

Theory of Generalized Landau Levels and Its Implications for Non-Abelian States

Zhao Liu^{1,2,*} Bruno Mera^{3,4,*} Manato Fujimoto^{5,6} Tomoki Ozawa⁴ and Jie Wang^{7,†}

¹*Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou 310058, China*

²*Zhejiang Key Laboratory of Micro-Nano Quantum Chips and Quantum Control, School of Physics, Zhejiang University, Hangzhou 310027, China*

³*Instituto de Telecomunicações and Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Avenida Rovisco Pais 1, 1049-001 Lisboa, Portugal*

⁴*Advanced Institute for Materials Research (WPI-AIMR), Tohoku University, Sendai 980-8577, Japan*

⁵*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*

⁶*Department of Applied Physics, The University of Tokyo, Hongo, Tokyo 113-8656, Japan*

⁷*Department of Physics, Temple University, Philadelphia, Pennsylvania 19122, USA*



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Landau levels (LLs) are cornerstones of the fractional quantum Hall effect, which is a paradigm of topologically ordered phases of matter. Predicting and understanding the emergence of fractional Chern insulators in a Chern band, to a large extent, relies on judging the similarity between the Chern band and LLs. In this work, we generalize the notion of LLs to Chern bands to allow for nonuniform quantum geometries while preserving certain geometric invariants. Mathematically, generalized LLs can be understood in terms of holomorphic curves and associated moving frames in the sense of Cartan. This formalism enables a thorough analytical discussion of their associated quantum geometries. In practical terms, generalized LLs allow for a systematic discussion of the geometric stability of fractional Chern insulators and provide useful guidance in material design. Approximating Chern bands as superpositions of a few generalized LLs and employing exact diagonalization, we numerically identify a universal geometric criterion for the non-Abelian Moore-Read state to be stabilized in Chern bands, providing useful guidance for searching for such phase in moiré materials. Moreover, a double-twisted bilayer graphene model at the second magic angle is used to suggest a possible experimental realization of generalized LLs and zero-field non-Abelian phases. More broadly, our results point to generalized LLs being an interdisciplinary research direction connecting Kähler geometry, topologically ordered phases, and moiré materials, which we expect will benefit the study of the geometric aspect of correlated phases in a more general sense.

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I. INTRODUCTION

Quantum geometry, an intrinsic and fundamental feature of quantum states, plays a significant role in various applications within both topological and nontopological quantum matter [1–3]. The influence of quantum geometry extends across a diverse range of phenomena, spanning from linear and nonlinear responses [4–9] to nonequilibrium physics [10–12], encompassing domains such as topological photonics and light-matter interactions [13–15], coherence bounds [16–21], emergent degrees of

freedom [22–24], the stability of quantum matter [25–30], and beyond.

Quantum geometric effects are particularly important in narrow band systems, such as moiré materials, due to the strong suppression of electrons' kinetic energies and enhanced interactions. Recent work pointed out the importance of engineering quantum geometry in enhancing the stability of fractional Chern insulators (FCIs) [31–36] in moiré flat bands: Concretely, the saturation of the trace bound [37] between Berry curvature and quantum metric [25] is found to imply that the Bloch band behaves as if it was the lowest Landau level (LLL) and, in particular, it implies exact FCI ground states at certain filling when short-ranged interactions are considered [38–43]. Such theoretical insight provides useful guidance in optimizing material's quantum geometric properties and enhancing the likelihood of LLL-type fractionalized phases in realistic materials. Representative examples include the twisted-MoTe₂ homobilayers and rhombohedral

*These authors contributed equally to this work.

†Contact author: jiewang.phy@gmail.com

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multilayer graphene materials, where both optical signature and transport evidence of FCIs were recently experimentally reported [44–48]; saturating the trace bound is found to be useful in identifying an optimal parameter region in terms of twisting angles and displacement fields [49–54].

The profound link between the trace bound and the exact fractionalized many-body ground state is founded on Kähler geometry—a mathematical discipline that studies the geometry of manifolds that are simultaneously Riemannian, symplectic, and complex, in a way such that these three structures are compatible. Recent theoretical work pointed out the geometric meaning of saturating the trace bound as a Kähler condition on the quantum geometry, which leads to holomorphic properties of the Bloch state [39–41]. These holomorphic properties are reminiscent of the holomorphic condition for the standard LLL in real space, however, now they emerge in momentum space and hold for a broader class of bands, allowing arbitrarily nonuniform quantum geometric distributions. The holomorphic properties implied from saturating the trace bound allow a thorough, model-independent determination of the Bloch wave function form and permit certain exact statements on the interacting physics that, typically, strongly rely on many-body numerics [42].

There is, however, a glaring lacking of quantum geometric criteria to judge the higher Landau-level (LL) behavior of Bloch Chern bands. One motivation to generalize the notion of higher LLs to Bloch Chern bands is the exploration of non-Abelian fractionalized states in moiré flat bands. Recently, signatures of non-Abelian orders have been experimentally reported in moiré materials [55,56]. In addition, numerics based on first-principle calculations pointed out small-angle twisted-MoTe₂ material as a concrete promising platform for non-Abelian phases [57–60], but general exploration of the stability of non-Abelian FCIs in the presence of nonuniform quantum geometry is still missing. Following the standard fractional quantum Hall paradigm, non-Abelian topological orders naturally exist in the first (LL index $n = 1$) rather than in the lowest ($n = 0$) LL [61]. As a consequence, the need for a systematic and direct comparison between the Bloch bands of moiré materials and the first LL has become urgent.

This work is devoted to fill this void by systematically extending the notion of *all* LLs to Bloch Chern bands and exploring their implications for non-Abelian topological orders in moiré materials. The “generalized Landau levels” we introduce in this work allow arbitrary quantum geometric fluctuations but, nevertheless, maintain a quantized value of their integrated trace of quantum metric as in the case of the standard LLs. The quantum geometries of individual, as well as that of filling several generalized LLs are explicitly derived. The results are naturally understood in terms of the theory of holomorphic curves and Cartan’s

method of moving frames [62,63], extending the previously known link between quantum geometric bound and Kähler geometry to a broader scope. Extracting the weights of a generic Chern band on generalized LLs quantifies the problem of “Landau-level mimicry.” In addition to laying down a mathematically rigorous foundation for the notion of generalized LLs, we also discuss their implication to non-Abelian moiré fractionalized states. Approximating Chern bands as a superposition of a few generalized LLs, we employ exact diagonalization and identify a region of material’s single-particle quantum geometry—described in terms of quantities that can be easily evaluated, such as the integrated trace of quantum metric and standard deviation of Berry curvature—that can stabilize the non-Abelian Moore-Read fractionalized phase. Our geometric criteria for the non-Abelian phases is general and may serve as a guiding principle for future material design. We expect the new notion of generalized LLs will strengthen the community’s awareness of the fundamental link between Kähler geometry, quantum geometry, and strongly correlated two-dimensional physics. Furthermore, we believe it will provide insights and implications not only to fractionalized physics in moiré materials, but will be impactful to the geometric aspects of correlated materials in a more general sense.

The structure of this paper is as follows. We begin by introducing the fundamental concepts of quantum geometry and band theory in Sec. II. Section III is pivotal, establishing the theoretical foundations of this work. It presents the proof of the integrated trace formula satisfied by the generalized LL basis states and explores various geometric quantities and their relationships. Holomorphic curves and moving frames are presented in that section to provide a unified gauge-invariant geometric formulation of the results. Section IV discusses the implications for non-Abelian fractionalized states and proposes a quantum geometric criterion for the Moore-Read phase, which represents another focal point of this work. We summarize key results, discuss open problems and suggest future directions in Sec. V.

II. QUANTUM GEOMETRY AND RELATED BAND THEORIES

In this section, we introduce the fundamental concept of the quantum geometry of Bloch bands and standard LLs for both single- and multiple-band settings, which we will use extensively in this paper. We recall the quantum geometric bounds involving the Berry curvature and quantum metric. We also interpret the quantum geometry in terms of the underlying geometry of complex projective spaces and Grassmannians. Finally, we clarify the relationships among various recently introduced notions, such as Kähler bands [39–41], ideal bands [42,64], and vortexable bands [43,65], and elucidate their implications for the stability of FCIs.

A. Quantum geometry and bounds

We consider Bloch states $|\psi_{nk}\rangle$ of two-dimensional (2D) electrons, where n is the band index and momentum $\mathbf{k}$ is the lattice translation quantum number. The state $|\psi_{nk}\rangle$ can carry internal components describing degrees of freedom such as spin, valley, and sublattice. Bloch wave functions are orthogonal in the sense of $\langle\psi_{nk}|\psi_{n'k'}\rangle \propto \delta_{n,n'}\delta_{\mathbf{k},\mathbf{k}'}$. LLs can be incorporated into this description by regarding $\mathbf{k}$ as magnetic translation quantum numbers. We denote the first quantized wave function by $\psi_{nk}(\mathbf{r}) = \langle\mathbf{r}|\psi_{nk}\rangle$. The cell-periodic functions $|u_{nk}\rangle$ are defined by

$$u_{nk}(\mathbf{r}) \equiv e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_{nk}(\mathbf{r}), \quad u_{nk}(\mathbf{r}) = \langle\mathbf{r}|u_{nk}\rangle, \quad (1)$$

which obey, independently of $\mathbf{k}$, the same real-space translation properties and, hence, can be understood as vectors in the same Hilbert space.

The overlaps $\langle u_{mk}|u_{nk}\rangle$ are, in general, nonzero, and they give rise to gauge-covariant quantum geometric quantities: the (non-Abelian) Berry curvature $\Omega_{mn}(\mathbf{k})$ that characterizes the non-Abelian phase factor accumulated in an infinitesimal rectangle in momentum space and the non-Abelian quantum metric $g_{mn}^{ab}(\mathbf{k})$, with a, b the momentum-space indices and m, n the band indices, which describes the decaying properties of quantum states via $|\langle u_{mk}|u_{n,k+\delta\mathbf{k}}\rangle|^2 = 1 - \sum_{a,b=x,y} g_{mn}^{ab}(\mathbf{k})\delta k_a\delta k_b + \mathcal{O}(|\delta\mathbf{k}|^3)$. The Berry curvature and quantum metric are, respectively, the imaginary antisymmetric and real symmetric parts of the gauge-covariant, Hermitian, positive semidefinite quantum geometric tensor,

$$\begin{aligned} \chi_{mn}^{ab}(\mathbf{k}) &\equiv \langle D^a u_{mk}|D^b u_{nk}\rangle, \\ &= g_{mn}^{ab}(\mathbf{k}) + \frac{i}{2}\epsilon^{ab}\Omega_{mn}(\mathbf{k}), \end{aligned} \quad (2)$$

where $A_{mn}(\mathbf{k}) = -i\langle u_{mk}|\partial_{\mathbf{k}}^a u_{nk}\rangle$ is the non-Abelian Berry connection and the covariant derivative $|D^a u_{nk}\rangle$ is denoted as

$$|D^a u_{nk}\rangle = |\partial_{\mathbf{k}}^a u_{nk}\rangle - i \sum_{m=1}^r |u_{mk}\rangle A_{mn}(\mathbf{k}). \quad (3)$$

Here the sum is restricted to the r bands under consideration.

The non-Abelian quantum geometric tensor χ_{mn}^{ab} becomes an Abelian quantum geometric tensor when either a single band or the fermionic many-body Slater determinant state constructed from the band complex is considered. For the latter, we have

$$\chi_{\text{det}}^{ab}(\mathbf{k}) = \sum_{m=1}^r \chi_{mm}^{ab}(\mathbf{k}) \equiv \text{Tr}[\chi^{ab}(\mathbf{k})], \quad (4)$$

where the trace Tr is over the band index. Mathematically, the determinant state describes the determinant line bundle

(the highest exterior power bundle) associated with the vector bundle of the occupied Bloch states. Physically, it is the many-body state obtained by fully filling the band with fermions. A direct proof of Eq. (4) and its relation to Plücker embedding can be found in Appendix B 1.

The fact that the Abelian quantum geometric tensor is a positive semidefinite Hermitian tensor implies a bound relating its real symmetric part (quantum metric) and imaginary antisymmetric part (Berry curvature),

$$\text{tr}_{\tilde{g}}g(\mathbf{k}) \geq 2\sqrt{\det g(\mathbf{k})} \geq |\Omega(\mathbf{k})|. \quad (5)$$

These bounds are valid for any “general trace” $\text{tr}_{\tilde{g}}(g) \equiv \sum_{a,b} \tilde{g}_{ab}g^{ab}$ defined with respect to a unimodular matrix satisfying $\sqrt{\det(\tilde{g})} = 1$, which is not necessarily the identity matrix. In later sections, we will review the physical implications from saturating the quantum geometric bounds. Notice that we have introduced distinct notations for two different traces: The notation $\text{tr}_{\tilde{g}}$ denotes the generalized trace over spatial indices, and the notation Tr denotes the trace over band indices, as shown in Eq. (4).

B. Standard Landau levels and ladder operators

Now we introduce the quantum geometries of standard LLs that are formed in a 2D electron gas pierced by a uniform magnetic field. These LLs can be regarded as Chern bands exhibiting the simplest quantum geometric properties. Because of the magnetic translation invariance, their quantum geometric quantities are uniform. For comparison with lattice systems such as moiré materials and tight-binding models, we find it convenient to formulate the LL problem in the symmetric gauge and use 2D wave vectors inside a 2D Brillouin zone to label the Bloch-like LL wave functions. In this formulation, the lowest LL wave function on the torus can be written in terms of the Weierstrass σ function [66–70],

$$\Phi_{0k}^{\text{LL}}(\mathbf{r}) = \sigma(\zeta/\sqrt{\mathcal{C}} + i\sqrt{\mathcal{C}}k)e^{i\tilde{k}\zeta - \frac{\mathcal{C}}{2}|k|^2 - \frac{1}{2\mathcal{C}}|\zeta|^2}, \quad (6)$$

where $\mathbf{k} = (k_x, k_y)$ labels the magnetic translation quantum number and one takes $\mathcal{C} = 1$ for the single-component lowest LL. The wave function Eq. (6) can also be utilized (by summing over translation partners) to represent the color-entangled lowest LL wave function with Chern number $\mathcal{C} > 1$ [71,72], which is the unique wave function of an ideal Kähler band with constant Berry curvature [73]. Without loss of generality, we only consider $\mathcal{C} > 0$, as those of negative Chern numbers are easily obtained by complex conjugation. Complex vectors w_a and w^a determine the complex coordinate ζ and complex momentum k in Eq. (6), respectively, through

$$\zeta = \sum_{a=x,y} w_a r^a, \quad k = \sum_{a=x,y} w^a k_a, \quad (7)$$

where $iw_a = \sum_b \varepsilon_{ab} w_b$ and $\varepsilon_{xy} = -\varepsilon_{yx} = 1$ is the anti-symmetric tensor. The anisotropy of LLs is determined by the effective mass tensor $m_{ab} = \sqrt{\det(m)} \tilde{g}_{ab}$ of electrons in Galilean-invariant LL models, where $\tilde{g}_{ab} = \bar{w}_a w_b + w_a \bar{w}_b$ is its unimodular part. The mass tensor depends on the semiconductor material and is also tunable by altering the orientation of the magnetic field [74–77]. In the isotropic case with $w_{x,y} = (1, i)/\sqrt{2}$, the complex coordinate and the complex momentum reduce to the standard ones $\zeta = (x + iy)/\sqrt{2}$ and $k = (k_x + ik_y)/\sqrt{2}$.

It is important to notice that, although the lowest LL wave function is a holomorphic function of real-space coordinates ζ (up to Gaussian factors), its “periodic” part $u_{0k}^{\text{LL}}(\mathbf{r}) \equiv e^{-ik \cdot \mathbf{r}} \Phi_{0k}^{\text{LL}}(\mathbf{r})$ is a holomorphic function of complex momentum up to Gaussian factors. This property resembles the explicit position-momentum duality of LLs [27,69]. The $u_{0k}^{\text{LL}}(\mathbf{r})$ is

$$u_{0k}^{\text{LL}}(\mathbf{r}) = \sigma(i\sqrt{C}k + \zeta/\sqrt{C}) e^{-ik\zeta - \frac{C}{2}|k|^2 - \frac{1}{2C}|\zeta|^2}. \quad (8)$$

Notably, the LL index can be raised and lowered by a momentum-space ladder operator via

$$u_{nk}^{\text{LL}}(\mathbf{r}) = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n u_{0k}^{\text{LL}}(\mathbf{r}), \quad (9)$$

where $u_{nk}^{\text{LL}}(\mathbf{r})$ is the periodic part of the n th LL wave function $\Phi_{nk}^{\text{LL}}(\mathbf{r})$ and the ladder operators $\hat{a}$ and $\hat{a}^\dagger$ are differential operators given in terms of holomorphic and antiholomorphic derivatives with respect to momentum,

$$\hat{a}^\dagger = \frac{i}{\sqrt{C}} \frac{\partial}{\partial k} - \frac{i\sqrt{C}}{2} \bar{k}; \quad \hat{a} = \frac{i}{\sqrt{C}} \frac{\partial}{\partial \bar{k}} + \frac{i\sqrt{C}}{2} k. \quad (10)$$

For LLs, the real-space formulation with Φ_{nk}^{LL} and the momentum-space formulation with u_{nk}^{LL} are equally good. However, the momentum-space formulation is more general in the sense that it can be generalized to other systems, including moiré models and tight-binding models: No matter whether the real space is a continuum or lattice, their momentum space in either case is a continuum in the thermodynamic limit [27,28,78,79]. Therefore, in what follows in this work, we will primarily focus on the “periodic part” of the Bloch wave function $u_k(\mathbf{r})$ and momentum-space ladder operators [80].

Since this work will be primarily focusing on momentum-space holomorphicity properties, we will use the notation z , which is often used for real-space complex coordinate, to denote the complex momentum-space coordinate $-i\sqrt{C}k \equiv z$. Note that z and k differ by scale, which is an invertible holomorphic transformation, so they determine the same complex structure. We will also denote by $\int$ the integration over the Brillouin zone acting on periodic functions. The convention is chosen so that the integration

of Berry curvature gives the Chern number: $\int \Omega(\mathbf{k}) = \mathcal{C} \in \mathbb{Z}$. The notations used in this work are summarized as follows:

$$\begin{aligned} \int f(\mathbf{k}) &\equiv \frac{1}{2\pi} \int f(\mathbf{k}) d^2\mathbf{k} \\ &= \frac{1}{2\pi} \int_{\text{BZ}^2} f(\mathbf{k}) dk_x \wedge dk_y = \frac{i}{2\pi} \int_{\text{BZ}^2} f(\mathbf{k}) dk \wedge d\bar{k}, \\ k &\equiv \sum_{a=x,y} w^a k_a, \quad z \equiv -i\sqrt{C}k, \quad \partial = \frac{\partial}{\partial z}, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}}, \end{aligned} \quad (11)$$

where $f(\mathbf{k})$ is any periodic function and $f(\mathbf{k}) dk_x \wedge dk_y$ is a 2-form. With the notation above, the ladder operators are rewritten as

$$\hat{a}^\dagger = \partial - \bar{z}/2, \quad \hat{a} = -\bar{\partial} - z/2. \quad (12)$$

Last but not least, although all standard LLs have the same Berry curvature, the quantum metric depends on the LL index. For the n th LL, the trace of quantum metric equals $2n + 1$ times of the Berry curvature: $\text{tr}_{\tilde{g}} g_n(\mathbf{k}) = (2n + 1) \Omega_n(\mathbf{k})$, where both $g_n(\mathbf{k})$ and $\Omega_n(\mathbf{k})$ are independent of $\mathbf{k}$ [81]. The trace is defined with respect to $\tilde{g}$ defining the anisotropy of LLs. Physically, this means that different LL states have distinct spatial spreading and wave function overlap. Such feature is crucial for the occurrence of non-Abelian and symmetry-breaking orders in higher LLs [82–84].

Integrating both sides of the local trace condition $\text{tr}_{\tilde{g}} g_n(\mathbf{k}) = (2n + 1) \Omega_n(\mathbf{k})$ over the Brillouin zone leads to

$$\int \text{tr}_{\tilde{g}} g_n(\mathbf{k}) = (2n + 1) \mathcal{C} \quad (13)$$

for the n th standard LL. While this step looks redundant for standard LLs, we emphasize that the integration form Eq. (13) of the trace condition is actually more general than the local form. One of the central achievements in our work is to systematically construct generalizations of standard LLs in the presence of nonuniform quantum geometries. As shown later, once the quantum geometric fluctuations are switched on, the local form of the trace condition is generally violated for $n \geq 1$ and only holds for the $n = 0$ state, due to holomorphicity. By contrast, the integration form Eq. (13) is preserved for arbitrary LL index n for the generalized LL states introduced in this work.

C. Band theory in the holomorphic setting

In this section, we review different classes of bands that are closely tied to the geometric bounds in Eq. (5). In these band classes, the notion of holomorphicity—either

momentum- or real-space holomorphicity, or both—is present, similar to the situation of the LLL. In particular, we will focus on the notions of Kähler bands [39–41], ideal bands [42,64], and vortexable bands [43,65].

1. Differential geometric perspective of Bloch bands

From a differential geometric perspective, Bloch bands can be interpreted as smooth maps from the Brillouin zone, a genus-1 smooth compact 2D manifold, to Kähler manifolds parametrizing certain spaces of quantum states. In the single-band case, the corresponding Kähler manifold is the projective space, and, more generally, in the multiple-band case it is the Grassmannian of the rank equal to the number of bands. The quantum geometric quantities of Bloch bands are naturally understood as the pullback of geometric quantities on the target Kähler manifold to the Brillouin zone: The Abelian quantum metric and Berry curvature are, respectively, the pullback of the Fubini-Study (FS) metric and symplectic form.

We begin by discussing the quantum geometry of a single band in this language. If the Hilbert space under consideration is N dimensional, the unnormalized single-band state $|u_k\rangle$ can be represented by a nonvanishing N -dimensional column vector

$$|u_k\rangle = Z(k) \equiv [Z_1(k), Z_2(k), \dots, Z_N(k)]^t. \quad (14)$$

There is a $\text{GL}(1; \mathbb{C}) = \mathbb{C}^*$ ambiguity in the state such that $Z(k)$ and $G(k)Z(k)$, for $G(k) \in \mathbb{C}^*$, a nonvanishing complex number, represent the same physical state. $Z(k)$ and all its equivalent vectors form a ray, which is a one-dimensional subspace of $\mathbb{C}^N$ to which $Z(k)$ belongs. The set of all rays forms the complex projective space consisting of all pure quantum states, denoted by $\mathbb{CP}^{N-1}$. The space $\mathbb{CP}^{N-1}$ is a Kähler manifold, which means that it is a complex manifold and also that the FS metric and symplectic forms are compatible through the complex structure. Furthermore, they are both locally determined by derivatives with respect to the complex coordinates and its complex conjugates of the Kähler potential (see Appendix A 1 for details).

The quantum geometry of electronic states as described by the Berry curvature and the quantum metric, is inherited from the geometry of the complex projective space through the map determined by the Bloch wave function. Regarding the state $|u_k\rangle$ as a map $P: \text{BZ}^2 \cong \mathbb{T}^2 \rightarrow \mathbb{CP}^{N-1}$ from the Brillouin zone BZ^2 to projective space, the associated Berry curvature and quantum metric are given, respectively, by the pullback of the FS symplectic form and metric. We use the notation P , because the state determined by the Bloch wave function is uniquely described by the orthogonal projector $P(k) = |u_k\rangle\langle u_k|/\langle u_k|u_k\rangle$.

The generalization to multiple bands requires the notion of Grassmannian manifold. Readers interested in

multi-band quantum geometry and Grassmannians can refer to Appendix A 1 for details.

The differential geometric perspective of Bloch bands plays a crucial role for the theory of Kähler bands [39–41]. The notion of Kähler bands, as introduced in Refs. [39–41], corresponds to the special case where the locally defined matrix $Z(k)$ can always be taken to be holomorphic with respect to local complex coordinates in the Brillouin zone, coming from a global complex structure. We will discuss more about Kähler bands in Sec. II C 2 below.

