question reduces to computing the entanglement entropy of the fixed-point density matrix for disks and annuli, as is done in Appendix H 4. Because $|G_{\text{ej}}| < |G|$ for any non-Abelian model, the bound on memory capacity in Eq. (10) is saturated if and only if G is Abelian.

Decohering into ρ^e —Unlike ρ^m , obtaining $\prod_v A_v^1$ is not straightforward. Naively, we attempt to decohere fluxes in a fashion analogous to the Abelian case, applying mixed unitary channels in terms of L_e^g , where $L_e^g|h\rangle = |gh\rangle$ on edge e . Since this does not commute with A_v^R , we define $L_e^R := (\dim R/|G|) \sum_{g \in G} \chi_R(g) L_e^g$ for each irreducible representation R of G , which gives $[L_e^R, A_v^R] = 0$. Therefore, we may use $\{L_e^R\}$ as the local channel performing projective measurements. But applying this channel generically cannot remove $\prod_p B_p^1$ unlike the Abelian case. The difference of charge decoherence and flux decoherence is also evident in the fusion rules of charges and fluxes. In non-Abelian quantum double models, it is no longer true that fusing two fluxes always results in a flux (e.g., in the S_3 quantum double [67]), whereas fusing two charges still always yields a charge.

While we leave a systematic study of generic flux-decohered non-Abelian states to future work, it is worth noting a related example. Reference [42] studied a non-Abelian quantum double model under a specific flux-decohering channel and found evidence that non-Abelian topological order can remain robust even at maximal decoherence. We expect that this robustness is tied to the nontrivial fusion structure of the particular fluxes being proliferated. More specifically, we conjecture that projecting onto the trivial charge pattern described by ρ^e requires proliferating all flux types rather than only a single flux species. At present, however, we do not have a general proof of this claim.

B. Secret-sharing models

What distinction between (charge) decohered Abelian and non-Abelian models can we make? One notable difference is the secret sharing [113–116], which refers to the scheme to distribute information (or secret) among a group in such a way that only nontrivial collaboration of multiple parties can decode the information. Let us partition the torus $\mathbb{T}^2$ into four disklike patches $ABCD$ as shown in Fig. 20. The question is, what can we learn given pieces of the state?

As a topological memory, two properties immediately follow: (i) Given any three parties (e.g., ρ_{ABC}), we can recover the entire state, and the knowledge is complete. (ii) No single party can recover any information. But what if we allow only the collaboration between two parties

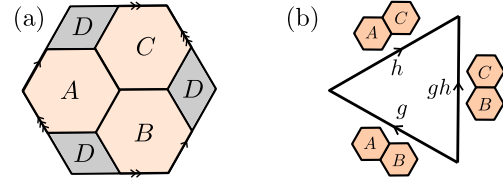


FIG. 20. (a) Partition a torus into four disklike pieces A , B , C , and D . The torus is visualized as a big hexagon with opposite edges identified, as in Fig. 1.2 in Ref. [117]. (b) A schematic drawing of the conjugacy classes detected by the collaboration of two parties (thickened noncontractible loops of the torus), $(A, B) \rightarrow C_g$, $(A, C) \rightarrow C_h$, and $(B, C) \rightarrow C_{gh}$.

(A, B) , (B, C) , and (C, A) ? The two parties can do any joint operations. Suppose we have enough copies of the state; then this corresponds to knowing the reduced density matrices ρ_{AB} , ρ_{BC} , and ρ_{CA} . To be concrete, we consider the torus partitioned into A , B , C , and D in Fig. 20.

This can be converted to a group-theoretic computation. Recall that

$$\begin{aligned} C_g &:= \{rgr^{-1} | r \in G\}, \\ C_{g,h} &:= \{(rgr^{-1}, rhr^{-1}) | r \in G\}, \end{aligned} \quad (60)$$

where $C_{g,h}$ is defined for $gh = hg$. Suppose we store the information in the extreme point $\rho^{C_{g,h}}$. AB knows C_g , AC knows C_h , and BC knows C_{gh} . When G is Abelian, $C_{g,h}$ is determined from g , h and, thus, knowing ρ_{AB} and ρ_{BC} is enough.

However, for non-Abelian cases, the collaboration between any two parties may not determine the message. Finding an example is equivalent to solving the following problem. Consider a finite group G which provides a triple (g, h, h') , $g, h, h' \in G$, such that

$$gh = hg, \quad gh' = h'g, \quad (61)$$

and $h' \in C_h$, $gh' \in C_{gh}$, which nonetheless, has $C_{g,h} \neq C_{g,h'}$.

Finding such a group G corresponds to finding a model with information not decodable with the collaboration of any two parties.

One such group is the four-element alternating group A_4 . It has 12 group elements, and there are three elements of order 2:

$$\alpha := (12)(34), \quad \beta := (13)(24), \quad \gamma := (14)(23). \quad (62)$$

$\alpha^2 = \beta^2 = \gamma^2 = 1$, $\{\alpha, \beta, \gamma\}$ are mutually commuting and

$$\alpha\beta = \gamma, \quad \gamma\alpha = \beta, \quad \beta\gamma = \alpha. \quad (63)$$

$\{\alpha, \beta, \gamma\}$ forms a conjugation class of A_4 . One can check: $C_{\alpha,\beta} = \{(\alpha, \beta), (\beta, \gamma), (\gamma, \alpha)\}$ and $C_{\alpha,\gamma} = \{(\alpha, \gamma), (\beta, \alpha), (\gamma, \beta)\}$. Thus, if we let $g = \alpha$, $h = \beta$, and $h' = \gamma$, all the

TABLE II. Secret-sharing properties for models with different groups G . “Yes” and “no” refer to the answers to whether the complete logical information on the torus can be learned.

G	ρ_{AB}	ρ_{AB}, ρ_{BC}	$\rho_{AB}, \rho_{BC}, \rho_{CA}$	ρ_{ABC}
Abelian	No	Yes	Yes	Yes
S_3	No	No	Yes	Yes
A_4	No	No	No	Yes

conditions that enable the nontrivial secret sharing are satisfied. This means that the two extreme points associated with $C_{\alpha,\beta}$ and $C_{\alpha,\gamma}$ can be distinguished only by the collaboration of three parties.

Do all decohered non-Abelian quantum doubles have this property? In fact, the decohered S_3 quantum double provides another interesting example sitting between any Abelian group and A_4 . In this model, we do not need ρ_{ABC} ; instead, the complete logical information can be learned given $\{\rho_{AB}, \rho_{BC}, \rho_{CA}\}$ but not from any two of the three. The result is summarized in Table II.

VII. HIGHER DIMENSION

Let us consider the 3D models based on an Abelian group $G = \mathbb{Z}_2$, adopted from the 3D toric code. We consider two reference states of the decohered 3D toric code

$$\rho^e = \prod_v \left(1 + \prod_{e \ni v} X_e\right), \quad \rho^m = \prod_p \left(1 + \prod_{e \in p} Z_e\right). \quad (64)$$

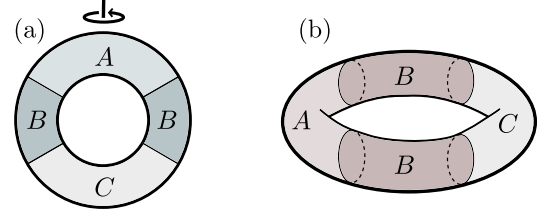
These reference states satisfy the 3D version of the axioms. By the 3D version of the axioms, we mean the conditions centered on bounded radius 3D balls, which are the rotated versions of BC or BCD in the 2D axiom figure.

These two fixed points ρ^e and ρ^m must be in two different phases. As discussed in Sec. IV, if the two states can be separated by a domain wall that preserves all the axioms **P0**, **M0**, and **M1** (i.e., if the domain is transparent), the TE must match. However, the topological entropies summarized in Table III show the mismatch, implying that two states belong to different topological mixed-state phases.

The physical meaning of the two types of topological entropy (γ_{particle} , γ_{flux}) in Fig. 21 is suggested by their names and is studied in ground states of topological order [12,118,119]. γ_{particle} measures the total quantum dimension of point particles, and γ_{flux} measures the total

TABLE III. Topological entropies for 3D models.

	ρ^{TC}	ρ^e	ρ^m	$\otimes_i \rho_i$
γ_{particle}	$\log \sqrt{2}$	$\log \sqrt{2}$	0	0
γ_{flux}	$\log \sqrt{2}$	0	$\log \sqrt{2}$	0


 FIG. 21. Definition of 3D TEs. (a) A partition of a sphere shell that defines $\gamma_{\text{particle}} := I(A:C|B)/2$ and (b) a partition of the solid torus that defines $\gamma_{\text{flux}} := I(A:C|B)/2$.

quantum dimension of flux loop excitations. Importantly, for any pure-state topological order in three dimensions, $\gamma_{\text{particle}} = \gamma_{\text{flux}}$. The values of TEs in Table III not only suggest that the two phases represented by the fixed points ρ^e and ρ^m are distinct mixed-state phases, they also suggest that they are different from any pure-state topological order in 3D, by our definition.

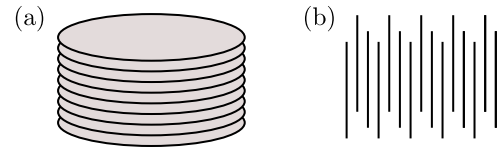
Now let us study the behavior of ρ^e and ρ^m as memory on closed manifolds. Consider the 3D manifold $M = S^1 \times S^2$. A simple computation shows

- (i) $\Sigma(M|\rho^e)$ is a simplex with two extreme points;
- (ii) $\Sigma(M|\rho^m)$ is a simplex with two extreme points.

In both sets, the two extreme points are orthogonal, and the entropy of each extreme point is identical. They both represent a classical memory. Can we tell the difference between the two classical memories? There is a simple answer: The secret-sharing schemes are different.

We can divide the manifold $S^1 \times S^2$ in two different ways, as illustrated in Fig. 22. For the classical memory $\Sigma(M|\rho^e)$, the information can be decoded on each thickened S^2 , but no information can be decoded in thickened S^1 . For the classical memory $\Sigma(M|\rho^m)$, the information can be decoded on each thickened S^1 , but no information can be decoded in thickened S^2 .

It is further possible to consider fracton models with decoherence. However, we expect such states to violate axiom **M1**. The intuition behind this is that **M1** guarantees a topological sense of smoothness (the information of excitation can be transported in a topological manner), but for fractons, the excitations are free to move only in submanifolds.


 FIG. 22. Partition $M = S^2 \times S^1$ into pieces to create secret sharing schemes. (a) Partition M into thickened S^2 by dividing S^1 into pieces, and (b) partition M into thickened S^1 by dividing S^2 into small disks.

VIII. CONCLUSION

In this work, we developed a systematic framework for understanding topological mixed states by formulating a set of fixed-point axioms. These axioms, grounded in the spirit of the entanglement bootstrap [10] and the recent study of Markov length [26], encapsulate fundamental features we expect of any mixed state exhibiting topological correlation or entanglements. Each axiom imposes a constraint reflecting the local recoverability, stability, and consistency of the topological mixed states.

From the bootstrap axioms, we could derive several topological invariants that partially characterize a given topological mixed-state phase. The primary diagnosis is the information convex set $\Sigma(M|\sigma)$, a set of density matrices locally indistinguishable from the fixed point σ on M . On a closed manifold M , Σ tells us the classical or quantum nature of information encoded in the topological mixed states. We identified classes of solvable mixed-state fixed points, showing that they satisfy all the axioms. We can prepare classes of classical fixed points efficiently via finite-depth local channels. The associated Σ is shown to exhibit classical topological memory, marking a sharp contrast to the fully coherent structure of pure topological states. Another important diagnosis is the topological entropy, which is a straightforward generalization of Levin-Wen topological entanglement entropy. Interestingly, the topological entropy gives a fundamental bound to the memory capacity.

Crucially, our axiomatic framework allows us to go beyond fixed points: By requiring the axioms to hold only under coarse-graining, we explored phases of topological mixed states and provided a notion of equivalence via transparent domain walls (Definition 6). This naturally leads to the concept of topological channel connectivity and generalizes traditional notions of phase equivalence. It successfully distinguishes between decohered Abelian topological orders and trivial product states—something the previous naive definition of two-way connectivity via FDL failed to capture. Furthermore, as long as *approximate* axioms hold, we proved that topological entropy remains invariant across the system up to exponentially small corrections. This further substantiates that our axioms play the role of a gap in mixed states.

We further investigated the richness of non-Abelian topological mixed states. While many of the classical mixed states are unentangled and separable, we demonstrated that they may nevertheless exhibit nontrivial features such as multiparty classical secret sharing and nonlocal storage of classical information. These properties are absent in decohered Abelian phases, highlighting a new kind of nontriviality in the mixed-state setting. Our results also suggest that there may exist an interesting hierarchy among decohered topological mixed states, which is different from that proposed in Ref. [120] based on the preparation complexity using measurements and feedback.

This paper takes a first important step toward the systematic understanding of topological mixed states, which inspires numerous interesting directions.

- (i) Our approach naturally motivates the study of quantities such as conditional mutual information on a sequence of coarse-grained partitions, each of which is further coarse-grained from the previous one. This class of regions is featured by the uniform rescaling of *all* regions involved, e.g., in concentric partitions centered at a disk, and its generalization related to axiom **M1**. See Sec. IV B for details. We note that, this way of scaling the regions, which naturally reflects the need for the study of RG behavior, is different from the three cases commonly studied in literature, i.e., “pairwise,” “local,” and “global”; see, e.g., Ref. [99]. We hope future literature will provide stronger bounds on such a partition for Gibbs states, states under decoherence, and ground states.
- (ii) Incorporating global symmetries—Definition 10 can be naturally extended to incorporate global symmetries: Given a topological channel, we can additionally impose a symmetry to be conserved on its truncated boundary. This will open a route to study mixed-state versions of symmetry-enriched topological orders as well as symmetry-protected topological states.
- (iii) Higher dimensions—While the axioms naturally extend to higher dimensions and we addressed some examples, a comprehensive classification of 3D and beyond remains an open challenge.
- (iv) RG flow and numerical study—More detailed numerical studies of RG behavior under the axioms could uncover critical points and universality classes in mixed-state settings. While path-integral formalism suggests that this criticality would be that of the same-dimensional classical models, a more systematic understanding of critical behavior is lacking, such as how finite-size scaling works.
- (v) Information robustness—Inspired by recent experimental realizations of the D_4 topological order [121,122], which remain coherent under maximal X decoherence [42,123], we suggest that tools in our framework, e.g., information convex set and memory capacity, may offer a more direct way to understand such robustness. For instance, our method may show that the convex set remains a Bloch sphere at maximal decoherence.
- (vi) Domain walls and mixed-pure interfaces—The study of domain walls between mixed and pure topological phases may yield insights into emergent quasiparticles and braiding, especially when viewed through the lens of information convex sets. This may explain what the right mixed-state analog of anyon condensation [124–126] is, by deriving the fundamental constraints on excitations tunneling through the domain walls.
- (vii) Chiral mixed-state phases—Can there exist mixed-state analogs of chiral topological order? If so, they

might necessarily violate the axioms by an amount that decays with system size, akin to how chiral fixed points cannot be exactly realized on finite-dimensional Hilbert spaces [17]. However, the modular commutator stops being a topological invariant in mixed-state setups; see Appendix E (and also Refs. [38,127]). One may need another chirality measure [128]. Furthermore, should domains between chiral and nonchiral mixed-state phases break more assumptions than **M1**? If so, perhaps one will be able to identify new fixed-point conditions [14].

