

Quantized Hall Drift in a Frequency-Encoded Photonic Chern Insulator

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The quantization of transport and its resilience to backscattering are key features for leveraging topological matter in applications that demand stringent noise mitigation, such as metrology and quantum information processing. Because of the bosonic nature of light, engineering such robust, “one-way” channels in synthetic photonic systems imposes the implementation of topological models with broken time-reversal symmetry; this is challenging, since photons possess neither an electric charge nor a magnetic moment. Here, we propose and demonstrate an approach to realizing photonic Chern insulators—topological insulators with broken time-reversal symmetry—by encoding a Haldane-like model in the synthetic frequency dimension of an optical fiber loop platform. The bands’ topology is assessed by reconstructing the Bloch state geometry across the Brillouin zone. We further highlight its consequences by measuring a driven-dissipative analog of the quantized transverse Hall conductivity. Our results open avenues for harnessing topologically protected light propagation in frequency-multiplexed photonic systems, with applications ranging from precision metrology to photonic quantum processors.

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I. INTRODUCTION

Chern insulators are a topological phase of matter with broken time-reversal symmetry ($\mathcal{T}$) [1,2]. A prominent example is provided by the quantum Hall effect, where the defining feature of topology is the emergence of backscattering-immune channels along the edges giving rise to quantized plateaus in the transverse conductivity [3,4]. These robust, quantized channels hold significant value for applications requiring extensive resilience against environmental noise, e.g., in metrology [5], quantum computing [6], and information processing [7].

Extending quantum Hall physics to photonics could enable the design of devices—classical or quantum—where the flow of light is similarly quantized and unidirectional, leading to enhanced performance and reduced footprint [8–10]. However, this endeavor currently faces a twofold challenge. On the one hand, breaking $\mathcal{T}$ through

magneto-optical effects is technically challenging, particularly at optical or near-infrared frequencies where the Verdet constant of most materials is very small. Therefore, implementing $\mathcal{T}$ -broken topological phases in these frequency bands has so far required either operating in a lasing regime [11,12] or mapping time onto spatial coordinates [13,14]. On the other hand, $\mathcal{T}$ -invariant photonic topological insulators, inspired by the spin [15,16] and valley Hall effects [17,18], are easier to realize but are inherently prone to backscattering [19]. Indeed, Kramers’ degeneracy theorem which precludes the hybridization of counterpropagating electrons in $\mathcal{T}$ -invariant topological matter does not apply to bosonic fields [20].

Moreover, transport quantization in driven-dissipative photonic systems arises from fundamentally different physical mechanisms compared to condensed matter, cold atoms, or unitary quantum walks [21]. In these latter systems, the particle number is conserved, and the quantization is typically associated with a well-defined band occupation. In contrast, driven-dissipative systems are inherently out of equilibrium, and a steady state can emerge from a complex interplay between coherent drive, dissipation, and the underlying band geometry. As a nonequilibrium steady state necessarily involves non-Hermitian effects, the observation of a quantized transport

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of light typically requires distinct theoretical and experimental frameworks to account for this driven-dissipative balance [22].

In this work, we introduce and demonstrate a novel approach to realizing photonic Chern insulators that do not rely on strong external magnetic fields and evidence the onset of a quantized Hall drift of light. We focus on the Haldane model [23], where $\mathcal{T}$ is broken by adding complex next-nearest-neighbor couplings to a honeycomb lattice. This model was first realized with cold atoms in optical lattices [24,25] and, more recently, has also been reported in the solid state in temporally modulated monolayers [26] and in twisted Moiré bilayers [27,28]. The key challenge in implementing Haldane phases, especially in the photonic context [29–31], lies in the strict requirements for intersite connectivity within the lattice.

To overcome this challenge, we capitalize on the concept of synthetic dimensions, which roots in the use of some internal degrees of freedom of a particle to mimic spatial coordinates [32,33]. Here, we make use of an optical fiber loop platform inspired by electro-optics frequency combs [34] to encode a honeycomb lattice in frequency space. Independent tuning of the intersite hopping amplitude and phase is achieved via suitable electro-optical modulation of the loop's refractive index [35–37]. Although our platform uses magneto-optical components (i.e., optical isolators and circulators) to improve the cavity lifetime, the breaking of time-reversal symmetry in frequency space, necessary to open a topological gap, is achieved by tuning the phase of the refractive index temporal modulation. This allows implementing a Haldane-like model across its topological phase diagram while maintaining clearly resolved band gaps. Taking profit of the driven-dissipative nature of our system, we realize a comprehensive study of the bulk topology of this Haldane-like model. First, we perform a detailed tomography of the Bloch modes to extract the Berry curvature over the Brillouin zone (BZ) and the resulting Chern number of each band. Then, we evidence an anomalous transverse drift of light in frequency space induced by a synthetic electric field. As a hallmark of quantum Hall physics, we show that the frequency integral of this transverse drift is quantized, thus providing a driven-dissipative analog of the quantum Hall conductivity.

The structure of the article is the following. In Sec. II, we give a short review of the Haldane model. Its implementation in the synthetic frequency setup is presented in Sec. III. In Sec. IV, we summarize our experimental results for the band structure, the tomography of the Bloch bands, and the extraction of the Berry curvature and for the anomalous displacement and the quantized Hall drift. Conclusions and perspectives are finally given in Sec. V. Details on the experimental system and the experimental protocols are given in the Appendixes.

II. THE HALDANE MODEL

The Haldane model describes a honeycomb lattice with complex next-nearest-neighbor (NNN) couplings. In the tight-binding approximation, it is described by a two-band Hamiltonian:

$$\mathcal{H}(\mathbf{k}) = \mathbf{h}(\mathbf{k}) \cdot \vec{\sigma}, \quad (1)$$

with $\mathbf{k}$ spanning the BZ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ the Pauli matrices. Eigenstates can be expressed as spinors:

$$|\psi_{\mathbf{k}}^{(\pm)}\rangle = \cos(\theta_{\mathbf{k}}^{(\pm)}/2)|a\rangle + \sin(\theta_{\mathbf{k}}^{(\pm)}/2)e^{i\phi_{\mathbf{k}}^{(\pm)}}|b\rangle, \quad (2)$$

with $\{|a\rangle, |b\rangle\}$ the sublattice basis, $\theta_{\mathbf{k}}^{(\pm)}$ and $\phi_{\mathbf{k}}^{(\pm)}$ the polar and azimuthal angles on the Bloch sphere, and superscripts $\pm$ identifying the lower and upper band.

In a honeycomb lattice with only nearest-neighbor (NN) couplings, as for graphene, inversion ($\mathcal{I}$) and $\mathcal{T}$ impose $h_z(\mathbf{k}) = 0$. This leads to the emergence of band-touching points in reciprocal space, labeled K and K' with $K' = -K$, around which the band dispersion is linear, similarly to Dirac fermions. These Dirac points carry remarkable topological signatures [38], evidenced by singularities in the Berry curvature $\Omega_{\mathbf{k}}^{(\pm)} = \nabla \times \langle \psi_{\mathbf{k}}^{(\pm)} | i\partial_{\mathbf{k}} | \psi_{\mathbf{k}}^{(\pm)} \rangle$.

The degeneracies at Dirac points can be lifted by breaking either $\mathcal{I}$ or $\mathcal{T}$. In the former case, e.g., with inequivalent atoms in the unit cell as in hexagonal boron nitride (h -BN), a constant mass term $h_z(\mathbf{k}) = \Delta$ emerges. The bands are, however, topologically trivial with a vanishing Chern number $\mathcal{C}^{(\pm)} = [1/(2\pi)] \int_{\text{BZ}} \Omega_{\mathbf{k}}^{(\pm)} \cdot d^2\mathbf{k}$, as $\mathcal{T}$ imposes $\Omega_{\mathbf{k}}$ to be antisymmetric in $\mathbf{k}$. In the latter case, the Berry curvature no longer necessarily integrates to zero, giving rise to topological bands with nonzero Chern number.

In the Haldane model, $\mathcal{T}$ is broken by adding complex NNN couplings with amplitude J' and opposite phase $\pm\phi_h$ for each sublattice. Differently from the original Haldane model, we consider only one pair of NNN coupling terms along $\hat{x}$ [Fig. 1(a)]. This simplification does not qualitatively modify the topological phase diagram shown in Fig. 1(b). The NNN coupling terms induce a $\mathbf{k}$ -dependent mass term $h_z(\mathbf{k}) = J' \sin[\mathbf{k} \cdot (\mathbf{R}_2 - \mathbf{R}_1)] \sin(\phi_h)$ which has the same amplitude but opposite sign at K and K' , leading to topologically nontrivial phases for $|\Delta| < |\sqrt{3}J' \sin(\phi_h)|$.

III. IMPLEMENTATION IN THE SYNTHETIC FREQUENCY DIMENSION

We realize our Haldane-like model using the frequency-periodic cavity modes of an optical fiber loop [Fig. 1(c)] to form a honeycomb lattice in the brick-wall geometry [Fig. 1(e)]. Unit cells are separated by the loop's free

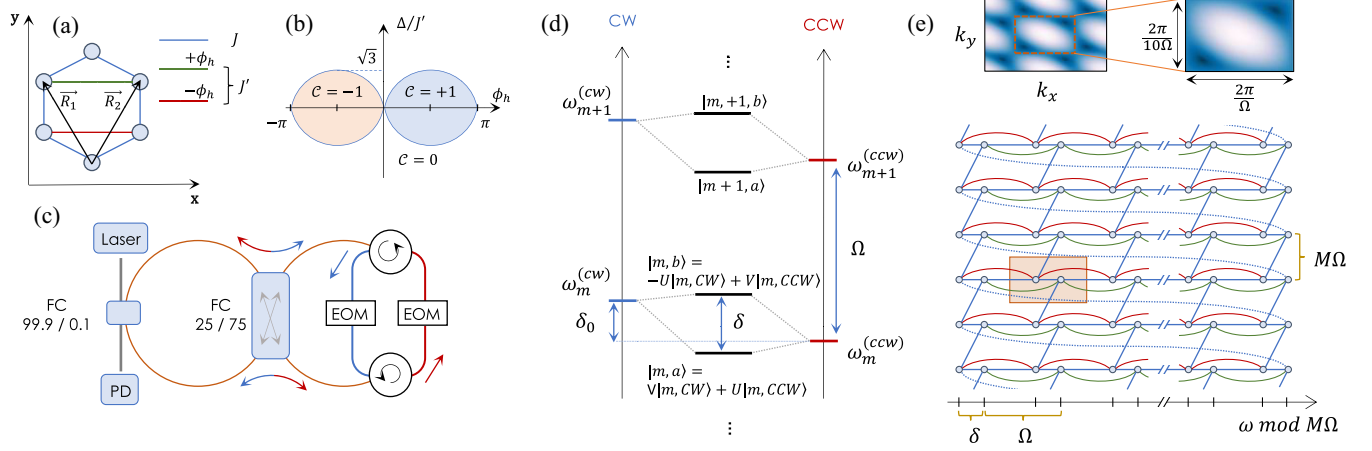


FIG. 1. Implementation of the Haldane-like model in the synthetic frequency dimension. (a) Schematic representation of our simplified Haldane model with NN couplings (blue lines) and a single pair of NNN couplings (red and green lines) of opposite phase ($\pm\phi_h$) for each sublattice. (b) Topological phase diagram, as a function of ϕ_h and Δ/J' , exhibiting three distinct phases with Chern number $\mathcal{C} = 0, \pm 1$. (c) Schematic representation of the experimental setup depicting a single loop whose CCW (red arrows) and CW (blue arrows) circulating modes are coupled with a 25/75 fiber coupler (FC). On the right side, a pair of optical circulators allows modulating CW and CCW modes individually. The loop is probed with a photodiode (PD), using a transmission line coupled to the main loop with a 99.9% transmission FC. (d) Optical mode structure for two pairs of supermodes $|i, a\rangle$ and $|i, b\rangle$ with $i = m, m+1$, formed from the coupling of CW and CCW modes split by a frequency δ_0 . (e) Representation of the brick-wall Haldane-like Hamiltonian with a folding period of $M\Omega$ —each site represents a supermode of the fiber loop. A unit cell is depicted in orange. Blue lines indicate NN couplings with dotted lines indicating the twisted boundary conditions. NNN couplings are indicated with red and green lines for the $|a\rangle$ and $|b\rangle$ sublattices, respectively (for the sake of clarity, boundary conditions are not depicted for NNN couplings). The insets above depicts the associated reciprocal space (left) and an enlargement of the first BZ.