2. Kähler bands

For a given band or, more generally, a band complex, saturation of the determinant bound $\sqrt{\det g(k)} = |\Omega(k)|/2$ and nonvanishing of the Berry curvature everywhere in the Brillouin zone mean that the quantum metric and the Berry curvature give the Brillouin zone the structure of a Kähler manifold, as induced by the Kähler manifold structure of the space of quantum states. This is the definition of a Kähler band [39–41]. The gauge-invariant condition for a band complex to be Kähler, with complex structure described locally by complex coordinates $\xi = \xi(k)$ (which can be nonlinear in k_x, k_y), is

$$Q(k) \frac{\partial P(k)}{\partial \bar{\xi}} P(k) = 0, \quad (15)$$

where $Q(k) = 1 - P(k)$ is the orthogonal complement projector of the band, supplemented with requiring positivity of the Berry curvature.

For a Kähler band (single band or band complex) the pullback of FS symplectic form and metric can be described very simply in terms of a “holomorphic gauge.” The condition Eq. (15) allows one to locally choose $Z(k)$ in Eq. (A4) to be holomorphic [40], or, equivalently,

$$\frac{\partial}{\partial \bar{\xi}} [|u_{1k}\rangle, \dots, |u_{rk}\rangle] = 0, \quad (16)$$

where ξ , noted above Eq. (15), denotes a local complex coordinate in the Brillouin zone for which the band defines a holomorphic map to the Grassmannian $\text{Gr}_r(\mathbb{C}^N)$. One can then write the Gram matrix of scalar products

$$S(k) = [\langle u_{ik}|u_{jk}\rangle]_{1 \leq i, j \leq r}, \quad (17)$$

and we have the Abelian Berry curvature and the quantum metric given by

$$\frac{\Omega}{2} = P^* \omega_{\text{FS}} = \frac{i}{2} h d\xi \wedge d\bar{\xi} \quad \text{and} \quad g = P^* g_{\text{FS}} = h |d\xi|^2, \quad (18)$$

where h is built out of second derivatives of the Kähler potential as $h = \{[\partial^2 \log \det S(k)]/[\partial \xi \partial \bar{\xi}]\}$.

Because it will be useful for later sections, we now discuss some aspects of Riemannian curvature appropriate to Kähler bands. In the Kähler setting, one has the Ricci form, expressed in terms of h as

$$\text{Ric}(\omega) = \frac{i}{2} \frac{\partial^2 \log h}{\partial \xi \partial \bar{\xi}} d\xi \wedge d\bar{\xi}. \quad (19)$$

The Ricci form determines the Gaussian/scalar curvature of the Kähler metric, given by

$$K = -2 \frac{\text{Ric}(\omega)}{\omega} = -\frac{1}{h} \frac{\partial^2 \log h}{\partial \xi \partial \bar{\xi}}. \quad (20)$$

The Euler characteristic of the Brillouin zone, being a genus-1 surface, is equal to zero and, as a consequence,

$$\chi(\text{BZ}^2) = \frac{1}{2\pi} \int_{\text{BZ}^2} K \omega = - \int_{\text{BZ}^2} \frac{\text{Ric}(\omega)}{\pi} = 0, \quad (21)$$

independent of the Kähler band.

3. Ideal bands

Ideal bands form a subset of Kähler bands satisfying the additional constraint: At each momentum $\mathbf{k}$, the trace of the quantum metric, with respect to some $\mathbf{k}$ -independent unimodular metric $\tilde{g}$, equals the Berry curvature which is positive; in other words, $\text{tr}_{\tilde{g}} g(\mathbf{k}) = \Omega(\mathbf{k}) > 0$ for every $\mathbf{k}$ [42]. In this case, the determinant bound is automatically saturated. There are also other equivalent definitions of ideal Kähler bands. For instance, a variational definition requires

$$\min_{\tilde{g}} \left(\int \text{tr}_{\tilde{g}} g(\mathbf{k}) \right) = \mathcal{C}, \quad (22)$$

which means the integrated trace minimized over the space of unimodular metrics $\tilde{g}$ must equal to the Chern number for an ideal band. Alternatively, a canonical way of defining the ideal band is by requiring the quantum geometric tensor to have a $\mathbf{k}$ -independent null vector satisfying $\sum_b \chi^{ab}(\mathbf{k}) w_b = 0$ for all $\mathbf{k}$ [42]. Such null vector is the complex vector w_a that determines the anisotropy of ideal bands $\tilde{g}_{ab} = \bar{w}_a w_b + w_a \bar{w}_b$ and antisymmetric tensor $\varepsilon_{ab} = -i(\bar{w}_a w_b - w_a \bar{w}_b)$. The complex vector w_a is uniquely determined by these two conditions up to a U(1) phase. The trace condition Eq. (22) is also referred to as the “ideal quantum geometry condition.” From now on we will simplify notation by using “tr” for the generalized trace defined with respect to the optimal unimodular metric $\tilde{g}$ whose integrated trace value is minimized.

Therefore, an ideal band is a Kähler band whose complex structure is momentum independent over the Brillouin zone and the one determined by the unimodular metric for which the trace bound is saturated. The

momentum-space holomorphicity further restricts the cell-periodic part of the Bloch wave function to be a holomorphic function of complex momentum k , or z , up to a normalization factor

$$|u_{0\mathbf{k}}\rangle = \tilde{\mathcal{N}}_{0\mathbf{k}} |\tilde{u}_{0\mathbf{k}}\rangle, \quad \text{for all } \mathbf{k} \in \text{BZ}^2, \quad (23)$$

where $|\tilde{u}_{0\mathbf{k}}\rangle$ is holomorphic: $|\partial \tilde{u}_{0\mathbf{k}}\rangle = 0$. The reason why we add a subscript “0” is that, later, we will generalize the notion of LL so that the ideal band fits as the $n=0$ generalized LL in an infinite tower analogous to LLs. Higher LLs will not be holomorphic in momentum, just as happened in the standard LL case, but they will obey a quantized integrated trace formula.

The momentum-space holomorphicity strongly restricts the possible form of wave function, once the Chern number is given [42,64,73]. To see this, we fix a coordinate $\mathbf{r}$, then the holomorphic wave function $\tilde{u}_{0\mathbf{k}}(\mathbf{r})$, if regarded as a function of $\mathbf{k}$, must satisfy appropriate boundary conditions consistent with the Chern number of the band—as such, they determine holomorphic sections of a holomorphic line bundle over the Brillouin zone. By the Riemann-Roch theorem, $\tilde{u}_{0\mathbf{k}}(\mathbf{r})$ may then be expressed in terms of $\mathcal{C}$ linearly independent functions, which are given by elliptic functions such as the Jacobi θ functions or the Weierstrass σ functions—in physical terms, they are essentially the standard lowest LL wave functions $u_{0\mathbf{k}}^{\text{LL}}(\mathbf{r})$. The type of functions, either θ functions or the σ function, depends essentially on the gauge choice, but they are ultimately equivalent. The most general form of all ideal (a single band, i.e., rank $r=1$) bands of Chern number $\mathcal{C} > 0$, including $\mathcal{C}=1$, is given by [42,64]

$$u_{0\mathbf{k}}(\mathbf{r}) = \mathcal{N}_{0\mathbf{k}} \sum_{\sigma=0}^{\mathcal{C}-1} \mathcal{B}(\mathbf{r} + \sigma \mathbf{a}_1) u_{0\mathbf{k}}^{\text{LL}}(\mathbf{r} + \sigma \mathbf{a}_1), \quad (24)$$

where one magnetic unit cell defining $u_{0\mathbf{k}}^{\text{LL}}$ corresponds to $\mathcal{C}$ lattice unit cells. Technically, this means we have chosen the gauge such that the Weierstrass σ function in $u_{0\mathbf{k}}^{\text{LL}}$ defined in Eq. (8) has its quasiperiodicity in $\mathcal{C}\mathbf{a}_1$ and $\mathbf{a}_2$, where $\mathbf{a}_{1,2}$ are the primitive lattice vectors. Holomorphicity also allows us to describe the fluctuations of the quantum geometry in terms of the normalization factors [42]: For a single ideal band, the Berry curvature and trace of quantum metric are [85]

$$\Omega_0(\mathbf{k}) = \text{tr}_{g_0}(\mathbf{k}) = \mathcal{C}(1 + \partial \bar{\partial} \log \mathcal{N}_{0\mathbf{k}}^{-2}), \quad (25)$$

where ∂ and $\bar{\partial}$ are defined in Eq. (11).

Ideal bands can be realized exactly or approximately in many different systems. The chiral magic-angle twisted bilayer graphene (TBG) is one important example for ideal bands [38,69,86]. Additionally, the $\mathcal{C}=1$ ideal bands are realized in an exact way as Dirac zero modes in a periodic

magnetic field [87], the lowest LL on curved manifolds [88], and quadratic band touching systems with strain [89], as well as the exactly flat band of the Kapit-Mueller model [90–93]. Moreover, higher Chern number ideal bands are exact in a couple of multilayer graphene and moiré graphene systems [94–102]. In addition, ideal bands are also approximated across many realistic materials, including, for example, the strained graphene [103,104] and the twisted-bilayer TMD material [52,53,105,106].

An important implication of ideal bands is the existence of exact model FCIs realized as short-ranged interactions regardless of the quantum geometric fluctuations (i.e., nonuniformity of Berry curvature and quantum metric) [38,42,64,94,95,107]. For instance, Laughlin $\nu = 1/3$ states can be stabilized by the shortest-ranged Trugman-Kivelson interaction $\sum_{i<j} \delta''(\mathbf{r}_i - \mathbf{r}_j)$ in ideal bands as exact zero modes, which necessarily must be the ground state because of the non-negativity of the many-body spectrum of repulsive interactions; generalization to all other model wave functions is straightforward [108–111]. Since the ideal quantum geometry Eq. (22) directly implies exact FCI ground states (if one ignores band dispersion and considers projected short-ranged interactions), engineering models or materials to approach the ideal band limit is supposed to enhance the stability of the fractionalized phase [49,50,54,112]. This observation provides theoretical insights of the experimental occurrence of FCIs in twisted MoTe_2 [52,53]. It is also worth emphasizing that ideal quantum geometry and short-ranged interaction are sufficient but not necessary conditions for FCIs; there are examples of stable FCIs in models violating the ideal conditions [113–115].

4. Vortexable bands

A related notion to Kähler bands is that of vortexability, which is a real-space property and was introduced in Ref. [43]. A vortexable band is defined to be any band complex (either a single band or multiple bands) that obeys the following condition, called the vortexability criterion:

$$(1 - \mathcal{P})\mathbf{z}(\mathbf{r})\mathcal{P} = 0, \quad \mathbf{r} = (x, y), \quad (26)$$

where $\mathcal{P}$ is the orthogonal projector onto the band. The vortex function $\mathbf{z}(\mathbf{r})$ defines a diffeomorphism $\mathbf{z}: \mathbb{R}^2 \rightarrow \mathbb{C}$. When $\mathbf{z}$ is a linear function of $\mathbf{r}$, the vortex function $\mathbf{z}(\mathbf{r})$ reduces to $\zeta(\mathbf{r})$ up to a constant shift, and the vortexable band reduces to ideal band. If the vortex function is instead a nonlinear function of $\mathbf{r}$, the momentum-space ideal quantum geometric condition Eq. (22) is violated and such general vortexable band is outside the ideal band class. One concrete example of the general nonideal vortexable band is the zero mode of Dirac fermion with periodically rotated velocity field in magnetic field [43]. Similar to the ideal bands, vortexable bands allow exact FCI model states constructed from the vortex functions,

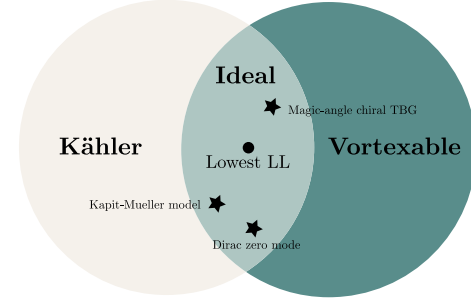


FIG. 1. The figure illustrates the relationships between Kähler, ideal, and vortexable bands. Kähler and vortexable bands are defined by momentum- and real-space holomorphicity, respectively, while ideal bands lie at their intersection, possessing both properties. Ideal bands can be viewed as “generalized lowest LLs,” and this work extends this concept to higher-LL analogs.

including the model Laughlin, Halperin, and Moore-Read states in a single band [38,42,68,94,107], as well as unprojected Jain states in multiple bands [65].

5. Relations and implications to lowest Landau-level physics

To summarize, Kähler and vortexable bands exhibit, respectively, momentum- and real-space holomorphicity. Not all Kähler bands are vortexable and, similarly, not all vortexable bands have momentum-space Kähler structure. Vortexability allows an explicit construction of first quantized fractional quantum Hall and FCI model wave functions within the Hilbert space of vortexable bands.

The set of ideal bands is precisely at the intersection of Kähler and vortexable bands. They hence exhibit both momentum- and real-space holomorphicity. We summarize the relationships among Kähler, ideal, and vortexable bands in Fig. 1. Because of this discussion, we will use the terms ideal band, ideal Kähler band, or ideal vortexable band interchangeably in what follows. Ideal bands can be regarded as generalized lowest LL states.

III. GENERALIZED LANDAU LEVELS AND QUANTIZED INTEGRATED TRACE FORMULA

In this work, we will start from the ideal band setting which, as discussed above, is at the intersection between Kähler and vortexable bands. Also, we will mainly focus on the single nondegenerate band case, meaning $P(\mathbf{k})$ is a rank-one projector, though multiple bands will be important for the intermediate discussions. The novelty in this work is that we will not be focusing on the holomorphic wave functions themselves, but rather on their higher-LL partners, which themselves are not holomorphic and do not saturate any of the local geometric bounds relating the quantum metric and Berry curvature considered above [cf. Eq. (5)]. We will point out the existence of a canonical basis constructed from higher-LL analogs of ideal bands,

provide invariants that characterize them, and show their precise quantization in analogy to what happens with LLs.

We begin by taking Chern number $\mathcal{C} = 1$ and consider adding modulations to the LL wave functions. We define an unnormalized “modulated Landau-level” basis,

$$e_{nk}(\mathbf{r}) = \mathcal{B}(\mathbf{r})u_{nk}^{\text{LL}}(\mathbf{r}), \quad n = 0, 1, \dots, \quad (27)$$

which is obtained by multiplying the standard periodic part of the LL states u_{nk}^{LL} by a same coordinate $\mathbf{r}$ -dependent modulating factor $\mathcal{B}(\mathbf{r})$. If such factor is periodic in real space, the resulting e_{nk} ’s still obey lattice magnetic translation symmetry as the standard LLs and hence represents a LL-type state; if $\mathcal{B}(\mathbf{r})$ is instead quasiperiodic with the opposite quasiperiodicity as $u_{nk}^{\text{LL}}(\mathbf{r})$, the modulated LL state $e_{nk}(\mathbf{r})$ can be real-space periodic and represent the cell-periodic part of a Bloch wave function. A concrete realization of e_{0k} with periodic $\mathcal{B}(\mathbf{r})$ includes Dirac zero modes in a periodic magnetic field [87]; e_{0k} with quasiperiodic $\mathcal{B}(\mathbf{r})$ can be realized as the zero mode of magic-angle chiral TBG [38,69,86].

The generalization to Chern number $\mathcal{C} > 1$ is straightforward. The $|e_{nk}\rangle$ will be a modulated color-entangled n th LL state: $e_{nk}(\mathbf{r}) = \sum_{\sigma=0}^{\mathcal{C}-1} \mathcal{B}(\mathbf{r} + \sigma\mathbf{a}_1)u_{nk}^{\text{LL}}(\mathbf{r} + \sigma\mathbf{a}_1)$, where the magnetic unit cell of u_{nk}^{LL} is defined on $\mathcal{C}\mathbf{a}_1 \times \mathbf{a}_2$, the same as in Eq. (24). The indices n are still raised up and down in an identical way as in Eqs. (28)–(30). From now on, unless specified, when we write modulated LLs or generalized LLs, they mean states of general Chern number $\mathcal{C} \geq 1$.

In general, the modulated basis is no longer orthonormal $\langle e_{mk}|e_{nk}\rangle \neq \delta_{mn}$; nevertheless, they transform in an identical way as LL states under the momentum-space ladder operators, because $\hat{a}$ and $\hat{a}^\dagger$ do not act on $\mathcal{B}(\mathbf{r})$,

$$\sqrt{n}|e_{nk}\rangle = \hat{a}^\dagger|e_{n-1,k}\rangle, \quad (28)$$

$$\sqrt{n}|e_{n-1,k}\rangle = \hat{a}|e_{nk}\rangle, \quad (29)$$

$$0 = \hat{a}|e_{0k}\rangle. \quad (30)$$

What will be relevant for the quantized trace formula below is the orthonormal basis, $\{|u_{mk}\rangle\}$, with $\langle u_{mk}|u_{nk}\rangle = \delta_{mn}$, obtained by applying the Gram-Schmidt orthogonalization process to the basis $\{|e_{nk}\rangle\}$ and setting $|u_{0k}\rangle \equiv \mathcal{N}_{0k}|e_{0k}\rangle$. This orthogonalization process yields

$$|u_{mk}\rangle = \sum_n |e_{nk}\rangle U_{nm}(\mathbf{k}), \quad (31)$$

where the $U_{nm}(\mathbf{k})$ ’s determine an upper-triangular matrix, i.e., $U_{nm}(\mathbf{k}) = 0$ for $n > m$. The $|u_{0k}\rangle$ is an ideal band Bloch wave function, the $|u_{1k}\rangle$ is a linear superposition of $|e_{0k}\rangle$ and $|e_{1k}\rangle$ with a $\mathbf{k}$ -dependent coefficient, and, in general, $|u_{nk}\rangle$ is a superposition of $|e_{mk}\rangle$ with $m = 0, \dots, n$.

We will be adding a subscript n to quantum geometric quantities to indicate that they are derived from $|u_{nk}\rangle$.

We now state one of the main results of this work:

Theorem 1. The integrated trace of the quantum metric of the orthonormal state $|u_n\rangle$ obeys the quantization formula

$$\int \text{tr} g_n(\mathbf{k}) = (2n+1) \int \Omega_n(\mathbf{k}) = (2n+1)\mathcal{C}, \quad (32)$$

for all $n \geq 0$. Here $\int$ stands for momentum-space integration, conveniently normalized as defined in Eq. (11).

This equality generalizes the standard LL trace formulas and remarks how the LL index gives an invariant regardless of quantum geometric fluctuations. Therefore, we term the orthonormal basis $\{|u_{nk}\rangle\}$ the generalized Landau levels. The zeroth generalized LL is the standard ideal band. We emphasize Eq. (32) does not mean $2n+1$ is a “topological invariant,” which is defined to be robust to any type of local perturbation (such as disorder). In contrast, the quantized trace formula in Eq. (32) is invariant under smoothly changing the modulation function $\mathcal{B}(\mathbf{r})$ and requires momentum-space “holomorphicity” of $|u_{0k}\rangle$ as a prerequisite, which will become clear in the next sections.

In the following, we first discuss a proof of the integrated trace formula motivated from the physical point of view by comparing and contrasting the orthonormal basis $|u_n\rangle$ to standard LL states. We then give a geometric interpretation and proof from a mathematical point of view, namely, in light of the theory of holomorphic curves and associated moving frames [62,63,116].

A. Generalized Landau levels, recursions, and invariants

1. Recursion of states

We begin by studying the structure of the non-Abelian interband Berry connection coefficients in the orthonormal basis $\{|u_{nk}\rangle\}$. We will show that, due to holomorphicity and orthogonality of the basis, the non-Abelian Berry connection coefficients admit a particular simple form in such basis. We define the holomorphic and antiholomorphic non-Abelian Berry connection coefficients as

$$A_{mn}(\mathbf{k}) \equiv -i\langle u_{mk}|\partial u_{nk}\rangle, \quad \bar{A}_{mn}(\mathbf{k}) \equiv -i\langle u_{mk}|\bar{\partial} u_{nk}\rangle, \quad (33)$$

where the momentum-space derivatives ∂ and $\bar{\partial}$ are defined in Eq. (11). The holomorphic and antiholomorphic connection are related by Hermitian conjugation in the orthonormal basis. A remarkable property is that, as proved in Appendix B 2, the holomorphic connection coefficient A_{mn} can only take nonzero values when $m = n$ or $m = n + 1$. Therefore, we parametrize the non-Abelian holomorphic connection coefficients as

$$\begin{aligned}
A_{nn}(\mathbf{k}) &= -i\frac{\bar{z}_k}{2} + i\alpha_{nk}; & n \geq 0, \\
A_{n,n-1}(\mathbf{k}) &= -i\mathcal{N}_{nk}^{-1}; & n \geq 1,
\end{aligned} \tag{34}$$

where $\mathcal{N}_{nk} \in \mathbb{R}$ is real and $\alpha_{nk} \in \mathbb{C}$. We note that, for the standard LLs, $\mathcal{N}_{nk} = 1/\sqrt{n}$ and $\alpha_{nk} = 0$. For notational simplicity, in what follows, we will occasionally omit the momentum subscript and simply denote $(|u_{nk}\rangle, \mathcal{N}_{nk}, \alpha_{nk})$ as $(|u_n\rangle, \mathcal{N}_n, \alpha_n)$.

It follows from Eq. (34) that the momentum-space raising operator $\hat{a}^\dagger$ will map $|u_n\rangle$ to a superposition of $|u_n\rangle$ and $|u_{n+1}\rangle$ only. Similarly, the lowering operator $\hat{a}$ will map state $|u_n\rangle$ to a superposition of $|u_n\rangle$ and $|u_{n-1}\rangle$. Using the ladder operators and Eq. (34), we arrive at the following recursion relation for the $|u_n\rangle$'s:

$$\mathcal{N}_n^{-1}|u_n\rangle = \hat{a}^\dagger|u_{n-1}\rangle + \alpha_{n-1}|u_{n-1}\rangle, \tag{35}$$

$$\mathcal{N}_n^{-1}|u_{n-1}\rangle = \hat{a}|u_n\rangle + \bar{\alpha}_n|u_n\rangle, \tag{36}$$

for all $n \geq 1$, and for $n = 0$ there is

$$0 = \hat{a}|u_0\rangle + \bar{\alpha}_0|u_0\rangle. \tag{37}$$

The state recursion relation above is one of the key results derived in this section. They show that, also for the Gram-Schmidt orthogonalized states, a ladder structure resembling that of the LL ladder operator representation is satisfied. Comparing with Eqs. (28)–(30), we see that the nontrivial information of the modulations $\mathcal{B}(\mathbf{r})$ and the quantum geometric fluctuations are encoded in the coefficients $\mathcal{N}_n$ and α_n .

The coefficients cannot be all independent. The ladder operators $\hat{a}, \hat{a}^\dagger$ appearing in Eqs. (35)–(37) satisfy the algebra $[\hat{a}, \hat{a}^\dagger] = 1$. This algebra imposes the following constraints on the coefficients:

$$\mathcal{N}_{n+1}^{-2} - \mathcal{N}_n^{-2} = 1 - (\partial\bar{\alpha}_n + \bar{\partial}\alpha_n), \tag{38}$$

$$\alpha_n - \alpha_{n-1} = \partial \log \mathcal{N}_n, \tag{39}$$

when $n \geq 1$, and when $n = 0$ there is

$$\mathcal{N}_1^{-2} = 1 - (\partial\bar{\alpha}_0 + \bar{\partial}\alpha_0), \tag{40}$$

$$\alpha_0 = \partial \log \mathcal{N}_0. \tag{41}$$

From Eqs. (38)–(41), we see that the coefficients $(\mathcal{N}_n, \alpha_n)$ are all uniquely determined by $\mathcal{N}_0$ in our problem. Explicitly, $(\mathcal{N}_n, \alpha_n)$ of all $n \geq 1$ are recursively determined from $\mathcal{N}_0$ as follows:

$$\alpha_n = \sum_{m=0}^n \partial \log \mathcal{N}_m, \tag{42}$$

$$\mathcal{N}_{n+1}^{-2} = 1 + \mathcal{N}_n^{-2} + \sum_{m=0}^n \partial\bar{\partial} \log \mathcal{N}_m^{-2}. \tag{43}$$

Recall that $\mathcal{N}_0$ is the normalization of the standard ideal band wave function, thereby it is uniquely determined by $\mathcal{B}(\mathbf{r})$. The concrete expression of $\mathcal{N}_0$ in terms of $\mathcal{B}(\mathbf{r})$ can be found in Refs. [42,64]. We leave derivation details of Eqs. (38)–(43) to Appendix B 3.