- (viii) Generalized symmetries—Our approach offers a potential route to derive emergent generalized symmetries, such as higher forms or noninvertible ones, directly from the axioms rather than postulating them externally.
- (ix) Gapless analogs of mixed-state phases: Our framework provides an information-theoretic approach to classifying gapped mixed-state phases of matter. An important open direction is to extend this framework beyond the gapped setting, toward a theory of mixed-state criticality and the recently discovered *information-critical phases* [129].
- (x) Classical preparability and braiding—In general, what obstructions exist for preparing certain mixed states using finite-depth local channels? We expect the answer lies in a clearer understanding of the emergent particles and their braiding. One possible route is to consider the information convex set for embedded subregions and locally embedded (i.e., immersed) regions. In parallel to the pure-state entanglement bootstrap, we expect that insight into braiding can be gained by deforming the region back to its original position, passing through a nontrivial path, and observing the nontrivial isomorphism (Appendix B) of the information convex set. In contrast to the braiding nondegeneracy derived from area law axioms, we expect the volume-law axioms to allow braiding degeneracy as the outcome.
- (xi) With the crucial assumption of spatial uniformity, we also propose a sharpened time-based definition of mixed-state topological phases; see Conjecture 1. Rather than invoking a single diverging Markov length, this refinement requires the finiteness of three distinct Markov (or correlation) lengths, thereby revealing a richer structure underlying mixed-state phase transitions. This perspective suggests a richer landscape of critical behavior, distinguished by which subset of the three length scales diverges. Identifying such transitions and uncovering their physical origin would sharpen our understanding of the mechanisms that separate mixed-state phases.

Note added. Ref. [130], which appeared after our preprint, independently studies the information convex set of mixed-state phases.

ACKNOWLEDGMENTS

J. Y. L. dedicates this paper to his daughter Seol Lee, catalyst and companion to this work. We thank Soonwon Choi, Tyler Ellison, Haeum Kim, Isaac Kim, Hyukjoon Kwon, Michael Levin, Yizhou Ma, and Shengqi Sang for discussions and feedback. B. S. thanks Tarun Grover, Meng Cheng, Timothy Hsieh, and Chong Wang for inspiring discussions about mixed states. The computational work is conducted through the Illinois Campus Cluster Program at the University of Illinois, Urbana-Champaign. T.-H. Y. is supported by the Taiwan-UIUC Scholarship program and Elite Dream Project Grant of Veterans and Dependents Foundation, Republic of China. B. S. acknowledges support from National Science Foundation under Award No. PHY-2337931 at UC Davis, and support in part by Perimeter Institute for Theoretical Physics. B. S. and J. Y. L. are supported by the IQIST fellowship and faculty startup grant at the University of Illinois, Urbana-Champaign. J. Y. L. acknowledges support from KIAS through the Quantum Universe Center scholar program.

DATA AVAILABILITY

Processed data and code used to reproduce the figures are openly available in Ref. [131].

APPENDIX A: MORE ON MIXED-STATE BOOTSTRAP

1. Global version of the axioms

In the main text, we stated **M0** and **M1** in local patches, making **P0** the only axiom that needs the global state. In fact, **P0** implies a global version of the other two axioms. By the global version of the axioms **M0** and **M1**, we mean the following.

- (i) **M0***— $I(AA': C) = 0$ for the partition in Fig. 23(a), where AA' is the complement of BC in the global system M .
- (ii) **M1***— $I(AA': C|B) = 0$ for the partition in Fig. 23(b), where AA' is the complement of BCD in the global system M .

Here, we explain the details.

Proposition 5. A reference state σ_M satisfies **P0**, **M0**, and **M1** if and only if it satisfies **P0**, **M0***, and **M1***.

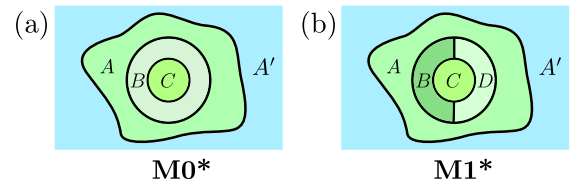


FIG. 23. Alternative axioms of **M0*** and **M1***. Here, ABC and $ABCD$ are the local patches that appear in the definition of **M0** and **M1**. AA' is the complement of the local region BC or BCD on the entire system.

Proof. ($\Leftarrow$) This follows immediately from the strong subadditivity, which says $I(AA': C) \geq I(A: C)$ and $I(AA': C|B) \geq I(A: C|B)$.

($\Rightarrow$) **P0** implies the existence of a local channel on BC to recover a missing disk C . [See Fig. 2(a).] It implies the vanishing of conditional mutual information $I(C: E|B) = 0$. Noticing that the condition is symmetric under the exchange of C and E , we immediately see that the state on region E can be recovered by a channel on BE . Applying this to the configurations in Fig. 23, namely, for the partition of the system into $ABCA'$, where $A' = M \setminus ABC(D)$, we find the following. **P0** implies that $\exists$ a quantum channel $\mathcal{E}_{A \rightarrow AA'}$ such that

$$\mathcal{E}_{A \rightarrow AA'}(\sigma_{ABC}) = \sigma_{AA'BC}, \quad (\text{A1})$$

for both Figs. 23(a) and 23(b). This implies that

- (i) for the case in Fig. 23(a), $I(A: C)_\sigma \geq I(AA': C)_\sigma$ — This is because mutual information cannot increase when applying a quantum channel to one party. Then, **M0** implies that $I(AA': C)_\sigma = 0$. Thus, **M0*** holds.
- (ii) for the case in Fig. 23(b),

$$I(A: C|B)_\sigma \geq I(AA': C|B)_\sigma,$$

because quantum channels on an unconditioned party cannot increase the conditional mutual information. Then, **M1** implies that $I(AA': C|B)_\sigma = 0$. Thus, **M1*** holds.

This completes the proof. ■

2. Mixed versus pure

In this appendix, we talk about two things relating the “mixed-state” to the “pure-state” entanglement bootstrap.

First, we called the entanglement bootstrap for gapped system as “pure-state” bootstrap. It is not because that the reference state need to be pure but for the reason that the state satisfies a local purity condition **A0**:

$$\text{axiom A0: } (S_{BC} + S_C - S_B)_\sigma = 0, \quad \text{for Fig. 23(a).} \quad (\text{A2})$$

This condition implies that the state is “locally pure” in the sense that $\sigma_{BC} = \rho_{B_L} \otimes |\varphi\rangle_{B_R C} \langle \varphi|$, for some decomposition of the Hilbert space $\mathcal{H}_B = (\mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R}) \oplus \dots$. The state $|\varphi\rangle$ purifies the reference state density matrix in C .

Second, as long as the local purity condition **A0** holds, the axioms **A1** and **M1** can be used interchangeably.

Lemma 2. Consider a state σ satisfying **A0** on regions centered at CD and D in Fig. 23(b), and suppose

$$I(A: C|B)_\sigma = 0. \quad (\text{A3})$$

Then, $(S_{BC} + S_{CD} - S_B - S_D)_\sigma = 0$; namely, **A1** holds on configurations centered at C .

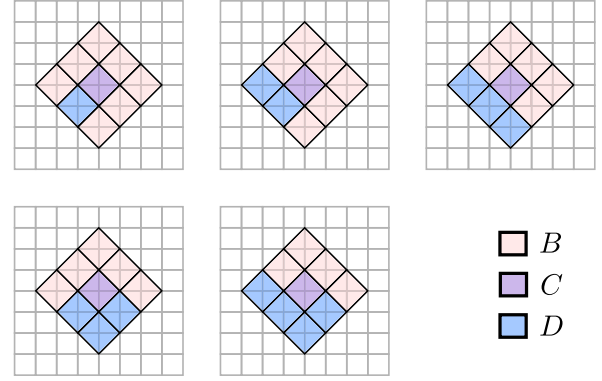


FIG. 24. The five partitions together with ones obtained from them by rotations, reflections, and swapping B and D is the complete list to check for **M1**.

Proof. The condition **A0** centered at CD and D gives

$$S_{ABCD} + S_{CD} = S_{AB} \quad \text{and} \quad (\text{A4})$$

$$S_{ABCD} + S_D = S_{ABC}. \quad (\text{A5})$$

They imply $S_{AB} - S_{ABC} = S_{CD} - S_D$ on σ . Plugging this into Eq. (A3), we get $(S_{BC} + S_{CD} - S_B - S_D)_\sigma = 0$. ■

Corollary. The axiom set $\{\mathbf{A0}, \mathbf{M1}\}$ defines the same entanglement bootstrap setup as $\{\mathbf{A0}, \mathbf{A1}\}$.

Thus, when the local purity condition holds, the axioms **A1** and **M1** can be used interchangeably.

3. Axioms on a discrete lattice

On a discrete lattice, such as a square lattice or a honeycomb, the set of partitions for the axioms can be stated precisely. The partition centered at a (coarse-grained) site is a finite set. We explicitly draw this set of partitions for the square lattice. The single configuration in Fig. 6(a) is the region to check for **P0** and **M0**. The five configurations in Fig. 24 with ones obtained from them by rotations, reflections, and $B \leftrightarrow D$ is the complete list to check for axiom **M1**.

4. Information convex set is convex

We show that $\Sigma(M|\sigma)$ defined in Definition 2 must be a compact convex set when M is a closed manifold. Here, σ is a fixed-point state on M , which satisfies all the axioms **P0**, **M0**, and **M1**. We note that any state $\rho_M \in \Sigma(M|\sigma)$ has an identical reduced density matrix as σ on any ball.

Lemma 3. For any $\rho_M \in \Sigma(M|\sigma)$ and any disklike region $\mathfrak{D} \subset M$,

$$\rho_{\mathfrak{D}} = \sigma_{\mathfrak{D}}. \quad (\text{A6})$$

This follows from **M1** and the definition of $\Sigma(M|\sigma)$. The proof is identical to that in Lemma 3.1 and Proposition 3.5 in Ref. [10], and, thus, we recall only the essence.

The axiom **M1** and the monotonicity of conditional mutual information under partial trace imply

$$I(A:C|B)_\rho = 0 \quad \text{for} \quad \begin{array}{c} \text{A} \quad \text{B} \quad \text{C} \\ \text{---} \end{array} \quad (\text{A7})$$

where the ABC , AB , and BC shown in the figure are disk regions in the bulk and BC is contained in a bounded radius disk. Thus, ρ_{ABC} is a quantum Markov state, and it is uniquely determined by ρ_{AB} and ρ_{BC} . We use the same idea to divide the state into smaller pieces, each of which is contained in some bounded radius disk, as appeared in Definition 2. As both $\rho_{\mathfrak{D}}$ and $\sigma_{\mathfrak{D}}$ are determined (in the same way) by the same density matrices on the marginals, they must be identical.

Proposition 6 (convex set). The $\Sigma(M|\sigma)$ in Eq. (4) is a compact convex set.

Proof. First, M is a finite lattice, and, thus, the state space (i.e., the set of all density matrices) of a finite-dimensional Hilbert space is a finite-dimensional compact convex set S . The set $\Sigma(M|\sigma)$ is compact, because it is obtained by adding a set of constraints that are linear equalities to the set S . Let us analyze the conditions one by one.

- (i) The first constraint is the local indistinguishability. For any bounded radius ball $b \subset M$, we require $\rho_b = \sigma_b$. These are clearly a set of linear constraints.
- (ii) This second constraint, local recoverability, is the requirement that **P0** is satisfied for an $\rho_M \in \Sigma(M|\sigma)$. Noticing that ρ_M reduced to an arbitrary disk must be identical to the reference state (Lemma 3), we see that the constraint can be rewritten as

$$\mathcal{E}_{B \rightarrow BC}^\sigma(\rho_{AB}) = \rho_{ABC}, \quad \forall \rho_{ABC} \in \Sigma(M|\sigma), \quad (\text{A8})$$

where $\mathcal{E}_{B \rightarrow BC}^\sigma(\cdot)$ is the Petz map, which is fixed by the reference state σ_{BC} on local disk BC [53] and is independent of the state ρ . These are another set of linear constraints.

Thus, $\Sigma(M|\sigma)$ is a compact convex set. ■

5. Role of **P0** in information convex set

We proposed that the information convex set $\Sigma(M|\sigma)$ captures the notion of physically robust information storage. But why is this the appropriate choice? In particular, one might ask: Why not relax the local recoverability (**P0**) in Definition 2? If we were to do so, we would arrive at an alternative set, but as we show, this modified set fails to retain key features essential for topological robustness.

Consider the alternative set

$$\tilde{\Sigma}(M|\sigma) = \{\rho_M | \rho_b = \sigma_b, \quad \forall b \subset M\}, \quad (\text{A9})$$

where b is any bounded radius disk and M is a closed manifold. $\tilde{\Sigma}$ is the set of states on M , which are locally

indistinguishable with the reference state. It is easy to show $\tilde{\Sigma}(M|\sigma)$ is a compact convex set that contains $\Sigma(M|\sigma)$. When the pure-state bootstrap axioms (**A0** and **A1**) apply, these two sets are identical. This is because **A0** on the local ball implies **P0**. However, in mixed-state setups, $\tilde{\Sigma}(M|\sigma)$ can be strictly larger, as we explain in examples below.

Example 10—Let $\sigma = \prod_{i \in \mathcal{S}} 1_i$. Here, $\mathcal{S}$ is the set of all qubits of the system. In this case, $\Sigma(M|\sigma) = \{\sigma_M\}$. On the other hand,

$$\frac{1 \pm \prod_{i \in \mathcal{S}} X_i}{2} \in \tilde{\Sigma}(M|\sigma). \quad (\text{A10})$$

This is because, as long as we trace out any single site, the state will be the maximally mixed state. Thus, $\tilde{\Sigma}$ is strictly larger than Σ .

Example 11—Let $M = \mathbb{T}^2$ and let $\sigma = \prod_v A_v^1$ be the dephased toric code, as in Example 3. In this case, $\Sigma(M|\sigma)$ is a simplex with four extreme points (Proposition 2). These extreme points are

$$\rho_{ab}^e \propto \prod_v A_v^1 (1 \pm \bar{X}_1)(1 \pm \bar{X}_2), \quad (\text{A11})$$

where $\bar{X}_1$ and $\bar{X}_2$ are noncontractible X string on the dual lattice along the noncontractible x, y cycles of the torus, respectively. On the other hand, we have more states in $\tilde{\Sigma}(M|\sigma)$, e.g.,

$$(1 + \bar{Z}_j) \prod_v A_v^1 (1 + \bar{Z}_j) \in \tilde{\Sigma}(M|\sigma), \quad (\text{A12})$$

where $\bar{Z}_j$ with $j = 1, 2$ are noncontractible Z strings along the noncontractible x, y cycles of the torus, respectively. The reason is that the noncontractible loop $\bar{Z}_j$ disappears after the partial trace to a local region, and, thus, the state in Eq. (A12) is locally indistinguishable from σ . The distinction between $\bar{X}_j$ and $\bar{Z}_j$ is that the former can be smoothly deformed when acting on $\rho \in \Sigma(M|\sigma)$, whereas the latter cannot.