spectral range Ω with the two sublattices formed from the hybridized supermodes, split by frequency δ , that emerge from coupling the quasiresonant clockwise (CW) and counterclockwise (CCW) modes [see Fig. 1(d) and Appendix A].

Intersite hoppings are tailored by independently modulating the CW and CCW fields with electro-optic phase modulators (EOMs) driven at frequencies corresponding to the frequency spacing between pairs of supermodes [36,39]. NN hoppings are realized by driving both EOMs with Fourier components at frequencies δ , $\Omega - \delta$, and $M\Omega + \delta$ (see Appendix C). The first two terms which describe coupling between adjacent frequency modes provide a first synthetic dimension. The second synthetic dimension is provided by the third, higher-frequency terms that couple modes separated by $2M + 1$ lattice sites [40–42]. The resulting synthetic lattice shown in Fig. 1(e) presents a reorganization of the modes in frequency space with a $M\Omega$ folding period. An effective brick-wall lattice with twisted boundary conditions (dotted lines) emerges. The associated reciprocal space and an enlargement of the first BZ are depicted in the top inset.

Finally, the NNN couplings of the Haldane-like model are implemented by including an additional component to the EOMs' driving field at frequency Ω with opposite phase $\pm\phi_{\text{mod}}$ for each EOM. For this model specifically, we work in a regime where the bare CW and CCW modes are not degenerate, so that each supermode has a dominant CW or

CCW character, i.e., $|U|^2 \gg |V|^2$ in Fig. 1(d). This is critically important to obtain opposite phases for the NNN couplings of the $|a, b\rangle$ sublattices, even though their splittings are equal $\omega_m^{(a/b)} - \omega_{m+1}^{(a/b)} = \Omega$. The value of ϕ_h is determined by the ratio U/V : If $U/V > 1$, $\phi_h = \phi_{\text{mod}}$, and if $U/V < 1$, $\phi_h = -\phi_{\text{mod}}$ [see Eq. (S.27) in Supplemental Material [43]].

IV. RESULTS

A. Band structure measurement

While edge states are the natural observable for topological photonics systems in real space [32,45], synthetic frequency dimensions offer direct access to the bulk energy bands through time-resolved transmission measurement using a continuous-wave laser [36]. The principle underlying this measurement (see Appendix C and Supplemental Material [43]) is that the periodicity of the eigenmodes in frequency gives rise to time-periodic pulse trains with superposed long ($T = [(2\pi)/\Omega]$) and short periods ($T' = [(2\pi)/(M\Omega)]$) components. Each time bin t_i probes a single point $\mathbf{k} = (t_i \bmod T, t_i \bmod T')$ within a BZ whose dimensions are inversely proportional to Ω and $M\Omega$, respectively, associated to the short- and long-range periodicity in frequency space. The transmitted signal exhibits intensity dips whenever the laser frequency is resonant with an energy band of the lattice.

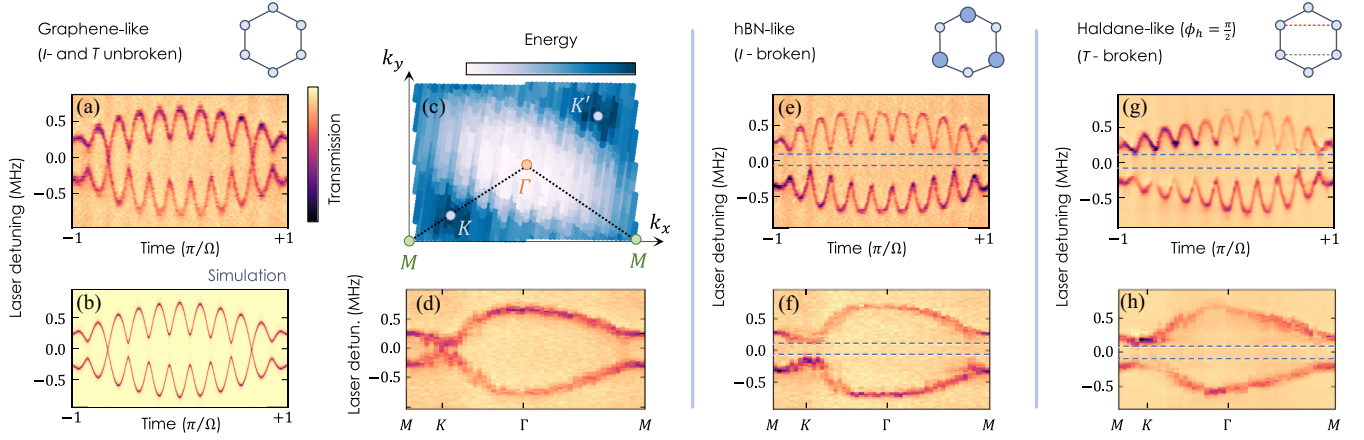


FIG. 2. Band structure measurements for graphenelike, *h*-BN-like, and Haldane-like models. (a),(e),(g) Band structure measurements for configurations emulating graphene (a), hexagonal boron nitride (e), and the Haldane model with $\phi_h = \pi/2$ (g). The band dispersion is obtained by measuring the time-resolved transmission of the loop as a function of time and laser detuning. (b) The simulation of the graphenelike configuration agrees with the experimental results in (a). (c) Lower band dispersion as a function of (k_x, k_y) ; a fine coverage of the BZ is obtained by scanning the phase of one driving component. (d),(f),(h) Band dispersion along a trajectory through high-symmetry points M - K - Γ - M indicated in (c) for graphenelike (d), *h*-BN-like (f), and the Haldane-like model (h). The latter two clearly show a band gap, indicated by dashed horizontal lines.

This periodic signal is monitored on a photodiode while scanning the frequency of the laser. We plot in Fig. 2 the result averaged over 150 periods of length T with $M = 10$. For the case with only NN coupling terms with equal amplitude, which emulates the physics of graphene [Fig. 2(a)], we observe two bands with one slow (period T) and ten fast (period T') oscillations, respectively, associated to scans of the BZ along k_x and k_y . The band-touching points associated to each Dirac point (K and K') are clearly visible, in very good agreement with theoretical simulations [Fig. 2(b)].

This measurement is then reproduced by varying the phase of the Fourier component with frequency $\Omega - \delta$ to follow slightly shifted trajectories in reciprocal space, thus producing a finer coverage of the BZ. In Fig. 2(c), we plot the extracted energy of the lower band, showing very clearly the position of the Dirac points in accordance with the theory [inset in Fig. 1(e)]. Then, by selecting data at time bins along the high-symmetry M - K - Γ - M trajectory (dashed line), it is possible to generate the band structure in Fig. 2(d).

Breaking $\mathcal{T}$ in this lattice, thus emulating the physics of *h*-BN, is realized by slightly shifting the EOM driving frequencies associated to NN coupling terms to $\delta + \Delta$, $\Omega - \delta - \Delta$, and $M\Omega + \delta + \Delta$ with $\Delta/2\pi = 200$ kHz. In the synthetic lattice, this detuning realizes a staggered on-site potential giving rise to a mass term $h_z(\mathbf{k}) = \Delta/2$. Accordingly, we observe at each Dirac point the opening of a band gap of $E_g \sim 175$ kHz [Figs. 2(e) and 2(f)], significantly larger than the optical linewidth.

Breaking $\mathcal{T}$ and realizing the Haldane-like Hamiltonian is achieved by adding the NNN component at frequency Ω . Using a phase of $\phi_h = \pi/2$, setting $\Delta = 0$, and working in

a regime where $|U|^2 \gg |V|^2$, we again open a significant band gap [Figs. 2(g) and 2(h)]. Interestingly, the intensity of each band changes as a function of $\mathbf{k}$, with the lower (upper) band reaching a vanishing intensity at K (K'); this is a direct consequence of the mass term $h_z(\mathbf{k})$, since the transmission measurement is performed along the CCW direction (see Appendix C).

B. Extraction of the Berry curvature

These transmission measurements further allow performing a full tomography of the eigenmodes across the BZ [46]. As the eigenmodes are linear combinations of CW and CCW modes but we measure the transmission along the CCW direction, the transmitted intensity is modulated in time at frequency δ and, for each time bin in the BZ, the phase of this beating exactly follows the azimuthal angle $\phi_{\mathbf{k}}$ in Eq. (2). The polar angle $\theta_{\mathbf{k}}$ indicating the eigenmodes' relative weight on the $|a, b\rangle$ sublattices is extracted from the intensity ratio between the two bands (see Appendix C).