2. Geometries of individual generalized Landau levels

In this section, we discuss quantum geometric quantities derived from the $|u_n\rangle$ band and illustrate the physical meaning of the recursion relations presented in the previous section. Although $|u_n\rangle$ of $n \geq 1$ itself is not holomorphic in momentum, it turns out to be beneficial to express quantum geometric quantities in terms of complex variables. For a general band $|u\rangle$, the Berry curvature and the trace of the quantum metric take the following forms in terms of holomorphic and antiholomorphic derivatives, whose derivation details can be found in Appendix B 7:

$$\Omega = \mathcal{C}[\langle \partial u | \partial u \rangle - \langle \bar{\partial} u | \bar{\partial} u \rangle], \tag{44}$$

$$\text{tr}g = \mathcal{C}[\langle \partial u | \partial u \rangle + \langle \bar{\partial} u | \bar{\partial} u \rangle - 2|A|^2], \tag{45}$$

where $A \equiv -i\langle u | \partial u \rangle$ is the holomorphic Abelian Berry connection.

Replacing $|u\rangle$ with $|u_n\rangle$, and using the connection elements in Eq. (34) and the state recursions Eqs. (35)–(37), one arrives at the explicit relation between geometric quantities of the $|u_n\rangle$ band and the normalization factors $\mathcal{N}_n$, which is one of the key results of this section. We have

$$\Omega_n = \mathcal{C}(\mathcal{N}_{n+1}^{-2} - \mathcal{N}_n^{-2}), \tag{46}$$

$$\text{tr}g_n = \mathcal{C}(\mathcal{N}_{n+1}^{-2} + \mathcal{N}_n^{-2}), \tag{47}$$

for all $n \geq 1$, and

$$\Omega_0 = \text{tr}g_0 = \mathcal{C}\mathcal{N}_1^{-2} \tag{48}$$

$$= \mathcal{C}(1 + \partial\bar{\partial} \log \mathcal{N}_0^{-2}) \tag{49}$$

for the zeroth state.

We comment that the above illustrates the physical meaning of Eqs. (38) and (40)—they are the Berry curvature formula for the n th generalized LL $|u_n\rangle$: Their right-hand sides are nothing but the standard definition of Berry curvature after noticing that α_n determines the diagonal components of the Berry connection; their left-hand sides are alternative expressions of Berry curvature in terms of normalization factors, consistent with Eq. (46).

For the zeroth state $|u_0\rangle$, holomorphicity directly yields its Berry curvature as discussed around Eq. (25) and in

Ref. [42]. In addition, Eq. (48) provides a new expression of Berry curvature of the ideal band, not in terms of its own normalization factor, but instead in terms of the first normalization factor $\mathcal{N}_1$. Later, we will see such relation can be extended to a more general one: The $\mathcal{N}_{N+1}^{-2}$ determines the Berry curvature of the fully filled lowest N generalized LLs, and products of normalizations, more precisely the logarithm of Eq. (53), determines the Kähler potential of the ideal band for the many-body determinant state.

We conclude this subsection by proving the quantized integrated trace formula, which is a consequence of the results derived above. Integrating the Berry curvature yields the Chern number, and hence, Eqs. (46) and (48) give the following quantized formula for normalization factors:

$$\int \mathcal{N}_n^{-2} = n, \quad \forall n \geq 1, \quad (50)$$

whose physical meaning is the quantization of Chern number of the determinant states constructed from filling the lowest n generalized LL states. Equation (50) determines the integrated value of the trace of quantum metric through Eqs. (47) and (48). Using these, we prove the quantized integrated trace formula in Theorem 1 for the generalized LLs $\{|u_n\rangle\}$ constructed here.

3. Geometries of filled lowest N generalized Landau levels

In this section, we further derive relations among $\mathcal{N}_n$ and α_n from geometric properties of the fully filled lowest generalized LLs. We will illustrate the meaning of Eq. (39) and derive a recursion relation for $\mathcal{N}_n$. The recursion relation shows that all normalization factors descend from the first normalization factor $\mathcal{N}_0$ and its derivatives and is consistent with Calabi's rigidity theorem, which will be discussed in more detail in the next section.

We will consider fully occupied lowest N generalized LLs by fermions. The many-body wave function Ψ_N at a fixed momentum is a Slater determinant, also known as a wedge product of $|u_n\rangle$, written as

$$\begin{aligned} |\Psi_N\rangle &= \text{Slaterdet}[|u_0\rangle, \dots, |u_{N-1}\rangle] \\ &= |u_0\rangle \wedge |u_1\rangle \wedge \dots \wedge |u_{N-1}\rangle, \end{aligned} \quad (51)$$

which is normalized to 1. Since $|u_n\rangle$ is a linear superposition of $|e_n\rangle$ with $n = 0, \dots, N-1$, the same state can be rewritten as wedge products of $|e_0\rangle$ to $|e_{N-1}\rangle$ up to a normalization factor,

$$|\Psi_N\rangle = \mathcal{N}_{\Psi_N} \times |e_0\rangle \wedge |e_1\rangle \wedge \dots \wedge |e_{N-1}\rangle. \quad (52)$$

The state recursion relations, Eqs. (35)–(37), give a concrete form for the normalization factor,

$$\mathcal{N}_{\Psi_N} = \prod_{n=0}^{N-1} (\sqrt{n!} \mathcal{N}_n)^{N-n}. \quad (53)$$

The many-body state $|\Psi_N\rangle$ is an ideal band. Because of holomorphicity, similar to Eq. (25), the Berry curvature of $|\Psi_N\rangle$, denoted as Ω_{Ψ_N} , is given by

$$\begin{aligned} \Omega_{\Psi_N}/\mathcal{C} &= N + \partial\bar{\partial} \log \mathcal{N}_{\Psi_N}^{-2}, \\ &= N + \sum_{n=0}^{N-1} (N-n) \partial\bar{\partial} \log \mathcal{N}_n^{-2}, \end{aligned} \quad (54)$$

where the total Chern number of $|\Psi_N\rangle$ is $N\mathcal{C}$. On the other side, the Berry curvature of the band complex must be the sum of that of individual bands,

$$\Omega_{\Psi_N}/\mathcal{C} = \sum_{n=0}^{N-1} \Omega_n/\mathcal{C} = \mathcal{N}_N^{-2}. \quad (55)$$

Equating Eqs. (54) and (55) gives the following recursion relation that must be satisfied by the normalization factors:

$$\mathcal{N}_N^{-2} = N + \sum_{n=0}^{N-1} (N-n) \partial\bar{\partial} \log \mathcal{N}_n^{-2}, \quad (56)$$

valid for all $N \geq 1$. This equation extends the relations Eqs. (48) and (49) to general $N \geq 1$ and again points out all normalization factors $\mathcal{N}_n$ are recursively and uniquely determined from $\mathcal{N}_0$ and its derivatives. Notice Eq. (56) is equivalent to Eq. (43).

B. Geometric formulation: Holomorphic curves and moving frames

The geometric formulation in this section offers an alternative way, using the theory of holomorphic curves and associated moving frames, to derive the results of the previous section. Here, we emphasize the underlying Kähler geometry of the problem and formulate it in a way that does not rely on specific gauge choices, other than it being holomorphic in the momentum variable. The gauge choices here are twofold. On the one hand, by multiplying all the wave functions by a smooth $\mathbf{r}$ -dependent phase factor, which alters the form of the wave functions and ladder operators introduced before, we can, for instance, change to a Landau gauge or to a more general gauge. Moreover, we can also multiply the Bloch wave function of the ideal Kähler band by a smooth momentum-dependent nonvanishing complex number and this does not change the band, since the latter only cares about the associated orthogonal projector. Indeed, by making use of the latter choice is how we achieve a holomorphic representative of the Bloch wave function, see the discussion around Eq. (16). Hereafter, by holomorphic gauge, we mean that

we take the Bloch wave function of the ideal Kähler band to be such that $[\partial/(\partial\bar{z})]|u_k\rangle = 0$. Note that the holomorphic gauge is not unique, because we can always multiply $|u_k\rangle$ by a nonvanishing holomorphic function and this does not change the band nor the holomorphicity of the resulting Bloch wave function. A specific holomorphic gauge was used before in Sec. II C 3 and denoted by $\tilde{u}_{0k}(\mathbf{r})$. Below, since there are no other types of gauges involved, we will drop the tilde to make the notation lighter.

Before proceeding to holomorphic curves, the reader may wish to recall the classical Frenet-Serret theory for curves in Euclidean space, which we present in Appendix A 3 for clarity.

1. Holomorphic curve and moving frames

The classical Frenet-Serret theory describes a curve, parametrized by a single real variable t , in $\mathbb{R}^n$. The geometry of such curve is completely specified, up to rigid motion, by the invariants determined by the associated moving frames. For instance, in Appendix A 3, the functions κ and τ are the invariants and $R(t)$ is an orthogonal moving frame. This picture can be generalized to ideal Kähler bands and provides a geometric interpretation of the quantized trace formula discussed above.

The important observation is that ideal Kähler bands are, up to normalization and phase, holomorphic functions of the complex momentum variable z , where complex momentum z is defined in Eq. (11). This means that an ideal Kähler band can be understood as a “holomorphic curve” [62,116], i.e., a holomorphic map from the Brillouin zone (a manifold of complex dimension one) to the projective space. A holomorphic curve looks like a curve in the sense that it is described by a single, yet complex, parameter z , which in our case describes the Brillouin zone, but more generally can be an arbitrary Riemann surface.

We denote the periodic part of the Bloch wave function of the ideal Kähler band in a holomorphic gauge by $|u_k\rangle$. We can attach, at each point of the Brillouin zone z , a natural holomorphic frame composed of the derivatives with respect to z of $|u_k\rangle$. Such moving frame is illustrated in Fig. 2 and is denoted by

$$S(z) = \left[|u_k\rangle, \frac{\partial}{\partial z} |u_k\rangle, \dots, \frac{\partial^n}{\partial z^n} |u_k\rangle, \dots \right], \quad (57)$$

where we used the notation $S(z)$ to stress that the frame is holomorphic in z . The frame $S(z)$, after the Gram-Schmidt orthogonalization process (denoted as GS below), gives the unitary frame—i.e., an orthogonal basis at each k —of generalized LL states $\{|u_{nk}\rangle\}$,

$$U(k) = \text{GS}[S(z)] = [|u_{0k}\rangle, |u_{1k}\rangle, \dots, |u_{nk}\rangle, \dots]. \quad (58)$$

The unitary frame $U(k)$ is distinguished in the sense that a different holomorphic gauge, by multiplying $|u_k\rangle$ by a

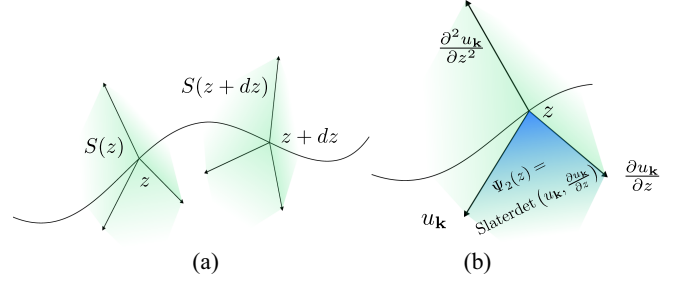


FIG. 2. Geometric picture of an ideal Kähler band as a holomorphic curve, the idea of moving frames, and the geometric interpretation of the associated tower of holomorphic maps. (a) Illustration of the holomorphic curve and moving frame. (b) Geometric interpretation of the holomorphic state $\Psi_{N=2}(z)$ —it is the two-dimensional subspace spanned by the first two elements of $S(z)$. Ψ_2 physically corresponds to fully filling the bands determined by $|u_{0k}\rangle$ and $|u_{1k}\rangle$.

holomorphic nonvanishing function of z , only changes $U(k)$ by a smooth scalar phase, see Appendix B 4. Hence, the associated bands, uniquely determined by the orthogonal projectors $|u_n\rangle\langle u_n|$ with $n \geq 1$ are uniquely determined from the knowledge of $|u_0\rangle\langle u_0|$, i.e., from the ideal Kähler band.

The rate of the change of the unitary frame along the holomorphic curve is described by the equation

$$d|u_n\rangle = \sum_m |u_m\rangle \theta_n^m, \quad (59)$$

where we have omitted the momentum labeling. Here $d = dz[\partial/(\partial z)] + d\bar{z}[\partial/(\partial\bar{z})]$, and θ is the pullback of the Maurer-Cartan 1-form on the unitary group,

$$\theta_n^m \equiv [U^{-1}dU]_n^m = \langle u_m | d | u_n \rangle, \quad (60)$$

which is closely related to the Berry connection in Eq. (33). The unitarity of $U(k)$ implies θ is a skew-Hermitian matrix. Moreover, holomorphicity of the ideal Kähler band together with the fact that $U(k)$ results from the Gram-Schmidt process hence differs from $S(z)$ by an upper-triangular matrix, forces θ to take a sparse tridiagonal form, just like the case of Frenet-Serret theory for the classical curves,

$$\theta = \begin{bmatrix} \theta_1^1 & \theta_1^2 & 0 & \cdots \\ \theta_2^1 & \theta_2^2 & \theta_2^3 & \cdots \\ 0 & \theta_3^2 & \theta_3^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (61)$$

Last but not least, θ_n^{n+1} is proportional to dz and $\theta_n^{n-1} = -\overline{\theta_{n-1}^n}$ is proportional to $d\bar{z}$. This implies that the unitary

moving frame obeys the “Frenet-Serret” equations along the holomorphic curve: For $n \geq 1$,

$$d|u_n\rangle = |u_{n-1}\rangle\theta_n^{n-1} + |u_n\rangle\theta_n^n + |u_{n+1}\rangle\theta_n^{n+1}, \quad (62)$$

and for $n = 0$,

$$d|u_0\rangle = |u_0\rangle\theta_0^0 + |u_1\rangle\theta_0^1, \quad (63)$$

where more details can be found in Appendix B 4.

The above Frenet-Serret equations for a holomorphic curve constitute the geometric interpretation of the state recursion relations of Eqs. (35)–(37). In the following, we will revisit the proof of the quantized integrated trace condition presented in Sec. III A 2 from the point of view of holomorphic curves.

2. Slater determinants, a tower of holomorphic maps, and geometric recursion relations

Associated with the ideal Kähler band determined by $|u_k\rangle$ [as mentioned above Eq. (57), we have chosen to use $|u_k\rangle$ to denote the ideal Kähler band $|u_{0k}\rangle$ in the holomorphic gauge], there is a tower of holomorphic maps induced by the Slater determinants

$$\begin{aligned} \Psi_N: z \mapsto \text{Slaterdet}\left(|u_k\rangle, \frac{\partial}{\partial z}|u_k\rangle, \dots, \frac{\partial^{N-1}}{\partial z^{N-1}}|u_k\rangle\right) \\ = |u_k\rangle \wedge \frac{\partial}{\partial z}|u_k\rangle \wedge \dots \wedge \frac{\partial^{N-1}}{\partial z^{N-1}}|u_k\rangle, \end{aligned}$$

with $N = 1, 2, \dots$. The induced maps to projective space are clearly holomorphic because $\Psi_N(z)$ is holomorphic in z , since $|u_k\rangle$ is so, and taking derivatives with respect to z and performing wedge products preserves holomorphicity. At each momentum, described by the complex variable z , $\Psi_N(z)$ is an N -particle fermionic state obtained by fully filling the lowest N generalized LL states $|u_0\rangle, \dots, |u_{N-1}\rangle$. Geometrically, these can be interpreted as representing the N -dimensional plane generated by the first N states, as illustrated in Fig. 2.

Each of these holomorphic maps can be interpreted as a rank- N degenerate Kähler band described by the orthogonal projector $P_N = \sum_{n=0}^{N-1} |u_n\rangle\langle u_n|$ carrying a multiband quantum metric γ_N and Berry curvature ω_N ,

$$\gamma_N = \text{Tr}(P_N dP_N dP_N) \quad \text{and} \quad \omega_N = -\frac{i}{2} \text{Tr}(P_N dP_N \wedge dP_N), \quad (64)$$

which, due to holomorphicity, saturate the quantum geometric inequalities in Eq. (5). In particular, we can show, using the Frenet-Serret equations, that

$$\begin{aligned} \gamma_N &= |\theta_{N-1}^N|^2 = \left| \langle u_N | \frac{\partial}{\partial z} |u_{N-1}\rangle \right|^2 |dz|^2, \\ \omega_N &= \frac{i}{2} \theta_{N-1}^N \wedge \overline{\theta_{N-1}^N} = \frac{i}{2} \left| \langle u_N | \frac{\partial}{\partial z} |u_{N-1}\rangle \right|^2 dz \wedge d\bar{z}. \end{aligned} \quad (65)$$

For a full derivation of the above equations, we refer the reader to Appendix B 6.

The ω_N 's or, equivalently, the γ_N 's, are invariants of the Kähler band determined by $|u_k\rangle$ and they are completely specified by the interband Berry connection coefficients θ_{n-1}^n . However, unlike the case of curves in Euclidean space, which carry extrinsic information, holomorphic curves only care about intrinsic data, meaning everything depends on the first fundamental form, which in this setting is precisely the quantum metric [117]. The intrinsic and extrinsic geometries of Bloch states is also considered in Ref. [118]. In particular, by Calabi's rigidity theorem [119], if two Kähler bands $|u_k\rangle$ and $|u'_k\rangle$ give rise to the same Kähler metric, then they differ, up to rescaling, by a k -independent unitary transformation, so the Kähler metrics ω_N of $N \geq 2$ should be related to ω_1 . Note that, if two Kähler bands $|u_k\rangle$ and $|u'_k\rangle$ differ by a k -independent unitary transformation, the resulting quantum metrics are the same, but the converse is not obvious. Indeed, this is the content of the recursion relation in Eq. (69) which is a local form of the so-called “Plücker relations” [62,63,116].

The n th state $|u_n\rangle$ does not determine a Kähler band because its quantum metric and Berry curvature do not saturate the quantum geometric bounds in Eq. (5), but its quantum metric, which we denote by g_n , satisfies

$$g_n = \langle du_n | (1 - |u_n\rangle\langle u_n|) | du_n \rangle = \gamma_n + \gamma_{n+1}, \quad (66)$$

which can be proved using the Frenet-Serret equations; the details can be found in Appendix B 6. Hence, when performing the integral of the trace with respect to the unimodular metric, one gets the sum of the integrals of the Berry curvatures ω_n and ω_{n+1} which are Chern numbers and, thus, quantization follows. However, to show the exact value of quantization, it is convenient to proceed further in the understanding of the Kähler geometry of the Ψ_N 's and, in particular, their relation to the one of the generating Kähler band $|u_0\rangle$. For that, one can use the Frenet-Serret equation together with the Maurer-Cartan structure equation,

$$d\theta + \theta \wedge \theta = 0, \quad (67)$$

or in components,

$$d\theta_n^m + \sum_l \theta_l^m \wedge \theta_n^l = 0, \quad (68)$$

to derive the following geometric recursion relations on the Kähler geometry in the Brillouin zone induced by the tower of maps Ψ_N [62]:

$$\text{Ric}(\omega_N) = \omega_{N-1} - 2\omega_N + \omega_{N+1}, \quad (69)$$

where $N \geq 1$ and $\text{Ric}(\omega_N)$ is the Ricci form associated with the Kähler form ω_N . Integrating the Ricci form yields the Euler characteristic of the Brillouin zone, see Eq. (21), which is zero independently of the metric chosen. This yields the following recursion between Chern numbers:

$$0 = \mathcal{C}_{N-1} - 2\mathcal{C}_N + \mathcal{C}_{N+1}, \quad (70)$$

with $\mathcal{C}_N \equiv \int_{\text{BZ}^2} \omega_N / \pi$. Using the fact that $\mathcal{C}_0 = 0$ (because Ψ_0 is the constant map) and $\mathcal{C}_1 = \mathcal{C}$ (the Chern number of the ideal Kähler band $|u_{0k}\rangle$), one arrives at $\mathcal{C}_n = n\mathcal{C}$. Therefore, we arrived at the quantized formula of integrated trace metric,

$$\int \text{tr} g_n = \mathcal{C}_n + \mathcal{C}_{n+1} = (2n+1)\mathcal{C}. \quad (71)$$

We comment that the above results all require the ω_N 's to be nonvanishing everywhere. LLs fulfill this condition. The existence of singularities (ramification points), which exist for finite-dimensional tight-binding models, will alter the above relations Eq. (70). We leave more detailed discussions about these singular points and related physics for future work.

C. Summary

In this section, we introduced the generalized Landau levels $\{|u_n\rangle\}$, which form an orthonormal canonical basis built from the ideal band. They can also be interpreted as forming a distinguished unitary moving frame over the holomorphic curve determined by the ideal band. The geometric invariants and their relationships can be understood either by drawing analogies to LLs or through the framework of holomorphic curves and moving frames.

From the viewpoint of LL analogy, the basis states $\{|u_n\rangle\}$ have a similar structure under the action of momentum-space ladder operators: the $\hat{a}^\dagger$ and $\hat{a}$, respectively, increase or decrease the level index at most by 1. The difference of the ladder operator representation is encoded in the normalization factors $\mathcal{N}_n$ and coefficients α_n which determine the connection. Notably, these coefficients completely determine the geometries of both individual and multiple generalized LLs in a concise way. All normalization factors are uniquely determined from the zeroth normalization factor $\mathcal{N}_0$. The geometries contained in normalization factors directly prove the quantized integrated trace formula.

Alternatively, we can also interpret an ideal Kähler band as a holomorphic curve. Associated with the holomorphic

curve there is a canonical, up to an overall phase factor, unitary moving frame built by applying Gram-Schmidt orthogonalization to the derivatives of a holomorphic representative of the ideal Kähler band with respect to the holomorphic momentum variable z . Physically, this means that associated with an ideal Kähler band there is a tower of bands, exactly similar to the case of LLs. The ladder structure is also naturally related to the application of a holomorphic derivative, cf. the expression of the momentum-space raising operator in Eq. (12). Because of the geometric properties of the distinguished frame and the fact that filling N of these bands, from $n = 0$ up to $n = N - 1$, produces a rank N ideal Kähler band, one can relate the metric of the individual u_n to the Kähler metric of filled bands, from which the quantization formula for the integrated trace follows.

Recently, the notion of “vortexability” has been generalized to higher-LL-like states via “higher vortexability” [65]. Here, we briefly clarify the relation between the higher-vortexable bands and generalized LLs and emphasize the novelty of our work. First of all, one can regard generalized LLs and higher-vortexable bands as the higher-LL generalization of ideal bands and vortexable bands, respectively; therefore, all generalized LLs fulfill the higher-vortexability criteria, and generalized LLs and higher-vortexable bands are related by the real-space diffeomorphism. However, we notice the higher vortexability introduced in Ref. [65] requires a couple of projectors and other vortexable bands which are initially unknown. By contrast, in this work, we start from the ideal bands, whose wave functions are already classified, and systematically construct all generalized LLs. Moreover, we point out the fundamental mathematical structure of all the generalized LLs as being holomorphic curves and Cartan’s moving frame—this enables a complete discussion of all geometric properties of the generalized LLs, among which the most noticeable is the quantized trace formula that is previously unknown in condensed-matter literature. Because of this precise meaning to the higher-LL analogs in terms of Kähler geometry, we expect that this can establish a prosperous bridge of communication between the study of geometric band theory and Kähler geometry in a more general sense.

Last but not least, we comment that because generalized LLs form a complete basis, they can be used to expand a general Chern band. One of the most interesting applications of this completeness is, given a Chern band, how to know the weight of each generalized LL component. Addressing this question quantifies the Landau-level mimicry and is important for understanding the geometric stabilities of Abelian and non-Abelian FCIs. To be more precise, any cell-periodic state of a Chern band can be decomposed as

$$|u_k\rangle = \sum_{n=0}^N C_{nk}^{\mathcal{B}} |u_{nk}^{\mathcal{B}}\rangle, \quad (72)$$

where, in general, $N = \infty$ and we assume $C_{nk}^{\mathcal{B}}$ is periodic under shifting the momentum by reciprocal lattice vectors $\mathbf{k} \rightarrow \mathbf{k} + \mathbf{b}$. In the above, we have included the superscript $\mathcal{B}$ to emphasize the fact that both the generalized LL basis $\{|u_n\rangle\}$ and the decomposition coefficients C_{nk} depend on the choice of the modulation function $\mathcal{B}(\mathbf{r})$. Viewing $\mathcal{B}(\mathbf{r})$ as a “gauge choice” and $|C_{nk}^{\mathcal{B}}|^2$ as the distribution function defined in the parameter space $(n, \mathbf{k})$, it is a well-defined question about what is the optimal gauge $\mathcal{B}_*$ that minimizes the spread of the distribution function, in the same spirit of Wannier function’s optimization problem [120]. Remarkably, if a band is known to be a superposition of finite number N of generalized LLs (can be termed as finite- N models), one can prove there exists an canonical algorithm to extract the momentum-dependent superposition coefficients $C_{nk}^{\mathcal{B}}$. See Appendix C for details about the wave function decomposition and how to use the Frenet-Serret equation to exactly decompose finite- N models back into generalized LL basis.