We have seen from examples above that $\tilde{\Sigma}(M|\sigma)$ can be strictly larger than $\Sigma(M|\sigma)$. We are ready to explain why such a convex set does not represent the set of “topologically protected” information. Suppose that the environment is capable of removing any local ball and resetting the state to a local product state. Then the information represented by states in $\tilde{\Sigma}(M|\sigma)$ cannot be recovered. Explicitly,

- (i) for the state in Eq. (A10), tracing out any qubit will result in the loss of the information;
- (ii) for the state in Eq. (A12), tracing out any qubit touching the (nondeformable) $\bar{Z}_j$ string will result in the loss of the information.

On the other hand, due to **P0**, the information in $\Sigma(M|\sigma)$ is always preserved by such a process. In fact, the original states can be recovered by the local channel surrounding

the hole. This is the main physical reason we prefer Definition 2 as the definition of information convex set, which captures the physical information storage that is robust.

6. Bound on relative entropy change

Consider a system partition into BCE where BC is a bounded region and E may be large, allowing $\dim \mathcal{H}_E \rightarrow \infty$. In general, it is possible to induce a global change in the density matrix by applying a local quantum channel $\mathcal{E}_C$, even if such a global change is inaccessible (negligible) if we trace out the complement of the local neighborhood of C . In other words,

$$D(\rho_{EBC} \| \mathcal{E}_C[\rho_{EBC}]) \gg D(\rho_{BC} \| \mathcal{E}_C[\rho_{BC}]) \quad (\text{A13})$$

may happen. This phenomenon happens in the context of SWSSB phase transition.

We explain why **P0** prevents this from happening. First, the conditional mutual information is monotonic under the action of quantum channels on an unconditioned region. Thus,

$$0 = I(E:C|B)_\rho \geq I(E:C|B)_{\mathcal{E}_C[\rho]} = 0. \quad (\text{A14})$$

Importantly, ρ_{BCE} and $\mathcal{E}_C[\rho_{BCE}]$ can be recovered from ρ_{BC} and $\mathcal{E}_C[\rho_{BC}]$, respectively, by the *same* quantum channel. Such a quantum channel can be chosen to be the Petz map [53] $\mathcal{E}_{B \rightarrow BE}^{\text{Petz}}(\cdot) := \rho_{BE}^{1/2} \rho_B^{-1/2}(\cdot) \rho_B^{-1/2} \rho_{BE}^{1/2}$. By the monotonicity of relative entropy, we must have $D(\rho_{BC} \| \mathcal{E}_C[\rho_{BC}]) \leq D(\rho_{EBC} \| \mathcal{E}_C[\rho_{EBC}])$, where the quantum channel is the partial trace Tr_E . The existence of the recovery map $\mathcal{E}_{B \rightarrow BE}^{\text{Petz}}$, on the other hand, implies $D(\rho_{EBC} \| \mathcal{E}_C[\rho_{EBC}]) \leq D(\rho_{BC} \| \mathcal{E}_C[\rho_{BC}])$. Thus,

$$D(\rho_{EBC} \| \mathcal{E}_C[\rho_{EBC}]) = D(\rho_{BC} \| \mathcal{E}_C[\rho_{BC}]). \quad (\text{A15})$$

Therefore, the local channel acting on C cannot induce a global change in the density matrix.

7. Sphere has zero memory capacity

We show that the information convex set on the sphere has a unique element. Thus, no robust information can be stored on a sphere. The memory capacity $\mathcal{Q}(S^2|\sigma) = 0$.

Proposition 7. Let S^2 be a sphere and σ be a reference state; then,

$$\Sigma(S^2|\sigma) = \{\sigma_{S^2}\}. \quad (\text{A16})$$

Proof. The proof is an analog of Proposition 3.7 of pure-state entanglement bootstrap [10], where in the latter case one can also show the reference state on the sphere is pure. Let the sphere S^2 be partitioned into A , B , and C such that B is the thickening of the equator. Thus, AB and BC are two

disks. Suppose $\rho_{ABC}, \sigma_{ABC} \in \Sigma(S^2)$. As both AB and BC are disks, Lemma 3 implies that $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$. Then, by **P0**, we have $I(A:C|B) = 0$ for both ρ and σ . Thus, the states are determined by the marginals on AB and BC . Thus, the two global states $\rho_{ABC} = \sigma_{ABC}$. This completes the proof. ■

APPENDIX B: INFORMATION CONVEX SET BEYOND CLOSED MANIFOLDS

In Appendixes A 4, A 5, and A 7, we discuss the properties of information convex sets on closed manifolds. A similar definition applies to regions with boundaries. When we discuss “boundaries” in the bootstrap context, there are two types: (i) physical boundary, beyond which the system is in a product state and does not mediate correlation or entanglement, and (ii) correlation boundary (generalizing the notion of entanglement boundary discussed in Refs. [10,13]), which is not. The boundary we shall be interested in is the correlation boundary.

In general, we allow Ω to be a region embedded in the background physical system equipped with a fixed-point state σ satisfying **P0**, **M0**, and **M1**. Let us make a few remarks about the definition.

- (1) We always consider a region that contains a finite number of coarse-grained sites. Each site has a finite Hilbert space dimension. Again, one can show that the set $\Sigma(\Omega)$ is a compact convex set.
- (2) Information convex sets associated with different regions are related to each other nontrivially. Notably

$$\forall \rho_{AB} \in \Sigma(AB), \quad \text{Tr}_B \rho_{AB} \in \Sigma(A). \quad (\text{B1})$$

- (3) Let D be a disk; then, $\Sigma(D) = \{\sigma_D\}$. This generalizes Lemma 3.
- (4) The purpose of having the full set of conditions near the entanglement boundary is to ensure that qubits located in far-separated parts of the entanglement boundary do not entangle with each other. This definition is a close analog to Definition C.1 in Ref. [10]. To see the similarity more clearly, we note that the properties inherited from the fixed-point state and the argument in Appendix A 1 imply the properties listed in Fig. 25. This further guarantees that the states on region Ω can be smoothly

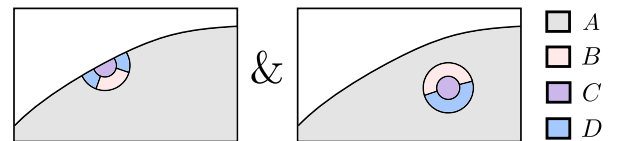


FIG. 25. Some properties of $\rho \in \Sigma(\Omega)$. $I(A:C)_\rho = 0$ and $I(A:C|B)_\rho = 0$ for the configurations shown. Here, the region B is disklike. In the left figure, B may fill in the blank region on the left, right, or both.

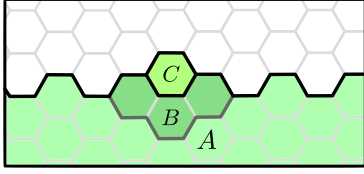


FIG. 26. What we mean by an elementary step. $\Omega = AB$ and $\Omega' = ABC$ are related by an elementary step. BC is within a bounded radius ball, and only part of C is shown.

extended, leading to the important *isomorphism* property, Eq. (B2), as we discuss below.

- (5) More broadly speaking, the definition can be further generalized to immersed (i.e., locally embedded) regions. The reason is that the first condition in Definition 2 puts consistency on local balls only. While we omit the discussion of immersed regions for simplicity, we point out that the information convex sets for such regions is known to give nontrivial constraints for braiding statistics [13,80] in pure-state entanglement bootstrap.

Let us explain the notion of “smooth deformation” of an embedded region. We say embedded regions Ω and Ω' can be smoothly deformed into each other (denoted as $\Omega \sim \Omega'$) if they are related by a finite sequence of elementary steps. Here, an elementary step is the process of adding or deleting a small ball near the entanglement boundary in a way that preserves the topology of the region, as shown in Fig. 26.

The important statement (known as the isomorphism theorem [10]) is the statement that

$$\forall \Omega \sim \Omega', \quad \Sigma(\Omega) \cong \Sigma(\Omega'). \quad (\text{B2})$$

Here, $\Omega \sim \Omega'$ (reads Ω is smoothly deformable to Ω') if two subsystems Ω and Ω' can be related by a finite sequence of elementary steps (Fig. 26). The meaning of isomorphism “ $\cong$ ” is explained in a few paragraphs, and for now, we just mention that an isomorphism of the information convex sets is obtained by a sequence of quantum channels that is determined by the elementary steps in the deformation process $\Omega \sim \Omega'$. It is, thus, sufficient to explain what an elementary step does [132]. Consider the following elementary steps for $\Omega = AB$ and $\Omega' = ABC$ as in Fig. 26. Note that $I(A:C|B)_\rho = 0$ for any $\rho \in \Sigma(ABC)$, following Definition 2. The nontrivial statements are the operations on the convex set of quantum states.

- (i) Elementary step of extension:

$$\mathcal{E}_{B \rightarrow BC}^\sigma(\rho_{AB}) = \rho_{ABC} \in \Sigma(ABC), \quad (\text{B3})$$

for any $\rho_{AB} \in \Sigma(AB)$. Here, $\mathcal{E}_{B \rightarrow BC}^\sigma$ is the Petz map defined from the reference state.

- (ii) Elementary step of restriction:

$$\text{Tr}_C(\rho_{ABC}) = \rho_{AB} \in \Sigma(AB), \quad (\text{B4})$$

for any $\rho_{ABC} \in \Sigma(ABC)$.

Importantly, the composition of $\mathcal{E}_{B \rightarrow BC} \circ \text{Tr}_C$ is an identity on the set $\Sigma(ABC)$, and $\text{Tr}_C \circ \mathcal{E}_{B \rightarrow BC}$ is an identity on the set $\Sigma(AB)$. Thus, the two sets must be isomorphic. This further implies the three conditions below, which underpin our notion of isomorphism “ $\cong$ ”.

- (1) (Geometry preserving) The two sets must be identical as geometrical convex sets in the sense that the convex combination commutes with the mapping:

$$\Phi[p\rho + (1-p)\lambda] = p\Phi(\rho) + (1-p)\Phi(\lambda). \quad (\text{B5})$$

- (2) (Distance preserving) The distance measure between elements is preserved:

$$D(\rho, \lambda) = D[\Phi(\rho), \Phi(\lambda)], \quad (\text{B6})$$

where $D(\cdot, \cdot)$ is any distance measure that is monotonic under quantum channels and Φ is the isomorphism map between the sets.

- (3) (Entropy difference preserving)

$$S(\rho) - S(\lambda) = S[\Phi(\rho)] - S[\Phi(\lambda)]. \quad (\text{B7})$$

These properties imply further conclusions, including the conservation of memory capacity:

$$\mathcal{Q}(\Omega|\sigma) = \mathcal{Q}(\Omega'|\sigma), \quad \forall \Omega \sim \Omega'. \quad (\text{B8})$$

Importantly, the isomorphism guarantees a sense of uniformity of many-body quantum states on the 2D manifold. This is because it is possible to deform a region Ω to another Ω' that is separated far apart from it. That the structure captured by the information convex sets of Ω and Ω' (e.g., the topological charges) is isomorphic implies that the system has an emergent homogeneity without invoking any symmetry (e.g., translation).

1. Annulus and memory capacity

Let X be an annulus that is embedded in a larger disk on which we have the reference state. Under **P0**, **M0**, and **M1**, we prove the following.

Lemma 4. The reference state reduced density matrix $\sigma_X := \text{Tr}_{\bar{X}}\sigma$ is an element of the information convex set of the annulus, $\Sigma(X)$. Moreover,

- (1) σ_X is a state with (absolute) minimum von Neumann entropy in the set $\Sigma(X)$;
- (2) σ_X is an extreme point of $\Sigma(X)$.

Proof. The fact that $\sigma_X \in \Sigma(X)$ follows immediately from the definition of information convex set (Definition 2).

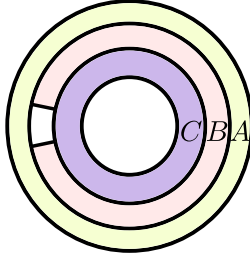


FIG. 27. Delete a hole (white rectangle) from a contractable annulus X to obtain $ABC \subset X$.

Now consider an arbitrary element of $\Sigma(X)$, denoted as α_X . For the partition in Fig. 27,

$$\begin{aligned} I(A:C|B)|_\sigma^\alpha &= S(\alpha_{AB}) + S(\alpha_{BC}) - S(\alpha_B) - S(\alpha_{ABC}) \\ &\quad - S(\sigma_{AB}) - S(\sigma_{BC}) + S(\sigma_B) + S(\sigma_{ABC}) \\ &= S(\alpha_X) - S(\sigma_X). \end{aligned} \quad (\text{B9})$$

This follows because of three facts. First, $S_{AB}|_\sigma^\alpha$ and $S_{BC}|_\sigma^\alpha$ are the same, and they are both the same with $S_X|_\sigma^\alpha$. This is a consequence of the isomorphism theorem. Second, $S_B|_\sigma^\alpha = 0$ because all states in the information convex set must be identical on any disklike subregion, e.g., B . Third, $S_{ABC}|_\sigma^\alpha = S_X|_\sigma^\alpha$. This is because creating a hole on the annulus X gives ABC , and punching such a hole cannot change the entropy difference between α and σ .

From Eq. (B9) and the fact that conditional mutual information is non-negative, we see that $S(\alpha_X) \geq S(\sigma_X)$ for all states in $\Sigma(X)$. In other words, σ_X is a state with the absolute minimum von Neumann entropy in the set $\Sigma(X)$.

Lastly, we show σ_X is an extreme point. Suppose σ_X is not an extreme point; then, there exists $\alpha_X, \beta_X \in \Sigma(X)$ such that $\sigma_X = p\alpha_X + (1-p)\beta_X$ with $p \in (0, 1)$ and $\alpha_X \neq \beta_X$. But then

$$\begin{aligned} S(\sigma_X) &> pS(\alpha_X) + (1-p)S(\beta_X) \\ &\geq \min\{S(\alpha_X), S(\beta_X)\}. \end{aligned} \quad (\text{B10})$$

This leads to a contradiction. This completes the proof. ■

With this, we have the following.

Proposition 8. For a contractable annulus X , the memory capacity of the information convex set equals twice the topological entropy:

$$\mathcal{Q}(X|\sigma) = 2\gamma_{\text{LW}}. \quad (\text{B11})$$

Proof. We apply the same proof strategy as Refs. [10,13]. The max-entropy state ρ_X^* of the information convex set $\Sigma(X|\sigma)$ is a Markov state having $I(A:C|B)_{\rho^*} = 0$ for the Levin-Wen partition of the annulus. On the other hand, the vacuum state σ_X is a state in $\Sigma(X|\sigma)$ that reaches the absolute minimum of entropy by Lemma 4. Thus,

$$\begin{aligned} \mathcal{Q}(X|\sigma) &= \max_{\rho, \lambda} (S(\rho_X) - S(\lambda_X)) \\ &= S(\rho_X^*) - S(\sigma_X) \\ &= I(A:C|B)_\sigma - I(A:C|B)_{\rho^*} \\ &= 2\gamma_{\text{LW}} - 0 \\ &= 2\gamma_{\text{LW}}. \end{aligned} \quad (\text{B12})$$

The first equality is the definition of memory capacity. The third equality uses the fact that any state in the information convex set of $X = ABC$ reduces to the same state on AB , BC , and B . This completes the proof. ■

In this perspective, the topological entropy γ_{LW} is a derived quantity that captures part of the richer structure.

APPENDIX C: ROBUSTNESS OF ISOMORPHISM

To analyze the robustness of the isomorphism away from the fixed points of topological mixed states, we allow relaxations in which the axioms may take small decaying values rather than being required to vanish exactly. In addition, we also relax the strict local equivalence condition of the information convex set. We, therefore, define a notion of approximate information convex set as a refinement to Definition 2.