The experimentally extracted values of $\phi_{\mathbf{k}}$, $\theta_{\mathbf{k}}$ across the BZ can be plotted on a Bloch sphere [Figs. 3(a)–3(e)]. For the case of graphene (a), the lower band's eigenstates $|\psi_{\mathbf{k}}^{(+)}\rangle$ stays confined to the vicinity of the equator, as expected for $h_z = 0$. For *h*-BN (b), the evolution is shifted to the northern hemisphere, evidencing the effect of the constant mass term $h_z = \Delta/2$. For Haldane-like lattices with $\phi_h = \pm\pi/2$ (c) and (d), the situation is drastically different: The opposite sign of the mass term $h_z(\mathbf{k})$ between K and K' makes $|\psi_{\mathbf{k}}^{(+)}\rangle$ explore the full Bloch sphere, indicating a nontrivial topology. The trajectory on the Bloch sphere is reversed when switching the sign of ϕ_h . When we add a

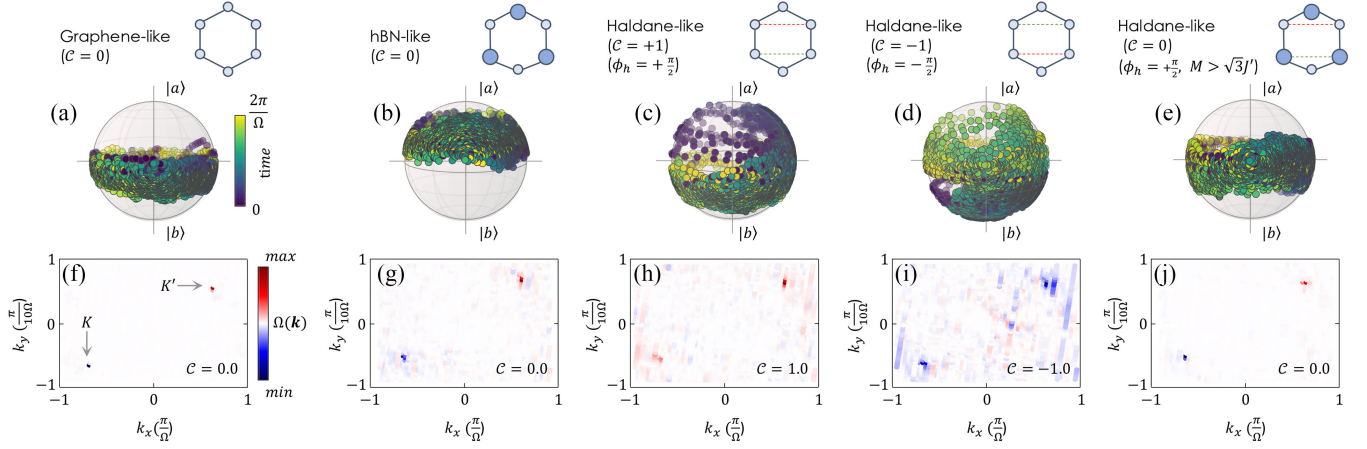


FIG. 3. Extraction of the Berry curvature and Chern number. (a)–(e) Tomographic measurement of the lower band's eigenstates $|\psi_{\mathbf{k}}^{(+)}\rangle$ across the full BZ for the cases of graphenelike (a) and h -BN-like (b) lattices, for the topologically nontrivial Haldane-like model with $\phi_h = +\pi/2$ (c) and $\phi_h = -\pi/2$ (d), and the topologically trivial Haldane-like model with $\phi_h = +\pi/2$ and $\Delta > \sqrt{3}J'$ (e). The north and south poles of the Bloch spheres correspond to the sublattice basis states $|a\rangle$ and $|b\rangle$, respectively. The color code refers to the time bin of each $\mathbf{k}$ point. (f)–(j) The Berry curvature, extracted using the Fukui method from the momentum-resolved eigenstates shown in (a)–(e), exhibits clear peaks at the positions of the Dirac point K and K' . The value of the Chern number obtained by integrating the Berry curvature over the BZ is indicated in the corner of each panel.

constant mass term $\Delta > \sqrt{3}J'$ (e), the Bloch sphere is no longer entirely covered.

In order to quantitatively assess the bands' topology, we take profit of this tomography to extract the value of the Berry curvature across the entire BZ [47]. To do this, we use the technique developed in Ref. [48] for discretized BZs. The result is plotted below each Bloch sphere, and the associated Chern number, extracted by summing $\Omega_{\mathbf{k}}$ over the BZ, is indicated in the corner.

For graphenelike lattices, the Berry curvature is negligible over the entire BZ except in the close vicinity of K and K' . Sharply peaked Berry curvatures with opposite signs are observed close to K and K' points, leading to a vanishing Chern number; this is consistent with $\mathcal{T}$ symmetry of the graphenelike lattice. For h -BN-like lattices, the situation is very similar as the Hamiltonian is still $\mathcal{T}$ symmetric but the peaks are broader in reciprocal space, as the Berry curvature spreads in $\mathbf{k}$ when a band gap opens. For the Haldane-like model, we clearly see that the two peaks now have the same sign, either positive for $\mathcal{C} = +1$ or negative for $\mathcal{C} = -1$, as a direct consequence of the broken $\mathcal{T}$. The extracted Chern numbers are consistent with the expected values for topologically nontrivial phases. In the presence of a constant mass term $\Delta > \sqrt{3}J'$, the peaks recover opposite signs: This gives a vanishing Chern number as expected for a topologically trivial phase.

C. Anomalous displacement and quantized Hall drift

One of the most representative consequences of topology in quantum Hall [3,4] and magnetic [49] systems is the emergence of plateaus in the transverse conductivity at values determined by the bands' Chern number. In such

conservative systems, the topological properties of transport can be probed by tuning the band filling through the chemical potential. In contrast, driven-dissipative photonic systems lack particle-number conservation, and transport manifests as the displacement of an out-of-equilibrium steady state resulting from the interplay between the drive properties (frequency, phase, amplitude, and spatial profile), losses, and the underlying band geometry. It is characterized by probing the spatial distribution of the steady-state field intensity induced by a continuous-wave coherent pump. In this context, the topological nature of a band manifests through the system's out-of-equilibrium response integrated over the entire bandwidth as the driving frequency is swept [22]. Here, we perform this task and evidence a quantized Hall drift of light in frequency space.

We implement a synthetic electric field by adding a detuning λ to the Fourier component associated to the NN coupling term along $\hat{y}$. In the rotating frame, this induces a linearly varying detuning with respect to the loop's eigenmodes. Within the synthetic lattice picture [Fig. 4(a)], this can be interpreted as a scalar potential $V(y) = -\lambda y$ giving rise to a synthetic electric field $\mathbf{E} = \lambda \hat{y}$ [50,51]. The transverse displacement [38] induced by this synthetic electric field is measured according to the theoretical proposals in Refs. [22,52] by injecting monochromatic light at a single lattice site and then detecting the center-of-mass displacement of the steady-state distribution of light in frequency space. In our synthetic-dimension framework, this is extracted by means of a heterodyne measurement [Fig. 4(b)], where the signal radiating from the CCW and CW modes of the loop is mixed with the laser shifted by 200 MHz. A typical heterodyne spectrum obtained by adding the CW and CCW signals' Fourier transform is

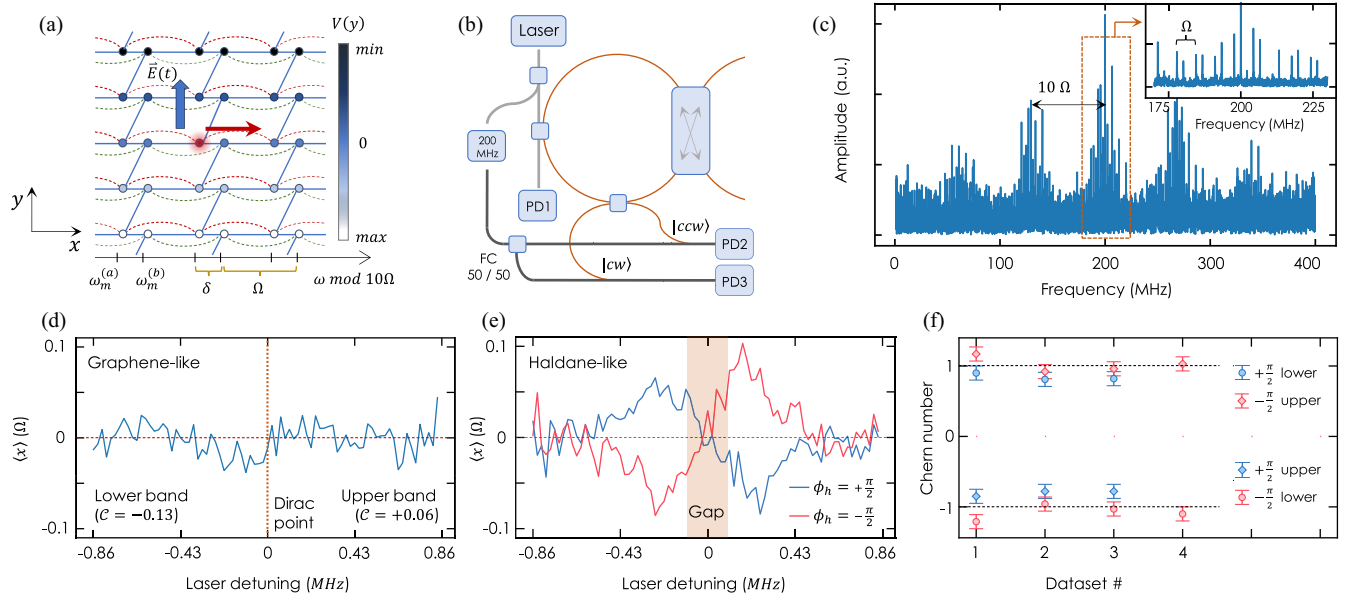


FIG. 4. Photonic analog of the transverse Hall conductivity in frequency space. (a) Schematic representation of the experimental protocol for measuring the transverse displacement. The effective scalar potential induced by the detuning along $\hat{y}$ is depicted by the blue color gradient. The driven site is indicated by the red spot and the displacement by the red arrow. (b) Experimental setup for the heterodyne measurement. The local oscillator consists of the laser signal shifted by 200 MHz and split in two to beat with both the CCW and CW modes, respectively, measured with photodiodes PD2 and PD3. (c) Example of a heterodyne spectrum exhibiting broad regions separated by 10Ω with a high density of narrowly spaced peaks. The inset shows an enlargement of the central region exhibiting pairs of peaks separated by Ω . (d),(e) Anomalous transverse displacement in units of the unit cell (Ω) as a function of the laser detuning for a graphenelike lattice (d) and two topologically distinct phases of the Haldane-like model (e). For the graphenelike lattice, we indicate the Chern number extracted by integrating the displacement over the laser detuning bandwidth corresponding to the lower and upper band. (f) Summary of the extracted Chern numbers extracted for the Haldane-like model where three measurements are taken with $\phi_h = +(\pi/2)$ (blue) and four measurements taken with $\phi_h = -(\pi/2)$ (red). Circles (diamonds) correspond to integrating over a bandwidth corresponding to the lower (upper) band. Taking the mean of the absolute value of all these points, we obtain $|\mathcal{C}|_{\text{mean}} = 0.95$ with a standard deviation of $\sigma = 0.14$. The error bars reflect an experimental uncertainty of ± 0.10 on the extracted Chern numbers, obtained by propagating the variance of the graphenelike measurement through Eq. (3) (see Appendix C).

presented in Fig. 4(c). Regions with closely spaced peaks are visible with a spacing of 10Ω , which correspond to distinct rows of the lattice; the inset gives a magnified view over the central region, where the spacing Ω between the pairs of peaks corresponds to the lattice periodicity along $\hat{x}$. From this spectrum, we extract the wave packet's center-of-mass position $\langle x \rangle = \sum [(\omega/\Omega) \bmod M\Omega] I_\omega / \sum I_\omega$, where the modulo $M\Omega$ serves to isolate the displacement along $\hat{x}$ (see Appendix C).