IV. NON-ABELIAN FRACTIONALIZATION

In this section, we utilize generalized LLs as a tool to study the geometric stability of FCIs, with a focus on non-Abelian phases. To be more concrete, “stability” here refers to the competition between fractionalized phases and non-fractionalized phases, such as Fermi liquids, charge density waves, or other conventional orders. In general, many factors influence this competition, including band dispersion, band mixing, and details of interactions. Therefore, theoretical studies at all levels, including general theories, first-principle calculations, model building, and many-body numerics, are crucial for comparisons with or guidance for real experiments [26,49–51,54,57,59,60,112,115,121–144]. In the context of fractional quantum Hall physics, non-Abelian fractionalized states can often occur in the first $n = 1$ LL with Coulomb interaction. Representative examples include the Moore-Read (MR) [145] and Read-Rezayi (RR) state [61].

Among the various factors mentioned above, the quantum geometry is one crucial factor determining the stability of fractionalized phases. To focus on the quantum geometric effect, we consider an “idealized” setup with an isolated and dispersionless band. Our construction of generalized higher-LL states allows us to systematically explore the non-Abelian fractionalization directly within the single-particle Hilbert space containing key features of first LL. In contrast to existing literature on the geometric stability of non-Abelian phases, which focus on the twisted MoTe_2 [57,59,60] or related toy models [58], our setup does not require a particular single-electron Hamiltonian that yields bands. We will directly employ a simple model of generalized LLs as the ansatz of the wave function of the isolated flat band. We use this model to propose a geometric criteria in terms of integrated trace of quantum metric $\int \text{tr} g$

and Berry curvature standard derivation δ_Ω favoring the Moore-Read state. We believe that our model captures a signature portion of the general single-particle Hilbert space of first Landau level and thereby our geometric stability criteria is supposed to be general and can serve as a necessary condition for the Moore-Read phase in general a Chern band. We will compare our geometric stability criteria to existing case studies available in literature.

A. Stability of the Moore-Read state

A representative class of non-Abelian states in the MR class includes the Pfaffian (Pf) state, its particle-hole conjugate—the anti-Pfaffian (aPf) state—and the more exotic particle-hole symmetric Pfaffian state (PH-Pf), all of which can, in principle, occur in a half filled $n = 1$ standard LL and can be interpreted as the superconducting paired state of composite fermions on top of the composite Fermi liquid state [146,147]. In what follows, we will use “Moore-Read” to denote the MR class that includes Pf, aPf, and PH-Pf states.

The MR states were initially motivated by the conformal-field-theory construction of fractional quantum Hall wave functions [145]. The Pf state is the densest zero-energy ground state of a three-body short-ranged repulsion in the lowest LL [110]. Moreover, numerical studies indicate Pf and aPf are favored by the Coulomb interaction in the $n = 1$ LL [84,148–150]. Although the MR states have been extensively studied in the context of LLs, their general stability in the presence of nonuniform quantum geometries is not explored much.

In what follows, we will consider the stability of the MR phase in the context of generalized LLs. We set Chern number $\mathcal{C} = 1$. Throughout our numerics, we will choose isotropic geometry such that complex momentum $k = (k_x + ik_y)/\sqrt{2}$, respecting triangular lattice translation symmetries, and the trace tr reduces to the standard one. Such standard trace is also often used in literature in studying twisted- MoTe_2 and moiré graphene. We take the modulation function $\mathcal{B}(\mathbf{r})$ to be periodic on a triangular lattice and parametrize it as

$$\mathcal{B}(\mathbf{r}) = 1 + \tilde{\omega} \sum_{m=1}^6 e^{i\mathbf{b}_m \cdot \mathbf{r}}, \quad (73)$$

where $\mathbf{b}_m$ ’s are the six shortest reciprocal lattice vectors and $\tilde{\omega}$ is the only parameter tuning the strength of quantum geometric fluctuations. The details of the Hamiltonian can be found in Appendix E.

The numerical hallmark of the Pf and aPf at finite system sizes is the sixfold nearly degenerate ground states on the torus geometry when the electron number N_e is even, which can clearly distinguish the MR phase from trivial charge density waves. We hence focus on periodic samples with even N_e . In the absence of quantum geometry

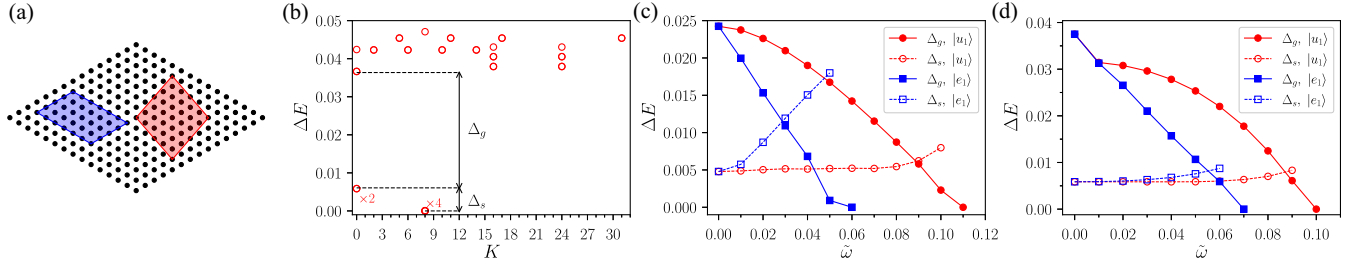


FIG. 3. (a) Tilted lattices for $N_e = 12$ (blue) and $N_e = 16$ (red) electrons. (b) A representative energy spectrum of the Coulomb interaction of $N_e = 16$ electrons in a half-filled first generalized LL $|u_1\rangle$ constructed by using modulation function $\mathcal{B}(\mathbf{r})$ of Eq. (73) with $\tilde{\omega} = 0.02$. K labels the momentum of the many-body states. The multiplicity of levels is given when their energy difference is hardly visible. Neutral gap and ground-state manifold energy splitting are denoted, respectively, as Δ_g and Δ_s . (c), (d) Ground-state splitting Δ_s and neutral gap Δ_g for the Coulomb interaction in the first generalized LL $|u_1\rangle$ and the first modulated LL $|e_1\rangle$, as a function of the quantum geometric fluctuation strength parameter $\tilde{\omega}$. (c) $N_e = 12$ electrons and (d) for $N_e = 16$ electrons.

fluctuation $\tilde{\omega} = 0$, any two-body translational-invariant interacting Hamiltonian has exact particle-hole symmetry at half filling, leading to the equal energy of the Pf and aPf states. Therefore, we expect 12-fold ground-state degeneracy in total in the thermodynamic limit for the MR phase stabilized by two-body interactions. However, due to finite-size effect, numerical simulations often report sixfold nearly degenerate ground-state manifold, corresponding to either the symmetric or the antisymmetric superposition of Pf and aPf states in a particle-hole symmetric system where ground states spontaneously break particle-hole symmetry [84,150–152]. Degeneracies and nature of ground states with and without particle-hole symmetry will be discussed in Appendix F 2.

To begin with, we study the MR state in the first generalized LL $|u_1\rangle$ and the first modulated LL $|e_1\rangle$. Similar to the situation in the standard $n = 1$ LL, we observe sixfold ground-state degeneracy below the critical quantum geometry fluctuation in our exact diagonalization study. For various system sizes of $N_e = 12, 14$, and 16 electrons, the momenta of the ground states match both those of Pf and aPf, which is a strong signature of the MR phase. A further confirmation from wave function overlap is presented in Appendix F 3. The sample geometry and a representative energy spectrum are displayed in Figs. 3(a) and 3(b), respectively. We denote the ground-state energy splitting as Δ_s , defined as the difference between the maximal and minimal energy of the six ground states. We also denote the neutral gap as Δ_g , defined as the difference between the minimal excited energy and the maximal ground-state energy. A stable non-Abelian phase can be characterized by large Δ_g and small Δ_s [Fig. 3(b)]. With increasing quantum geometry fluctuation by increasing $\tilde{\omega}$, the neutral gap Δ_g monotonically decays. As shown in Figs. 3(c) and 3(d), Δ_g becomes even smaller than Δ_s at sufficiently strong $\tilde{\omega}$ and finally vanishes with the absence of sixfold ground-state degeneracy at $\tilde{\omega} \sim 0.1$ (see Appendix F 4 for similar results of the RR state). We also compare the evolution of Δ_g and Δ_s with $\tilde{\omega}$ for $|e_1\rangle$ and

$|u_1\rangle$. For the latter, Δ_g survives up to larger $\tilde{\omega}$ [Figs. 3(c) and 3(d)], suggesting that the putative MR phase is generally enhanced in the Schmidt orthogonalized state $|u_1\rangle$ as compared to $|e_1\rangle$.

Next, we study the geometric stability of the MR state against the superposition of two generalized LLs u_m and u_n . Although the single-particle Hilbert space consists of linearly superposing them with $\mathbf{k}$ -dependent superposition coefficients, we study a simpler yet solvable model by assuming the superposition coefficients are $\mathbf{k}$ independent. In this case, the band wave function can be expressed as

$$u_{\mathbf{k}}(\mathbf{r}) = \sqrt{\lambda} u_{m\mathbf{k}}(\mathbf{r}) + \sqrt{1 - \lambda} e^{i\phi} u_{n\mathbf{k}}(\mathbf{r}), \quad (74)$$

where λ and ϕ do not vary with $\mathbf{k}$. We term this model the $u_m - u_n$ model and it captures a subspace of the full Hilbert space spanned by two generalized LLs. We will focus on the $u_0 - u_1$, $u_2 - u_1$, and $u_3 - u_1$ model for the purpose of realizing the MR phase because they contain the first LL while being enriched by geometric fluctuation and other LL components. The quantum geometric properties of the general $u_m - u_n$ model can be found in Appendix D.

This setup has a controlled limit where the MR phase is guaranteed to occur: In the limit of $\lambda = 0$ and $\tilde{\omega} = 0$, the above wave function reduces to the standard $n = 1$ LL wave function, which is known to stabilize the MR phase under Coulomb interaction. We then explore the stability of the observed sixfold ground-state degeneracy against λ and ϕ at fixed quantum geometric fluctuation strength $\tilde{\omega}$. As a representative example, we display in Fig. 4 the neutral gap Δ_g and the ground-state splitting Δ_s at $\tilde{\omega} = 0.02$ in the (λ, ϕ) parameter space for the $u_0 - u_1$, $u_2 - u_1$, and $u_3 - u_1$ model of $N_e = 12$ electrons, respectively [153]. We can observe a clear tendency that the sixfold ground-state degeneracy with Pf and aPf momenta becomes worse when the weight $1 - \lambda$ of u_1 decreases. For $\tilde{\omega} = 0.02$, we find the largest λ that allows decent sixfold ground-state degeneracy with $\Delta_g \geq \Delta_s$ is about 0.07, 0.18, and 0.06 for the $u_0 - u_1$, $u_2 - u_1$, and $u_3 - u_1$ model, respectively. These results

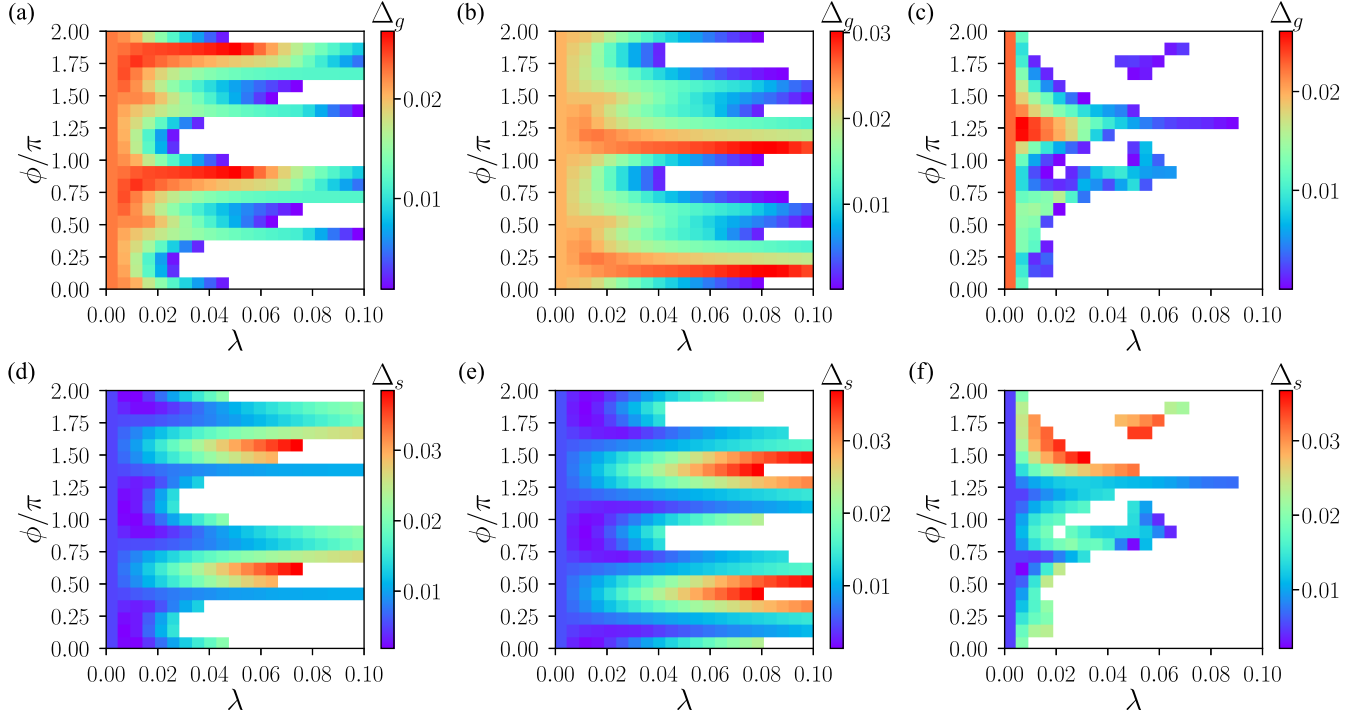


FIG. 4. The many-body gap Δ_g (the first row) and ground-state splitting Δ_s (the second row) at $\nu = 1/2$ in the $u_0 - u_1$ [(a) and (d)], $u_2 - u_1$ [(b) and (e)], and $u_3 - u_1$ model [(c) and (f)] for $N_e = 12$ electrons with Coulomb interaction. The modulation parameter are taken as $\tilde{\omega} = 0.02$ for all subfigures. In those white regions of the $\lambda - \phi$ parameter space, there is no sixfold ground-state degeneracy found numerically at MR momenta.

indicate that the dominant u_1 is essential for the stabilization of the MR phase. We also emphasize $\int \text{tr} g$ being close to 3 is not a sufficient condition for the MR phase; for instance, a nearly equal weight mixing of $|u_1\rangle$ and $|u_2\rangle$ can give $\int \text{tr} g \approx 3$, but the wave function is no longer dominated by $|u_1\rangle$ as seen in Fig. 8(c).

In the above, we have fixed $\tilde{\omega}$ and studied the conditions in terms of (λ, ϕ) that allows the MR phase. In the following, we convert model parameters (λ, ϕ) into

physical parameters $(\int \text{tr} g, \sigma_\Omega)$ and vary $\tilde{\omega}$ to obtain a geometric stability criteria for the MR state. Here $\int \text{tr} g$ is the integrated trace value and σ_Ω is the standard deviation of Berry curvature. Concretely, for each fixed $\tilde{\omega}$, we first get the union of $\int \text{tr} g$ data or σ_Ω data obtained from the three $u_m - u_n$ models ($u_0 - u_1$, $u_2 - u_1$, and $u_3 - u_1$), whose evolution with respect to $\tilde{\omega}$ can be found in Figs. 5(a) and 5(b). We find that the window of $\int \text{tr} g$ favoring the MR phase becomes narrower with increasing

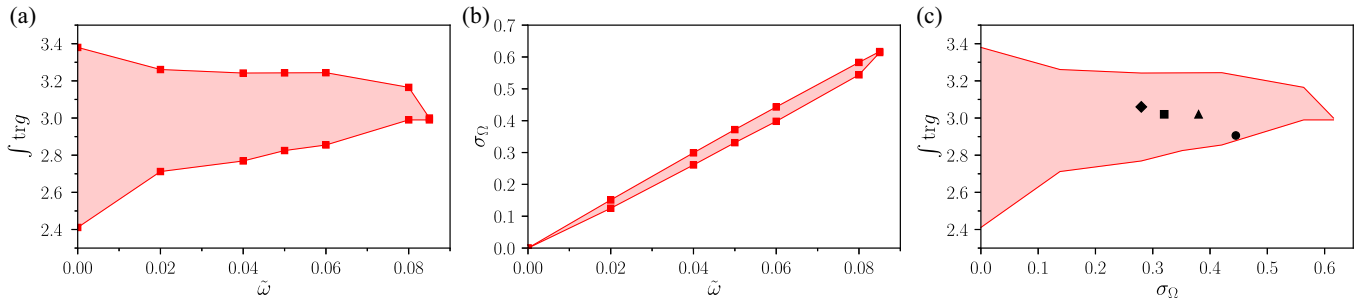


FIG. 5. The windows of (a) $\int \text{tr} g$ and (b) σ_Ω as a function of $\tilde{\omega}$, in which we find sixfold ground-state degeneracy with $\Delta_g \geq \Delta_s$ for $N_e = 12$ with Coulomb interaction. The windows are evaluated based on the energy spectra for the half filled $u_0 - u_1$, $u_2 - u_1$, and $u_3 - u_1$ models. (c) Combining (a) and (b), we give the region of the stable MR phase in the quantum geometric parameter space spanned by σ_Ω and $\int \text{tr} g$. The markers indicate the materials and models in which the MR phase and the corresponding quantum geometric quantities were reported, including the twisted-MoTe₂ homobilayers with twist angle 2° (square) and 2.14° (diamond) [57], skyrmion Chern band model (triangle) [58], and the double-TBG model in Sec. IV B (dot). Result for $N_e = 14$ electrons can be found in Appendix F in Fig. 10.

$\tilde{\omega}$ [Fig. 5(a)], while the scope of σ_Ω is almost located along a straight line [Fig. 5(b)]. We can thus combine Figs. 5(a) and 5(b) to determine a region in terms of $(\sigma_\Omega, \int \text{tr}g)$ in which the MR phase is stable, as displayed in Fig. 5(c). The geometric region for the MR phase on larger system size with $N_e = 14$ can be found in Appendix F in Fig. 10. This region provides a quantum geometric criterion for the existence of the MR phase in our $u_m - u_n$ models. In what follows, we will argue such geometric criterion is a necessary condition for the MR phase.

Compared to the interacting physics in realistic models such as twisted-MoTe₂ homobilayers, the setup of the interacting problem in our $u_m - u_n$ model has a couple of simplifications as summarized below. (i) First of all, we work in the exact flat band limit by ignoring single-particle dispersion, which can originate from bare band dispersion as well as Hartree-Fock renormalization. (ii) Second, we consider a single-band projected interaction and ignore multiband mixing effects. (iii) Third, we have assumed spacial isotropy and have only retained the leading Fourier modes in $\mathcal{B}(\mathbf{r})$. (iv) Last but not least, we have neglected the momentum dependence of the superposition coefficients λ and ϕ ; therefore, the $u_m - u_n$ model only captures a subspace of the Hilbert space spanned by two generalized LLs.

We will argue many of the simplifications mentioned above are conditions that favor fractionalized phases. Therefore, the geometric criteria we propose for the MR phase, arguably, sets the largest window. The argument goes as follows. (i) Weak band dispersion is generally regarded as important for fractionalized phases that are strongly correlated phases driven by interactions; suppressing single-particle dispersion helps to stabilize fractionalized phases over trivial phases, such as Fermi liquids or density waves. (ii) A single isolated band is important for a flat Chern band to behave like an isolated LL, which also facilitates fractionalized phases. Band mixing often leads to competing phases. (iii) Based on the observation in LLs that increasing the strength of anisotropy can drive phase transitions from fractional quantum Hall to classically ordered phases [74], the isotropy condition also favors fractionalized phases. Higher-order Fourier modes can be shown to have Gaussian-suppressed contributions to the form factor, hence they arguably have negligible effects to interacting physics [42]. (iv) We expect allowing momentum-dependent superposition coefficients deviates flat bands from being LL-like, therefore also disfavors the fractionalized phases.

Based on the argument above, we expect that the geometric criteria obtained from our $u_m - u_n$ models in Fig. 5(c) is the general geometric stability criterion for the MR phase within the consideration of Coulomb interaction. Recent first-principle calculations indicated consecutive Chern bands in the twisted-MoTe₂ homobilayers at small twist angle (about 2°) [129]. Motivated by this, a couple of

theoretical studies pointed out the possibility of realizing the MR phase at small-angle twisted-MoTe₂ homobilayers [57,59,60] as well as related toy models [58]. We collect from these works the single-particle geometric quantities ($\int \text{tr}g$ and σ_Ω) allowing the MR phase and demonstrate them in Fig. 5(c). Remarkably, we find they are all bounded within the window we propose based on the $u_m - u_n$ model. A practical message of our result is that, for a given model or material, if its single-particle geometric quantities, including the integrated trace value of $\int \text{tr}g$ and Berry curvature fluctuation σ_Ω , are outside the window we propose in Fig. 5(c), the likelihood of realizing the MR phase is low, if Coulomb interaction is considered.

B. Approximately first generalized Landau level in a graphene-based Moiré model

In this section, we present a concrete graphene-based moiré model hosting approximately $n = 1$ generalized LL at zero magnetic field. In a certain parameter region, the model exhibits a nearly flat band, which is separated by a finite (but small) band gap from remote bands and exhibits suitable quantum geometric properties for non-Abelian states.

We first describe the twisting and stacking geometry of this model. The model consists of four layers of graphene sheets, and we will denote their twisting angle as $(\theta, 0, 0, \theta)$. This corresponds to a configuration where a TBG sheet is stacked on top of another TBG sheet. In addition to the twisting angles, we also need to describe their relative stacking. We denote the layers as $[(1, t), (1, b), (2, t), (2, b)]$. Here (1, 2) indicates the first or the second TBG sheet, and (t, b) refers to the top or bottom layer within each TBG sheet. As illustrated in Fig. 6(a), the two TBG sheets are stacked with a lateral shift such that $[(1, b), (2, t)]$ forms a Bernal stacking configuration.