Definition 12 (approximate information convex set). For a subsystem Ω and a relaxation $\mathbf{q} \geq 0$, we define the approximate information convex set as

$$\Sigma_{\mathbf{q}}(\Omega) := \{\rho_\Omega | (1), (2)\}. \quad (\text{C1})$$

- (i) $\|\rho_b - \sigma_b\|_1 \leq \mathbf{q}$ for bounded radius balls $b \subset \Omega$.
- (ii) $I(E:C|B)_\rho \leq \mathbf{q}$ for configurations in Fig. 4.

With this definition stated, we now analyze whether deformations of the information convex set within the physical system preserve its structure under errors decaying with the length scale. If so, the topological data extracted from a local region remain invariant across the system, thereby ensuring that the information convex set is stable under small, decaying perturbations.

Now we discuss a toy problem that the axioms are satisfied exactly on the left part of the system and is noisy on the right side of the system with axioms violated with errors up to $\delta(r) \leq \mathcal{O}(1)e^{-r/\xi}$, as shown in the geometry in Fig. 28(a). Suppose we transport the states on region Ω to the noisy region (to Ω') and back by quantum channels Φ_{forward} and Φ_{backward} . We ask if the information is approximately preserved. This corresponds to an estimation of the error $\mathbf{q}$ in

$$\Phi_{\text{backward}} \circ \Phi_{\text{forward}}: \Sigma(\Omega) \rightarrow \Sigma_{\mathbf{q}}(\Omega). \quad (\text{C2})$$

Using tools of quantum information theory like Fawzi-Renner inequality [133], Alicki-Fannes-Winter inequality [134,135], and Fuchs–Van de Graaf inequality [136] and

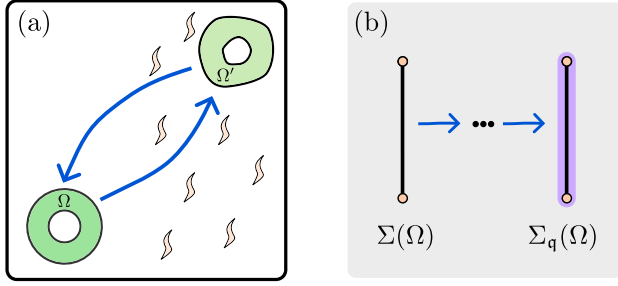


FIG. 28. On the robust (approximate) isomorphism of information convex sets.

the idea of merging quantum Markov states [115,137], we are able to come up with an estimation:

$$q(r) \leq \text{poly}(r) \exp\left(-\frac{r}{c\xi}\right), \quad (\text{C3})$$

where $c \in \mathbb{R}_+$ is an $O(1)$ constant that depends on the details of the deformation path but is independent of the length scale r . In particular, the structure of the information convex set is preserved up to exponentially suppressed corrections, so that violations of the axioms induce only a tiny “blurring” of the convex set geometry as shown in Fig. 28(b). We refer to a future work [138] for details. Importantly, the degree of $\text{poly}(r)$ in Eq. (C3) is still unclear. If one can bound the degree of the polynomial, then the convex set structure is still preserved under certain milder power-law decay violations of the axioms.

APPENDIX D: BOUND ON MEMORY CAPACITY

In this appendix, we prove a general theorem (Theorem 1) which gives an upper bound of memory capacity in terms of the topological entropy γ_{LW} . This implies the bound on capacity in the main text (Proposition 1). We start with a useful lemma.

Lemma 5. Consider regions A, B, C , and D embedded in M as in Fig. 29(a). Then,

$$I(A:C|B)_\rho = 2\gamma_{\text{LW}}, \quad \forall \rho \in \Sigma(M|\sigma). \quad (\text{D1})$$

Proof. Let us relabel D as $D_1 D_2$ and A as $A_1 A_2$ as shown in Fig. 29(b). First, we note that

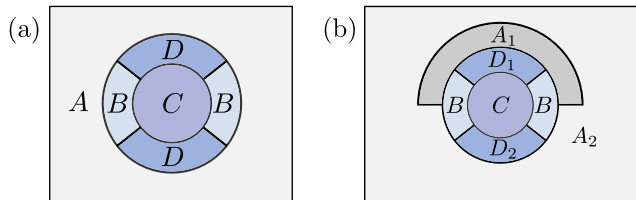


FIG. 29. Partition the closed manifold M into a ball BCD (with topology as shown) and the complement A .

$$\begin{aligned} I(A:C|B)_\rho &\geq I(A_1:C|B)_\rho \\ &= 2\gamma_{\text{LW}}. \end{aligned} \quad (\text{D2})$$

The inequality follows from the strong subadditivity. Second,

$$\begin{aligned} I(A:C|B)_\rho - 2\gamma_{\text{LW}} &= I(A_1 A_2:C|B)_\rho - I(A_1:C|B)_\sigma \\ &= I(A_1 A_2:C|B)_\rho - I(A_1:C|B)_\rho \\ &= I(A_2:C|A_1 B)_\rho \\ &\leq I(A_2:CD_1|A_1 B)_\rho \\ &= 0. \end{aligned} \quad (\text{D3})$$

The “=” in the first line uses the definition of γ_{LW} , noticing that A_1, B, C is the Levin-Wen partition of an annulus $A_1 B C$; the annulus is embedded in a disk. The second line uses that $\rho_M \in \Sigma(M)$ is identical to the reference state σ on the disk containing $A_1 B C$. The third line is obtained by the chain rule (i.e., expansion and trivial rewriting) of the conditional mutual information. The fourth line is due to the strong subadditivity. The last line is by axiom **M1** with the choice of the local ball $(A_1 B, CD_1, D_2)$.

By Eq. (D3), $I(A:C|B)_\rho \leq 2\gamma_{\text{LW}}$, while Eq. (D2) says $I(A:C|B)_\rho \geq 2\gamma_{\text{LW}}$. Thus, Eq. (D1) holds. ■

Theorem 1. Suppose σ_M is a reference state on closed 2D manifold M , satisfying all the axioms **P0**, **M0**, and **M1**. The memory capacity satisfies

$$\mathcal{Q}(M|\sigma) \leq 2n(M)\gamma_{\text{LW}}, \quad (\text{D4})$$

where γ_{LW} is the topological entropy and $n(M)$ is an integer, such that

- (i) if M is orientable, i.e., $M = \#g\mathbb{T}^2$, $n(M) = 2g$;
- (ii) if M is nonorientable, $M = \#h\mathbb{RP}^2$, $n(M) = h$.

Proof. Let us start with a basic topology observation. Let M^* be the manifold with boundary obtained by removing a ball from M . Then, M^* can be converted to a disk by cutting $n(M)$ handles. See Fig. 30(a) for the meaning of cutting a handle.

Now, let us discuss the quantum aspect of removing a ball and cutting a handle. Removing an isolated ball does not change the entropy difference:

$$S(\rho_M) - S(\lambda_M) = S(\rho_{M^*}) - S(\lambda_{M^*}), \quad \rho, \lambda \in \Sigma(M). \quad (\text{D5})$$

This follows from condition **P0** on the two states ρ and λ and the fact that their local reduced density matrices are identical.

On the other hand, cutting a handle can lower the entropy difference by an amount. Let $ABC \subset M$ as shown in Fig. 30(b). For $\rho, \lambda \in \Sigma(M)$,

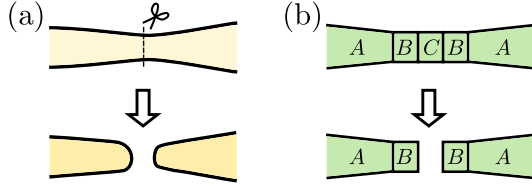


FIG. 30. Cut handle and how it affects the entropy difference. In (b), only part of the region A is shown.

$$\begin{aligned} S_{ABC}|_\lambda^\rho - S_{AB}|_\lambda^\rho &= I(A:C|B)|_\lambda^\rho \\ &\leq I(A:C|B)_\lambda \\ &\leq 2\gamma_{\text{LW}}. \end{aligned} \quad (\text{D6})$$

The first line is because ρ and λ are identical on BC . The second line follows because $I(A:C|B)_\rho$ is non-negative. The last line follows from Lemma 5.

This implies that the entropy difference $S_{ABC}|_\lambda^\rho$ and $S_{AB}|_\lambda^\rho$ may be different, but their difference is upper bounded, due to the fact that AB is obtained by ABC by cutting a handle. But, on a disk, we have a unique state, by Lemma 3. Thus, the memory capacity of M can be at most $2n(M)\gamma_{\text{LW}}$. This completes the proof. ■

This proof is elementary. The idea is inspired by the equally elementary proof by Kim [88] as well as the uniqueness of the ground state of the invertible state [139]. In the pure-state context, Kim's bound is stricter than ours, and the bound saturates for Abelian anyon theory; however, it is not optimal when there are non-Abelian anyons. The more advanced pure-state bootstrap technique can give a precise prediction on the memory capacity [13]. The analog of such results in the mixed-state setup remains an open problem.

APPENDIX E: QUANTITIES THAT FAIL TO BE INVARIANT ON TOPOLOGICAL MIXED STATES

We demonstrate why certain topological invariants on the ground state of topological order are no longer invariant under smooth deformation of the regions for topological mixed states. We show that **P0**, **M0**, and **M1** cannot guarantee the invariance of

- (i) the Kitaev-Preskill entropy [78]

$$\gamma_{\text{KP}} := S_{AB} + S_{BC} + S_{CA} - S_A - S_B - S_C - S_{ABC}; \quad (\text{E1})$$

- (ii) the modular commutator [127]

$$J(A, B, C) = i\text{Tr}(\rho_{ABC}[\ln \rho_{AB}, \ln \rho_{BC}]), \quad (\text{E2})$$

where the partition A, B, C in question is that in Fig. 31(a). We construct the counterexample as follows. We start with a fixed-point state σ on a certain coarse-grained lattice,

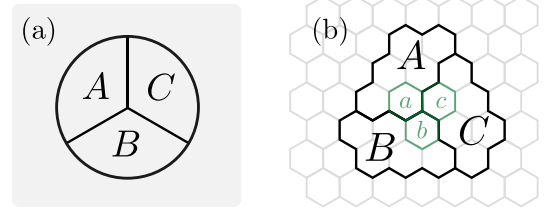


FIG. 31. A coarse-grained lattice where A, B, C form a useful partition of the disk. $a \subset A$, $b \subset B$, and $c \subset C$.

where σ satisfies all three axioms on this coarse-grained lattice as exemplified in Fig. 31(b). Let us further suppose that σ has γ_{KP} and $J(A, B, C)$ to be the same value anywhere on the coarse-grained lattice.

Next, we stack a mixed state λ_{abc} on the three coarse-grained sites a, b , and c in Fig. 31(b) to obtain

$$\tilde{\sigma} = \sigma \otimes \lambda_{abc}. \quad (\text{E3})$$

By choosing λ_{abc} appropriately, we can shift γ_{KP} and $J(A, B, C)$ for $\tilde{\sigma}$ relative to those for σ . In particular,

$$\begin{aligned} \lambda_{abc}^\gamma &= \frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|), \\ \lambda_{abc}^J &= \frac{1}{8} \left[I + \frac{1}{2}(X_a X_b + Z_b Z_c + X_a Y_b Z_c) \right] \end{aligned} \quad (\text{E4})$$

decrease γ_{KP} by $\log 2$ and J by $\simeq 0.3$, respectively. These shifts occur only for tripartitions (A, B, C) whose triple-intersection point coincides with the center of the abc cell.

Furthermore, for any state λ_{abc} supported on the three adjacent coarse-grained cells, the stacked state $\tilde{\sigma} = \sigma \otimes \lambda_{abc}$ satisfies the axioms on the coarse-grained lattice. Because a, b , and c are adjacent cells, when it comes to the calculation of mutual information or conditional mutual information between *separated* regions, their correlation never appears; furthermore, λ_{abc} is in a tensor product with σ , which already satisfies the axioms.

Finally, it is easy to see why, in the pure-state version of the entanglement bootstrap, such counterexamples do not exist. In this context, we are allowed only to stack pure state $|\lambda_{abc}\rangle$ for preserving axioms the pure-state axioms; suppose that we tensor with a mixed state at a local region, and then the axioms **A0** centered at the region will be violated by twice the entropy of the mixed state. Such a pure state does not give any correction to γ_{KP} and J . In particular, $J(A, B, C)_{|\psi_{ABC}\rangle} = i\langle [\ln \rho_{AB}, \ln \rho_{BC}] \rangle = i\langle [\ln \rho_C, \ln \rho_A] \rangle = 0$ for any tripartite pure state $|\psi_{ABC}\rangle$ [91].

APPENDIX F: DECOHERENCE AND ENTROPY CONDITIONS

In this appendix, we give details on the calculations of axiom violations. In Appendix F 1, we analytically compute the violations of various entropy conditions at the

mixed-state fixed point in the vicinity of a domain wall between ρ^e and $\otimes_i \rho_i$. Such a fixed-point state is a stabilizer state, and, thus, we borrow well-known tools to compute the entanglement entropy [140]. In Appendix F 2, we explain the numerical method to compute the entropy for arbitrary error rate $p \in [0, 1]$ from the nontrivial state, following the technique in Ref. [26]. This also explains the method to obtain Fig. 14. In Appendix F 3, we explain why the quantum channel from a trivial state to maximally dephased toric code ρ^e must be fragile.

1. Fixed point of the domain wall and the violation of M1

a. Stabilizer state entropy

Let us first make a general remark on the computation of entanglement entropy of stabilizer states. Let $\{S_i\}_{i=1}^k$ be a set of stabilizer generators. Each of them is a product of Pauli operators, $S_j^2 = 1$; they mutually commute $[S_i, S_j] = 0$, and they are independent as generators of the group by multiplication. A stabilizer state associated with $\{S_i\}_{i=1}^k$ on n qubits is

$$\rho[\{S_i\}_{i=1}^k] := \frac{1}{2^n} \prod_{j=1}^k (1 + S_j). \quad (\text{F1})$$

The entanglement entropy of a stabilizer state is $S(\rho) = (n - k) \log 2$. In other words, computing the entropy corresponds to finding the number of independent stabilizer generators.

b. Domain wall configuration

We consider the dry boundary and the wet boundaries in Fig. 15. On the upper half-plane, we have the fixed point $\alpha = \prod_v (1 + A_v)$, and on the lower half-plane, we have $\beta = \prod_i |+\rangle^{\otimes N}$. As both states are stabilizer states, we need only to specify the stabilizer generators. This is shown in Fig. 32. On the upper half-plane, each vertex has a stabilizer generator $A_v = \prod_{e \ni v} X_e$. On the lower half-plane, the stabilizer generator is X_e on each link e . On the domain wall lies the difference.

- (i) For each dry boundary site, we do not associate any stabilizer generator; see Fig. 32(a).
- (ii) For the wet boundary, for each boundary vertex v' we have a stabilizer generator that is the product of X_e on three adjacent links pointing left, right and up; see Fig. 32(b).

Based on this knowledge of stabilizer generators, one can compute the von Neumann entropy of any finite region, and, thus, we can compute the correction to the axioms explicitly. Nonetheless, we mention an alternative way to compute the conditional mutual information $I(A:C|B)_\rho$, which may be more intuitive.