The expected anomalous displacement is limited to only a fraction of a unit cell, notably because of the relatively strong dissipation of our system. In order to accurately extract such a small quantity, we make use of frequency modulation to create a time-varying synthetic electric field $\lambda(t) = \lambda_0 \sin(\Omega_\lambda t)$. The modulation frequency $[\Omega_\lambda/(2\pi)] = 100$ kHz is slow enough to allow the system to follow its instantaneous out-of-equilibrium steady state. We then perform a quadrature demodulation of the heterodyne signal with $\lambda(t)$ as the reference signal to isolate the Fourier component of $\langle x \rangle$ that oscillates exactly at Ω_λ . Since the displacement is obtained by driving the system at

one specific frequency at a time, it is sensitive to several contributions that are not linked to the Berry curvature, including the quantum metric (see Supplemental Material [43]). The transverse displacement associated solely to the Berry curvature—the so-called anomalous displacement—is then isolated by combining measurements obtained by driving different sublattice sites, which allow canceling out all non-Berry contributions, according to the specific procedure discussed in Appendix C.

The results are presented in Figs. 4(d) and 4(e) as a function of the laser detuning. As the linewidth of each state in the bands is smaller than the overall width of the bands, the observed frequency-dependent anomalous displacement probes the Berry curvature averaged over states spectrally overlapping with the laser and provides evidence of an anomalous Hall effect [22,47,53,54]. For the Haldane-like model in Fig. 4(e), we observe significant displacements when the laser is resonant with eigenmodes with a strong Berry curvature, i.e., at the top of the lower band and the bottom of the upper band. This displacement has opposite signs for the upper and lower bands, as

expected, and gets reversed when switching the topological phase. In contrast, no displacement is observed for the case of a graphenelike lattice [Fig. 4(d)].

While this anomalous displacement is of geometrical origin, a topologically quantized Chern number results from integrating the Berry curvature over the BZ. Here, it can be obtained by integrating the transverse displacement over the laser detuning:

$$\mathcal{C} = -\frac{\gamma^2}{4\pi^2\lambda_0} \int_{\text{band}} d\omega_L \langle \Delta x \rangle_{\omega_L} \int \frac{d^2\mathbf{k}}{(\epsilon_{\mathbf{k}} - \omega_L)^2 + (\gamma/2)^2} \quad (3)$$

with $\epsilon_{\mathbf{k}}$ the band dispersion, $\langle \Delta x \rangle_{\omega_L}$ the displacement of the center-of-mass position along $\hat{x}$ measured with a laser detuning ω_L , and λ_0 the amplitude of the oscillating synthetic electric field (see Supplemental Material [43]).

The result of such a procedure is presented in Fig. 4(d) for a measurement on a graphenelike lattice and in Fig. 4(f) for seven different measurements on Haldane-like lattices. Within the experimental uncertainty and the approximations made, values compatible with 0 and ± 1 are obtained, in accordance with the theory and with the geometry of the band states previously measured in Fig. 3. This quantization of the photonic transverse Hall drift is a clear signature of its robustness and universality, endowed by the bands' underlying topology.

V. CONCLUSION

Our work demonstrates a novel approach to realizing photonic Chern insulators. In particular, the remarkable tunability and the intrinsic driven-dissipative nature of our platform has allowed us to perform a thorough characterization of the bulk topological properties, including the measurement of the band structures and Berry curvature. Importantly, we have shown the onset of a driven-dissipative analog of the quantized transverse Hall conductivity, thus paving the way to exploring the consequences of quantization in open quantum systems.

We envision that our platform will allow investigating driven-dissipative topological models that extend beyond the three physical dimensions [35] or involve non-Abelian gauge fields [55]. It also offers new avenues for further exploring the consequences of topologically nontrivial bands, including directional light-matter interaction phenomena in chiral environments [56–58], the emergence of one-way topological edge modes when translational symmetry in frequency space is broken [35,59,60], and their peculiar coupling to bulk states [61]. On a longer run, extending our platform to the quantum nonlinear regime opens a promising route to emulating topological quantum matter with fluids of light [62–64].

It is important to point out that, although our system makes use of magneto-optical elements (i.e., optical isolators and circulators), these are not inherently essential, as their main purpose is to facilitate the control of the

electromagnetic field and reduce spurious crosstalk. By replacing the CW and CCW propagating modes of a single fiber loop with a recently proposed [65] and experimentally demonstrated [31,66] configuration consisting of a pair of coupled fiber ring resonators, a similar model could be implemented without the need for those components. This will eventually allow integrating on a chip a frequency-encoded photonic Chern insulator, e.g., using modulated microring resonators [67].

From a technological perspective, our work opens new pathways, inspired by topological band theory, to robustly engineer the flow of light in frequency space. This advancement holds significant potential for a wide range of applications involving multifrequency dynamics, including pulsed lasers [68,69], frequency combs [70,71], signal processing [72,73], sensing [74,75], time crystals [76,77], neural networks [78,79], and analog quantum simulation [80–82].

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DATA AVAILABILITY

The data that support the findings of this article are openly available [83].

APPENDIX A: EXPERIMENTAL SETUP

1. Optical setup

A schematic of the optical setup is presented in Fig. 5(a). We use a grade 3 Rio Orion laser emitting with a central wavelength of 1542.27 nm and with a linewidth of 3.1 kHz. A tunable optical attenuator is used to limit the optical power in the cavity and mitigate nonlinear effects. The output of the laser is split by a fiber coupler with 1% sent toward the main loop and the rest continuing to a 200 MHz frequency shifter to be used as a local oscillator (LO) in the heterodyne measurement. A 99.9:0.1 fiber coupler is used as the entry port to the main loop. The transmitted

intensity is measured with a 250 MHz bandwidth InGaAs photodiode connected to a digitizer. This signal is used for measuring band structures and performing wave-function tomography.

All optical components in the setup are polarization maintaining to ensure perfect alignment with the optical axes of the electro-optic modulators; a polarizer is used to further ensure proper polarization filtering. A semiconductor optical amplifier (SOA) is used to amplify the optical signal inside the cavity in order to compensate for the losses caused by the different components inside the loop and ensure a high quality factor ($Q \sim 4 \times 10^9$). The resulting finesse of the cavity is $\mathcal{F} \sim 120$. A 100 GHz bandwidth filter is used to suppress the amplification of undesired modes.

Inside the loop, a 75:25 fiber coupler is used to hybridize the two directions of propagation inside the cavity. As described in the main text, the resulting supermodes form the lattice sites in the synthetic frequency dimension. They are given by

$$\begin{aligned} |m, a\rangle &= U|m, \text{CCW}\rangle + V|m, \text{CW}\rangle, \\ |m, b\rangle &= V|m, \text{CCW}\rangle - U|m, \text{CW}\rangle \end{aligned} \quad (\text{A1})$$

split by $\delta = 2\sqrt{g^2 + (\delta_0/2)^2}$, with $g \sim 1.1$ MHz the strength of the effective coupling induced by the FC and $\delta_0 = \omega_m^{(\text{CW})} - \omega_m^{(\text{CCW})}$ the splitting between the CW and CCW uncoupled modes. The precise value of this splitting changes depending on the measurement (see Appendix B). On the other hand, the lattice periodicity $\omega_m^{(a/b)} - \omega_{m+1}^{(a/b)}$ is constant and is given by the free spectral range: $\Omega/2\pi \sim 6.6$ MHz.

In order to efficiently implement NN and NNN coupling terms—the latter with opposite phases for different sublattices—we separate the CW and CCW propagating modes with a pair of circulators. This allows time modulating independently CW and CCW modes with iXBlue lithium niobate electro-optic phase modulators. Each branch also includes an isolator to ensure unidirectional propagation in between the circulators (see Appendix C1 for details on the signal sent to the EOMs). It is important to note here that, while circulators and isolators break time-reversal symmetry due to magneto-optics effects, they do so in real space and not in frequency space. As such, they play no role in our implementation of a T -broken topological phase. The experiment could be adapted to make no use of these components, as their only role is to provide a better signal-to-noise ratio.

A second 99.9:0.1 fiber coupler is used as an output port to realize the heterodyne measurement. The use of a second port allows probing simultaneously the CW and CCW modes. The LO for the heterodyne measurement is provided by the laser shifted by 200 MHz and amplified with an erbium-doped fiber amplifier (EDFA). This LO is then split in half with a 50:50 fiber coupler and then mixed (using 99:1 fiber couplers) with the signal radiating from the CW and CCW modes output ports. Two InGaAs photodiodes with a bandwidth of 600 MHz are used for collecting these mixed signals.

2. Electrical setup

A schematic of the full electrical setup, used to control and monitor the platform's state, can be seen in Fig. 5(b). The electrical signal from the photodiode measuring the transmission signal is split in two toward

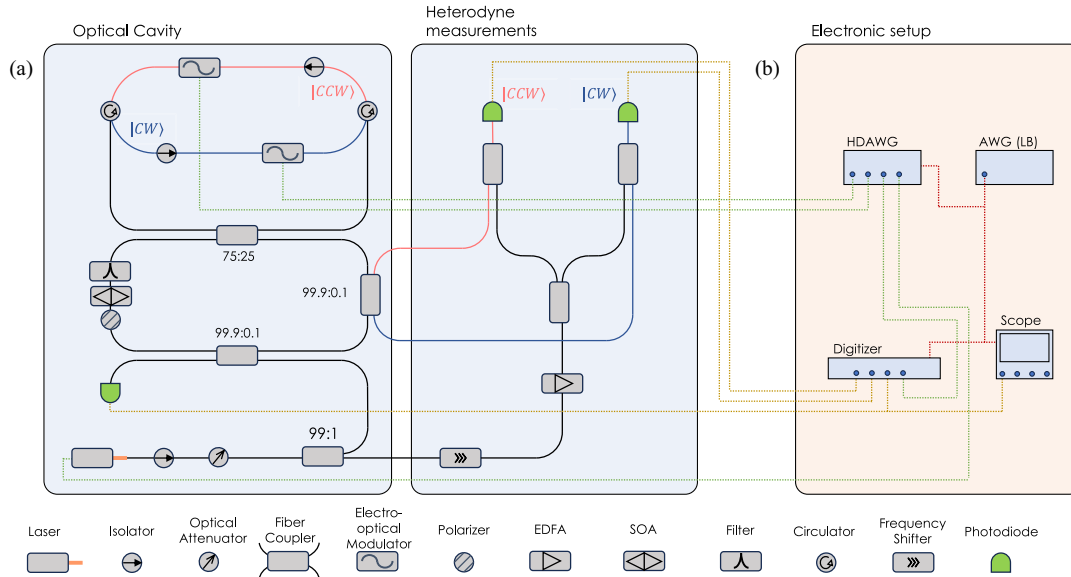


FIG. 5. Schematic representation of the experimental setup. (a) Optical setup, including the cavity and the heterodyne measurement, and (b) electrical setup, including electrical controllers and digitizers.

a 2-GHz-bandwidth Rigol oscilloscope and a Tektronix 6 Series Low Profile digitizer (with a bandwidth of 2.5 GHz and a sampling rate of 25 GS s^{-1}). This allows probing the signal over two different time windows, each with high temporal resolution; the window measured on the oscilloscope is used as a calibration probe and the window measured on the digitizer is used to register the actual experiment data (see Appendix B).