The single-electron Hamiltonian in a single valley is given as

$$H_1 = \begin{pmatrix} 0 & D_1^\dagger \\ D_1 & 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} D_0 & \Gamma_1 \\ 0 & D'_0 \end{pmatrix}, \quad (75)$$

where H_1 is written in the sublattice basis $[A, B]$, and D_1 describes the intersublattice coupling written in the layer basis. Since the second TBG is twisted in the opposite direction to the first TBG, $D'_0 = \mu_x D_0 \mu_x$ with the Pauli matrices in layer space μ . Here D_0 is the standard chiral operator for the chiral model of TBG [86],

$$D_0 = \begin{pmatrix} -i\bar{\partial}_r & \alpha U(\mathbf{r}) \\ \alpha U(-\mathbf{r}) & -i\bar{\partial}_r \end{pmatrix}, \quad U(\mathbf{r}) = \sum_{n=0}^2 e^{i\left(\frac{2\pi n}{3} - \mathbf{q}_n \cdot \mathbf{r}\right)},$$

where α is proportional to the inverse of twist angle, and $\mathbf{q}_j$ are the three primitive moiré vectors. We have followed the

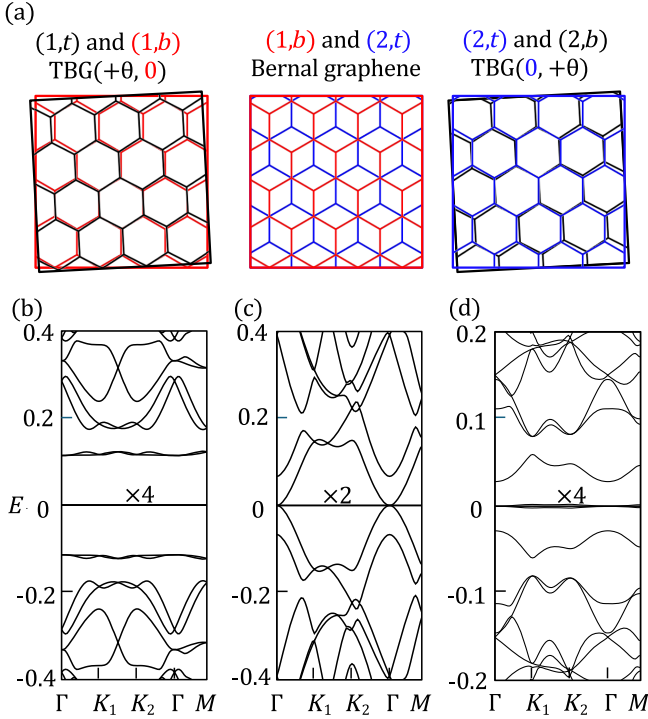


FIG. 6. (a) The lattice structure of the double-TBG model consisting of Bernal stacked two TBG sheets. Here layers $[(1, t), (1, b)]$ and $[(2, t), (2, b)]$ are from individual TBG sheets with the relative twist angles given by $(\theta, 0)$ and $(0, \theta)$, respectively. The layers $[(1, b), (2, t)]$ form Bernal bilayer graphene. (b)–(d) The band structures of Eq. (75) with parameters $(\alpha, \gamma_1, \beta) = (0.586, 8, 1)$, $(\alpha, \gamma_1, \beta) = (0.586, 8, 0)$, and $(\alpha, \gamma_1, \beta) = (2.221, 8, 0)$, respectively. (b) The band structure for the unphysical limit $\beta = 1$ at the first magic angle, where fourfold exactly degenerate bands are seen. (c), (d) At the physical limit $\beta = 0$, at the first and second magic angles, respectively. Compared to (c), (d) still has nearly flat bands with a finite band gap to remote bands. These bands can be split by applying displacement fields due to their different layer spinor structures.

notation of Ref. [86]. To differentiate from the momentum-space derivative notation used in this work, we have added subscript $\mathbf{r}$ for real-space holomorphic and antiholomorphic derivatives above. The interlayer tunneling matrix Γ_1 is given by $\Gamma_1 = \begin{pmatrix} 0 & \gamma_1 \\ \beta\gamma_1 & 0 \end{pmatrix}$, where the $\beta\gamma_1$ component corresponds to the intersublattice coupling between $(1, t)$ and $(2, b)$, i.e., the top layer of the first TBG and the bottom layer of the second TBG. Physically, β should be smaller than 1; in the realistic situation, β should be nearly 0.

The model is known to support twofold exact zero modes per valley and sublattice for each momentum $\mathbf{k}$ at the magic angles at chiral limit with unphysical parameter $\beta = 1$ [65] [see Fig. 6(b)]. The two sublattice polarized zero modes are

$$|\mathcal{U}_{0\mathbf{k}}\rangle = \begin{pmatrix} |u_{0\mathbf{k}}\rangle \\ 0 \end{pmatrix}, \quad |\mathcal{U}_{1\mathbf{k}}\rangle = \begin{pmatrix} |u_{1\mathbf{k}}\rangle \\ -\gamma_1^{-1}\mu_x|u_{0\mathbf{k}}\rangle \end{pmatrix},$$

and the fact that they are zero modes follows from the zero mode equations $D_0|u_{0\mathbf{k}}\rangle = 0$ and $D_0|u_{1\mathbf{k}}\rangle = \mu_x|u_{0\mathbf{k}}\rangle$. The $|u_{0\mathbf{k}}\rangle$ is the standard ideal band wave function of magic-angle chiral TBG with lowest LL features [38, 42, 69, 86, 154–156], and we can readily show $|u_{1\mathbf{k}}\rangle = \hat{a}^\dagger|u_{0\mathbf{k}}\rangle$, where $\hat{a}^\dagger$ is the momentum-space ladder operator introduced in Eq. (12). At sufficiently large γ_1 , the second zero mode $|\mathcal{U}_{1\mathbf{k}}\rangle$ polarizes to the first TBG sheet and becomes identical to the generalized $n = 1$ LL discussed above. For finite γ_1 , the $|\mathcal{U}_{0\mathbf{k}}\rangle$ is still the ideal band, but $|\mathcal{U}_{1\mathbf{k}}\rangle$ deviates from the generalized first LL. For example, the integrated trace of $|\mathcal{U}_{0\mathbf{k}}\rangle$ and $|\mathcal{U}_{1\mathbf{k}}\rangle$ at the first magic angle $\alpha = 0.586$ and $\gamma_1 = 8$ with $\beta = 1$ as $\int \text{tr} g_0 = 1.0$ and $\int \text{tr} g_1 = 2.98$, respectively. Importantly, the exact flatness of degenerate zero modes requires not only chiral approximation, magic angle, but also $\beta = 1$ which does not hold for realistic material. Instead, for realistic materials, β is nearly zero. Nevertheless, one can ask what are the properties of the model with more realistic parameters. We notice at the first magic angle, in the more realistic limit $\beta = 0$, the band becomes dispersive as shown in Fig. 6(c).

The novelty of this section is to point out that, at the second magic angle, the model allows an isolated band that is well approximated by the $n = 1$ generalized LL even in the more realistic limit with $\beta = 0$. In Fig. 6(d), we compute the band structure at the second magic angle $\alpha = 2.221$ with parameters $(\gamma_1, \beta) = (8, 0)$. The bands near charge neutrality are almost flat and there is a finite gap from the remote bands. The survival of flat bands results from the fact that the deviation from the $\beta = 1$ limit is suppressed by the large moiré coupling α compared to the case at the first magic angle.

Since the $|\mathcal{U}_{1\mathbf{k}}\rangle$ has a nonzero second component at finite γ_1 , the two sublattice polarized flat bands can, in principle, be separated with the help of vertical displacement fields. However, it is important to notice that there is a trade-off: The displacement field will enhance both the band gap and the bandwidth. In other words, the band has already become quite dispersive when the band gap exceeds the interaction strength upon adding a sufficiently large displacement field. At small displacement field, we find the flat band retains character of the first generalized LL. Its integrated trace and the standard derivation of Berry curvature are given as follows:

$$\int \text{tr} g_1 = 2.91, \quad \sigma_\Omega = 0.45. \quad (76)$$

On the other side, reducing the interaction strength by increasing the dielectric constant can make the band gap effectively larger. We hence proceed to neglect the band dispersion and assume valid single-band projection (which is not justified). We show numerical evidence of the non-Abelian MR state within this setting. In Fig. 7, we provided

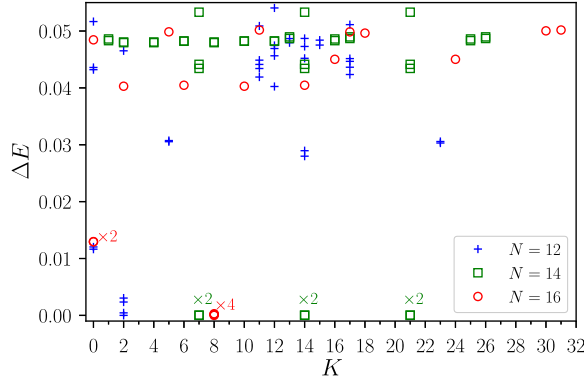


FIG. 7. The energy spectrum of Coulomb interaction projected to the first conduction band of the double-TBG model at the second magic angle with $\beta = 0$ for $N_e = 12, 14$, and 16 electrons. The ground-state degeneracy pattern is exactly what one would expect if the MR state is stabilized.

the exact diagonalization spectrum for $N_e = 12, 14, 16$ electrons at half filling. For all particle numbers considered, a sixfold degeneracy in the right momentum sectors are observed, which is a hallmark of the MR phase.

To conclude, the double-TBG model exhibits nearly flat bands of the first generalized LL type at zero magnetic field. The single-particle quantum geometric indicators fall within the range demonstrated in Fig. 5(c), and many-body calculations show evidence for the MR phase. We emphasize the purpose of studying this model is not to make connection to realistic experiments, but as a concrete example to show the consistency of the quantum geometric criterion we proposed. In the following, we summarize the unrealistic aspects of the model considered above. First, although we have eliminated unphysical tunneling, the model still requires the chiral limit. The chiral limit requires the vanishing of the intrasublattice moiré tunneling, which is unrealistic in real materials. Second, the model considered here requires the second magic angle, which is near 0.25° . In reality, the relaxation effect is stronger at smaller angle where the strain and domain wall effect become important [157,158], which have not been taken into account in the current model. Importantly, also as mentioned above, it is difficult in this model to isolate the approximately $n = 1$ generalized LL and meanwhile keep it flat. Therefore, the band projection, which is used to obtain the sixfold degeneracy, is hard to be justified, although we argue reducing interaction strength can effectively enhance the band gap. Last but not least, the Hartree-Fock renormalization presumably will also strongly modify the single-particle dispersion and affect the stability of non-Abelian states. Given the challenge of realizing non-Abelian topological order in experiments, we anticipate that applying a small magnetic field could help stabilize it [159]. We leave a more detailed exploration of this model for future work.

V. CONCLUSION, DISCUSSION, AND OUTLOOK

In this work, we generalize the notion of ideal Chern bands, i.e., the generalized lowest LL, to higher-LL partners. We term them, including the standard ideal band, as generalized LLs. The generalized LLs are systematically constructed, starting from the standard ideal band, by applying ladder operators in momentum space and Gram-Schmidt orthogonalization. Generalized LLs resemble the standard LLs in many ways, while allowing for nonuniform quantum geometries. Single- and multiple-band quantum geometries of generalized LLs are explicitly determined in a concise form, with the results summarized in Appendix A in Table I. Particularly notable is the quantized integrated trace value that determines the LL index. Such quantization justifies the $\int \text{tr} g = 3$ condition for the first LL mimicry, which was widely used in the literature [57–60]. The generalized LLs are also understood as describing a distinguished unitary moving frame over the holomorphic curve on the space of quantum states determined by the ideal band.

As a complete orthonormal basis, the generalized LLs span the entire Hilbert space of Chern bands, including LL- and Bloch-type Chern bands. The $u_m - u_n$ model is supposed to capture a large portion of the single-particle Hilbert space of LL-type bands. Exact diagonalization within such models provides an arguably widest range of geometric quantities permitting the non-Abelian MR phase. Remarkably, the quantum geometry of existing models and materials that host MR states all fall into our geometric window, which may hence serve as a guiding rule in future material engineering for searching the MR phase in moiré flat bands. We further propose a concrete double moiré graphene model at small twist angle to realize an isolated first generalized LL. This model clearly improves previous graphene-based examples by removing the necessity for remote layer hopping, although more work is required to connect it to realistic materials. Moreover, geometric stability of the Read-Rezayi phase is briefly discussed.

We expect plenty of intriguing future directions can emanate from the notion of generalized LLs. First of all, generalized LLs can be exploited as a powerful tool to study the quantum geometric stability of strongly correlated physics in narrow bands, not only FCIs, but also beyond. Notable examples include fractional topological insulators and anyon exciton states, where the former are time-reversal-invariant topologically ordered states, while the latter involve simultaneous fractionalization and quantum coherence, both of which clearly are beyond the scope of FCIs. Recently, signature of fractional topological insulators has been reported in moiré materials [55] and an anyon exciton insulator has been reported in bilayer graphene [160]; however, their microscopic mechanisms remain unclear. In addition, the realization of an anyon exciton state in moiré materials deserves a future experimental and

theoretical attempt, where quantum geometry nonuniformity can be one crucial factor in determining their competition with other phases. The framework of generalized LLs is a useful tool in theoretically discussing and numerically studying important energetics of these exotic quantum phases. Along this line, a systematic study of the analog of Haldane's pseudopotentials [109] in generalized LLs is an important direction of investigation.

It is also worthy to highlight that, due to the generality of the generalized LL framework discussed in this work, our theory is not constrained to continuum systems such as moiré or Landau levels. Rather, the generalized LL applies to lattice systems too. We envision generalized LL in lattice systems is useful in designing fractionalized phases on optical lattices realized by ultracold neutral atoms, where a few-particle Laughlin state was just recently reported [161]. On the other side, lattice systems can also exhibit a fundamental difference compared to continuum systems. One hint is from the mathematical point of view: generalized LLs realized in lattice models will inevitably encounter ramification points—momentum-space singularities where quantum geometric quantities vanish (γ_N , for some $N > 1$) [62,63]. The ramification points will enrich the recursion formulas provided in this work and alter the invariants. While ramification points can also occur in the continuum case under certain circumstances, in the lattice models they are necessary. Studying the new physics associated with ramification points lies beyond the scope of Landau levels in continuum, with lattice models providing an ideal framework for its exploration.

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APPENDIX A: REVIEW OF USEFUL CONCEPTS

In this section, we provide more detailed reviews on relevant concepts, such as Kähler geometry for single and multiple bands, Plücker embedding, and classical Frenet-Serret theory for curves. We also summarize the key formulas derived in this work in Table I.

1. Kähler geometry for single and multiple bands

In the main text, we introduced Kähler manifolds for a single band

$$|u_{\mathbf{k}}\rangle = Z(\mathbf{k}) \equiv [Z_1(\mathbf{k}), Z_2(\mathbf{k}), \dots, Z_N(\mathbf{k})]^t. \quad (\text{A1})$$

In this case, the local Kähler potential φ is given by

$$\varphi = \log \left(1 + \sum_i |W_i|^2 \right) \quad (\text{A2})$$

in terms of the homogeneous complex coordinates $W_i \equiv Z_i/Z_1$ ($2 \leq i \leq N$), where $Z_1 \neq 0$. The FS symplectic form and metric are determined by

$$\omega_{\text{FS}} = \frac{i}{2} \sum_{ij} g_{i\bar{j}} dW_i \wedge d\bar{W}_j \quad \text{and} \quad g_{\text{FS}} = \sum_{ij} g_{i\bar{j}} dW_i d\bar{W}_j, \quad (\text{A3})$$

with $g_{i\bar{j}} = [(\partial^2 \varphi)/(\partial W_i \partial \bar{W}_j)]$, $i, \bar{j} = 2, \dots, N$.

The generalization to multiple bands requires the notion of a Grassmannian manifold. The periodic state of a rank- r band complex—i.e., a band having r linearly independent states per momentum—can be locally described by

$$[|u_{1\mathbf{k}}\rangle, \dots, |u_{r\mathbf{k}}\rangle] = \begin{bmatrix} Z_{11}(\mathbf{k}) & \cdots & Z_{1r}(\mathbf{k}) \\ \vdots & \ddots & \vdots \\ Z_{N1}(\mathbf{k}) & \cdots & Z_{Nr}(\mathbf{k}) \end{bmatrix}. \quad (\text{A4})$$

In this case, arbitrary linear combination among the $|u_{n\mathbf{k}}\rangle$'s does not change the band and hence the $N \times r$ matrix $Z(\mathbf{k}) = [Z_{mn}(\mathbf{k})]_{1 \leq m \leq N, 1 \leq n \leq r}$ determines the bands up to multiplication on the right by an $r \times r$ invertible matrix

TABLE I. Summary of geometries for individual generalized LLs $|u_n\rangle$ and filled lowest N generalized LLs Ψ_N . The “expression” for metric and Ricci curvature refer to their trace part.

Geometric data	Component	Expression	Differential form
Connection coefficients	A_{nn} $A_{n,n-1}$	$-i\bar{z}_k/2 + i\alpha_n$ $-i\mathcal{N}_n^{-1}$	$\theta_n^n = iA_{nn}dz$ $\theta_{n-1}^n = iA_{n,n-1}dz$
Curvature of $ u_n\rangle$	Ω_n	$\mathcal{C}(\mathcal{N}_{n+1}^{-2} - \mathcal{N}_n^{-2})$	$\omega_{n+1} - \omega_n = (i/2)\Omega_n dz \wedge d\bar{z}$
Metric of $ u_n\rangle$	g_n	$\mathcal{C}(\mathcal{N}_{n+1}^{-2} + \mathcal{N}_n^{-2})$	$\gamma_{n+1} + \gamma_n = (\text{tr} g_n) dz ^2$
Curvature of Ψ_N	Ω_{Ψ_N}	$\mathcal{C}\mathcal{N}_N^{-2}$	$\omega_N = (i/2)\Omega_{\Psi_N} dz \wedge d\bar{z}$
Metric of Ψ_N	g_{Ψ_N}	$\mathcal{C}\mathcal{N}_N^{-2}$	$\gamma_N = \theta_{N-1}^N ^2$
Ricci curvature of Ψ_N	R_{Ψ_N}	$\partial\bar{\partial}\log\mathcal{N}_N^{-2}$	$\text{Ric}(\omega_N) = \omega_{N-1} - 2\omega_N + \omega_{N+1}$

with entries smooth functions in the Brillouin zone. In particular, we can locally choose $Z(\mathbf{k})$ such that $Z(\mathbf{k})^\dagger Z(\mathbf{k}) = I_r$, which is to say that the $|u_{nk}\rangle$'s form an orthonormal basis for the band at $\mathbf{k}$. The projector $P(\mathbf{k}) = \sum_{n=1}^r |u_{nk}\rangle\langle u_{nk}|$, where the right-hand side assumes an orthonormal basis choice, uniquely determines the band. Furthermore, $P(\mathbf{k})$ determines a map from the Brillouin zone torus to the Grassmannian $\text{Gr}_r(\mathbb{C}^N)$, a manifold which, as a set, consists of all r -dimensional subspaces in $\mathbb{C}^N$ that are in bijection with the $N \times N$ orthogonal projection matrices of rank r . When $r=1$, the Grassmannian $\text{Gr}_r(\mathbb{C}^N)$ reduces to $\mathbb{C}P^{N-1}$. Similar to what happens in the $r=1$ case, here the Abelian Berry curvature and the (Abelian) quantum metric, given by the traces over band indices of their non-Abelian counterparts, are, respectively, the pullback of (twice) the FS symplectic form and metric by the map $P: \text{BZ}^2 \rightarrow \text{Gr}_r(\mathbb{C}^N)$.

Just like $\mathbb{C}P^{N-1} = \text{Gr}_1(\mathbb{C}^N)$, the Grassmannian $\text{Gr}_r(\mathbb{C}^N)$, for general r , is also a Kähler manifold, with the Fubini-Study symplectic form taking the role of the Kähler (symplectic) form. What this means is that these spaces carry the structure of complex manifolds in such a way that the Fubini-Study metric and symplectic forms are compatible, locally determined by derivatives of a local Kähler potential. Assuming that the first $r \times r$ block of Z is invertible (this defines a chart in the Grassmannian), we can by multiplication by the inverse of this block on the right bring Z to the form

$$Z = \begin{bmatrix} I_r \\ W \end{bmatrix}, \quad (\text{A5})$$

where W is an $(N-r) \times r$ matrix analog to the homogeneous coordinates for the case $r=1$. The local Kähler potential is given by

$$\varphi = \log \det(I_r + W^\dagger W), \quad (\text{A6})$$

with I_r denoting the $r \times r$ identity matrix, and we have that the Fubini-Study symplectic form and metric are determined by

$$\begin{aligned} \omega_{\text{FS}} &= \frac{i}{2} \sum_{i,j,k,l} \frac{\partial^2 \varphi}{\partial W_{ij} \partial \bar{W}_{kl}} dW_{ij} \wedge d\bar{W}_{kl}, \\ g_{\text{FS}} &= \sum_{i,j,k,l} \frac{\partial^2 \varphi}{\partial W_{ij} \partial \bar{W}_{kl}} dW_{ij} d\bar{W}_{kl}. \end{aligned} \quad (\text{A7})$$

2. Plücker embedding

The Plücker embedding is an isometric holomorphic embedding,

$$\text{Gr}_r(\mathbb{C}^N) \hookrightarrow \mathbb{P}\Lambda^r \mathbb{C}^N \cong \mathbb{C}P_r^{N-1}. \quad (\text{A8})$$

Here $\Lambda^r \mathbb{C}^N$ means the r th exterior power of $\mathbb{C}^N$, which physically is the Hilbert space of r -particle fermionic states built of $\mathbb{C}^N$ interpreted as the single-particle Hilbert space. The Plücker embedding assigns to a subspace of dimension r ,

$$E = \text{span}\{|u_1\rangle, \dots, |u_r\rangle\} \subset \mathbb{C}^N,$$

the associated r -particle state corresponding to filling the r states spanning E , namely, the state determined by

$$\text{Slaterdet}(|u_1\rangle, \dots, |u_r\rangle) = |u_1\rangle \wedge \dots \wedge |u_r\rangle.$$

In the local coordinates defined above for the Grassmannian $\text{Gr}_r(\mathbb{C}^N)$ and for $\mathbb{C}P_r^{(N)-1}$, this map simply takes the $(N-r) \times r$ matrix Z , defined in Eq. (A5), and maps it to $\binom{N}{r} - 1$ ratios of size $r \times r$ minors $M_{k_1, \dots, k_r} / M_{1, \dots, r}$, with

$$M_{k_1, \dots, k_r} = \sum_{j_1, \dots, j_r} \sum_{\sigma \in \mathcal{S}_r} \text{sgn}(\sigma) Z_{k_{\sigma(1)} 1} \dots Z_{k_{\sigma(r)} r},$$

which is obviously holomorphic in W , which only appears in the numerator in products of its matrix elements. Being an isometric embedding means, in particular, that the metric obtained by restriction of the Fubini-Study metric in

$\mathbb{C}P^{(N)-1}$ yields the Fubini-Study metric in $\text{Gr}_r(\mathbb{C}^N)$. This is readily understood from the fact that

$$||\text{Slaterdet}(|u_1\rangle, \dots, |u_r\rangle)||^2 = \det(\langle u_i | u_j \rangle)_{1 \leq i, j \leq r},$$

which yields that the restriction of the Kähler potential in $\mathbb{C}P^{(N)-1}$ to the image is the Kähler potential of the Grassmannian—hence implying that the map is an isometry, i.e., it preserves the Kähler structure. In particular, this implies the useful result that, if we have a complex of r bands determined by a projector,

$$P(\mathbf{k}) = \sum_{n=1}^r |u_{nk}\rangle \langle u_{nk}|,$$

we can form the r -particle state,

$$|\Psi(\mathbf{k})\rangle = \text{Slaterdet}(|u_{1k}\rangle, \dots, |u_{rk}\rangle),$$

and we have the equality between the quantum metric determined by $P(\mathbf{k})$ and that determined by $|\Psi_{\mathbf{k}}\rangle$,

$$\text{tr}(PdPdP) = \langle d\Psi | (1 - |\Psi\rangle \langle \Psi|) | d\Psi \rangle, \quad (\text{A9})$$

where we assumed states $|u_{nk}\rangle$ are orthonormal and $d = dk_a \partial_k^a$. The imaginary part, the relation between Berry curvatures in Eq. (4), follows from the same reasoning.

3. Classical Frenet-Serret theory for curves

The Frenet-Serret theory describes the dynamics of a particle moving along a curve in a n -dimensional Euclidean space. Without loss of generality and for illustration purposes, we take $n = 3$. Denoting the particle's position at time t as $\mathbf{r}(t)$, its velocity, acceleration, and jerk are, respectively,

$$S(t) = \left[\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \frac{d^3\mathbf{r}}{dt^3} \right], \quad (\text{A10})$$

where for simplicity we will take the velocity to have unit norm $|d\mathbf{r}/dt|^2 = 1$. We denote the velocity by $\mathbf{v} = d\mathbf{r}/dt$. Moreover, we will denote the vector normal to the curve as $\mathbf{n}$ and the third vector orthogonal to $\mathbf{v}$ and $\mathbf{n}$, known as the binormal, as $\mathbf{b}$. In general, the velocity $d\mathbf{r}/dt$, the acceleration $d^2\mathbf{r}/dt^2$, and the jerk $d^3\mathbf{r}/dt^3$ are not orthogonal; nevertheless, they define the local frame. One can take a Gram-Schmidt orthogonalization to get an orthogonal frame by the rotation matrix,

$$R(t) = \text{Gram-Schmidt}[S(t)] = [\mathbf{v}, \mathbf{n}, \mathbf{b}], \quad (\text{A11})$$

where

$$\mathbf{n} = \frac{\frac{d\mathbf{v}}{dt} - \left(\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \right) \mathbf{v}}{\left| \frac{d\mathbf{v}}{dt} - \left(\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \right) \mathbf{v} \right|} = \frac{\frac{d\mathbf{v}}{dt}}{\left| \frac{d\mathbf{v}}{dt} \right|}, \quad \mathbf{b} = \pm \mathbf{v} \times \mathbf{n}, \quad (\text{A12})$$

where we used the derivative of the condition $\mathbf{v}^2 = 1$ to note $\mathbf{v} \cdot d\mathbf{v}/dt = 0$ in order to simplify the expression. The vectors $\mathbf{v}, \mathbf{n}, \mathbf{b}$ are called a Frenet-Serret frame for the curve $\mathbf{r}(t)$. The Frenet-Serret equations give the rate of change of the frame $R(t)$,

$$\frac{dR}{dt} = \left[\frac{d\mathbf{v}}{dt}, \frac{d\mathbf{n}}{dt}, \frac{d\mathbf{b}}{dt} \right] = [\mathbf{v}, \mathbf{n}, \mathbf{b}] \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}, \quad (\text{A13})$$

where the functions $\kappa(t)$ and $\tau(t)$ are geometric invariants of the curve and they are known, respectively, as the “curvature” and “torsion” of the curve. Note that $\kappa = |d\mathbf{v}/dt|$ is just the magnitude of the acceleration of the curve. The expression for the torsion can be derived too, but it will not be important in what follows and we omit it. Importantly, these geometric invariants completely determine the curve up to rigid motion, i.e., a global translation and rotation in Euclidean space.