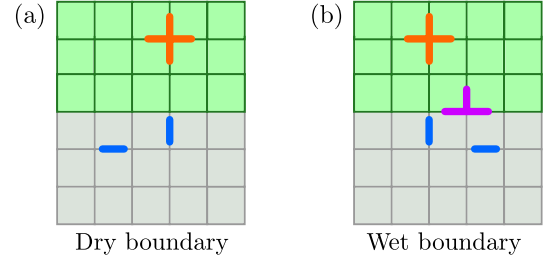


FIG. 32. The stabilizer generators for the dry and wet boundaries. There are three different types of stabilizer generators. Each of them is a product of Pauli X on the colored links. The red ones are centered at a vertex in the upper half-plane. The blue ones are on a single (horizontal or vertical) link on the lower half-plane. Compared to the dry boundary, the wet boundary has an additional purple stabilizer for each boundary site.

Consider a stabilizer state ρ_{ABC} . Its reduced density matrices $\{\rho_{AB}, \rho_{BC}, \rho_B\}$ must all be stabilizer states. Then,

$$I(A:C|B)_\rho = (k_B + k_{ABC} - k_{AB} - k_{BC}) \log 2 \quad (\text{F2})$$

due to the cancellation of the Hilbert space dimensions, where k_X is the number of stabilizer generators associated with region X . Note that the stabilizer generators associated with ρ_{AB} can be chosen to be the set $\mathcal{S}_{A|B} \cup \mathcal{S}_B$, where $\mathcal{S}_B$ are the stabilizer generators associated with ρ_B and $\mathcal{S}_{A|B}$ is the subset of stabilizers associated with ρ_{AB} which are supported on AB and cannot be reduced to B by multiplication. Similarly, the set of stabilizer generators associated with ρ_{BC} can be chosen to be $\mathcal{S}_{C|B} \cup \mathcal{S}_B$. $|\mathcal{S}_{A|B}| = k_{AB} - k_B$ and $|\mathcal{S}_{C|B}| = k_{BC} - k_B$. Also note $\mathcal{S}_{ABC}^{\text{loc}} := \mathcal{S}_{A|B} \cup \mathcal{S}_B \cup \mathcal{S}_{C|B}$ is a set of independent stabilizer generators, $\mathcal{S}_{ABC}^{\text{loc}} \subset \mathcal{S}_{ABC}$. Physically speaking, $\mathcal{S}_{ABC}^{\text{loc}}$ is the subset of stabilizer generators of ρ_{ABC} that one can infer from local regions AB and BC . By simple counting

$$k_{ABC} + k_B - k_{AB} - k_{BC} = |\mathcal{S}_{ABC}| - |\mathcal{S}_{ABC}^{\text{loc}}|, \quad (\text{F3})$$

one can see that the computation of conditional mutual information $I(A:C|B)_\rho$ of the stabilizer state ρ_{ABC} is boiled down to finding the number of stabilizer generators that cannot be inferred from local regions.

From this reasoning, we verified the value of γ_{dry} and γ_{wet} in Table I and Fig. 17. The preserved axioms on the boundary, shown in Fig. 16, are verified with the same consideration.

2. Numerical method and data

We explain the numerical method to obtain the data in Fig. 14 in the main text. We need to calculate the von Neumann entropy of a local region at any error rate $p \in [0, 1/2]$. Especially, we want to be able to do such a computation at the spatial boundary of decoherence.

Following the method in Ref. [26], since states with different anyonic configurations are orthogonal to each other, we have

$$S\left(\sum_i p_i \rho_i\right) = \sum_i p_i S(\rho_i) + H(\{p_i\}), \quad (\text{F4})$$

where $H(\{p_i\}) := -\sum_i p_i \log p_i$ denotes the Shannon entropy and subscript i denotes the anyon configuration excited by errors. p_i is the error probability that gives the anyon configuration labeled by i . The von Neumann entropy is the same across each ρ_i . For the no-error event corresponding to the state ρ_0 , the axioms are already satisfied. The first terms in Eq. (F4) would be canceled when it comes to the calculation of the conditional mutual information; therefore, even for a noisy case, this contribution would disappear. To calculate the second term, we can use the Monte Carlo method:

$$H(\{p_i\}) = -\sum_i p_i \log(p_i) = -\mathbb{E}_{p_i}[\log(p_i)]. \quad (\text{F5})$$

Thus, we can sample the anyonic configuration from the error probability p at each edge. After getting the configuration i , we can get p_i by contracting a tensor network. The sampling process is exact, and no Markov chain Monte Carlo procedure is needed. Note that the variance is reduced if we calculate the violations of **P0** and **M1** for the same samples $\{p_{i,ABC}\}$:

$$I(A:C|B) = -\mathbb{E}_{p_i}[\log(p_{i,AB}) + \log(p_{i,BC}) - \log(p_{i,B}) - \log(p_{i,ABC})], \quad (\text{F6})$$

where $\{p_{i,AB}, p_{i,B}, p_{i,BC}\}$ are computed from the same samples as $\{p_{i,ABC}\}$.

The computation of probabilities for a given anyon configuration is done by the contraction of a tensor network, as in Ref. [26]. To illustrate how the data in Fig. 14 are obtained, we write the explicit tensors for $r = 3$ for the calculation of probabilities related to the **M1** violation δ_1 in Fig. 14:

$$p_{i,ABC} = \text{[Diagram: A 10x10 grid of blue squares with red lines connecting them. A path of yellow squares is highlighted, starting from the top left and ending at the bottom right, representing a specific anyon configuration.]}, \quad (\text{F7})$$

$$p_{i,AB} =$$

$$= \frac{1}{2} \cdot \text{[Diagram: A 10x10 grid of blue squares with red lines. A path of yellow squares is highlighted, similar to Fig. F7 but with a different configuration.]}. \quad (\text{F8})$$

The upper middle tensor in the first line is $\delta_{\sum_i a_i \equiv 1 \pmod{2}}$. Now consider a valid set of $\{a_i\}$; the number of the degree left for the inner links is only two and both of their weights are one, so it is 2 times the original tensor network:

$$p_{i,B} = \text{[Diagram: A 10x10 grid of blue squares with red lines. A path of yellow squares is highlighted, similar to Fig. F7 but with a different configuration.]}, \quad (\text{F9})$$

$$p_{i,BC} = \text{[Diagram: A 10x10 grid of blue squares with red lines. A path of yellow squares is highlighted, similar to Fig. F7 but with a different configuration.]}. \quad (\text{F10})$$

Even though the partition AB crosses the boundary, because region AB is not simply connected as shown in Fig. 14(a), we still need to take into account the possible excitation of this closed loop stabilizer in the lower half-plane. The sample sizes for $r = [4, 5, 6, 7]$ for Fig. 14 are $[10^7, 2.5 \times 10^6, 2.5 \times 10^5, 5 \times 10^4]$, respectively.

Finally, the local tensors are

$$\text{[Diagram: A blue square with a cross inside]} = \delta_{i+j+k+l \equiv 0 \pmod{2}}, \quad (\text{F11})$$

$$\text{[Diagram: A yellow square with a cross inside]} = \delta_{i+j+k+l \equiv 1 \pmod{2}}, \quad (\text{F12})$$

$$\text{[Diagram: A red circle on a line]} = \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix}. \quad (\text{F13})$$

For three-leg tensors, they are $\delta_{i+j+k \equiv 0,1 \pmod 2}$ depending whether there is an anyon there. Finally, the uncontracted tensor legs will be summed over in the end. No approximation is used for Fig. 14, but approximate frameworks such as boundary matrix product states (MPSs) or tensor renormalization group methods [141,142] may help us investigate larger system sizes.

3. From product state to ρ^e and fragility

Starting from $|+\rangle^{\otimes N}$, the density matrix under $\prod_{e \in p} Z_e$ decoherence on every plaquette for the toric code is

$$\rho_u \propto \sum_{l_\perp \in \mathcal{S}_d} \left[\prod_{v \in \partial l_\perp} (1-2p) \right] X^{l_\perp}, \quad (\text{F14})$$

where p is the error rate of B_p^f and $\mathcal{S}_d$ is the set of all possible strings on the dual lattice. In this appendix, we investigate if the state is in the same phase as the dephased toric code ρ^e , above a certain criticality.

For this purpose, we are interested in computing $I(A:C|D)_{\rho_u}$ in Fig. 33(a). Although there is no efficient way to get the von Neumann entropy, we can calculate the Rényi-2 entropy as follows:

$$\exp(-S^{(2)}(\rho_u)) = \text{Tr}(\rho_u^2) = 2^{-N} \times \sum_{l_\perp} (1-2p)^{2|\partial l_\perp|}. \quad (\text{F15})$$

Here, the sum runs over all configurations $l_\perp$.

This computation can be done efficiently using a tensor network, where the $r = 3$ case is illustrated as in Figs. 33(a) and 33(b). The local tensors are

$$\begin{aligned} \text{[Green square with blue cross]} &= \delta_{i+j+k+l \equiv 0 \pmod 2} \\ &+ (1-2p)^2 \delta_{i+j+k+l \equiv 1 \pmod 2}. \end{aligned} \quad (\text{F16})$$

Note that here the computation is simpler than that in Appendix F 2, as the weight of the excitations is on the anyons themselves instead of being on the edges, so we can sum over all the configurations with one tensor network without doing Monte Carlo. For three-leg tensors, we just replace the delta functions with $\delta_{i+j+k \equiv 0,1 \pmod 2}$, respectively, and the same for two-leg tensors. Finally, the indices of uncontracted legs are summed over as before.

Contracting this tensor network numerically yields the Rényi-2 entropy $S^{(2)}$ as a function of the probability p of applying $\prod_{e \in p} Z_e$ to each plaquette. The result is shown in Fig. 33(c).

a. Another argument, with bulk γ_{LW}

Another indirect way to justify the fragility of the channel preparing the state ρ^e is through the theoretical computation of the Rényi-2 LW topological entropy of the bulk. First, the Rényi-2 entropy of Eq. (F15) can be rewritten as

$$S^{(2)}(\rho) = N \times \log 2 - \ln \mathcal{Z}, \quad \mathcal{Z} = \sum_{l_\perp \in \mathcal{S}_d} e^{-\beta |\partial l_\perp|}, \quad (\text{F17})$$

where $\beta = \ln[1/(1-2p)^2]$. Note that $\mathcal{Z}$ is equivalent to the partition function of the two-dimensional Ising gauge theory since

$$\mathcal{Z} = \sum_{l_\perp \in \mathcal{S}_d} e^{-\beta |\partial l_\perp|} = \sum_{\sigma} e^{-\beta \sum_p \prod_{e \in p} \sigma_e}, \quad (\text{F18})$$

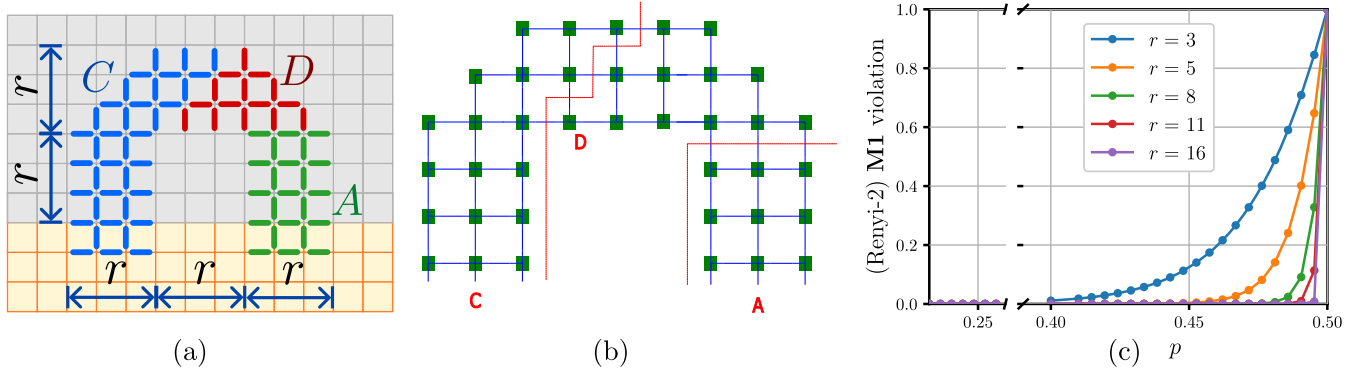


FIG. 33. Fragility of the wet boundary. (a) Decoherence on the upper half-plane, where the Z decoherence acts on the gray links. The subsystems A , C , and D are shown at $r = 3$. For $p = 0.5$, the decohered state (F14) has a wet boundary between ρ^e and the product state. (b) The tensor network to calculate Rényi-2 version of $I(A:C|D)$, which diagnoses a Rényi-2 analogue of the **M1** violation at the boundary $\delta_1^{(2)} := S_{AD}^{(2)} + S_{DC}^{(2)} - S_D^{(2)} - S_{ADC}^{(2)}$ (known as $2\gamma_{\text{wet}}$ in Sec. IV C). (c) A plot of Rényi-2 **M1** violation $\delta_1^{(2)}$, in units of $\log 2$, as a function of the error rate p and subsystem sizes r .

where $\sigma = \{\sigma_e\}$ lives in the links of the dual lattice. Let $F_R = -\ln \mathcal{Z}_R$ be the free energy of the region R . Then, the (Rényi-2) topological entropy is given as

$$\gamma_{\text{LW}}^{(2)} = \frac{1}{2}(F_{AB} + F_{BC} - F_B - F_{ABC}), \quad (\text{F19})$$

where the partition is that in Eq. (8). However, since the Wilson-loop expectation value obeys an area law, at any finite temperature $\gamma_{\text{LW}}^{(2)}$ should vanish for large enough system size. At zero temperature, the exact twofold topological degeneracy (from loop) provides F_{ABC} smaller than $F_{AB} + F_{BC} - F_B$ by $\log 2$ but not at any finite temperature.

APPENDIX G: APPLICATION OF PHASE EQUIVALENCE

We discuss a few applications of the mixed-state phases equivalence as in Definition 6.

1. Phase distinction beyond topological entropy

We provide examples of mixed-state fixed points which can be distinguished even though they have identical topological entropy.

The key idea is to look at the information convex set of a certain region and use the isomorphism theorem. If two mixed-state fixed points $\alpha \sim \beta$, they can be joined by the transparent wall (Definition 6). Therefore, according to the isomorphism theorem [Eq. (B2)], the information convex sets for a given region embedded (or immersed) in these states must be isomorphic. We argue that such statements should also apply to the approximate information convex sets Definition 12. The following examples are based on this approach.

Example 12 (toric code versus Ising anyon)—As a warm-up example, we explain why the toric-code fixed point ρ^{TC} [111] and the Ising anyon fixed point ρ^{Ising} [6,8] are different by their information convex sets. Note that both theories have identical topological entropy $\gamma_{\text{LW}} = \log 2$ and, thus, topological entropy cannot distinguish them.