The electrical signals driving the EOMs and tuning the laser frequency are provided by a Zurich Instruments high-definition arbitrary waveform generator (HDAWG). A low-bandwidth arbitrary waveform generator, generating a low-frequency square wave, is also used to trigger all our electrical instruments simultaneously.

APPENDIX B: CALIBRATION PROCEDURE

The calibration consists in determining the values of Ω (the FSR) and $\delta = 2\sqrt{g^2 + (\delta_0/2)^2}$ (the splitting between the two sublattices) with δ_0 the CW-CCW frequency splitting. Accurate determination of these two frequencies is critical for driving the EOMs with the appropriate signal in order to not induce spurious on-site energies and/or synthetic electric fields. Hereafter, we detail the procedure for calibrating each frequency.

1. Calibration of the free spectral range Ω

The FSR is the easiest frequency splitting to calibrate, as it does not fluctuate significantly in time. Hence, it can be measured a single time prior to the experiment and be kept identical for all measurements. While a good first approximation can be made using the fact that $\Omega = [(2\pi c)/(nL)]$, this does not yield a precise enough value. To increase the precision, we send the following signals to the EOMs:

$$V^{(\text{cw/ccw})}(t) = \frac{V_{\text{nnn}}}{2} e^{-i\Omega' t} + \text{c.c.}$$

This results in the coupling of next-nearest neighbors which are exactly split by a frequency Ω . Since each sublattice is coupled independently by this driving field and no coupling between sublattices is introduced, the effective bands emerging from each frequency mode are identical and correspond to that of a lattice of identical sites with a unique hopping strength. This leads to band dispersions of the form

$$\omega(t) = -2J' \cos(\Omega' t).$$

A slight detuning between Ω' and Ω results, in the rotating frame, in a linearly varying frequency detuning between the drive and the frequency eigenmodes and, thus, to an effective synthetic electric field. This effective field leads to asymmetric features with respect to $k = 0$ in the measurement of the dispersion relation. Hence, we can tune Ω' until we reach symmetric bands (Fig. 6). This procedure

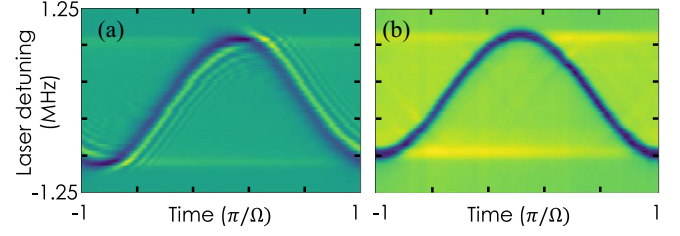


FIG. 6. Band dispersions obtained when calibrating the FSR with $|\Omega' - \Omega| \approx 40 \text{ kHz}$ in (a) and $|\Omega' - \Omega| < 2 \text{ kHz}$ in (b).

can be further optimized by changing the modulation frequency to $N\Omega$, thus enhancing by a factor N the accuracy of the calibration. This led to a value of $\Omega = 2\pi \times 6.628 \text{ MHz}$ with an uncertainty of less than 5 kHz when modulating up to $N = 10$.

2. Calibration of the sublattice spacing δ

Calibration of the sublattice spacing δ is much more challenging, as thermal fluctuations continuously modify the relative length of the CW and CCW branches between the circulators, hence changing the frequency splitting between the hybridized modes on a timescale of the order of several hundred milliseconds. This relatively slow drift was achieved by passively stabilizing the setup thermally; this drift characteristic time is much larger than typical measurement times (approximately 10–100 μs). Moreover, for the Haldane-like model measurements, we need to work in a regime with a very specific detuning between CW and CCW modes such that the frequency eigenmodes in Eq. (A1) have $U \neq V$. This allows modulating the two sublattices independently—using the modulators in the CW and CCW branches—which is a necessary requirement for engineering NNN couplings with opposite phase for the two sublattices. When implementing the Haldane-like model, we typically operate in a regime where $|U|^2/|V|^2 \sim 20$ such that we can still have significant NN couplings (which scale as UV) with moderate EOM drive amplitude.

Hence, the challenge consists in fixing a specific and exact value of δ , such that the EOM drives do not induce undesirable variations of the effective on-site energies, while thermal fluctuations continuously modify the relative length of the CW and CCW branches, hence continuously changing δ_0 .

To do this, we developed a two-step protocol that is implemented for every measurement realized in this work. This protocol consists in separating the acquisition process in two distinct time frames separated by 100 ms (which is typically faster than the typical thermal fluctuations rate); this dual-time-frame sequence repeats continuously with a period imposed by the low-bandwidth AWG which triggers all other electrical instruments. The first time frame is used to calibrate the value of δ , while the second is dedicated to the measurement itself. Each time frame has a duration of 50 ms during which the laser frequency scans a bandwidth

comparable to the FSR in order to measure at least two band structures. During the first time frame used for calibration, the EOMs are driven with the following signal:

$$V_{nn}^{(cw/ccw)}(t) = \pm \frac{1}{2} V^{(0)} [e^{-i\delta' t} - e^{-i(\Omega - \delta') t}] + \text{c.c.} \quad (\text{B1})$$

with $+$ and $-$ corresponding to the modulation of the CW and CCW modes, respectively, and Ω obtained following the procedure described above. This driving field induces a coupling between NN lattice sites with identical strength $V^{(0)}$. Upon such driving, the effective band structures follow the following dispersion:

$$\omega_{\pm}(t) = \mp \sqrt{2J[1 + \cos(ka)] + (\bar{\Delta}/2)^2}, \quad (\text{B2})$$

where J is a coupling coefficient proportional to $V^{(0)}$ and $\bar{\Delta} = \delta' - \delta$ is the detuning between the driving frequency and the sublattice spacing (whose effect can be viewed as a staggered on-site potential). This dispersion relation is gapped whenever $\bar{\Delta} \neq 0$; hence, probing the density of states (DOS) during this time frame (to identify moments where the DOS is not gapped) allows identifying when thermal fluctuations lead to $\delta = \delta'$ (Fig. 7). Therefore, it is possible to define a value of δ' with which we want to work and then wait until thermal fluctuations lead to a gapless DOS, indicating perfect matching between the driving frequencies and the loop's eigenmode splitting, i.e., $\bar{\Delta} = 0$. Then, in the second time frame, the EOMs are driven with the fields described in the main text for each experimental part, using δ' as the definition of the sublattice spacing. For graphenelike and h -BN-like measurements, δ' is chosen such that the CW and CCW modes are degenerate, i.e., $\delta_0 = 0$. This is because these models do not require NNN couplings. However, to implement the Haldane-like model, δ' is chosen such that $\delta_0 \approx 1.1$ MHz. This corresponds to values of $|U| = 0.82$ and $|V| = 0.18$ —a regime where $|U|^2/|V|^2 \sim 20$. This regime allows for a good balance between the amplitudes of the NN and NNN couplings and ensures that the NNN hopping phase ϕ_h is exactly equal to the phase of the modulated signal sent to the EOMs, ϕ_{mod} (see Supplemental Material [43]). As δ drifts over a much longer timescale than that of individual measurements, this regime does not fluctuate throughout an entire experiment, and the value of ϕ_h remains stable.

More specifically, the procedure for acquiring data exactly when $\bar{\Delta} = 0$ is the following. The electrical response from the photodiode measuring the transmitted signal is split in half with one part going to the oscilloscope and the other to the digitizer (see the above section on the electrical setup). We ensure that the time delays for triggering the scope and the digitizer are such that the former acquires signal during the first time frame while the latter acquires during the second time frame; separating

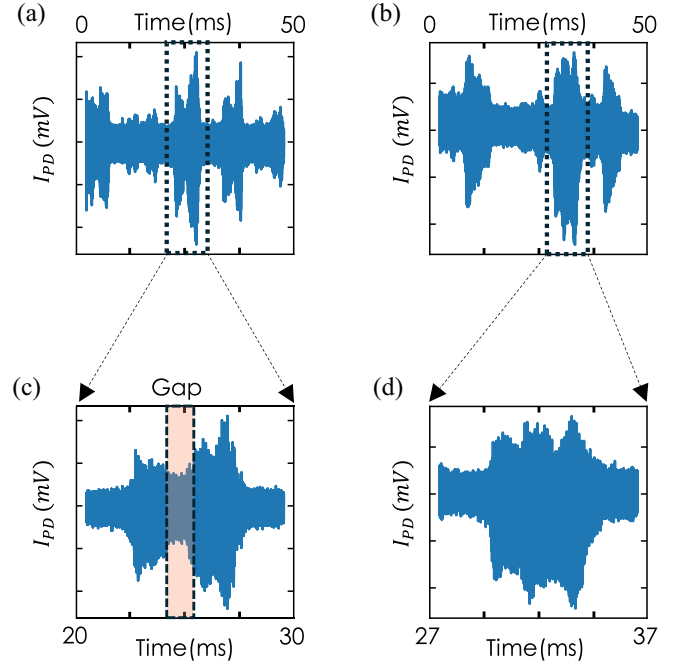


FIG. 7. Density of states can be observed visually from the transmission signal on both the digitizer and the oscilloscope when averaging the signal over a time window much bigger than the period of a BZ. (a) shows densities of states that exhibit a gap, and (c) is an enlargement of one of them. This describes a regime where $\delta' \neq \delta$. Meanwhile, (b) depicts densities of states that are gapless, which can be more easily seen when enlarging (d). In this case, we have $\delta' \approx \delta$, and we can acquire the data from the digitizer.

the signal in two distinct time windows ensures a high temporal resolution for each measurement. As soon as we observe a gapless density of states on the oscilloscope, we stop the acquisition, which automatically transfers the memory buffer of the digitizer (containing the experimental signal measured upon good experimental conditions) to a computer where it can be analyzed as described below.

APPENDIX C: EXPERIMENTAL PROTOCOL

1. Band structure measurements

The band structures shown in Fig. 2 in the main text are obtained by modulating the laser frequency with a staircase triangular waveform with frequency 10 Hz and amplitude 0.075 V. Upon this modulation, we probe 50 ms time windows (as described in the calibration procedure), each time window corresponding to the descending part of a modulation period. These modulation parameters allow the scan of around one FSR of the optical fiber loop. Each step of the staircase has a duration of 130 μ s, during which the laser frequency is kept constant. At the beginning of each laser step, we wait 5 μ s in order for the system to reach a steady state before acquiring data.