The matrix appearing on the right-hand side is $R^{-1}dR/dt$ and is the pullback under the map $t \mapsto R(t)$ of the Maurer-Cartan 1-form $R^{-1}dR$ on $\text{O}(3)$. Because of that, it is a skew-symmetric matrix, which follows from differentiating the equation $R'R = I$. More generally, for a curve in $\mathbb{R}^n$, by Gram-Schmidt orthogonalization of $S(t) = [d\mathbf{r}/dt, \dots, d^n\mathbf{r}/dt^n]$, we get an orthogonal frame $R(t) = [\mathbf{u}_1, \dots, \mathbf{u}_n]$. The rate of change of $R(t)$ is described by

$$\frac{d\mathbf{u}_i}{dt} = \sum_{j=1}^n \mathbf{u}_j \theta_i^j, \quad (\text{A14})$$

where

$$\theta = [\theta_i^j]_{1 \leq i, j \leq n} \equiv R^{-1} \frac{dR}{dt} = \left[\mathbf{u}_i \cdot \frac{d\mathbf{u}_j}{dt} \right]_{1 \leq i, j \leq n}. \quad (\text{A15})$$

In general, from R being a rotation matrix, we know that $\theta^t = -\theta$. The specific form of the frame R built from Gram-Schmidt orthogonalization of S yields that θ has a very sparse tridiagonal form, namely,

$$\theta = \begin{bmatrix} 0 & \theta_2^1 & 0 & \cdots \\ \theta_1^2 & 0 & \theta_3^2 & \cdots \\ 0 & \theta_2^3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad (\text{A16})$$

yielding the Frenet-Serret equations

$$\frac{d\mathbf{u}_i}{dt} = \mathbf{u}_{i-1} \theta_i^{i-1} + \mathbf{u}_{i+1} \theta_i^{i+1}. \quad (\text{A17})$$

4. Summary of key formulas

We summarize the geometric quantities and their inter-relationships, for individual and filled generalized LLs, in Table I.

APPENDIX B: PROOF DETAILS

In this section, we provide proofs for certain results appearing in the main text. We will work with holomorphic functions that are not normalized. For notational simplicity, we will omit the momentum labeling of state.

1. Quantum geometries of filled bands

In the main text, we discussed the relation between quantum geometries of filled bands and the non-Abelian geometric quantities associated with the band complex, in Eq. (4). It can be understood as a consequence of the fact that Plücker embedding is an isometric embedding [39–41]. In the following, we provide a direct proof, which is perhaps more accessible to physicists.

The basic idea used in this direct computation is that the connection and quantum metric can be identified by expanding the overlap functions $f_{mn}(\mathbf{k}, \delta\mathbf{k}) \equiv \langle u_{mk} | u_{n, \mathbf{k} + \delta\mathbf{k}} \rangle$ to first and second order,

$$f_{mn}(\mathbf{k}, \delta\mathbf{k}) = 1 + \delta k \text{tr} A_{mn}(\mathbf{k}) + O(|\delta\mathbf{k}|^2), \quad (\text{B1})$$

$$|f_{mn}(\mathbf{k}, \delta\mathbf{k})|^2 = 1 - \delta k^2 g_{mn}(\mathbf{k}) + O(|\delta\mathbf{k}|^3), \quad (\text{B2})$$

where we have tensor contraction notation $\delta k \text{tr} A = \sum_a \delta k_a \text{tr} A^a$ and $\delta k^2 g_{mn} = \sum_{ab} \delta k_a \delta k_b g_{mn}^{ab}$. We will denote the determinant state as

$$\mathcal{U}_k = \det_{i,j \in [1,r]} u_{ik}(\mathbf{r}_j), \quad (\text{B3})$$

whose form factor $\mathcal{F}$ and the Abelian quantum metric g_{det} are derived to be

$$\mathcal{F}(\mathbf{k}, \delta\mathbf{k}) \equiv \langle \mathcal{U}_k | \mathcal{U}_{k+\delta\mathbf{k}} \rangle = \det_{mn} f_{mn}(\mathbf{k}, \delta\mathbf{k}), \quad (\text{B4})$$

$$|\mathcal{F}(\mathbf{k}, \delta\mathbf{k})|^2 = 1 - \delta k^2 g_{\text{det}}(\mathbf{k}) + O(|\delta\mathbf{k}|^3). \quad (\text{B5})$$

We will now derive Eq. (4) from direct expansion. First of all, one can obtain the expansion for f_{mn} and $\mathcal{F}$ as

$$f_{mn}(\mathbf{k}, \delta\mathbf{k}) = \delta_{mn} + i\delta k A_{mn} + \frac{i\delta k^2}{2} (\nabla A_{mn} + i\langle \nabla u_m | \nabla u_n \rangle),$$

and

$$\begin{aligned} \mathcal{F}(\mathbf{k}, \delta\mathbf{k}) &= 1 + i\delta k \text{tr} A, \\ &+ \frac{\delta k^2}{2} [-(\text{tr} A)^2 + \text{tr} A^2 + i\text{tr} \nabla A - \text{tr} \langle \nabla u | \nabla u \rangle], \end{aligned}$$

where $\nabla_a \equiv \partial_{k_a}$ and identity $\det(I + tA) = 1 + t\text{tr}(A) + (t^2/2)[(\text{tr} A)^2 - \text{tr} A^2] + O(t^3)$ is used. Since the leading-

order expansion of the overlap function is defined to be the connection, the above shows the trace of the non-Abelian connection, which proves the imaginary part (for Berry curvature) of Eq. (4), i.e., $\Omega_{\text{det}} = \text{Tr} \Omega$ traced over band indices. To get the metric relation, one computes $|\mathcal{F}|^2$,

$$|\mathcal{F}(\mathbf{k}, \delta\mathbf{k})|^2 = 1 - \delta k^2 [\text{Tr} \langle \nabla u | \nabla u \rangle - \text{Tr} A^2], \quad (\text{B6})$$

which, following Eq. (B5), proves the real part of Eq. (4), i.e., $g_{\text{det}}^{ab} = \text{Tr} g^{ab}$ where the trace acts on the band indices. Hence, we finished the proof of Eq. (4) in a more direct way compared to the reasoning from Plücker embedding.

2. Holomorphic connection

Notice from Eq. (12) that the antiholomorphic partial derivative is closely related to the LL lowering operator; therefore, $|\bar{\partial} u_{nk}\rangle$ is at most a linear superposition of $|u_{mk}\rangle$ with $0 \leq m \leq n$. As a result, $\bar{A}_{mn}(\mathbf{k})$ must vanish identically for any $m > n$. Since $\bar{A}$ is Hermitian to A , we arrive at $A_{mn}(\mathbf{k}) = 0$ for any $m < n$. Similarly, $|\partial u_{nk}\rangle$ is at most a linear superposition of $|u_{mk}\rangle$ of $0 \leq m \leq n+1$. Hence, the holomorphic coefficients A_{mn} must vanish identically for $m > n+1$. We, therefore, conclude that the holomorphic connection coefficient A_{mn} can only take nonzero values when $m = n$ or $m = n+1$. We have chosen a gauge such that the off-diagonal component $A_{n, n-1}(\mathbf{k})$ is purely imaginary.

3. Constraints from the ladder algebra

In this section, we provide derivation details for Eqs. (38) and (39). Using the state recursion Eqs. (35)–(37), one can derive the action of $\hat{a}\hat{a}^\dagger|u_n\rangle$ and $\hat{a}^\dagger\hat{a}|u_n\rangle$ on a given generalized Landau-level state $|u_n\rangle$, with $n \geq 1$, as follows:

$$\begin{aligned} \hat{a}\hat{a}^\dagger|u_n\rangle &= -\mathcal{N}_n^{-1}\alpha_n|u_{n-1}\rangle + (\mathcal{N}_{n+1}^{-2} + |\alpha_n|^2 + \bar{\partial}\alpha_n)|u_n\rangle, \\ &- (\mathcal{N}_{n+1}^{-1}\bar{\alpha}_{n+1} + \bar{\partial}\mathcal{N}_{n+1}^{-1})|u_{n+1}\rangle, \end{aligned}$$

and

$$\begin{aligned} \hat{a}^\dagger\hat{a}|u_n\rangle &= (\partial\mathcal{N}_n^{-1} - \mathcal{N}_n^{-1}\alpha_{n-1})|u_{n-1}\rangle, \\ &+ (\mathcal{N}_n^{-2} + |\alpha_n|^2 - \partial\bar{\alpha}_n)|u_n\rangle, \\ &- \mathcal{N}_{n+1}^{-1}\bar{\alpha}_n|u_{n+1}\rangle. \end{aligned}$$

Since their difference must be $|u_n\rangle$ because of the algebra $[\hat{a}, \hat{a}^\dagger] = 1$, the above two equations yield the two recursions Eqs. (38) and (39).

When $n = 0$, we instead have

$$\begin{aligned} \hat{a}\hat{a}^\dagger|u_0\rangle &= (\mathcal{N}_1^{-2} + |\alpha_0|^2 + \bar{\partial}\alpha_0)|u_0\rangle, \\ &- (\mathcal{N}_1^{-1}\bar{\alpha}_1 + \bar{\partial}\mathcal{N}_1^{-1})|u_1\rangle, \end{aligned} \quad (\text{B7})$$

and

$$\hat{a}^\dagger \hat{a} |u_0\rangle = (|\alpha_0|^2 - \partial \bar{\alpha}_0) |u_0\rangle - \mathcal{N}_1^{-1} \bar{\alpha}_0 |u_1\rangle, \quad (\text{B8})$$

which gives Eq. (40) after using $[\hat{a}, \hat{a}^\dagger] = 1$. In order to derive Eq. (41), we first of all rewrite Eq. (37) as follows:

$$\hat{a} \left(\mathcal{N}_0 e^{-\frac{1}{2}z\bar{z}} |\tilde{u}_0\rangle \right) = -\bar{\alpha}_0 \left(\mathcal{N}_0 e^{-\frac{1}{2}z\bar{z}} |\tilde{u}_0\rangle \right), \quad (\text{B9})$$

where $z = -i\sqrt{C}k$ and we have replaced $|u_0\rangle$ by $\mathcal{N}_0 \exp(-\frac{1}{2}z\bar{z}) |\tilde{u}_0\rangle$, where $|\tilde{u}_0\rangle$ is a holomorphic function annihilated by $\hat{a}$ or $\bar{\partial}$. Plug into the expression of $\hat{a}$ one arrives at Eq. (41).

4. Frenet-Serret equation and the effect of holomorphic gauge transformations

In this section, we derive the Frenet-Serret equation. Moreover, we also prove that the orthonormal basis $\{|u_n\rangle\}$ is canonical in the sense that the basis elements do not mix upon gauge transformations of the ideal Kähler band Bloch wave function. The derivation of the Frenet-Serret formula is parallel to the discussion of state recursion presented in Sec. III A 1. Here, we follow the notation used in Sec. III B 1.

The Gram-Schmidt process is equivalent to multiplication on the right by an upper-triangular matrix $U(\mathbf{k}) = S(z)B(\mathbf{k})$, where $U(\mathbf{k})$ and $S(z)$ are defined in Eqs. (58) and (57). From now on, we omit the momentum variable. The standard ideal Kähler band in a holomorphic gauge is denoted as $|u_{\mathbf{k}}\rangle$, the holomorphic gauge condition being $[\partial/(\partial\bar{z})]|u_{\mathbf{k}}\rangle = 0$. Explicitly, the n th orthonormal state $|u_n\rangle$ is given by

$$|u_n\rangle = \sum_{m=0}^{\infty} \frac{\partial^m |u\rangle}{\partial z^m} B_n^m = \sum_{m=0}^n \frac{\partial^m |u\rangle}{\partial z^m} B_n^m.$$

Using the fact that $|u\rangle$ is holomorphic in z and the fact that the inverse of an upper-triangular matrix is also an upper-triangular matrix, we get the following:

$$\begin{aligned} d|u_n\rangle &= dz \sum_{m=0}^n \frac{\partial^{m+1} |u\rangle}{\partial z^{m+1}} B_n^m + \sum_{m=0}^n \frac{\partial^m |u\rangle}{\partial z^m} dB_n^m \\ &= dz \sum_{m=0}^n \sum_{l=0}^{m+1} |u_l\rangle (B^{-1})^l_{m+1} B_n^m \\ &\quad + \sum_{m=0}^n \sum_{l=0}^m |u_l\rangle (B^{-1})^l_m dB_n^m, \end{aligned}$$

where in the derivation we have used the fact that $|u\rangle$ is holomorphic. Since B is an upper-triangular matrix, its inverse is also upper triangular. From this, we obtain two important pieces of information:

- (i) $\theta_n^m = 0$, for any $m > n + 1$;
- (ii) θ_n^{m+1} is a $(1, 0)$ -form, i.e., it only contains dz .

This discussion allows one to derive the Frenet-type formulas, in Eqs. (62) and (63), which are in parallel to the state recursion relations shown in Eqs. (35)–(37).

Next, we want to understand how gauge transformations of $|u\rangle$ affect the $|u_n\rangle$'s. Suppose we choose a different holomorphic gauge by performing the transformation

$$|u_{\mathbf{k}}\rangle \rightarrow G(z)|u_{\mathbf{k}}\rangle, \quad (\text{B10})$$

where $G(z)$ is holomorphic and nonvanishing. Then, we have

$$\frac{\partial^{n-1}}{\partial z^{n-1}} |u\rangle \rightarrow \sum_{m=0}^{n-1} \binom{n-1}{m} G^{(m)}(z) \frac{\partial^{n-1-m}}{\partial z^{n-1-m}} |u\rangle, \quad (\text{B11})$$

hence the orthogonal projector P_N determined by $[|u\rangle, \dots, [(\partial^{N-1})/(\partial z^{N-1})]|u\rangle]$ is the same as the one determined after gauge transformation $[G(z)|u\rangle, \dots, [(\partial^{N-1})/(\partial z^{N-1})](G(z)|u\rangle)]$. This is consistent with the fact that the map to projective space induced by $|u\rangle \wedge \dots \wedge [(\partial^{N-1})/(\partial z^{N-1})]|u\rangle$ is invariant under this choice and with the fact that the Plücker map which sends an orthogonal projector of rank N —the set of which is the Grassmannian of N planes—to the state corresponding to a Slater determinant of a basis determining is an embedding. In terms of orthogonal projectors, we have

$$|u_n\rangle = \frac{(1 - P_n) \frac{\partial^n |u\rangle}{\partial z^n}}{\|(1 - P_n) \frac{\partial^n |u\rangle}{\partial z^n}\|}. \quad (\text{B12})$$

Since in the above transformation $(1 - P_n)$ is left invariant, because P_n is so, and since this orthogonal projector will kill all the terms in Eq. (B11) except the one containing $[(\partial^n)/(\partial z^n)]|u\rangle$, it is clear that all generalized Landau-level states acquire the same phase under holomorphic gauge transformation, Eq. (B10),

$$|u_{nk}\rangle \longrightarrow \frac{G(z)}{|G(z)|} |u_{nk}\rangle, \quad \text{for all } n \geq 0, \quad (\text{B13})$$

hence proving that the associated bands determined by $|u_n\rangle\langle u_n|$ are uniquely determined from $|u_0\rangle\langle u_0|$.

5. Geometry of filled bands

Here we provide a derivation of Eq. (65). We consider the map induced by Ψ_N to projective space. The pullback of the Fubini-Study form with respect to this map, which we will denote by ω_N , is equal to the pullback of the Fubini-Study form in the Grassmannian under the orthogonal projector $P_N(\mathbf{k}) = \sum_{n=0}^{N-1} |u_{nk}\rangle\langle u_{nk}|$ (because the Plücker embedding, induced by Ψ_N , is an isometric embedding of the Grassmannian onto its image in projective space in the main text). Hence,

$$\begin{aligned}
\omega_N &= -\frac{i}{2} \text{Tr}(P_N dP_N \wedge dP_N) \\
&= -\frac{i}{2} \sum_{n=0}^{N-1} \sum_{m=N}^{\infty} \langle du_n | u_m \rangle \wedge \langle u_m | du_n \rangle \\
&= \frac{i}{2} \sum_{n=0}^{N-1} \sum_{m=N}^{\infty} \theta_m^n \wedge \bar{\theta}_n^m \\
&= \frac{i}{2} \theta_N^{N-1} \wedge \bar{\theta}_{N-1}^N = \frac{i}{2} \theta_{N-1}^N \wedge \bar{\theta}_{N-1}^N, \quad (\text{B14})
\end{aligned}$$

where we used the Frenet-Serret formulas. The quantum metric, denoted by γ_N , is, due to holomorphicity, just

$$\gamma_N = |\theta_{N-1}^N|^2 = \left| \langle u_N | \frac{\partial}{\partial z} | u_{N-1} \rangle \right|^2 |dz|^2. \quad (\text{B15})$$

6. Geometry of the generalized Landau levels and relation to the geometry of filled bands

Here we derive Eq. (66). The quantum metric g_n , associated with u_{nk} , is given by

$$\begin{aligned}
g_n &= \langle du_n | (1 - |u_n\rangle\langle u_n|) | du_n \rangle \\
&= \sum_{n \neq m} \langle du_n | u_m \rangle \langle u_m | du_n \rangle \\
&= |\theta_{n+1}^n|^2 + |\theta_{n-1}^n|^2 = \gamma_{n+1} + \gamma_n. \quad (\text{B16})
\end{aligned}$$

Although it is not needed for the proof of our main result, we will write the Berry curvature associated with $|u_{nk}\rangle$,

$$\begin{aligned}
F_n &= \sum_{m \neq n} \langle du_n | u_m \rangle \wedge \langle u_m | du_n \rangle \\
&= -\theta_{n+1}^n \wedge \bar{\theta}_n^{n+1} - \theta_{n-1}^n \wedge \bar{\theta}_n^{n-1} \\
&= -\theta_{n+1}^n \wedge \bar{\theta}_n^{n+1} + \theta_{n-1}^n \wedge \bar{\theta}_n^{n-1} = 2i(\omega_{n+1} - \omega_n). \quad (\text{B17})
\end{aligned}$$

7. Geometries in holomorphic frame

In this section, we provide details for Eqs. (44) and (45). The definitions of the Berry curvature and quantum metric components for a given Bloch state $|u_k\rangle$ are

$$\begin{aligned}
\Omega(\mathbf{k}) &= \varepsilon_{ab} \langle \partial_k^a u | \partial_k^b u \rangle, \\
g^{ab}(\mathbf{k}) &= \frac{1}{2} \langle \partial_k^a u | \partial_k^b u \rangle - \frac{1}{2} A_k^a A_k^b + (a \leftrightarrow b), \quad (\text{B18})
\end{aligned}$$

where $A_k^a = -i \langle u_k | \partial_k^a u \rangle$ is the connection. Here, we will derive equivalent expressions in terms of holomorphic derivatives. Following the convention of the main text, introducing momentum-space complex coordinates,

$$z \equiv -i\sqrt{C} w^a k_a, \quad \partial \equiv \partial_z = iC^{-\frac{1}{2}} w_a^* \partial_k^a, \quad (\text{B19})$$

such that $[\partial, z] = 1$, where repeated indices are summed implicitly. Inversely, in terms of holomorphic coordinates, the momentum-space derivative is given by

$$\partial_k^a = -i\sqrt{C}(w^a \partial - w^{a*} \bar{\partial}). \quad (\text{B20})$$

Plugging this into the expression of curvature and metric and using $w_a w^a = 0$, $w_a w^{a*} = 1$, one arrives at Eqs. (44) and (45) where $\langle \partial u |$ denotes the Hermitian conjugate of $|du\rangle$, and the holomorphic connection is given by

$$A \equiv i w_a^* A^a / \sqrt{C} = -i \langle u | du \rangle. \quad (\text{B21})$$

8. Ricci form of filled states

Here we will prove the result in Eq. (69). The Ricci form for $\omega_N = (i/2) \theta_{N-1}^N \wedge \bar{\theta}_{N-1}^N = (i/2) h dz \wedge d\bar{z}$ (of course, $h = |\langle u_N | [\partial/(\partial z)] | u_{N-1} \rangle|^2$), denoted $\text{Ric}(\omega_N)$ is defined as $i/2$ of the curvature of the Chern connection on the Hermitian holomorphic line bundle $(T^*BZ^2)^{(1,0)}$, with the metric described by

$$\langle dz, dz \rangle = h^{-1}. \quad (\text{B22})$$

Accordingly,

$$\text{Ric}(\omega_N) = \frac{i}{2} \partial \bar{\partial} \log h. \quad (\text{B23})$$

Because the Chern connection coincides with the Levi-Civita connection (the unique torsion-free connection compatible with the metric) and θ_{N-1}^N is a unitary frame in $(T^*BZ^2)^{(1,0)}$, obtaining the connection coefficient is equivalent to determining the 1-form φ such that

$$d\theta_{N-1}^N = \varphi \wedge \theta_{N-1}^N, \quad (\text{B24})$$

and the Ricci form is just $\text{Ric}(\omega_N) = id\varphi/2$. Now the pullback of the Maurer-Cartan 1-form satisfies

$$d\theta = -\theta \wedge \theta, \quad (\text{B25})$$

or, in components,

$$d\theta_n^m = -\sum_k \theta_k^m \wedge \theta_n^k. \quad (\text{B26})$$

The above equations imply, setting $m = N + 1$ and $n = N$,

$$\begin{aligned}
d\theta_{N-1}^N &= -\sum_{n=0}^{\infty} \theta_n^N \wedge \theta_{N-1}^n \\
&= -\sum_{n=0}^{\infty} \theta_{N-1}^N \wedge \theta_{N-2}^{N-1} - \theta_N^N \wedge \theta_{N-1}^N \\
&= (\theta_{N-1}^{N-1} - \theta_N^N) \wedge \theta_{N-1}^N, \quad (\text{B27})
\end{aligned}$$

from which $\varphi = \theta_{N-1}^{N-1} - \theta_N^N$. Using the Maurer-Cartan equation once again,

$$\begin{aligned} d\theta_{N-1}^{N-1} &= -\theta_{N-2}^{N-1} \wedge \theta_{N-1}^{N-2} - \theta_N^{N-1} \wedge \theta_{N-1}^N \quad \text{and} \\ d\theta_N^N &= -\theta_{N-1}^N \wedge \theta_N^{N-1} - \theta_{N+1}^N \wedge \theta_N^{N+1}, \end{aligned} \quad (\text{B28})$$

from which we get the recursion relation

$$\text{Ric}(\omega_N) = \frac{id\varphi}{2} = \omega_{N-1} - 2\omega_N + \omega_{N+1}. \quad (\text{B29})$$

APPENDIX C: FINITE- N MODELS AND THE CANONICAL DECOMPOSITION ALGORITHM

In this section, we discuss consequences following the completeness of generalized LL basis. In particular, we define “LL mimicry” in terms of generalized LLs. Additionally, we discuss the finite- N model, which refers to a state constructed by linearly superposing finite number N of generalized LL with general superposition coefficients that are allowed to have momentum $\mathbf{k}$ dependence. We prove there exists an algorithm to exactly extract the superposition coefficients in such model, based on the ladder relation or, equivalently, the Frenet-Serret equation satisfied by the generalized LL states.