Consider the information convex set of the annulus. $\Sigma(\text{annulus}|\rho^{\text{TC}})$ is a simplex with four extreme points. In contrast, $\Sigma(\text{annulus}|\rho^{\text{Ising}})$ has three extreme points. Because of this difference, there cannot be a transparent wall connecting these two fixed points, when we treat them as mixed-state fixed points. It follows that

$$\rho^{\text{TC}} \not\sim \rho^{\text{Ising}}. \quad (\text{G1})$$

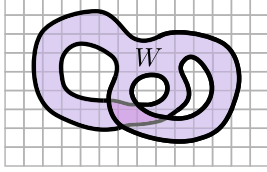
Example 13 (semion versus ρ^e)—We show that the chiral semion fixed point ρ^s and the decohered toric-code fixed point ρ^e are in different phases:

$$\rho^s \not\sim \rho^e. \quad (\text{G2})$$

This does not follow from the topological entropy, because $\gamma_{\text{LW}} = \log \sqrt{2}$. The information convex set of the annulus cannot distinguish them either because

$$\Sigma(\text{annulus}|\rho^s) \cong \Sigma(\text{annulus}|\rho^e) \cong \text{interval}. \quad (\text{G3})$$

An interval is a convex set with a pair of extreme points. Such extreme points are mutually orthogonal, and due to the Abelian nature of the theory, no distinction can be extracted from the entropy difference either. The justification of Eq. (G2) lies in the information convex set of an immersed punctured torus W :



$$\begin{aligned} \Sigma(W|\rho^e) &= \text{tetrahedron} \\ \Sigma(W|\rho^s) &= \text{Bloch ball} \end{aligned} \quad (\text{G4})$$

For ρ^e the information convex set associated with W is purely classical, with four extreme points. In contrast, for ρ^s , the information convex set forms a Bloch ball with a sphere of extreme points. This completes the argument.

More generally, there is a clear contrast of the pure-state bootstrap reference state σ^{pure} satisfying **A0** and **A1** and the decohered G quantum double (i.e., classical loop state) ρ^m in Eq. (48).

- (i) There exists a state in $\Sigma(W|\rho^m)$ such that its reduced density matrix to annuli $X \subset W$ and $Y \subset W$ (as in Fig. 34) can simultaneously be identical to the vacuum state $\rho^m|_X, \rho^m|_Y$.
- (ii) For σ^{pure} , there cannot be any state in $\Sigma(W|\sigma^{\text{pure}})$ which simultaneously agrees with the vacuum upon reduced to both X and Y , unless the topological entropy $\gamma_{\text{LW}} = 0$. This is due to the entropic uncertainty relations forced by information theoretic constraints [13,92].

This distinction arises because braiding is necessarily nondegenerate in the pure-state case [13].

The double semion and the toric code have isomorphic information convex sets for all immersed regions. Thus, they cannot be distinguished by the shapes of the information convex sets. This is expected as the information convex set detects only the fusion properties rather than

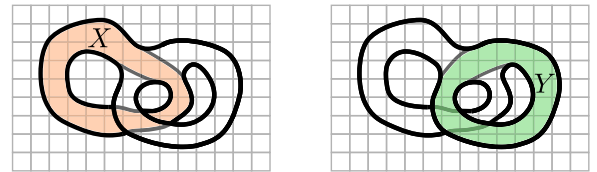


FIG. 34. Two annuli X and Y are subsets of the immersed punctured torus W . Both X and Y are embedded on the physical system. X and Y correspond to two noncontractible annuli of the punctured torus ($X, Y \subset W$).

braiding statistics. Nonetheless, braiding statistics *can* be detected by the automorphism of the information convex sets induced by a nontrivial deformation of the immersed region. Applying the idea in Ref. [80], we can distinguish this pair of phases. (The extraction of topological S -matrix and the proof of braiding nondegeneracy in Ref. [90] needed both the string operators and immersed regions which can distinguish the two phases in the context of pure-state bootstrap. However, for mixed-state phases, the approach in Ref. [80] Appendix B is useful to bypass the need of string operators.)

Example 14 (ρ^e versus ρ^f)—Is it true that the e - and f -decohered toric codes belong to two different phases? While previous literature argued an answer based on strong symmetry [23], no proof has been provided in the context when there are perturbations breaking the exact 1-form strong symmetry. Our approach opens a possibility to prove such a statement. We left it as an open problem for now, and details will be available in Ref. [138].

APPENDIX H: QUANTUM DOUBLE CALCULATIONS

Our notation of quantum double is similar to previous literature, e.g., Ref. [65]. The quantum double model [111] is a generalization of the toric-code model, which allows a non-Abelian group G . (See Ref. [65] for a clear review.) The lattice model has a tensor product Hilbert space on the edges of a lattice. The Hilbert space on each edge e is $\mathcal{H}_e := \{|g\rangle | g \in G\}$, which forms an orthonormal basis for the Hilbert space of dimension $|G|$. For visual representation convenience, we put an orientation on each edge such that, upon flipping the orientation, we require

$$\left| \begin{array}{c} \uparrow \\ a \end{array} \right\rangle = \left| \begin{array}{c} \downarrow \\ \bar{a} \end{array} \right\rangle. \quad (\text{H1})$$

On each oriented edge e , we define the following operators:

$$T_e^g: T_e^g|h\rangle = \delta_{g,h}|h\rangle, \quad (\text{H2})$$

$$L_e^g: L_e^g|h\rangle = |gh\rangle, \quad (\text{H3})$$

$$\bar{L}_e^g: \bar{L}_e^g|h\rangle = |h\bar{g}\rangle, \quad (\text{H4})$$

where $\bar{g} := g^{-1}$. For an Abelian group, $\bar{L}_e^g = (L_e^g)^\dagger = L_e^{\bar{g}}$. However, for non-Abelian group, in general, $\bar{L}_e^g \neq L_e^{\bar{g}}$.

Equipped with single-edge operators, we define operators acting on the oriented two-cellular complex where $\partial e = \sum_{v \in e} \epsilon_{e,v} v$ and $\partial p = \sum_{e \in p} \epsilon_{e,p} e$, where $\epsilon = \pm 1$ and $\partial^2 p = 0$. For instance, if $e \in E$ is oriented from vertex v_1 to v_2 , one takes $\epsilon_{e,v_1} = 1$ and $\epsilon_{e,v_2} = -1$. With this understanding, define

$$\begin{aligned} A_v^g \left| \begin{array}{c} \uparrow \\ a \\ \downarrow \\ b \end{array} \right\rangle &= \left| \begin{array}{c} \uparrow \\ ga \\ \downarrow \\ gb \end{array} \right\rangle, \\ B_s^g \left| \begin{array}{c} \uparrow \\ c \\ \downarrow \\ a \end{array} \right\rangle &= \delta_{g,abcd} \left| \begin{array}{c} \uparrow \\ c \\ \downarrow \\ a \end{array} \right\rangle. \end{aligned} \quad (\text{H5})$$

Here, v refers to a vertex, and s is a face p with a starting position chosen. What if the orientation of the edges are different from the figure? This is explained by Eq. (H1). For example, if the edge orientation for b is toward v instead of away from v , we can first reverse the orientation, apply A_v^g , and reverse it back as follows:

$$\begin{aligned} A_v^g |a_+, b_-, c_+\rangle &= A_v^g |a_+, \bar{b}_+, c_+\rangle = |ga_+, g\bar{b}_+, gc_+\rangle \\ &= |ga_+, b\bar{g}_-, gc_+\rangle, \end{aligned} \quad (\text{H6})$$

where the subscript is $\epsilon_{e,v}$. The case for B_s^g can be understood similarly.

While A_v^g is unitary, B_s^g is a projector. Using these operators, we define commuting projectors

$$A_v^R := \sum_{g \in G} \frac{\dim R}{|G|} \chi_R(g^{-1}) A_v^g, \quad B_p^C := \sum_{g \in C} B_p^g, \quad (\text{H7})$$

where $R \in (G)_{\text{ir}}$ and $C \in (G)_{\text{cj}}$ are labels for irreducible representation and conjugacy class of G , respectively. The projector properties of B_p^C

$$B_p^C B_p^{C'} = \delta_{C,C'} B_p^C \quad (\text{H8})$$

are obvious, while the projector property of A_v^R that is

$$A_v^R A_v^{R'} = \delta_{R,R'} A_v^R \quad (\text{H9})$$

is verified in Appendix H 1. These two sets of projectors $\{A_v^R\}$ and $\{B_p^C\}$ are commuting:

$$[A_v^R, B_p^C] = 0, \quad \forall v, p. \quad (\text{H10})$$

Of importance are the projectors corresponding to the identity irreducible representation $\mathbf{1} \in (G)_{\text{ir}}$ and $\mathbf{1} \in (G)_{\text{cj}}$:

$$A_v^{\mathbf{1}} = \frac{1}{|G|} \sum_{g \in G} A_v^g \quad \text{and} \quad B_p^{\mathbf{1}} = B_p^{\mathbf{1}}, \quad (\text{H11})$$

where $\mathbf{1} \in G$ is the identity element.

For a given 2D lattice, we can write the quantum double Hamiltonian as

$$H = -\sum_v A_v^{\mathbf{1}} - \sum_p B_p^{\mathbf{1}}. \quad (\text{H12})$$

If we take $G = \mathbb{Z}_2$, we get the toric code.

1. Projector properties

Given a group G , for all elements $g \in G$, denote the unitary matrix of the irreducible representation of g by $\Gamma_R(g)$. We have the great orthogonality relation:

$$\sum_{g \in G} \Gamma_R^{ij}(g) \bar{\Gamma}_{R'}^{i'j'}(g) = \frac{|G|}{\dim R} \delta_{R,R'} \delta_{i,i'} \delta_{j,j'}, \quad (\text{H13})$$

where the bar on top of Γ means complex conjugation. By summing over (i, i', j, j') , we obtain the first orthogonality relation for the characters $\chi_R(g) := \text{Tr} \Gamma_R(g)$:

$$\frac{1}{|G|} \sum_{g \in G} \chi_R(g) \bar{\chi}_{R'}(g) = \delta_{R,R'}. \quad (\text{H14})$$

Finally, we also have the second orthogonality relation:

$$\sum_{R \in (G)_{ir}} \chi_R(C) \bar{\chi}_{R'}(C') = \frac{|G|}{|C|} \delta_{C,C'}. \quad (\text{H15})$$

Consider a set of operators $\{\mathcal{O}_g\}_{g \in G}$ such that $\mathcal{O}_g \mathcal{O}_h = \mathcal{O}_{gh}$. We define

$$e_R^{ij} = \frac{\dim R}{|G|} \sum_{g \in G} \bar{\Gamma}_R^{ij}(g) \mathcal{O}_g. \quad (\text{H16})$$

With straightforward but lengthy algebra, we see

$$\begin{aligned} e_R^{ij} e_{R'}^{i'j'} &= \frac{\dim R \cdot \dim R'}{|G|^2} \sum_{g,g'} \bar{\Gamma}_R^{ij}(g) \bar{\Gamma}_{R'}^{i'j'}(g') \mathcal{O}_g \mathcal{O}_{g'} \\ &= \frac{\dim R \cdot \dim R'}{|G|^2} \sum_{g,r} \bar{\Gamma}_R^{ij}(g) \bar{\Gamma}_{R'}^{i'j'}(g^{-1}r) \mathcal{O}_r \\ &= \frac{\dim R \cdot \dim R'}{|G|^2} \sum_{g,r,k} \bar{\Gamma}_R^{ij}(g) \Gamma_{R'}^{ki'}(g) \bar{\Gamma}_{R'}^{kj'}(r) \mathcal{O}_r \\ &= \frac{\dim R}{|G|} \sum_{r,k} \delta_{R,R'} \delta_{i,k} \delta_{j,i'} \bar{\Gamma}_R^{kj'}(r) \mathcal{O}_r \\ &= \delta_{R,R'} \delta_{j,i'} e_R^{i'j'}. \end{aligned} \quad (\text{H17})$$

Now, let us define $e_R = \sum_i e_R^{ii}$. It follows that

$$e_R e_{R'} = \delta_{R,R'} e_R. \quad (\text{H18})$$

With Eq. (H18), we can see that each $\{A_v^R\}_{R \in (G)_{ir}}$ in Eq. (H7) is an orthogonal set of projectors because A_v^g is a valid choice of $\mathcal{O}_g$.

2. Classical loop configurations

Consider the quantum double model on a lattice M with a number of faces, links, and vertices, where the Hilbert space on each link is $\mathcal{H}_e = \text{span}\{|g\rangle | g \in G\}$ on each link.

Let $|c\rangle$ be a product state in the $\{|g\rangle_e\}$ basis and with zero flux on every plaquette (i.e., $B_p^1|c\rangle = |c\rangle$, $\forall p$). We say $|c\rangle \sim |c'\rangle$ if $|c'\rangle = \prod_v A_v^{g_v}|c\rangle$ by the action of unitary vertex terms $A_v^{g_v}$, with some $g_v \in G$. Using local zero-flux condition, the following can be checked.

(i) Suppose M is a connected tree—i.e., it has no faces and no closed loops formed by links; see Fig. 35(a). Then, it is always true that $|c\rangle \sim |1, 1, \dots, 1\rangle$. Furthermore, the number of configurations in the equivalent class is $|G|^{V-1}$ as the number of vertices (V) of a connected tree equals the number of links plus one.

Suppose M is a union of k connected trees; then, it is still true that $|c\rangle \sim |1, 1, \dots, 1\rangle$. But the number of configurations in the equivalent class is $|G|^{V-k}$.

(ii) When M is a disk or a sphere, we always have $|c\rangle \sim |1, 1, \dots, 1\rangle$. Thus, there is a unique equivalence class under $\sim$.

To see why this is true, we first turn the lattice into a tree in the following way. We remove a number of links in such a way that every step removes a face until we are left with 0 or 1 face. The resulting links form a connected tree, as in Fig. 35(b). The important observation is that the group elements on the links in the tree determine the full configuration on the lattice of the disk or sphere. This is because the group elements on the removed links can be inferred from the tree by zero-flux constraints of the faces. From the property of connected trees, we see it is always possible to convert $|c\rangle \sim |1, 1, \dots, 1\rangle$ by the action of vertex terms. Furthermore, the number of configurations in the equivalence class equals $|G|^{V-1}$, where V is the number of vertices of the original lattice.

Similarly, if M is the union of k disjoint disks, $|c\rangle \sim |1, 1, \dots, 1\rangle$, and the number of configurations in the equivalent class is $|G|^{V-k}$.

(iii) When M is the torus, the equivalence class of $|c\rangle$ is labeled by $C_{g,h}$. Here, the set $C_{g,h} := \{(rg\bar{r}, rh\bar{r}) | r \in G\}$, defined when $gh = hg$, $g, h \in G$. Let $\mathcal{S}_{g,h}$ be the set of configurations equivalent to that shown in Fig. 36. Furthermore, the sizes of such sets satisfy

$$|\mathcal{S}_{g,h}| = |C_{g,h}| \cdot |\mathcal{S}_{1,1}|, \quad |\mathcal{S}_{1,1}| = |G|^{V-1}. \quad (\text{H19})$$

To see why these statements are true, we first remove a number of links in such a way that every step removes a face until we are left with a unique face of the torus. The remaining graph is a tree attached to a square frame, with periodic boundary conditions, as Fig. 35(c). It is then easy to see that the nontrivial loop and the zero-flux constraints give rise to the equivalence classes $\mathcal{S}_{g,h}$ stated. The condition $gh = hg$ follows from the zero-flux constraint. The application

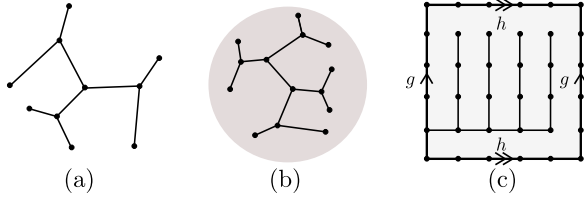


FIG. 35. An illustration of trees and lattices. (a) A connected tree. (b) A tree on a sphere obtained by removing links according to a certain rule. (c) A tree that is attached to a rectangular frame on a torus.

of star terms $\prod_v A_v^r$ can result in $(g, h) \rightarrow (rg r^{-1}, rhr^{-1})$. That is why $C_{g,h}$ captures the equivalence classes. The presence of the tree does not provide extra equivalence classes, and it increases all $|S_{g,h}|$ only by the same factor. This justifies the statement.