The coupling between the different frequency eigenmodes that underlies the synthetic dimension scheme is ensured by driving the EOMs with a signal formed from Fourier components with frequencies equal to the frequency splittings between each pairs of modes to be coupled and amplitudes proportional to the corresponding hopping amplitude. The general formula describing the signal sent to the EOMs is the following:

$$V^{(\text{cw/ccw})}(t) = \pm \frac{1}{2} \left[V_{\text{nn}}^{(1)} e^{-i\delta t} + V_{\text{nn}}^{(2)} e^{-i((\Omega-\delta)t + \phi_{\text{shift}})} + V_{\text{nn}}^{(3)} e^{-i(10\Omega + \delta)t} \right] + \frac{V_{\text{nnn}}}{2} e^{-i(\Omega t \pm \phi_{\text{mod}})} + \text{c.c.} \quad (\text{C1})$$

with $+$ and $-$ corresponding to the modulation of the CW and CCW modes, respectively. Specifically, $V_{\text{nn}}^{(i)}$ with $i = 1, 2, 3$ represent the nearest-neighbor hoppings of the honeycomb lattice; in particular, the coupling $V_{\text{nn}}^{(3)}$ to the 21st neighbor is used to simulate the second dimension. Components V_{nnn} are linked to the next-nearest-neighbor hoppings of the Haldane model; V_{nnn} is taken to be zero when emulating graphenelike and h -BN-like Hamiltonians. The h -BN-like Hamiltonian is realized by introducing slight detunings of the NN couplings:

$$\begin{aligned} \delta &\rightarrow \delta + \Delta, \\ \Omega - \delta &\rightarrow \Omega - \delta - \Delta, \\ M\Omega + \delta &\rightarrow M\Omega + \delta + \Delta \end{aligned} \quad (\text{C2})$$

with $\Delta/2\pi = 200$ kHz.

Throughout each frequency step of the laser scan, we measure the time-resolved transmitted signal with a high-bandwidth photodiode. This signal associated to the resonances in the lower ($+$) and upper ($-$) bands follows the relation (see theoretical derivation of the time-resolved transmission in Supplemental Material [43])

$$\begin{aligned} \frac{I^{(+)}(t)}{|F|^2} &\sim 1 - 2\kappa \text{Re} \left\{ \frac{f_{\mathbf{k}}^{(+)}(t)}{\frac{\gamma}{2} - i(\Delta\omega_L - \epsilon_{\mathbf{k}}^{(+)})} \right\} \bigg|_{(k_x, k_y)}, \\ \frac{I^{(-)}(t)}{|F|^2} &\sim 1 - 2\kappa \text{Re} \left\{ \frac{f_{\mathbf{k}}^{(-)}(t)}{\frac{\gamma}{2} - i(\Delta\omega_L - \epsilon_{\mathbf{k}}^{(-)})} \right\} \bigg|_{(k_x, k_y)} \end{aligned} \quad (\text{C3})$$

with numerators $f_k^{(\pm)}$ given by

$$\begin{aligned} f_{\mathbf{k}}^{(+)} &= U^2 \cos^2\left(\frac{\theta_{\mathbf{k}}}{2}\right) + UV \sin\left(\frac{\theta_{\mathbf{k}}}{2}\right) \cos\left(\frac{\theta_{\mathbf{k}}}{2}\right) e^{-i\delta t} e^{i\phi_{\mathbf{k}}}, \\ f_{\mathbf{k}}^{(-)} &= U^2 \sin^2\left(\frac{\theta_{\mathbf{k}}}{2}\right) - UV \cos\left(\frac{\theta_{\mathbf{k}}}{2}\right) \sin\left(\frac{\theta_{\mathbf{k}}}{2}\right) e^{-i\delta t} e^{i\phi_{\mathbf{k}}}, \end{aligned} \quad (\text{C4})$$

where we have used the shorthand $\theta_{\mathbf{k}} = \theta_{\mathbf{k}}^{(+)}$ and $\phi_{\mathbf{k}} = \phi_{\mathbf{k}}^{(+)}$ and we have taken into account that $\theta_{\mathbf{k}}^{(-)} = \pi - \theta_{\mathbf{k}}^{(+)}$ and $\phi_{\mathbf{k}}^{(-)} = \pi + \phi_{\mathbf{k}}^{(+)}$. These quantities carry information on the polar and azimuthal angles $\theta_{\mathbf{k}}^{(\pm)}$ and $\phi_{\mathbf{k}}^{(\pm)}$ of the Bloch eigenstates that is useful for extracting the Berry curvature (see Appendix C2).

In Eqs. (C3) and (C4), F and $\Delta\omega_L$ are the driving field amplitude and detuning with respect to the loop's closest eigenmodes, respectively, γ and κ are the loop's total decay rate and input-output-coupling ratio, respectively, and each time t is linked to a $\mathbf{k}$ point in the effective BZ using the relations

$$\begin{aligned} t &\mapsto (k_x, k_y), \\ k_x &= t + \frac{\phi_{\text{shift}}}{\Omega} \mod \frac{2\pi}{\Omega}, \\ k_y &= t \mod \frac{2\pi}{10\Omega}. \end{aligned} \quad (\text{C5})$$

The denominators in Eq. (C3) therefore lead to Lorentzian intensity dips whenever the laser is resonant with the lower or upper band dispersions $\epsilon_k^{(\pm)}$ at crystal momentum k_x, k_y . Hence, the effective BZ in the time domain has periodicity $(2\pi)/\Omega$ along k_x and $(2\pi)/(10\Omega)$ along k_y . The measurement of intensity dips at time t indicates that the laser is resonant with an eigenmode with effective crystal momentum $\mathbf{k} = (k_x, k_y)$.

Moreover, the effect of adding a phase ϕ_{shift} to the Fourier component $V_{\text{nn}}^{(2)}$ results in a shift of the definition of k_x . This allows the time trace monitored on the photodiode to follow horizontally shifted trajectories in reciprocal space. This is best seen by considering the mapping of the time bins of each measurement on the photodiode onto the BZ: This is presented in Fig. 8(a) for $\phi_{\text{shift}} = 0$ (each red dot represents a single time bin with a time separation dictated by the sampling rate of the digitizer) and in Fig. 8(b) for five distinct values $\phi_{\text{shift}} = 1.25(\pi/10), 0.625(\pi/10), 0, -0.625(\pi/10), -1.25(\pi/10)$. In order to obtain such a denser covering of the Brillouin zone, the

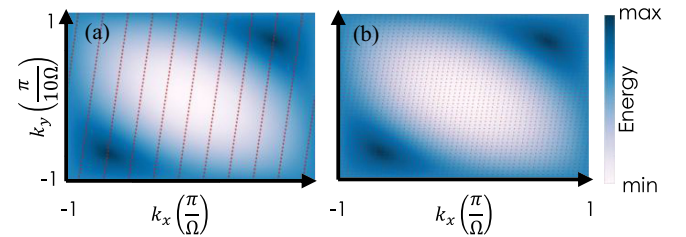


FIG. 8. Points sampled on the complete 2D BZ. The color plot in the background shows the theoretical dispersion of the lower band for graphene. The points sampled by using a single value of ϕ_{shift} value are depicted with red dots in (a), and the combined sampling using five different values of ϕ_{shift} is shown in (b).

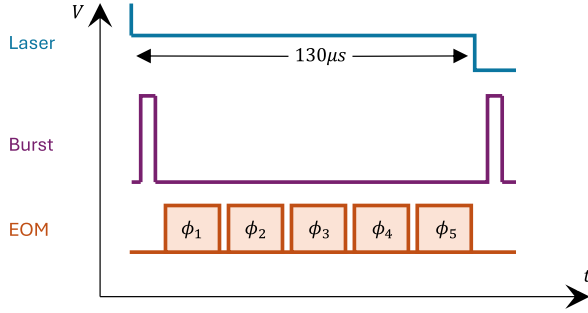


FIG. 9. Electrical signals sent to modulate the frequency of the laser (blue) and the EOMs (orange) and sent directly to the digitizer (purple) to realize band structure measurements.

EOM signal through each laser frequency step is partitioned in five equally spaced time regions, separated by 3 μ s and lasting the equivalent of 150 BZs. In each region, the EOM signal has one specific value of the phase ϕ_{shift} among the five used in Fig. 8(b). A schematic of the pulse sequence used is depicted in Fig. 9; a burst signal is also generated by the HDAWG at the beginning of each laser step and monitored on the digitizer to ensure that we can precisely identify the temporal position of each laser step in our data analysis.

For each ϕ_{shift} region of each laser step, we average the photodiode signal over the 150 trajectories through the BZ to obtain a single averaged trajectory across the BZ with a high signal-to-noise ratio. Then, by scanning the laser frequency and vertically stacking these averaged signals measured at each step, we can construct the full band structure. Figures 2(a), 2(e), and 2(g) in the main text show this construction when stacking only the signal associated to the region with $\phi_{\text{shift}} = 0$. Figure 2(c) in the main text presents the value of the lower band energy measured at each time bin in Fig. 8(b). Finally, Figs. 2(d), 2(f), and 2(h) in the main text present the data measured at specific time bins along the high-symmetry trajectory M - K - Γ - M .

2. Berry curvature measurement

a. Extraction of $\phi_{\mathbf{k}}$

The extraction of $\phi_{\mathbf{k}}$ is realized using the procedure introduced in Ref. [46]. Let us assume we want to extract $\phi_{\mathbf{k}}$ for the lower band. Equations (C3) show that the transmitted intensity ($I_{pd}^{(+)}$) is time modulated with frequency δ and phase $\phi_{\mathbf{k}}$. Specifically, each of the 150 BZs on a given laser step will have transmission peaks happening at the same times with respect to the start of each BZ, but the intensity of these peaks will vary between different BZs.

Hence, for each $\mathbf{k}$ point, we want to measure how the transmitted intensity varies over the different BZs and extract the phase of this modulation. To obtain a precise value of the transmitted intensity at each time bin corresponding to a given $\mathbf{k}$ point, we sum the intensity measured

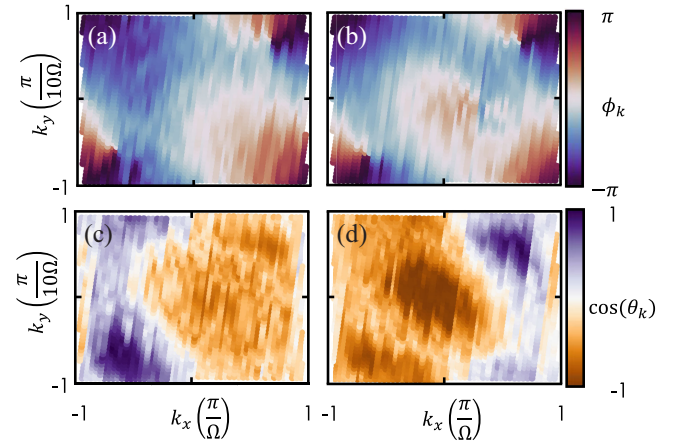


FIG. 10. 2D Mapping of extracted phases $\phi_{\mathbf{k}}$ and $\theta_{\mathbf{k}}$ that are used in Fig. 3 in the main text for the Haldane-like model datasets with $\phi_h = \pm(\pi/2)$. (a),(b) show the mapping of $\phi_{\mathbf{k}}$ for $\phi_h = +(\pi/2)$ and $\phi_h = -(\pi/2)$, respectively, whereas (c),(d) show the mapping of $\cos(\theta_{\mathbf{k}})$ with respect to the same data as (a),(b), respectively.

at this specific time bin (with respect to the beginning of the step) over all the laser steps covering the lower band; this corresponds to integrating the measured signal for a given time bin over each laser detuning. This allows properly taking into account the spectral linewidth of each eigenmode. Then, for every $\mathbf{k}$ point we evaluate how the transmitted intensity varies over the different BZs by taking the Fourier component at δ of the signal created from juxtaposing the 150 different intensity values measured at time bins corresponding to this $\mathbf{k}$ point. Taking the argument of this demodulated signal at every $\mathbf{k}$ yields $\phi_{\mathbf{k}}$.