1. Landau-level mimicry

As discussed in the main text, due to the completeness of generalized LLs, any cell-periodic state of a Chern band can be uniquely decomposed as follows:

$$|u_{\mathbf{k}}\rangle = \sum_{n=0}^N C_{n\mathbf{k}}^{\mathcal{B}} |u_{n\mathbf{k}}^{\mathcal{B}}\rangle, \quad (\text{C1})$$

where typically the number of generalized LLs involved N is ∞ . We have included the superscript $\mathcal{B}$ to emphasize the fact that both the generalized LL basis $\{|u_n\rangle\}$ and the decomposition coefficients $C_{n\mathbf{k}}$ depend on the choice of the modulation function $\mathcal{B}(\mathbf{r})$. Note that the topological properties and boundary condition of two sides must be identical: Chern number of the left side $|u_{\mathbf{k}}\rangle$ is identical to those on the right-hand side $|u_{n\mathbf{k}}\rangle$. Moreover, if $|u_{\mathbf{k}}\rangle$ is a LL-type state obeying magnetic translation symmetry, then $\mathcal{B}(\mathbf{r})$ is a periodic function; and if $|u_{\mathbf{k}}\rangle$ is a Bloch-type state obeying standard Bloch translation symmetry, then $\mathcal{B}(\mathbf{r})$ must be quasiperiodic.

By definition, a single generalized state $|u_n\rangle$ constructed from $\mathcal{B}_*(\mathbf{r})$ has the optimal distribution function sharply peaked at n for any momentum $\mathbf{k}$, supposing the gauge choice $\mathcal{B}$ used for decomposition has been chosen to be identical as the modulation function $\mathcal{B}_*$ used for constructing the input state. Thereby, we propose, for a general single-band wave function, the minimal spread of the distribution function $p_n \equiv \int |C_{n\mathbf{k}}|^2$ can serve as a quantitative criterion

of how its quantum geometry is approximated by a LL-type state. In particular, if there is a single peak in the distribution under the optimal decomposition gauge, the integer part of the mean value $n_* \equiv \sum_n n p_n$ approximates the LL index. This generalizes the $n = 0$ ideal quantum geometry criterion [38,42] to general n in a quantitative way.

2. The finite- N model and its decomposition

With the optimal gauge $\mathcal{B}_*$, the wave function coefficients are supposed to be concentrated near n_* ; thereby, it makes sense to give a cutoff on level index n and consider wave function with finite N . Moreover, although in the most general case, N is required to be taken to infinity for the representation Eq. (72) to be exact, we anticipate that models with finite N have the ability to well approximate a large class of wave functions of low integrated trace value. We thus define a “finite- N ” model to be Eq. (72) with N being a finite positive integer. The superposition coefficients $C_{n\mathbf{k}}^{\mathcal{B}}$ in the finite- N model are allowed to be momentum dependent. Remarkably, as shown below, we are able to construct a practical algorithm to obtain the decomposition of a Chern band into generalized LLs by assuming the finite- N model.

We will denote the N -dimensional Hilbert space at $\mathbf{k}$ spanned by $|u_{0\mathbf{k}}\rangle \dots |u_{N-1\mathbf{k}}\rangle$ as $\mathcal{H}_{N\mathbf{k}}$,

$$\mathcal{H}_{N\mathbf{k}} \equiv \text{span}\{|u_{0\mathbf{k}}\rangle, \dots, |u_{N-1\mathbf{k}}\rangle\}. \quad (\text{C2})$$

The recursion relations Eqs. (35)–(37) yield that the space $\mathcal{H}_{N\mathbf{k}}$ is preserved by the action of the lowering operator $\hat{a}$ or any of its positive powers, i.e.,

$$\hat{a}\mathcal{H}_{N\mathbf{k}} \in \mathcal{H}_{N\mathbf{k}}. \quad (\text{C3})$$

If at some momentum $\mathbf{k}$, operator $\hat{a}$ to some power n acting on the input state $|u\rangle$ annihilates the state, it means $\hat{a}^{n-1}|u\rangle$ is proportional to $|u_0\rangle$ at that point, and $|u_n\rangle$ can be constructed successively from the raising ladder operators $\hat{a}^\dagger$. A more general situation is that $\hat{a}^{n'}$ s with $n = 0, \dots, N-1$ do not annihilate the input state $|u\rangle$. In this case, the important observation is that the generated N vectors

$$|v_{n\mathbf{k}}\rangle \equiv \hat{a}^n |u_{\mathbf{k}}\rangle \quad (\text{C4})$$

fully span the Hilbert space $\mathcal{H}_{N\mathbf{k}}$, although $|v_n\rangle$'s are, in general, not orthonormal.

Additionally, the action of the raising operator $\hat{a}^\dagger$ maps $|u_{\mathbf{k}}\rangle$ to a vector living in the Hilbert space $\mathcal{H}_{N+1,\mathbf{k}}$, which is composed of $\mathcal{H}_{N\mathbf{k}}$ and $|u_{N\mathbf{k}}\rangle$,

$$\hat{a}^\dagger |u_{\mathbf{k}}\rangle \in \mathcal{H}_{N+1,\mathbf{k}} = \mathcal{H}_{N\mathbf{k}} \oplus \text{span}\{|u_{N\mathbf{k}}\rangle\}. \quad (\text{C5})$$

Hence, projecting out $|v_0\rangle, \dots, |v_{N-1}\rangle$ from $\hat{a}^\dagger |u_{\mathbf{k}}\rangle$ yields $|u_{N\mathbf{k}}\rangle$ uniquely up to an undetermined $\mathbf{k}$ -dependent $U(1)$

phase. In this way, the information of $\mathcal{B}$ is extracted as it is contained in an N th generalized LL state $|u_{Nk}\rangle$. With $|u_{Nk}\rangle$ in hand for every $\mathbf{k}$, according to the recursion relation Eq. (36), successively applying the lowering ladder operator followed by Gram-Schmidt orthogonalization in each step yields recursively from $|u_{N-1}\rangle$ to $|u_0\rangle$, up to undetermined $\mathbf{k}$ -dependent $U(1)$ phases. This procedure can be formulated as

$$|u''_{n-1,k}\rangle = \hat{a}|u_{n,k}\rangle, \quad (C6)$$

$$|u'_{n-1,k}\rangle = |u''_{n-1,k}\rangle - |u_{n,k}\rangle\langle u_{n,k}|u''_{n-1,k}\rangle, \quad (C7)$$

$$|u_{n-1,k}\rangle = |u'_{n-1,k}\rangle / \sqrt{|\langle u'_{n-1,k}|u'_{n-1,k}\rangle|}, \quad (C8)$$

for $n = N, N-1, \dots, 1$. In this way, the superposition weights are extracted from the wave function overlap $|C_{nk}|^2 = |\langle u_{nk}|u_k\rangle|^2$.

In the above, we have discussed how to extract superposition weight $|C_{nk}|^2$ from the finite- N model in an exact manner. Since individual generalized LL is a special case of the finite- N model, our exact algorithm easily reduces to the geometric criteria for single generalized LL. Comparing with Ref. [65], the canonical algorithm discussed here is simpler, more practical, and more general. We notice that, in practice, due to the fact that numerics replaces continuous derivatives with finite difference, the numerical error is accumulated in each step using the ladder operators. The larger the N is, the more times ladder operators are applied and more error is accumulated. Nevertheless, the error can be well controlled if one takes the momentum-space mesh to be dense enough.

In the end, we comment on extending the above discussion to a general Chern band, which involves infinite rather than finite N and has potential applications to moiré materials. One idea for decomposing an infinite- N model is to set a maximal $N_{\max}$ and approximate it as a finite- $N_{\max}$ model, which is supposed to be a good approximation if the genuine weights $|C_{nk}|^2$ are indeed small when $n > N_{\max}$. However, whether the weight extracted from the approximated finite- N model converges or not as one increases $N_{\max}$ and the general guiding principle behind it are currently not systematically explored. We leave more systematic exploration to future work.

APPENDIX D: THE $u_m - u_n$ MODEL

We will discuss the quantum geometric properties of the $u_m - u_n$ model in this section. The $u_m - u_n$ model is a special subset of the $N = 2$ finite- N model because the superposition coefficients are momentum independent in the $u_m - u_n$ model. The $u_m - u_n$ model is important for the discussion on non-Abelian fractionalization in Sec. IV.

Concretely, the cell-period state of the $u_m - u_n$ model takes the following form:

$$|u_k\rangle = \sqrt{\lambda}|u_{mk}\rangle + \sqrt{1-\lambda}e^{i\phi}|u_{nk}\rangle, \quad (D1)$$

where $\lambda \in [0, 1] \in \mathbb{R}$ and $\phi \in [0, 2\pi) \in \mathbb{R}$ parametrize the momentum-independent superposition coefficients. Clearly, $|u_k\rangle$ is normalized to unit.

1. Geometries of the model

We will denote the quantum metric and Berry curvature of $|u_k\rangle$ as $g(\mathbf{k})$ and $\Omega(\mathbf{k})$ and those for individual bands as $g_m(\mathbf{k})$ and $\Omega_m(\mathbf{k})$. Without loss of generality, we assume $l = n - m \geq 1$. We will leave all relevant derivation details to Appendix D 2. The momentum dependence in Berry curvature, quantum metric, and normalization factors will be omitted for notational simplicity.

a. When $|m - n| \geq 2$

In this case, the Berry curvature Ω can be proved to be a simple linear combination from Berry curvatures of $|u_m\rangle$ and $|u_n\rangle$,

$$\Omega = \lambda\Omega_m + (1 - \lambda)\Omega_{m+l}. \quad (D2)$$

Clearly, the standard deviation of Berry curvature, defined as

$$\sigma_\Omega \equiv (\langle \Omega^2 \rangle - \langle \Omega \rangle^2)^{\frac{1}{2}}, \quad (D3)$$

has no dependence on ϕ . In the above, $\langle \dots \rangle$ means average over the first Brillouin zone.

Nevertheless, the integrated quantum metric of $|u_k\rangle$ is slightly more complicated. Its analytical expression is given by

$$\begin{aligned} \int \text{tr} g/C &= \lambda \int \text{tr} g_m/C + (1 - \lambda) \int \text{tr} g_{m+l}/C \\ &+ 2\lambda(1 - \lambda) \int \left| \sum_{p=1}^l \partial \log \mathcal{N}_{m+p}^{-1} \right|^2. \end{aligned} \quad (D4)$$

Same as the Berry curvature formula, the first line is a linear superposition of the integrated trace value contributed from u_m and u_n , respectively. However, $\int \text{tr} g$ also has a nonlinear contribution. Such nonlinear term is zero in the limit of vanishing geometric fluctuation, i.e., when the generalized LLs reduce to standard LLs. More important, the integrated trace $\int \text{tr} g$, as seen above, is independent of the relative superposition phase ϕ .

b. When $|m - n| = 1$

In this case, even the Berry curvature Ω will have nontrivial dependence on λ ,

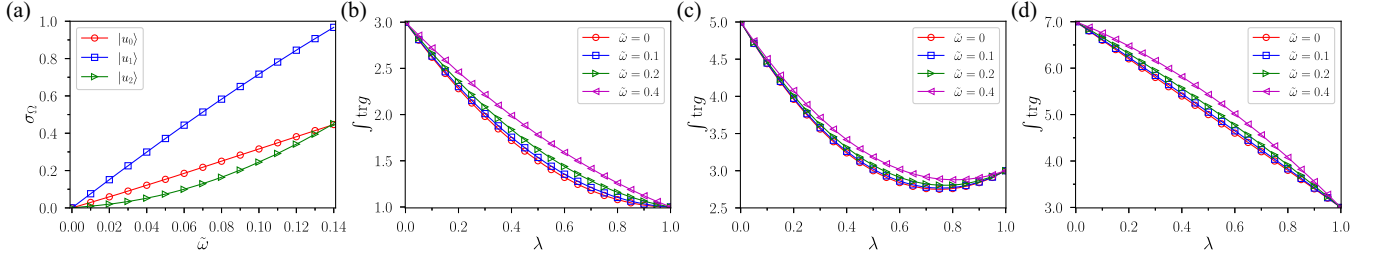


FIG. 8. (a) The variance of Berry curvature for $|u_0\rangle$, $|u_1\rangle$, and $|u_2\rangle$ as a function of the modulation parameter $\tilde{\omega}$. (b)–(d) $\int \text{tr} g$ for the (b) $u_0 - u_1$, (c) $u_1 - u_2$, and (d) $u_1 - u_3$ model at different values of $\tilde{\omega} = 0, 0.1, 0.2$, and 0.4 .

$$\Omega/C = \lambda \Omega_m/C + (1 - \lambda) \Omega_{m+1}/C + \sqrt{\lambda(1 - \lambda)} [e^{i\phi} \partial \mathcal{N}_{m+1}^{-1} + e^{-i\phi} \bar{\partial} \mathcal{N}_{m+1}^{-1}]. \quad (\text{D5})$$

However, since normalization factors $\mathcal{N}_{m+1,k}$ are periodic functions under shifting momentum by reciprocal lattice vectors $\mathbf{k} \rightarrow \mathbf{k} + \mathbf{b}$, the second line vanishes identically after integrating over the first Brillouin zone. Therefore, the Chern number of $|u_k\rangle$ is still C .

The expression of integrated trace value for the quantum metric for this case is found to be

$$\int \text{tr} g/C = \lambda \int \text{tr} g_m/C + (1 - \lambda) \int \text{tr} g_{m+1}/C + 2\lambda(1 - \lambda) \int [|\partial \log \mathcal{N}_{m+1}^{-1}|^2 - \mathcal{N}_{m+1}^{-2}], \quad (\text{D6})$$

which after using Eqs. (32) and (50) is simplified to

$$\int \text{tr} g/C = 2(m + 1)\lambda^2 - 2(m + 2)\lambda + (2m + 3) + 2\lambda(1 - \lambda) \int |\partial \log \mathcal{N}_{m+1}^{-1}|^2. \quad (\text{D7})$$

Notice that, in this case, even in the limit of vanishing geometric fluctuation, the integrated quantum metric still has nonlinear dependence on λ .

c. Integrated trace value: Phase insensitivity, nonlinearity, and lower bound

We first draw the conclusion that the $\int \text{tr} g$ of the $u_m - u_n$ model, either $|m - n| \geq 1$ or $|m - n| \geq 2$, is independent of the relative phase ϕ . Such feature is not only proved in Eqs. (D6) and (D7) where details can be found in Appendix D 2, but also can be straightforwardly numerically verified. Nevertheless, we emphasize the form factor of the $u_m - u_n$ model and hence the interacting problem of the model do have nontrivial ϕ dependence, as shown in Sec. IV.

Second, we notice that the quantum geometric fluctuation, i.e., the momentum dependence of $\mathcal{N}_{nk}$, generally enhances the integrated trace value. This means, for any fixed λ , the $\int \text{tr} g$ is always lower bounded by the value

obtained from linearly superposing two usual LLs whose quantum geometric fluctuation is zero. It is also notable that, increasing the weight of u_{m+1} in the $u_m - u_{m+1}$ model, by decreasing λ , does not necessarily always enhance $\int \text{tr} g$.

The analysis presented above is consistent with the numerical examination listed in Fig. 8, where the enhancement of $\int \text{tr} g$ by quantum geometric fluctuation [Figs. 8(b)–8(d)] and the nonmonotonic behavior of $\int \text{tr} g$ with λ [Fig. 8(c)] can be seen clearly. In the numerical calculation, we parametrize the modulation function $\mathcal{B}(\mathbf{r})$ as in Eq. (73). The standard LLs correspond to the special case of $\tilde{\omega} = 0$.

Last but not least, we discuss the lower bound of the integrated trace value of the $u_m - u_n$ model. Since, at any fixed λ , the $\int \text{tr} g$ of the $u_m - u_n$ model is lower bounded by that value of superposed standard LLs, and since $\int \text{tr} g$ of the standard LL has a simple form whose global minimal value is easy to obtain, we arrive at the following exact lower bound of $\int \text{tr} g$ of the $u_m - u_n$ model:

$$\int \text{tr} g/C \geq \begin{cases} 2m + 1, & n - m \geq 2, \\ \frac{3n}{2} - \frac{1}{2n}, & n - m = 1, \end{cases} \quad (\text{D8})$$

which is tighter than the trace bound $\int \text{tr} g/C \geq 1$.

Recently, fundamental bounds relating the energy gap of an insulator to its “quantum weight” were derived [21,162]. For noninteracting insulators, the quantum weight reduces to the ground-state integrated trace of the quantum metric. Following this, the lower bound Eq. (D8) gives a tighter upper bound of the energy gap of insulators in terms of Chern number, when the ground state is well approximated by the $u_m - u_n$ model.

2. Derivation details

In this section, we use Eqs. (44) and (45) to derive the Berry curvature and trace of quantum metric for the $u_m - u_n$ model, whose wave function is given as a linear superposition of two generalized Landau-level bases u_m and u_n with momentum-independent coefficients α, β that are normalized to one $|\alpha|^2 + |\beta|^2 = 1$,

$$|u_k\rangle \equiv \alpha|u_{mk}\rangle + \beta|u_{nk}\rangle. \quad (\text{D9})$$

We first discuss $n = m + l$ with $l \geq 2$. In this case, the following band-off-diagonal overlaps are all zero, seen from the state recursion relations: $\langle \partial u_m | \partial u_n \rangle$, $\langle \partial u_m | \bar{\partial} u_n \rangle$, $\langle \bar{\partial} u_m | \partial u_n \rangle$, and $\langle \bar{\partial} u_m | \bar{\partial} u_n \rangle$. Moreover, connection is a decoupled sum $A = |\alpha|^2 A_m + |\beta|^2 A_n$, where A_m is the diagonal element A_{mm} . Consequently, the Berry curvature is simply a decoupled sum $\Omega = |\alpha|^2 \Omega_m + |\beta|^2 \Omega_n$ and the trace of the metric is found to be

$$\text{tr} g = |\alpha|^2 \text{tr} g_m + |\beta|^2 \text{tr} g_n + 2C|\alpha\beta|^2(A_m - A_n)^2.$$

The final formula for integrated trace, shown in Eq. (D4), is derived by noticing $|A_m - A_n| = |\alpha_m - \alpha_n|$ and Eq. (39) copied below,

$$\alpha_n - \alpha_{n-1} = -\partial \log \mathcal{N}_n^{-1}. \quad (\text{D10})$$

We proceed to discuss the case with $n = m + 1$. For simplicity, we introduce notations $X_{mn} \equiv \langle \partial u_m | \partial u_n \rangle$ and $Y_{mn} \equiv \langle \bar{\partial} u_m | \bar{\partial} u_n \rangle$ and we denote their diagonal elements as $X_m \equiv X_{mm}$. After some algebra, we arrive at the following useful formula:

$$\begin{aligned} \langle \partial u | \partial u \rangle &= |\alpha|^2 X_m + |\beta|^2 X_{m+1} + (\bar{\alpha}\beta X_{m,m+1} + \text{c.c.}), \\ \langle \bar{\partial} u | \bar{\partial} u \rangle &= |\alpha|^2 Y_m + |\beta|^2 Y_{m+1} + (\bar{\alpha}\beta Y_{m,m+1} + \text{c.c.}), \end{aligned}$$

which gives the analytical expression of the Berry curvature, following Eq. (44),

$$\begin{aligned} \Omega/C &= |\alpha|^2 \Omega_m/C + |\beta|^2 \Omega_{m+1}/C, \\ &+ \mathcal{N}_{m+1}^{-1} [\bar{\alpha}\beta(\alpha_m - \alpha_{m+1}) + \text{c.c.}], \end{aligned} \quad (\text{D11})$$

which yields Eq. (D5). The second line can be proved to vanish identically after the integration, following Eq. (D10), due to the periodicity of normalization factors. Therefore, the Chern number of the model is preserved to be $\mathcal{C}$.

Next, we derive the trace of the quantum metric by using Eq. (45). We first of all prove the integrated trace of the metric is independent of the relative phase of α, β . This is equivalent to proving terms linear to $\alpha\bar{\beta}$ or $\bar{\alpha}\beta$ vanish after integration. After some algebra, we obtain the linear coefficient of $\bar{\alpha}\beta$, given as follows:

$$\begin{aligned} i\mathcal{N}_{m+1}^{-1} [(A_m + A_{m+1}) - 2(|\alpha|^2 A_m + |\beta|^2 A_{m+1})], \\ = i\mathcal{N}_{m+1}^{-1} (|\alpha|^2 - |\beta|^2)(A_{m+1} - A_m), \\ = \mathcal{N}_{m+1}^{-1} (|\alpha|^2 - |\beta|^2) \partial \log \mathcal{N}_{m+1}^{-1}, \\ = (|\alpha|^2 - |\beta|^2) \partial \mathcal{N}_{m+1}^{-1}, \end{aligned} \quad (\text{D12})$$

which clearly vanishes after integration because of the periodicity of normalizations. Similarly, terms linear to $\alpha\bar{\beta}$

also vanish after integration. Equation (D10) is used in the above derivation.

Having proved integrated trace is independent of the relative phase, we proceed to complete the derivation of the remaining terms. After some algebra, we arrive at the expression for the integrated trace, given as follows:

$$\begin{aligned} \int \text{tr} g/C &= |\alpha|^2 \int \text{tr} g_m/C + |\beta|^2 \int \text{tr} g_{m+1}/C + 2|\alpha\beta|^2 \\ &\times \int [|A_m - A_{m+1}|^2 - \mathcal{N}_{m+1}^{-2}], \\ &= |\alpha|^2 \int \text{tr} g_m/C + |\beta|^2 \int \text{tr} g_{m+1}/C + 2|\alpha\beta|^2 \\ &\times \int [|\partial \log \mathcal{N}_{m+1}^{-1}|^2 - (m+1)], \end{aligned} \quad (\text{D13})$$

where we used again Eq. (D10) and the quantized invariant $\int \mathcal{N}_{m+1}^{-2} = m+1$ derived in the main text. Therefore, we have finished the derivation details for Eqs. (D6) and (D7).

APPENDIX E: NUMERICAL DETAILS

In this section, we provide necessary numerical details. We describe in detail how to construct generalized LLs numerically, the interacting Hamiltonians, and the lattice geometry used in exact diagonalization.

1. Construction of generalized Landau levels

Here, we present numerical details of constructing the modulated LL and generalized LL states.

We denote the magic unit cell basis vectors as $\mathbf{a}_1$ and $\mathbf{a}_2$, which encloses an area of $|\mathbf{a}_1 \times \mathbf{a}_2| = 2\pi l_B^2$ with l_B being the magnetic length. In this work, we focus on the triangular lattice with

$$\mathbf{a}_{1,2} = a \left(\pm \frac{\sqrt{3}}{2}, \frac{1}{2} \right), \quad a = l_B \sqrt{\frac{4\pi}{\sqrt{3}}}. \quad (\text{E1})$$

The corresponding reciprocal vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are defined to satisfy $\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi \delta_{ij}$.