3. The memory capacity of the torus

Recall the definition of the memory capacity in Eq. (7):

$$\mathcal{Q}(M|\sigma) \equiv \max_{\rho, \lambda \in \Sigma(M|\sigma)} S(\rho) - S(\lambda). \quad (\text{H20})$$

We explain the group theoretical computation that leads to Eq. (56), namely, $\mathcal{Q}(\mathbb{T}^2|\rho^m) = \log |G| + \log |G_{\text{cj}}|$, where $|G_{\text{cj}}|$ is the number of conjugacy classes of G . First, given that the extreme points are orthogonal and that the entropies of the extreme points are given by $S(\rho^{C_{g,h}}) = S(\rho^{C_{1,1}}) + \log |C_{g,h}|$, the minimum entropy is, $S(\rho^{C_{1,1}})$ and the maximum entropy is, $S(\rho^*) = S(\rho^{C_{1,1}}) + \log(\sum |C_{g,h}|)$, where the sum runs over different $g, h \in G$ such that $gh = hg$ which gives inequivalent $C_{g,h}$. This maximum entropy can be obtained by optimizing the convex sum $\rho = \sum_{(g,h)} p_{g,h} \rho^{C_{g,h}}$ over $p_{g,h}$.

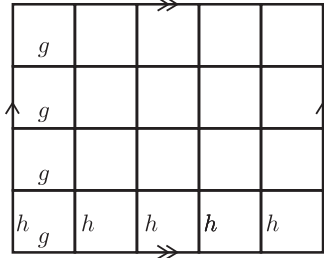


FIG. 36. A representative configuration of the torus in the equivalent class $S_{g,h}$, where $g, h \in G$ and $gh = hg$. On each link, we assign a group element as shown. We omit the label of the group element whenever it is the identity $1 \in G$. Each horizontal link is assigned a right arrow and each vertical link is assigned an up arrow, and we omit the drawing of them.

To proceed, let E_g be the centralizer group $\{x \in G | xgx^{-1} = g\}$. Note that two elements $h_1, h_2 \in E_g$ with $h_2 = sh_1s^{-1}$ for some $s \in E_g$ yield the same set $C_{g,h_1} = C_{g,h_2}$. Hence, one sums $|C_{g,h}|$ over h only in a choice of representatives of the conjugacy classes in E_g and then sums over g over a choice of representatives of the conjugacy classes in G .

Let $H := \{h^{(i)}\}_{i=1}^k$ be the set of representatives of those conjugacy classes, $k = |(E_g)_{\text{cj}}|$; then,

$$\sum_{h \in H} |C_{g,h}| = \sum_{i=1}^k \frac{|G|}{|E_{g,h^{(i)}}|}, \quad (\text{H21})$$

where we used the orbit-stabilizer theorem, where $E_{g,h} := \{x \in G | xgx^{-1} = g, xhx^{-1} = h\}$. On the other hand, the size of the conjugacy class for $h^{(i)}$ in E_g is

$$|\text{conj class of } h^{(i)} \text{ in } E_g| = \frac{|E_g|}{|E_{g,h^{(i)}}|}. \quad (\text{H22})$$

Since the union of these conjugacy classes exhausts E_g , we have

$$\sum_{i=1}^k \frac{|E_g|}{|E_{g,h^{(i)}}|} = |E_g|. \quad (\text{H23})$$

Therefore,

$$\sum_h |C_{g,h}| = \frac{|G|}{|E_g|} \sum_{i=1}^k \frac{|E_g|}{|E_{g,h^{(i)}}|} = |G|. \quad (\text{H24})$$

Let $|G_{\text{cj}}|$ be the number of conjugacy classes of G . Then, the sum over $\sum_{g,h} |C_{g,h}|$ equals to $|G| \cdot |G_{\text{cj}}|$.

4. Levin-Wen topological entropy

We derive the Levin-Wen topological entropy $\gamma_{\text{LW}} = \log \sqrt{|G|}$ [Eq. (59) in the main text], for the fixed point $\rho^m = \prod_p B_p^1$ of the quantum double. To do so, we apply the tools in Appendix H2. The important observation is that the wave function ρ^m is an equal weight convex combination of zero-flux classical loop configurations. We thus have the following.

- (i) Let D be the union of k disjoint disks. Then, $S(\rho_D^m) = (V_D - k) \log |G|$, where V_D is the number of vertices in D .
- (ii) Let X be an annulus (embedded in a disk). Then, $S(\rho_X^m) = (V_X - 1) \log |G|$. The reason is that the zero-flux conditions of the hole of the annulus effectively make the region a single disk.

Now we are ready to compute the Levin-Wen topological entropy:

$$\gamma_{\text{LW}} = \frac{1}{2} I(A:C|B)_{\rho^m}, \quad \text{for } \begin{array}{c} \text{B} \\ \text{A} \quad \text{C} \\ \text{B} \end{array}. \quad (\text{H25})$$

According to the discussion above, we can first shrink the lattice on the annulus ABC into a noncontractible loop and a tree attached to it. Let us call the resulting graph as $\mathbb{E}$, which is arranged in such a way that, when $\mathbb{E}$ is restricted to AB, BC , it becomes a single tree, and when it is reduced to B it becomes two disjoint trees.

We then compute

$$\begin{aligned} I(A:C|B)_{\rho^m} &= (V_{AB} - 1) \log |G| + (V_{BC} - 1) \log |G| \\ &\quad - (V_B - 2) \log |G| - (V_{ABC} - 1) \log |G| \\ &= E_{AB} \log |G| + E_{BC} \log |G| \\ &\quad - E_B \log |G| - (E_{ABC} - 1) \log |G| \\ &= \log |G|. \end{aligned} \quad (\text{H26})$$

Here, the first equality follows from the entropy counting from the number of vertices as explained above, where V_Y is the number of vertices in region Y . About the second equality, E_{AB}, E_{BC}, E_B , and E_{ABC} are the number of links of the graph $\mathbb{E}$; they are related to the number of vertices due to the tree topology (with one or two components or with an extra loop). The last equality follows from $E_{AB} + E_{BC} = E_B + E_{ABC}$. Thus, the topological entropy $\gamma_{\text{LW}} = \log \sqrt{|G|}$.

-
- [1] M. B. Hastings and X.-G. Wen, *Quasiadiabatic continuation of quantum states: The stability of topological ground-state degeneracy and emergent gauge invariance*, *Phys. Rev. B* **72**, 045141 (2005).
 - [2] X. Chen, Z.-C. Gu, and X.-G. Wen, *Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order*, *Phys. Rev. B* **82**, 155138 (2010).
 - [3] N. Schuch, D. Pérez-García, and I. Cirac, *Classifying quantum phases using matrix product states and projected entangled pair states*, *Phys. Rev. B* **84**, 165139 (2011).
 - [4] X. Chen, Z.-C. Gu, and X.-G. Wen, *Complete classification of one-dimensional gapped quantum phases in interacting spin systems*, *Phys. Rev. B* **84**, 235128 (2011).
 - [5] B. Zeng, X. Chen, D.-L. Zhou, X.-G. Wen *et al.*, *Quantum Information Meets Quantum Matter* (Springer, New York, 2019).
 - [6] A. Kitaev, *Anyons in an exactly solved model and beyond*, *Ann. Phys. (Amsterdam)* **321**, 2 (2006).
 - [7] M. A. Levin and X.-G. Wen, *String-net condensation: A physical mechanism for topological phases*, *Phys. Rev. B* **71**, 045110 (2005).
 - [8] E. Rowell, R. Stong, and Z. Wang, *On classification of modular tensor categories*, *Commun. Math. Phys.* **292**, 343 (2009).

- [9] L. Fidkowski and A. Kitaev, *Effects of interactions on the topological classification of free fermion systems*, *Phys. Rev. B* **81**, 134509 (2010).
- [10] B. Shi, K. Kato, and I. H. Kim, *Fusion rules from entanglement*, *Ann. Phys. (Amsterdam)* **418**, 168164 (2020).
- [11] B. Shi and I. H. Kim, *Entanglement bootstrap approach for gapped domain walls*, *Phys. Rev. B* **103**, 115150 (2021).
- [12] J.-L. Huang, J. McGreevy, and B. Shi, *Knots and entanglement*, *SciPost Phys.* **14**, 141 (2023).
- [13] B. Shi, J.-L. Huang, and J. McGreevy, *Remote detectability from entanglement bootstrap I: Kirby's torus trick*, *SciPost Phys.* **18**, 126 (2025).
- [14] T.-C. Lin and J. McGreevy, *Conformal field theory ground states as critical points of an entropy function*, *Phys. Rev. Lett.* **131**, 251602 (2023).
- [15] I. H. Kim, X. Li, T.-C. Lin, J. McGreevy, and B. Shi, *Conformal geometry from entanglement*, *SciPost Phys.* **18**, 102 (2025).
- [16] I. H. Kim and D. Ranard, *Classifying 2D topological phases: Mapping ground states to string-nets*, *arXiv:2405.17379*.
- [17] X. Li, T.-C. Lin, J. McGreevy, and B. Shi, *Strict area law entanglement versus chirality*, *Phys. Rev. Lett.* **134**, 180402 (2025).
- [18] For chiral topological orders, we refer to Ref. [17] for more in-depth discussions.
- [19] J. Y. Lee, Y.-Z. You, and C. Xu, *Symmetry protected topological phases under decoherence*, *Quantum* **9**, 1607 (2025).
- [20] R. Fan, Y. Bao, E. Altman, and A. Vishwanath, *Diagnostics of mixed-state topological order and breakdown of quantum memory*, *PRX Quantum* **5**, 020343 (2024).
- [21] J. Y. Lee, C.-M. Jian, and C. Xu, *Quantum criticality under decoherence or weak measurement*, *PRX Quantum* **4**, 030317 (2023).
- [22] Y. Bao, R. Fan, A. Vishwanath, and E. Altman, *Mixed-state topological order and the errorfield double formulation of decoherence-induced transitions*, *Phys. Rev. Lett.* **136**, 220402 (2026).
- [23] Z. Wang, Z. Wu, and Z. Wang, *Intrinsic mixed-state topological order*, *PRX Quantum* **6**, 010314 (2025).
- [24] Y. Zhao and D. E. Liu, *Extracting error thresholds through the framework of approximate quantum error correction condition*, *Phys. Rev. Res.* **6**, 043258 (2024).
- [25] J. Y. Lee, *Exact calculations of coherent information for toric codes under decoherence: Identifying the fundamental error threshold*, *Phys. Rev. Lett.* **134**, 250601 (2025).
- [26] S. Sang and T. H. Hsieh, *Stability of mixed-state quantum phases via finite Markov length*, *Phys. Rev. Lett.* **134**, 070403 (2025).
- [27] R. Niwa and J. Y. Lee, *Coherent information for calderbank-shor-steane codes under decoherence*, *Phys. Rev. A* **111**, 032402 (2025).
- [28] R. Ma and C. Wang, *Average symmetry-protected topological phases*, *Phys. Rev. X* **13**, 031016 (2023).
- [29] S. Sang, Y. Zou, and T. H. Hsieh, *Mixed-state quantum phases: Renormalization and quantum error correction*, *Phys. Rev. X* **14**, 031044 (2024).

- [30] A. Lyons, *Understanding stabilizer codes under local decoherence through a general statistical mechanics mapping*, [arXiv:2403.03955](#).
- [31] A. Coser and D. Pérez-García, *Classification of phases for mixed states via fast dissipative evolution*, *Quantum* **3**, 174 (2019).
- [32] H. Xue, J. Y. Lee, and Y. Bao, *Tensor network formulation of symmetry protected topological phases in mixed states*, [arXiv:2403.17069](#).
- [33] R. Ma and A. Turzillo, *Symmetry protected topological phases of mixed states in the doubled space*, [arXiv:2403.13280](#).
- [34] R. Sohal and A. Prem, *A noisy approach to intrinsically mixed-state topological order*, *PRX Quantum* **6**, 010313 (2025).
- [35] T. D. Ellison and M. Cheng, *Toward a classification of mixed-state topological orders in two dimensions*, *PRX Quantum* **6**, 010315 (2025).
- [36] Z. Zhang, U. Agrawal, and S. Vijay, *Quantum communication and mixed-state order in decohered symmetry-protected topological states*, *Phys. Rev. B* **111**, 115141 (2025).
- [37] Y.-H. Chen and T. Grover, *Separability transitions in topological states induced by local decoherence*, *Phys. Rev. Lett.* **132**, 170602 (2024).
- [38] Y.-H. Chen and T. Grover, *Symmetry-enforced many-body separability transitions*, *PRX Quantum* **5**, 030310 (2024).
- [39] Y. Guo and Y. Ashida, *Two-dimensional symmetry-protected topological phases and transitions in open quantum systems*, [arXiv:2311.12619](#).
- [40] K. Su, Z. Yang, and C.-M. Jian, *Tapestry of dualities in decohered quantum error correction codes*, *Phys. Rev. B* **110**, 085158 (2024).
- [41] T.-C. Lu, *Disentangling transitions in topological order induced by boundary decoherence*, *Phys. Rev. B* **110**, 125145 (2024).
- [42] P. Sala and R. Verresen, *Stability and loop models from decohering non-Abelian topological order*, *Phys. Rev. Lett.* **134**, 250403 (2025).
- [43] Z. Li, D. Lee, and B. Yoshida, *How much entanglement is needed for emergent anyons and fermions?*, *Phys. Rev. X* **15**, 021090 (2025).
- [44] Y. Xu and C.-M. Jian, *Average-exact mixed anomalies and compatible phases*, *Phys. Rev. B* **111**, 125128 (2025).
- [45] Z. Li and R. S. K. Mong, *Replica topological order in quantum mixed states and quantum error correction*, [arXiv:2402.09516](#).
- [46] Y.-H. Chen and T. Grover, *Unconventional topological mixed-state transition and critical phase induced by self-dual coherent errors*, [arXiv:2403.06553](#).
- [47] Z. Wang and L. Li, *Anomaly in open quantum systems and its implications on mixed-state quantum phases*, *PRX Quantum* **6**, 010347 (2025).
- [48] S. Lu, P. Zhu, and Y.-M. Lu, *Bilayer construction for mixed state phenomena with strong, weak symmetries and symmetry breakings*, [arXiv:2411.07174](#).
- [49] A.-R. Negari, T. D. Ellison, and T. H. Hsieh, *Spacetime Markov length: A diagnostic for fault tolerance via mixed-state phases*, [arXiv:2412.00193](#) [Phys. Rev. B (to be published)].
- [50] Y. Ogata, *Mixed state topological order: Operator algebraic approach*, [arXiv:2501.02398](#).
- [51] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2010).
- [52] E. H. Lieb and M. B. Ruskai, *Proof of the strong subadditivity of quantum-mechanical entropy*, *J. Math. Phys. (N.Y.)* **14**, 1938 (1973).
- [53] D. Petz, *Monotonicity of quantum relative entropy revisited*, *Rev. Math. Phys.* **15**, 79 (2003).
- [54] O. Fawzi and R. Renner, *Quantum conditional mutual information and approximate Markov chains*, *Commun. Math. Phys.* **340**, 575 (2015).
- [55] M. Junge, R. Renner, D. Sutter, M. M. Wilde, and A. Winter, *Universal recovery maps and approximate sufficiency of quantum relative entropy*, *Ann. Henri Poincaré* **19**, 2955 (2018).
- [56] O. Ogunnaike, J. Feldmeier, and J. Y. Lee, *Unifying emergent hydrodynamics and lindbladian low-energy spectra across symmetries, constraints, and long-range interactions*, *Phys. Rev. Lett.* **131**, 220403 (2023).
- [57] P. Sala, S. Gopalakrishnan, M. Oshikawa, and Y. You, *Spontaneous strong symmetry breaking in open systems: Purification perspective*, [arXiv:2405.02402](#).
- [58] L. A. Lessa, R. Ma, J.-H. Zhang, Z. Bi, M. Cheng, and C. Wang, *Strong-to-weak spontaneous symmetry breaking in mixed quantum states*, *PRX Quantum* **6**, 010344 (2025).
- [59] D. Gu, Z. Wang, and Z. Wang, *Spontaneous symmetry breaking in open quantum systems: Strong, weak, and strong-to-weak*, *Phys. Rev. B* **112**, 245123 (2025).
- [60] J. Kim, E. Altman, and J. Y. Lee, *Error threshold of syk codes from strong-to-weak parity symmetry breaking*, [arXiv:2410.24225](#).
- [61] C. Zhang, Y. Xu, J.-H. Zhang, C. Xu, Z. Bi, and Z.-X. Luo, *Strong-to-weak spontaneous breaking of 1-form symmetry and intrinsically mixed topological order*, [arXiv:2409.17530](#).
- [62] B. Buča and T. Prosen, *A note on symmetry reductions of the Lindblad equation: Transport in constrained open spin chains*, *New J. Phys.* **14**, 073007 (2012).
- [63] V. V. Albert and L. Jiang, *Symmetries and conserved quantities in Lindblad master equations*, *Phys. Rev. A* **89**, 022118 (2014).
- [64] By defects, we mean both pointlike defects in the sense of Refs. [65,66] and line defects, such as domain walls [67–69]; we discuss these further in Sec. IV.
- [65] H. Bombin and M. A. Martin-Delgado, *Family of non-Abelian Kitaev models on a lattice: Topological condensation and confinement*, *Phys. Rev. B* **78**, 115421 (2008).
- [66] B. J. Brown, S. D. Bartlett, A. C. Doherty, and S. D. Barrett, *Topological entanglement entropy with a twist*, *Phys. Rev. Lett.* **111**, 220402 (2013).
- [67] S. Beigi, P. W. Shor, and D. Whalen, *The quantum double model with boundary: Condensations and symmetries*, *Commun. Math. Phys.* **306**, 663 (2011).
- [68] B. Shi and I. H. Kim, *Domain wall topological entanglement entropy*, *Phys. Rev. Lett.* **126**, 141602 (2021).
- [69] M. Buican, R. Geiko, M. Moses, and B. Shi, *An algebraic theory of gapped domain wall partons*, [arXiv:2506.22544](#).