Examples of the extraction of $\phi_{\mathbf{k}}$ for the Haldane-like model (presented in Fig. 3 in the main text) are presented in Figs. 10(a) and 10(b).

b. Extraction of $\theta_{\mathbf{k}}$

To extract $\theta_{\mathbf{k}}$, we now consider the time-integrated transmitted intensity over the 150 BZs that is used in the band structure measurements (as in Fig. 2 in the main text). Doing so, the time-dependent oscillation at frequency δ averages out, and the integrated dip in the transmitted intensity, when the frequency of the driven input field is resonant to the energy of a band, is given by

$$\begin{aligned} T^{(+)} &\equiv 1 - \frac{\bar{I}_{pd}^{(+)}}{|F|^2} \approx 4\kappa \frac{U^2 \cos^2(\frac{\theta_{\mathbf{k}}}{2})}{\gamma}, \\ T^{(-)} &\equiv 1 - \frac{\bar{I}_{pd}^{(-)}}{|F|^2} \approx 4\kappa \frac{U^2 \sin^2(\frac{\theta_{\mathbf{k}}}{2})}{\gamma}. \end{aligned} \quad (\text{C6})$$

Then, we consider the difference between the intensities of the two bands, normalized by their sum:

$$\begin{aligned}
\frac{T^{(+)} - T^{(-)}}{T^{(+)} + T^{(-)}} &= \frac{\frac{4\kappa U^2}{\gamma} (\cos^2(\frac{\theta_{\mathbf{k}}}{2}) - \sin^2(\frac{\theta_{\mathbf{k}}}{2}))}{\frac{4\kappa U^2}{\gamma} (\cos^2(\frac{\theta_{\mathbf{k}}}{2}) + \sin^2(\frac{\theta_{\mathbf{k}}}{2}))} \\
&= \cos^2\left(\frac{\theta_{\mathbf{k}}}{2}\right) - \sin^2\left(\frac{\theta_{\mathbf{k}}}{2}\right) \\
&= \cos(\theta_{\mathbf{k}}). \tag{C7}
\end{aligned}$$

Hence, we obtain that $\theta_{\mathbf{k}} = \arccos[(T^{(+)} - T^{(-)})/(T^{(+)} + T^{(-)})]$.

Examples of the extraction of $\cos(\theta_{\mathbf{k}})$ for the Haldane-like model (used in Fig. 3 in the main text) are presented in Figs. 10(c) and 10(d).

c. Mapping on the Bloch sphere

Once we have extracted both polar and azimuthal angles using the methods described above, it is very straightforward to map the eigenstates on a Bloch sphere. The order with which the different points cover the Bloch sphere is chosen to be the ordering of all the points with respect to their value of k_x .

d. Extraction of the Berry curvature

The expression for the Berry curvature is defined as

$$\Omega_{\mathbf{k}} = \nabla_{\mathbf{k}} \times \langle \psi_{\mathbf{k}} | i \nabla_{\mathbf{k}} | \psi_{\mathbf{k}} \rangle \tag{C8}$$

with $|\psi_{\mathbf{k}}\rangle = \cos(\theta_{\mathbf{k}}/2)|a\rangle + \sin(\theta_{\mathbf{k}}/2)e^{i\phi_{\mathbf{k}}}|b\rangle$. This is difficult to evaluate from experimental data, as it involves numerical derivatives over a discrete number of points presenting experimental fluctuations. Instead, we use a technique developed by Fukui, Hatsugai, and Suzuki [48] for discretized BZs. We first consider the following quantity:

$$U_{\mathbf{k}}^{(\mu)} = \frac{\langle \psi_{\mathbf{k}} | \psi_{\mathbf{k}+\delta\mathbf{k}_{\mu}} \rangle}{|\langle \psi_{\mathbf{k}} | \psi_{\mathbf{k}+\delta\mathbf{k}_{\mu}} \rangle|} \tag{C9}$$

with $\mathbf{k}$ a discrete point sampled in the BZ (here, a specific time bin) and $\delta\mathbf{k}_{\mu}$ the vectors linking adjacent $\mathbf{k}$ points.

This quantity is easy to calculate at each $\mathbf{k}$, as we have extracted both polar and azimuthal angles at each point; thus, we know the eigenstate at each $\mathbf{k}$.

We can then refer to the 2D mapping described previously to determine the distribution of the $\mathbf{k}$ points sampled across the BZ, partition this latter into plaquettes, and identify the pairs of $\mathbf{k}$ points for which the scalar products have to be calculated.

The Berry curvature flux through a plaquette is given by

$$\Omega_{\mathbf{k}} = \frac{1}{i} \ln \left[U_{\mathbf{k}}^{(2)} U_{\mathbf{k}+\delta\mathbf{k}_2}^{(1)} U_{\mathbf{k}+\delta\mathbf{k}_1}^{(2)} U_{\mathbf{k}}^{*(1)} \right], \tag{C10}$$

where the plaquette described by this equation is schematically shown in Fig. 11. Finally, the Chern number is

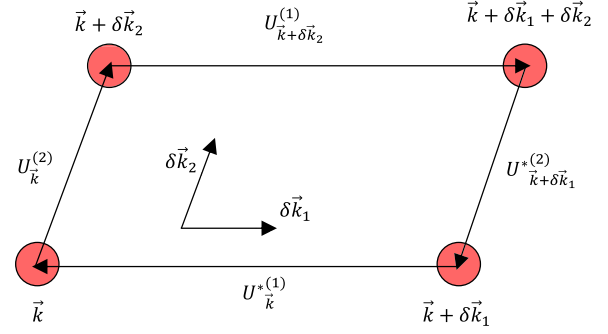


FIG. 11. Schematic representation of the plaquette considered in the extraction of $\Omega_{\mathbf{k}}$. The red circles represent sampled points.

then simply obtained by summing the extracted Berry curvature over all plaquettes forming the BZ:

$$\mathcal{C} = \frac{1}{2\pi} \sum_{\mathbf{k} \in \text{BZ}} \Omega_{\mathbf{k}}. \tag{C11}$$

3. Anomalous transverse displacement and quantized Hall drift

To obtain anomalous transverse displacement measurements, we slightly change the signals sent to the laser and to the EOMs. The laser frequency is now modulated by a staircase triangular waveform of frequency 20 Hz and amplitude 0.75 V, with steps of 60 μs (see Fig. 12). The general formula describing the signal sent to the EOMs is now the following:

$$\begin{aligned}
V^{(\text{cw/ccw})}(t) &= \pm \frac{1}{2} \left[V_{\text{nn}}^{(1)} e^{-i\delta t} + V_{\text{nn}}^{(2)} e^{-i[(\Omega-\delta)t]} \right. \\
&\quad \left. + V_{\text{nn}}^{(3)} e^{-i(10\Omega+\delta+\lambda_t)t} \right] \\
&\quad + \frac{V_{\text{nnn}}}{2} e^{-i(\Omega t \pm \phi_h)} + \text{c.c.}, \tag{C12}
\end{aligned}$$

where $\lambda_t = \lambda_0 \sin(\Omega_{\lambda} t)$ simulates an oscillating electric potential $V = \lambda_t y$ with amplitude $\lambda_0/2\pi = 10$ kHz and

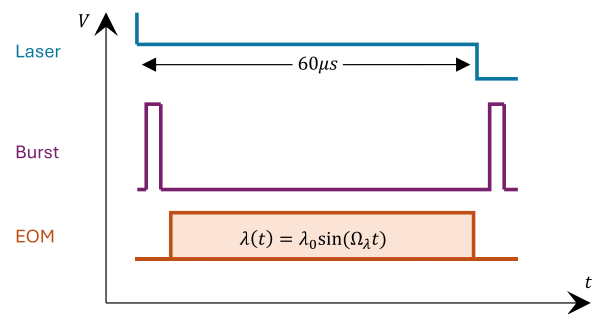


FIG. 12. Electrical signals sent to modulate the frequency of the laser (blue), to modulate the EOMs (orange), and sent directly to the digitizer (purple) to realize anomalous displacement measurements.

frequency $\Omega_\lambda/2\pi = 100$ kHz. This acts as a uniform electric field on the lattice along $\hat{y}$:

$$\mathbf{E}(t) = -\nabla V = -\lambda_t \hat{y}. \quad (\text{C13})$$

The fact that this field is oscillating rather than constant is critically important for two main reasons. First, it enables a much better signal-to-noise ratio by mimicking the effect of a lock-in amplifier; by later demodulating our measured displacements with respect to the oscillation frequency, we are able to filter out most of the external noise that would pollute our signals. As the displacements observed are on the order of a tenth of a unit cell, this filtering is essential to our measurements. Second, as shown in numerical simulations (see Supplemental Material Sec. II [43]), very strong zero-field contributions to the displacement typically dominate the displacement when using a constant synthetic electric field. As such, by demodulating the extracted signal at the field's oscillating frequency, we can easily filter out such dc components that do not depend on the field. Finally, in order to mitigate interband mixing, which would impact the quantization of the transverse drifts, the value of λ_0 , the amplitude of the modulated field, is chosen to be an order of magnitude smaller than the band gaps (approximately $2\pi \times 100$ kHz).