- [70] L. Zou and J. Haah, *Spurious long-range entanglement and replica correlation length*, *Phys. Rev. B* **94**, 075151 (2016).
- [71] L. H. Santos, J. Cano, M. Mulligan, and T. L. Hughes, *Symmetry-protected topological interfaces and entanglement sequences*, *Phys. Rev. B* **98**, 075131 (2018).
- [72] D. J. Williamson, A. Dua, and M. Cheng, *spurious topological entanglement entropy from subsystem symmetries*, *Phys. Rev. Lett.* **122**, 140506 (2019).
- [73] D. T. Stephen, H. Dreyer, M. Iqbal, and N. Schuch, *Detecting subsystem symmetry protected topological order via entanglement entropy*, *Phys. Rev. B* **100**, 115112 (2019).
- [74] K. Kato and F. G. S. L. Brandão, *Toy model of boundary states with spurious topological entanglement entropy*, *Phys. Rev. Res.* **2**, 032005(R) (2020).
- [75] B. Shi, *A softer notion of circuit invariant for topological phases* (to be published).
- [76] I. H. Kim, M. Levin, T.-C. Lin, D. Ranard, and B. Shi, *Universal lower bound on topological entanglement entropy*, *Phys. Rev. Lett.* **131**, 166601 (2023).
- [77] Using purification and strong subadditivity, one can prove that $S(BC) + S(C) - S(B) \geq I(A:C), I(A:C|B)$.
- [78] A. Kitaev and J. Preskill, *Topological entanglement entropy*, *Phys. Rev. Lett.* **96**, 110404 (2006).
- [79] J.-L. Huang, J. McGreevy, and B. Shi, *Knots and entanglement*, *SciPost Phys.* **14**, 141 (2023).
- [80] B. Shi, *Immersed figure-8 Annuli and Anyons*, *Ann. Henri Poincaré* **27**, 51 (2026).
- [81] S. Bravyi, M. B. Hastings, and S. Michalakis, *Topological quantum order: Stability under local perturbations*, *J. Math. Phys. (N.Y.)* **51**, 093512 (2010).
- [82] B. Shi and Y.-M. Lu, *Characterizing topological order by the information convex*, *Phys. Rev. B* **99**, 035112 (2019).
- [83] I. H. Kim, T.-C. Lin, D. Ranard, and B. Shi, *Strict area law implies commuting parent Hamiltonian*, *arXiv:2404.05867*.
- [84] M. Horodecki, P. W. Shor, and M. B. Ruskai, *Entanglement breaking channels*, *Rev. Math. Phys.* **15**, 629 (2003).
- [85] C. Castelnovo and C. Chamon, *Topological order and topological entropy in classical systems*, *Phys. Rev. B* **76**, 174416 (2007).
- [86] M. Levin and X.-G. Wen, *Detecting topological order in a ground state wave function*, *Phys. Rev. Lett.* **96**, 110405 (2006).
- [87] In the context of the mixed-state phase, the Levin-Wen topological entropy is well defined due to axiom **M1**. On the other hand, the Kitaev-Preskill topological entanglement entropy [78] stops being a topological invariant in the mixed-state context; see Appendix E.
- [88] I. H. Kim, *Long-range entanglement is necessary for a topological storage of quantum information*, *Phys. Rev. Lett.* **111**, 080503 (2013).
- [89] It means that the only anyon that has trivial braiding statistics (“transparent”) with all nontrivial anyons is the vacuum.
- [90] B. Shi, *Verlinde formula from entanglement*, *Phys. Rev. Res.* **2**, 023132 (2020).
- [91] I. H. Kim, B. Shi, K. Kato, and V. V. Albert, *Modular commutator in gapped quantum many-body systems*, *Phys. Rev. B* **106**, 075147 (2022).
- [92] Y. Zhang, T. Grover, A. Turner, M. Oshikawa, and A. Vishwanath, *Quasiparticle statistics and braiding from ground-state entanglement*, *Phys. Rev. B* **85**, 235151 (2012).
- [93] I. H. Kim, X. Li, T.-C. Lin, J. McGreevy, and B. Shi, *Chiral Virasoro algebra from a single wavefunction*, *Ann. Phys. (Amsterdam)* **471**, 169849 (2024).
- [94] Note that, when we generalize this to non-Abelian cases, charge and flux labels correspond to irreducible representation and conjugacy classes, respectively.
- [95] D. S. Dummit and R. M. Foote, *Abstract Algebra*, 3rd ed. (Wiley, New York, 2004).
- [96] M. Levin, *Physical proof of the topological entanglement entropy inequality*, *Phys. Rev. B* **110**, 165154 (2024).
- [97] H.-S. Kim, I. H. Kim, and D. Ranard, *Learning state preparation circuits for quantum phases of matter*, *arXiv:2410.23544*.
- [98] D. Ranard (2025), private communication about on-going work with Isaac H. Kim, Alexei Kitaev and Milo Moses.
- [99] C.-F. Chen and C. Rouzé, *Quantum Gibbs states are locally Markovian*, *arXiv:2504.02208*.
- [100] J. M. Hammersley and P. Clifford, *Markov fields on finite graphs and lattices* (1971), <https://ora.ox.ac.uk/objects/uuid:4ea849da-1511-4578-bb88-6a8d02f457a6>.
- [101] H. Casini, M. Huerta, and R. C. Myers, *Towards a derivation of holographic entanglement entropy*, *J. High Energy Phys.* **05** (2011) 036.
- [102] A. W. Ludwig, *Infinite hierarchies of exponents in a diluted ferromagnet and their interpretation*, *Nucl. Phys. B* **330**, 639 (1990).
- [103] J. Cardy, *Topics in the conformal field theory of disordered systems*, in *New Theoretical Approaches to Strongly Correlated Systems*, edited by A. M. Tsvelik (Springer, Netherlands, Dordrecht, 2001), pp. 163–171.
- [104] V. Gurarie and A. W. W. Ludwig, *Conformal algebras of two-dimensional disordered systems*, *J. Phys. A* **35**, L377 (2002).
- [105] There is another approach closely related to FDLG but it uses local Lindbladian [26]. It may not be completely equivalent to the local channel condition.
- [106] M. Levin, *Protected edge modes without symmetry*, *Phys. Rev. X* **3**, 021009 (2013).
- [107] I. H. Kim and B. J. Brown, *Ground-state entanglement constrains low-energy excitations*, *Phys. Rev. B* **92**, 115139 (2015).
- [108] B. Shi, *Seeing topological entanglement through the information convex*, *Phys. Rev. Res.* **1**, 033048 (2019).
- [109] T. Lan, J. C. Wang, and X.-G. Wen, *Gapped domain walls, gapped boundaries, and topological degeneracy*, *Phys. Rev. Lett.* **114**, 076402 (2015).
- [110] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, *Topological quantum memory*, *J. Math. Phys. (N.Y.)* **43**, 4452 (2002).
- [111] A. Y. Kitaev, *Fault-tolerant quantum computation by anyons*, *Ann. Phys. (Amsterdam)* **303**, 2 (2003).
- [112] A mixed unitary channel is a quantum channel that can be expressed as a convex combination of unitary operations.
- [113] A. Shamir, *How to share a secret*, *Commun. ACM* **22**, 612 (1979).

- [114] R. Cleve, D. Gottesman, and H.-K. Lo, *How to share a quantum secret*, *Phys. Rev. Lett.* **83**, 648 (1999).
- [115] K. Kato, F. Furrer, and M. Murao, *Information-theoretical analysis of topological entanglement entropy and multipartite correlations*, *Phys. Rev. A* **93**, 022317 (2016).
- [116] L. Fiedler, P. Naaijken, and T. J. Osborne, *Jones index, secret sharing and total quantum dimension*, *New J. Phys.* **19**, 023039 (2017).
- [117] W. P. Thurston and S. Levy, *Three-Dimensional Geometry and Topology* (Princeton University Press, Princeton, NJ, 2000), Vol. 1.
- [118] T. Grover, A. M. Turner, and A. Vishwanath, *Entanglement entropy of gapped phases and topological order in three dimensions*, *Phys. Rev. B* **84**, 195120 (2011).
- [119] B. Swingle and J. McGreevy, *Mixed s -sourcery: Building many-body states using bubbles of nothing*, *Phys. Rev. B* **94**, 155125 (2016).
- [120] N. Tantivasadakarn, A. Vishwanath, and R. Verresen, *Hierarchy of topological order from finite-depth unitaries, measurement, and feedforward*, *PRX Quantum* **4**, 020339 (2023).
- [121] N. Tantivasadakarn, R. Verresen, and A. Vishwanath, *Shortest route to non-Abelian topological order on a quantum processor*, *Phys. Rev. Lett.* **131**, 060405 (2023).
- [122] M. Iqbal, N. Tantivasadakarn, R. Verresen, S. L. Campbell, J. M. Dreiling, C. Figgatt, J. P. Gaebler, J. Johansen, M. Mills, S. A. Moses, J. M. Pino, A. Ransford, M. Rowe, P. Siegfried, R. P. Stutz, M. Foss-Feig, A. Vishwanath, and H. Dreyer, *Non-Abelian topological order and anyons on a trapped-ion processor*, *Nature (London)* **626**, 505 (2024).
- [123] P. Sala, J. Alicea, and R. Verresen, *Decoherence and wavefunction deformation of d_4 non-abelian topological order*, *arXiv:2409.12948*.
- [124] A. Kitaev and L. Kong, *Models for gapped boundaries and domain walls*, *Commun. Math. Phys.* **313**, 351 (2012).
- [125] L. Kong, *Anyon condensation and tensor categories*, *Nucl. Phys. B* **886**, 436 (2014).
- [126] L. Kong, Z.-H. Zhang, J. Zhao, and H. Zheng, *Higher condensation theory*, *arXiv:2403.07813*.
- [127] I. H. Kim, B. Shi, K. Kato, and V. V. Albert, *Chiral central charge from a single bulk wave function*, *Phys. Rev. Lett.* **128**, 176402 (2022).
- [128] S. Vardhan, B. Shi, I. H. Kim, and Y. Zou, *Chirality, magic, and quantum correlations in multipartite quantum states*, *arXiv:2503.10764*.
- [129] A. Vijay and J. Y. Lee, *Information critical phases under decoherence*, *arXiv:2512.22121*.
- [130] S. Sang, L. A. Lessa, R. S. K. Mong, T. Grover, C. Wang, and T. H. Hsieh, *Mixed-state phases from local reversibility*, *arXiv:2507.02292*.
- [131] T.-H. Yang, B. Shi, and J. Y. Lee, *Mixed-state topological phases: figures and calculations*, *GitHub repository*, https://github.com/xiaotai-yang/mixed_topo (2026).
- [132] The proof of the following statement is nontrivial, but it is essentially identical to the argument in Appendix C in Ref. [10]. We, thus, omit the proof here.
- [133] O. Fawzi and R. Renner, *Quantum conditional mutual information and approximate markov chains*, *Commun. Math. Phys.* **340**, 575 (2015).
- [134] K. M. R. Audenaert, *A sharp continuity estimate for the von Neumann entropy*, *J. Phys. A* **40**, 8127 (2007).
- [135] A. Winter, *Tight uniform continuity bounds for quantum entropies: Conditional entropy, relative entropy distance and energy constraints*, *Commun. Math. Phys.* **347**, 291 (2016).
- [136] C. A. Fuchs and J. van de Graaf, *Cryptographic distinguishability measures for quantum mechanical states*, *arXiv:quant-ph/9712042*.
- [137] I. H. Kim (unpublished).
- [138] B. Shi, T. H. Yang, and J. Y. Lee (unpublished).
- [139] A. Kitaev, *Toward a topological classification of many-body quantum states with short-range entanglement*, *Simons Center for Geometry and Physics*, https://scgp.stonybrook.edu/video_portal/video.php?id=1835 (2011).
- [140] A. Hamma, R. Ionicioiu, and P. Zanardi, *Bipartite entanglement and entropic boundary law in lattice spin systems*, *Phys. Rev. A* **71**, 022315 (2005).
- [141] Z. Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang, and T. Xiang, *Coarse-graining renormalization by higher-order singular value decomposition*, *Phys. Rev. B* **86**, 045139 (2012).
- [142] M. Levin and C. P. Nave, *Tensor renormalization group approach to two-dimensional classical lattice models*, *Phys. Rev. Lett.* **99**, 120601 (2007).