In contrast with the band structure measurements, the EOM signal is now partitioned into a single time frame lasting the equivalent of 370 BZs per laser step (approximately $56 \mu\text{s}$). Through each of these frequency steps scanned by the laser, we now consider the time-resolved heterodyne signal measured with two high-bandwidth photodiodes, each probing the signal radiating from both directions from the output coupler shown in Fig. 5. Each of these heterodyne signals is then split into different time windows of $1 \mu\text{s}$ associated to a time value t_n (see Fig. 13). For each time window, the Fourier transform gives a spectrum centered at $\omega_0/2\pi = 200$ MHz—the frequency of the LO—with peaks corresponding to the locations of the supermodes in the frequency space. We then define the displacement for a given t_n as

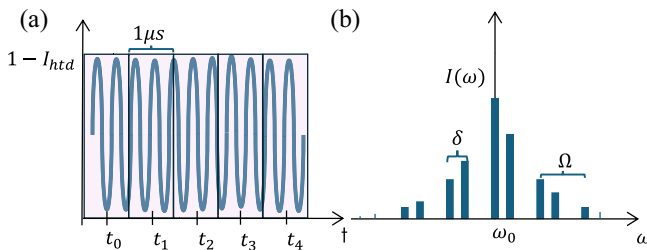


FIG. 13. Schematic representation of the windowed Fourier transform process to measure the anomalous displacement. (a) depicts the partitioning of the heterodyne signal I_{htd} into different time windows, with the Fourier transform of a given window shown in (b).

$$\langle \Delta x \rangle(t_n) = \sum_{\omega} \frac{\left[\frac{(\omega - \omega_0)}{\Omega} \bmod(M\Omega) \right] I_{\omega}}{\sum_{\omega} I_{\omega}}, \quad (\text{C14})$$

where ω indexes the unit cell such that I_{ω} represents the total intensity inside of a given unit cell. Following our encoding scheme sketched in Fig. 4(a), taking the frequency modulo $M\Omega$ (with $M = 10$ in our experiment) serves to discard the motion along $\hat{y}$ and isolate the displacement along the $\hat{x}$ direction. In the experiment, there are only three rows of the lattice that have enough light intensity to contribute non-negligibly to the displacement. Redoing this for each time window yields a time-dependent displacement $\langle x \rangle(t)$. We then demodulate this displacement with respect to the oscillating synthetic electric field's frequency:

$$\langle \Delta x \rangle = \frac{1}{\xi} \int_0^{56 \mu\text{s}} dt \langle \Delta x \rangle(t) \sin(\Omega_\lambda t), \quad (\text{C15})$$

where $\xi = \int_0^{56 \mu\text{s}} dt \sin^2(\Omega_\lambda t)$ is a normalization constant. We then repeat this process for every laser frequency step of a given band.

Finally, to obtain the displacement results presented in Fig. 4, we must remove spurious, non-Berry contributions to the transverse displacement which arise from the quantum metric and the asymmetric shape of the band dispersion. These terms are both significant and detrimental in the context of the quantization of transport (see Supplemental Material [43]).

To eliminate these unwanted contributions, we first take a measurement with the desired value of ϕ_h and then we subtract this measurement with a second one obtained with $-\phi_h$. Since the non-Berry terms are even under an inversion of the topological phase, whereas the Berry contributions are odd, taking the half-difference between the displacements from these two distinct measurements cancels out the non-Berry terms, isolating only the Berry curvature-dependent contributions.

An example of this two-step procedure is presented in Fig. 14. Figure 14(a) shows the conditions in which the first measurement is taken, in the regime $|U|^2 \gg |V|^2$. As light is injected in the CCW direction, in this regime the band structure emerging from the $|a\rangle$ sublattice (measured along the transmission line) is brighter. Therefore, the first heterodyne measurement is performed when the drive is resonant with this sublattice. Figure 14(b) shows the conditions of the second measurement. We now work in a regime where $|U|^2 \ll |V|^2$, leading to a bright band structure emerging from the $|b\rangle$ sublattice; the heterodyne spectrum is now measured when the drive is resonant with this sublattice. Both measurements are performed with the same ϕ_{mod} ; hence, the reversal of the ratio $|U|/|V|$ leads to a change of sign of ϕ_h [see Eq. (S.26) in Supplemental Material [43]] as required for canceling out non-Berry contributions.

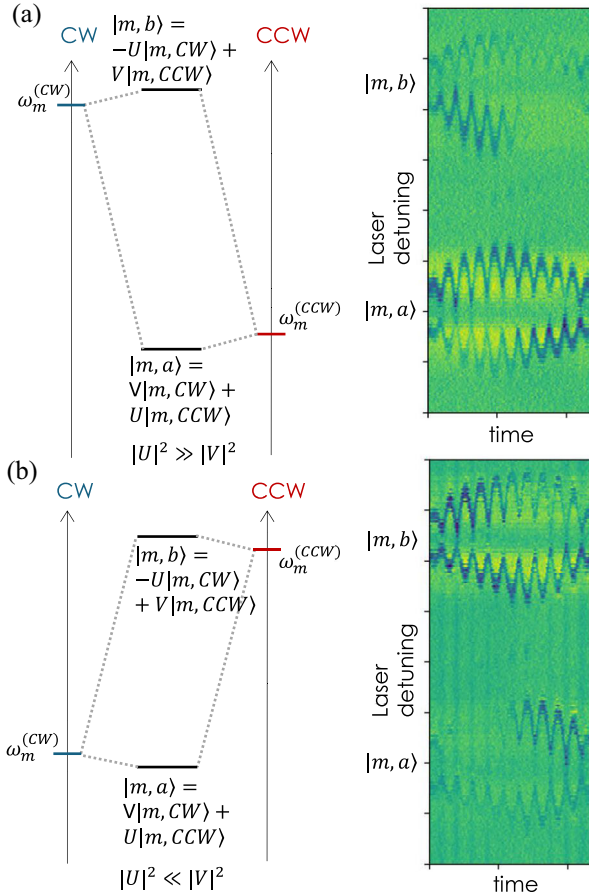


FIG. 14. (a) Probing only the CCW direction, when thermal fluctuations make it so that $|U|^2 \gg |V|^2$, we have that the band structure of the $|a\rangle$ site is more visible. (b) When $|U|^2 \ll |V|^2$, the band structure of the $|b\rangle$ site is more visible instead.

As a result, if $\langle \Delta x \rangle_a$ and $\langle \Delta x \rangle_b$ denote the displacements obtained for both configurations (i.e., with the laser aligned to an $|a\rangle$ or $|b\rangle$ mode, respectively), then the contribution to the displacement coming from the Berry curvature is given by

$$\langle \Delta x \rangle = \frac{\langle \Delta x \rangle_a - \langle \Delta x \rangle_b}{2}. \quad (\text{C16})$$

Figures 15(a) and 15(b) depict the measurements obtained in each individual configuration, and Fig. 15(c) depicts the half-subtraction described above. These anomalous displacements $\langle \Delta x \rangle$ are the ones depicted in Fig. 4 and the ones used to extract Chern numbers.

The Chern number for a given band is then extracted using the formula (see theoretical derivation of the anomalous displacement and its relation to the Chern number in Supplemental Material [43])

$$C = -\frac{\gamma^2}{4\pi^2\lambda_0} \int_{\text{band}} d\omega_L \langle \Delta x \rangle_{\omega_L} \int \frac{d^2\mathbf{k}}{(\epsilon_{\mathbf{k}} - \omega_L)^2 + (\gamma/2)^2}, \quad (\text{C17})$$

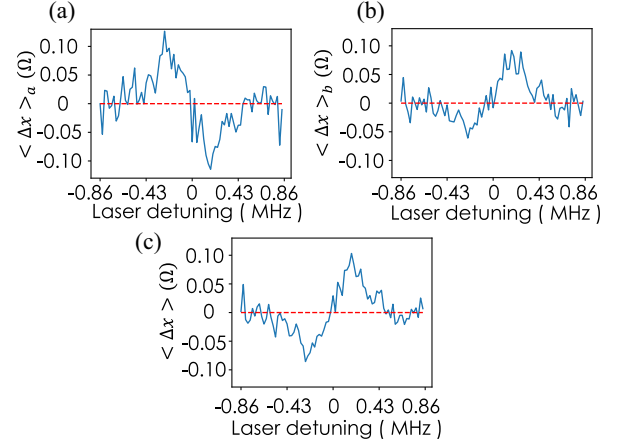


FIG. 15. Example of measured displacement obtained when the pump laser is aligned to an $|a\rangle$ site (a), to a $|b\rangle$ site (b), and the resulting displacement when using Eq. (C16) (c). These measurements were taken with $\phi_h = -(\pi/2)$. In (a), we see the non-Berry terms interfering positively with the Berry curvature-dependent contributions, leading to a greater displacement amplitude, whereas the non-Berry terms interfere destructively with the topology-dependent contributions in (b), leading to a smaller displacement amplitude. In (c), these non-Berry contributions are canceled, and we observe only the anomalous displacement.

where ω_L is the frequency corresponding to a step of the laser scan and the integral is performed across the spectral width of the band of interest; γ is the average linewidth of the band; $\epsilon_{\mathbf{k}}$ is the energy of the band; and $\langle \Delta x \rangle_{\omega_L}$ is the displacement measured above for a given laser step in units of the unit cell size.

To experimentally extract γ for a given measurement, we need to look at the corresponding band structure presented in Fig. 14. Indeed, γ is defined as the average linewidth of the band, which we can obtain by measuring the full width at half maximum (FWHM) at $\mathbf{k}$ points where the band dispersion reaches an extremum and average these values. However, we cannot consider the transmission signal on the whole step of the laser scan, as the field modulation has the effect of mixing different eigenstates together, which leads to an artificial broadening of the bands. Therefore, we instead consider the transmission peaks in time windows of 1 μs where the synthetic electric field modulation is minimal. By adding the transmission peaks found in the Brillouin zones corresponding to these time windows for each laser step, we obtain band structures. From them, we can then straightforwardly extract γ for a given measurement. This procedure is done independently on both experiments with the laser pump aligned to $|a\rangle$ and $|b\rangle$ sites, yielding two different γ values for each band. The one used in Eq. (C17) for a given band is then simply the average of those two values.

In the measurements of the band structures in Fig. 2 in the main text and of the Berry curvatures in Fig. 3 in the

main text, it was desirable to have the longest lifetime possible inside of the cavity. We therefore tuned the amplification level of the SOA to the maximum level compatible with the absence of lasing. In contrast, for the displacement measurements, we slightly reduced the amplification level of the SOA to a regime with greater loss in order to mitigate finite-lattice-size effects arising from the twisted boundary conditions along x and reduce the impact of the modulation frequency Ω_λ . As such, for the anomalous transverse displacements, we had $\gamma/2\pi \sim 115$ kHz; whereas we had $\gamma/2\pi \sim 50$ kHz for the other kinds of measurements. A similar procedure is also done to experimentally determine the corresponding $\epsilon_{\mathbf{k}}$ of a displacement measurement. Finally, the integral over $d^2\mathbf{k}$ is done experimentally by the change of variable $d^2\mathbf{k} = 2\pi\Omega d\tau$, where $d\tau = 320$ ps is the digitizer's sampling rate (in this equation, $\mathbf{k}$ is dimensionless, as it is defined for a normalized unit cell).

To get an estimate of the uncertainty on these extracted Chern numbers, we first calculate the variance in the displacement for graphenelike lattice in Fig. 4(d) in the main text. As we expect a theoretical transverse drift of 0 for this model, the variance in this measurement characterizes the uncertainty on the measured transverse displacement for each model. Then, we propagate this uncertainty by injecting the value found for the variance into Eq. (C17). Finally, taking the square root of this yields the propagated standard deviation and an experimental uncertainty on the Chern number of ± 0.10 . This estimate on the uncertainty seems reasonable when comparing to the standard deviation of $\sigma = 0.14$ that is found when looking at all the experimental measurements taken in Fig. 4(f).

It is important to note that our extraction procedure for the Chern number from the experimental data relies on a few approximations, namely, a linear response in the amplitude of the applied force and a slow modulation of this force. As such, it is likely that any discrepancies can be explained by these approximations adding systematic uncertainties to our measurements, making the true uncertainty on the measurements slightly greater.

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