We denote the wave function of the n th conventional Landau level as $|\Phi_{nk}^{\text{LL}}\rangle$ and the cell-periodic part as $|u_{nk}^{\text{LL}}\rangle$. They are related by

$$\langle \mathbf{r} | \Phi_{nk}^{\text{LL}} \rangle = e^{ik \cdot \mathbf{r}} \langle \mathbf{r} | u_{nk}^{\text{LL}} \rangle. \quad (\text{E2})$$

We make a special gauge choice, following Ref. [42], such that the boundary conditions of $|\Phi_{nk}^{\text{LL}}\rangle$ are similar to those of lattice Bloch bands. Denoting the guiding center operator as $\mathbf{R}$, which obeys $[\mathbf{R}^a, \mathbf{R}^b] = -i\epsilon^{ab} l_B^2$, the states transform as follows:

$$e^{iq \cdot R} |\Phi_{nk}^{\text{LL}}\rangle = e^{i\mathbf{q} \times \mathbf{k}} |\Phi_{n, \mathbf{k} + \mathbf{q}}^{\text{LL}}\rangle, \quad (\text{E3})$$

$$|\Phi_{n, \mathbf{k} + \mathbf{b}}^{\text{LL}}\rangle = \eta_b e^{i\mathbf{b} \times \mathbf{k}} |\Phi_{nk}^{\text{LL}}\rangle, \quad (\text{E4})$$

where $\eta_b = (-1)^{m+n+mn}$ for $\mathbf{b} = m\mathbf{b}_1 + n\mathbf{b}_2$. Under this gauge, the form factors of conventional Landau levels are

$$\begin{aligned} f_{mn}^{kk'}(\mathbf{b}) &= \langle \Phi_{mk}^{\text{LL}} | e^{i(\mathbf{k} - \mathbf{k}' - \mathbf{b}) \cdot \mathbf{r}} | \Phi_{nk'}^{\text{LL}} \rangle \\ &= \langle u_{mk}^{\text{LL}} | e^{-i\mathbf{b} \cdot \mathbf{r}} | u_{nk'}^{\text{LL}} \rangle \\ &= \eta_b e^{i\mathbf{b} \times (\mathbf{k} + \mathbf{k}') \cdot \mathbf{b}} e^{i\mathbf{k} \times \mathbf{k}' \cdot \mathbf{b}} g_{mn}(\mathbf{k} - \mathbf{k}' - \mathbf{b}), \end{aligned} \quad (\text{E5})$$

where

$$\begin{aligned} g_{n+m, n}(\mathbf{q}) &= g_{n, n+m}^*(-\mathbf{q}) \\ &= \sqrt{\frac{n!}{(n+m)!}} L_n^m(qq^*) (iq)^m \exp\left(-\frac{1}{2} qq^*\right), \end{aligned}$$

$q = (q_x + iq_y)/\sqrt{2}$, and $L_n^m(x)$ is the generalized Laguerre polynomial. In the following, for notational simplicity, we will denote $f_{nn'}^{kk'}(\mathbf{0})$ as $f_{nn'}^{kk'}$.

The modulated Landau-level Bloch states and their periodic parts are

$$\langle \mathbf{r} | \Phi_{nk} \rangle = \mathcal{B}(\mathbf{r}) \langle \mathbf{r} | \Phi_{nk}^{\text{LL}} \rangle, \quad (\text{E6})$$

$$\langle \mathbf{r} | e_{nk} \rangle = \mathcal{B}(\mathbf{r}) \langle \mathbf{r} | u_{nk}^{\text{LL}} \rangle, \quad (\text{E7})$$

where $\mathcal{B}(\mathbf{r}) = \sum_{\mathbf{b}} w_{\mathbf{b}} \exp(i\mathbf{b} \cdot \mathbf{r})$ is the modulation function with the Fourier modes $w_{\mathbf{b}}$. We can expand $|\Phi_{nk}\rangle$ under the orthonormal and complete basis $\{|\Phi_{nk}^{\text{LL}}\rangle\}$ as

$$\begin{aligned} |\Phi_{nk}\rangle &= \mathcal{B}(\mathbf{r}) |\Phi_{nk}^{\text{LL}}\rangle = \sum_m \langle \Phi_{mk}^{\text{LL}} | \mathcal{B}(\mathbf{r}) | \Phi_{nk}^{\text{LL}} \rangle |\Phi_{mk}^{\text{LL}}\rangle \\ &= \sum_m \left(\sum_{\mathbf{b}} w_{\mathbf{b}} f_{mn}^{kk}(-\mathbf{b}) \right) |\Phi_{mk}^{\text{LL}}\rangle. \end{aligned} \quad (\text{E8})$$

Then, we have

$$|e_{nk}\rangle = \sum_m \left(\sum_{\mathbf{b}} w_{\mathbf{b}} f_{mn}^{kk}(-\mathbf{b}) \right) e^{-i\mathbf{k} \cdot \mathbf{r}} |\Phi_{mk}^{\text{LL}}\rangle.$$

Because

$$\begin{aligned} e^{-i\mathbf{k} \cdot \mathbf{r}} |\Phi_{mk}^{\text{LL}}\rangle &= \sum_{m', \mathbf{k}' \in \text{1BZ}} |\Phi_{m'\mathbf{k}'}^{\text{LL}}\rangle \langle \Phi_{m'\mathbf{k}'}^{\text{LL}} | e^{-i\mathbf{k} \cdot \mathbf{r}} | \Phi_{mk}^{\text{LL}} \rangle \\ &= \sum_{m'} |\Phi_{m'\mathbf{0}}^{\text{LL}}\rangle f_{m'm}^{\mathbf{0k}}, \end{aligned} \quad (\text{E9})$$

we arrive at the final expression of the modulated LL states,

$$\begin{aligned} |e_{nk}\rangle &= \sum_m \left(\sum_{\mathbf{b}} \sum_{m'} w_{\mathbf{b}} f_{m'\mathbf{n}}^{\mathbf{k}\mathbf{k}'}(-\mathbf{b}) f_{mm'}^{\mathbf{0k}} \right) |\Phi_{m\mathbf{0}}^{\text{LL}}\rangle \\ &= \sum_m \left(\sum_{\mathbf{b}} w_{\mathbf{b}} f_{mn}^{\mathbf{0k}}(-\mathbf{b}) \right) |\Phi_{m\mathbf{0}}^{\text{LL}}\rangle. \end{aligned} \quad (\text{E10})$$

The generalized LL basis states can be obtained straightforwardly by applying Schmidt orthogonalization to Eq. (E10). Numerically, we keep the leading terms of $w_{\mathbf{b}}$: $w_{\mathbf{0}} = 1$ and $w_{\mathbf{b}} = \tilde{w}$ for the shortest $\mathbf{b}$. We also choose a cutoff for the sum over the Landau-level index m in Eq. (E10). The required cutoff increases with w , which we vary between 10 and 80.

Note that $|u_{nk}\rangle$ is expanded under the $\mathbf{k}$ -independent basis $\{|\Phi_{m\mathbf{0}}^{\text{LL}}\rangle\}$, so it is convenient for computing the derivatives with respect to $\mathbf{k}$. Numerically, the momentum-space derivatives, including the holomorphic and antiholomorphic derivatives (equivalently, the implication of the momentum-space ladder operators), are achieved by the covariant derivatives with $\mathbf{q}_a \rightarrow 0$,

$$|D^a u_k\rangle \approx \frac{1}{2|\mathbf{q}_a|} \left(\frac{|u_{k+\mathbf{q}_a}\rangle}{\langle u_k | u_{k+\mathbf{q}_a} \rangle} - \frac{|u_{k-\mathbf{q}_a}\rangle}{\langle u_k | u_{k-\mathbf{q}_a} \rangle} \right). \quad (\text{E11})$$

2. Interacting Hamiltonian

We first consider a band projected density-density two-body interaction

$$H = \frac{1}{2\mathcal{A}} \sum_{\mathbf{q}} v_{\mathbf{q}} : \rho_{\mathbf{q}} \rho_{-\mathbf{q}} :, \quad (\text{E12})$$

normal ordered with respect to the electron's vacuum. In the above, $\mathcal{A}$ is the real-space area of the 2D system, and $v(\mathbf{q})$ is the Fourier transform of the interaction potential. For the bare Coulomb interaction, we have $v_{\mathbf{q}} \propto 2\pi/|\mathbf{q}|$. In the presence of screening, $v(\mathbf{q})$ decays faster with $|\mathbf{q}|$. We adopt the screening description in Ref. [163] to stabilize the RR states (Fig. 12 in Appendix F 4). The energies of both bare and screened Coulomb interactions are given in units of $[(e^2)/(4\pi\epsilon a)]$, where $-e$ is the electron's charge and ϵ is the dielectric constant.

Suppose a band Bloch state can be expanded by the conventional Landau-level Bloch-like states as

$$|\Phi_{\mathbf{k}}\rangle = \sum_n \alpha_{n\mathbf{k}} |\Phi_{n\mathbf{k}}^{\text{LL}}\rangle, \quad (\text{E13})$$

whose periodic part is $|u_{\mathbf{k}}\rangle = \sum_n \alpha_{n\mathbf{k}} |u_{n\mathbf{k}}^{\text{LL}}\rangle$. The matrix element of the band projected density operator $\rho_{\mathbf{q}}$ is

$$\begin{aligned} \langle \Phi(\mathbf{k}) | \rho_{\mathbf{q}} | \Phi(\mathbf{k}') \rangle &= \sum_{m, n} \alpha_{m, \mathbf{k}}^* \alpha_{n, \mathbf{k}} \langle \Phi_{m\mathbf{k}}^{\text{LL}} | e^{i\mathbf{q} \cdot \mathbf{r}} | \Phi_{n\mathbf{k}'}^{\text{LL}} \rangle \\ &= \delta_{\mathbf{k} - \mathbf{k}' - \mathbf{b}, \mathbf{q}} \sum_{m, n} \alpha_{m, \mathbf{k}}^* \alpha_{n, \mathbf{k}} f_{mn}^{kk'}(\mathbf{b}) \\ &\equiv \delta_{\mathbf{k} - \mathbf{k}' - \mathbf{b}, \mathbf{q}} f_{\mathbf{b}}^{kk'}, \end{aligned}$$

TABLE II. Sample geometries used in our numerical simulations. The vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ give the periodic boundaries of the sample. The two integers (m, n) in the second and the third columns mean $\mathbf{T}_i = m\mathbf{a}_1 + n\mathbf{a}_2$, where $\mathbf{a}_{1,2}$ are given in Eq. (E1).

Number of $\mathbf{k}$ points N_s in the 1BZ	$\mathbf{T}_1$	$\mathbf{T}_2$
16	(4, 0)	(0, 4)
24	(4, 0)	(0, 6)
25	(5, 0)	(0, 5)
28	(6, 4)	(2, 6)
32	(6, 2)	(2, 6)

where $f_b^{k,k'}$ is the band form factor. Then, we can find the second quantization form of the two-body interaction as

$$H = \sum_{k_1, \dots, k_4 \in \text{1BZ}} V_{k_1 k_2 k_3 k_4} c_{k_1}^\dagger c_{k_2}^\dagger c_{k_3} c_{k_4}, \quad (\text{E14})$$

where $c_k^\dagger$ creates an electron in $|\Phi_k\rangle$ and the matrix elements are

$$V_{k_1 k_2 k_3 k_4} = \frac{1}{2\mathcal{A}} \sum_b v_{k_1 - k_4 - b} f_b^{k_1 k_4} f_{\delta b - b}^{k_2 k_3}, \quad (\text{E15})$$

with $\delta b \equiv \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4$ a reciprocal lattice vector. For the $u_m - u_n$ model, the coefficients $\alpha_{n,k}$ in Eq. (E13) can be obtained from Schmidt orthogonalization of Eq. (E8) at point $\mathbf{k}$.

Similarly, the band projected three-body interaction can be written as

$$H_{3b} = \frac{1}{6\mathcal{A}^2} \sum_{q_1, q_2} v_{q_1, q_2} : \rho_{q_1} \rho_{q_2} \rho_{-q_1 - q_2} :, \quad (\text{E16})$$

which, under second quantization, takes the form of

$$H_{3b} = \sum_{k_1, \dots, k_6 \in \text{1BZ}} V_{k_1, \dots, k_6} c_{k_1}^\dagger c_{k_2}^\dagger c_{k_3}^\dagger c_{k_4} c_{k_5} c_{k_6}. \quad (\text{E17})$$

The matrix element

$$V_{k_1, \dots, k_6} = \frac{1}{6\mathcal{A}^2} \sum_{b, b'} v_{k_1 - k_6 - b, k_2 - k_5 - b'} f_b^{k_1 k_6} f_{b'}^{k_2 k_5} f_{\delta b - b - b'}^{k_3 k_4},$$

where $\delta b = \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4 - \mathbf{k}_5 - \mathbf{k}_6$. For the parent repulsion of the Pf state, we have $v_{q_1, q_2} \propto -|q_1|^2 |q_1 - q_2|^4$.

3. Tilted sample geometry

In the system sizes tractable by exact diagonalization, robust fractionalized states, especially the non-Abelian ones, often require the geometry of the sample to be isotropic, i.e., close to the 2D limit. Otherwise, some

charge-ordered phases could be favored. In our numerical simulations, we carefully choose tilted samples whose aspect ratio is close to 1. The details of tilted samples can be found in Refs. [164, 165]. In Table II, we summarize the sample geometries used in this work.

APPENDIX F: MORE ON NON-ABELIAN STATES

In this section, we provide more discussions on the non-Abelian topologically ordered state studied in this work. We present numerical results for $N_e = 14$ particles, showing consistency with the $N_e = 12$ particle results presented in the main text. In addition, we discuss the nature of the non-Abelian ground state in the presence of nonuniform quantum geometry, i.e., whether they are Pfaffian state or anti-Pfaffian or others. We also present numerical studies on the stability of Read-Rezayi parafermion states in generalized LLs, which occurs at a filling fraction compared to Moore-Read families.

1. Moore-Read stability for $N_e = 14$ electrons

In Fig. 4 of the main text, we have shown the neutral gap Δ_g and the ground-state splitting Δ_s at $\tilde{\omega} = 0.02$ in the (λ, ϕ) parameter space for the $u_0 - u_1$, $u_2 - u_1$, and $u_3 - u_1$ models. The system size in that case is $N_e = 12$ electrons. Here we show some data for $N_e = 14$ electrons also at $\tilde{\omega} = 0.02$. As displayed in Fig. 9, we find similar behavior of the sixfold degeneracy with increasing λ —it becomes worse when the weight of u_1 decays.

We also extract the quantum geometric region favoring the stable MR phase from the data of $N_e = 14$ electrons, using the same method as in the main text. As shown in Fig. 10, the result is very similar to that of $N_e = 12$ electrons (Fig. 5), and the previously reported models and materials hosting the MR phase are still located in the region. This further supports the validity of the single-particle quantum geometric criterion proposed by us for the MR phase.

2. Nature of the ground states

Numerically, we have observed the sixfold ground-state degeneracy consistent with the MR phase in finite systems. A remaining question is: Are ground states Pf or aPf? In the following, we will consider this issue by studying the particle-hole (PH) symmetry in our model.

The PH transformation exchanges the fermionic creation and annihilation operators c_k and $c_k^\dagger$. It also exchanges the occupation number 0 and 1 at each $\mathbf{k}$ point in the Fock basis of the many-body state and complex conjugate scalars of the Hilbert space. Then, an arbitrary translational-invariant two-body interaction $H = \sum_{k_1, \dots, k_4} V_{k_1 k_2 k_3 k_4} c_{k_1}^\dagger c_{k_2}^\dagger c_{k_3} c_{k_4}$ is transformed to

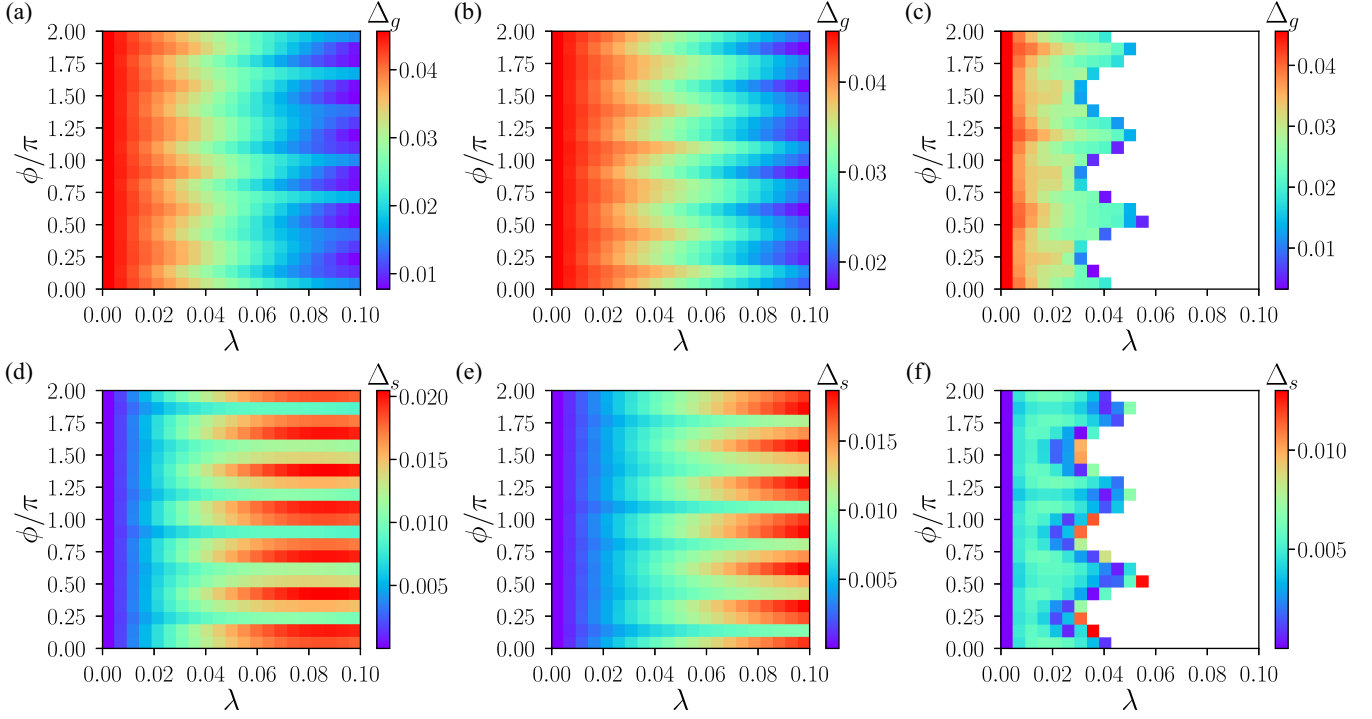


FIG. 9. The many-body gap Δ_g (the first row) and ground-state splitting Δ_s (the second row) at $\nu = 1/2$ in the $u_0 - u_1$ [(a) and (d)], $u_2 - u_1$ [(b) and (e)], and $u_3 - u_1$ model [(c) and (f)] for $N_e = 14$ electrons with Coulomb interaction. The modulation parameters are taken as $\tilde{\omega} = 0.02$ for all subfigures. In those white regions of the (λ, ϕ) parameter space, there is no sixfold ground-state degeneracy with MR momenta.

$$H_{\text{PH}} = H + \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \left(n_{\mathbf{k}} - \frac{1}{2} \right),$$

$$\epsilon_{\mathbf{k}} = \sum_{\mathbf{k}'} (V_{\mathbf{k}\mathbf{k}'\mathbf{k}\mathbf{k}'} + V_{\mathbf{k}'\mathbf{k}\mathbf{k}'\mathbf{k}} - V_{\mathbf{k}\mathbf{k}'\mathbf{k}'} - V_{\mathbf{k}'\mathbf{k}\mathbf{k}\mathbf{k}'}), \quad (\text{F1})$$

where $n_{\mathbf{k}} = c_{\mathbf{k}}^\dagger c_{\mathbf{k}}$ and $V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} = V_{\mathbf{k}_4\mathbf{k}_3\mathbf{k}_2\mathbf{k}_1}^*$ is used. Note that the PH transformed Hamiltonian differs from the

original one by an additional dispersion $\epsilon_{\mathbf{k}}$. In standard LLs, due to the magnetic translation symmetry, $\epsilon_{\mathbf{k}} = 0$, guaranteeing any translational-invariant two-body interaction in a half filled LL is exactly PH symmetric. As discussed in Sec. IV A, such exact PH symmetry in the standard LLs gives cat states of Pf and aPf when the ground state spontaneously breaks PH symmetry; at finite system sizes, sixfold degeneracy is detected in the MR phase,

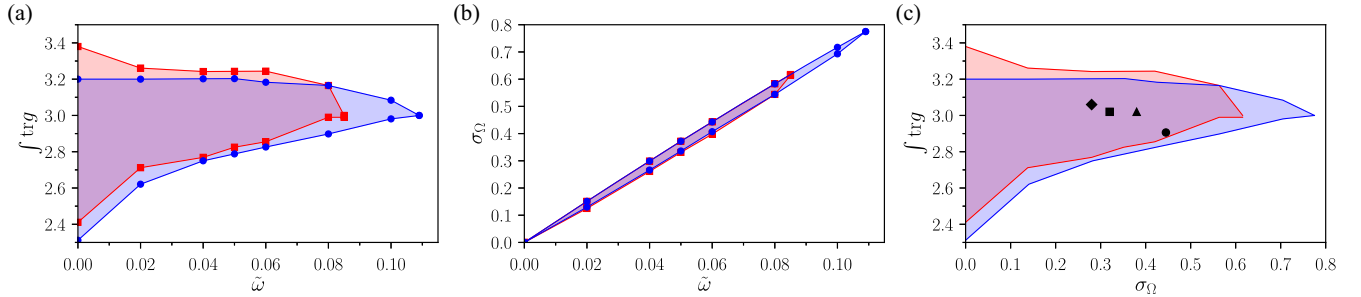


FIG. 10. The quantum geometric regions favoring the MR phase for $N_e = 14$ electrons with Coulomb interaction (blue shade). For convenience of comparison, here we also present the result of $N_e = 12$ electrons (red shade), which has been given in Fig. 5 of the main text. (a), (b) The windows of (a) $\int \text{trg}$ and (b) σ_Ω as a function of $\tilde{\omega}$ in which we find sixfold ground-state degeneracy with $\Delta_g \geq \Delta_s$. The windows are evaluated based on the energy spectra for the half filled $u_0 - u_1$, $u_2 - u_1$, and $u_3 - u_1$ models. (c) Combining (a) and (b), we give the region of the stable MR phase in the quantum geometric parameter space spanned by σ_Ω and $\int \text{trg}$. The markers indicate the materials and models in which the MR phase and the corresponding quantum geometric quantities were reported, including the twisted MoTe₂ homobilayers with twist angle 2° (square) and 2.14° (diamond) [57], skyrmion Chern band model (triangle) [58], and the double-TBG model in Sec. IV B (dot).

while one should expect 12-fold degeneracy in the thermodynamic limit.

When the quantum geometric fluctuation is turned on, both ϵ_k and the electron's occupations are nonuniform in the Brillouin zone; therefore, the PH symmetry is explicitly broken. We denote the energy difference of Pf and aPf as $E_\Delta \equiv \langle H \rangle_{\text{Pf}} - \langle H \rangle_{\text{aPf}}$, where $\langle \mathcal{O} \rangle_\Psi$ means the expectation value of operator $\mathcal{O}$ evaluated with respect to a many-body state Ψ . Performing PH transformation, E_Δ can be rewritten as $\langle H \rangle_{\text{Pf}} - \langle H_{\text{PH}} \rangle_{\text{Pf}}$, which following Eq. (F1) is given by

$$E_\Delta = -\sum_k \epsilon_k \left(\langle n_k \rangle_{\text{Pf}} - \frac{1}{2} \right). \quad (\text{F2})$$

When the system size is sufficiently large, one can replace the lattice summation to continuous integration. In such limit, Eq. (F2) can be further simplified to

$$E_\Delta = -\mathcal{A}e_\Delta, \quad (\text{F3})$$

$$e_\Delta = \frac{1}{4\pi^2} \int \epsilon_k \left(\langle n_k \rangle_{\text{Pf}} - \frac{1}{2} \right) d^2k, \quad (\text{F4})$$

where e_Δ is obtained by integration over the first Brillouin zone and has a converged value in the thermodynamic limit, which is, in general, nonzero. Therefore, asymptotically the energy difference E_Δ of Pf and aPf states has a linear scaling with particle number and the slope of the scaling is determined by e_Δ . When PH symmetry is exact, e_Δ is forced to be zero.

This analysis allows us to construct a phenomenological classical Ising model to describe the competition between the Pf and aPf at large system sizes,

$$h_{\text{Ising}} = \frac{1}{2} E_\Delta \sigma_3 + \left[\langle \text{Pf} | H | \text{aPf} \rangle \sigma^+ + \text{H.c.} \right], \quad (\text{F5})$$

where $\sigma^+ = \sigma_1 + i\sigma_2$ and $\sigma_{1,2,3}$ are Pauli matrices in the bases of Pf and aPf states. Because of the fact that Pf and aPf are of distinct topological order, they are orthogonal in the thermodynamic limit. Therefore, the off-diagonal terms decay exponentially when increasing the system size. Therefore, we expect that either the Pf or the aPf state is selected out by the linearly growing E_Δ for sufficiently large systems in generic flat bands outside standard LLs. We probe the quantum-geometry-induced selection of the Pf or aPf state in Appendix F3.

3. Ground-state weight for the Moore-Read phase

In this section, we probe the quantum-geometry-induced selection of the Pf or aPf state discussed in Sec. F2. We compute the weight of the Coulomb ground state projected to $|u_1\rangle$ on the Pf and aPf states, as displayed in Fig. 11(a). Here the Pf state at $\tilde{\omega} > 0$ [Eq. (73)] is defined as the exact zero-energy mode of the three-body Pfaffian parent

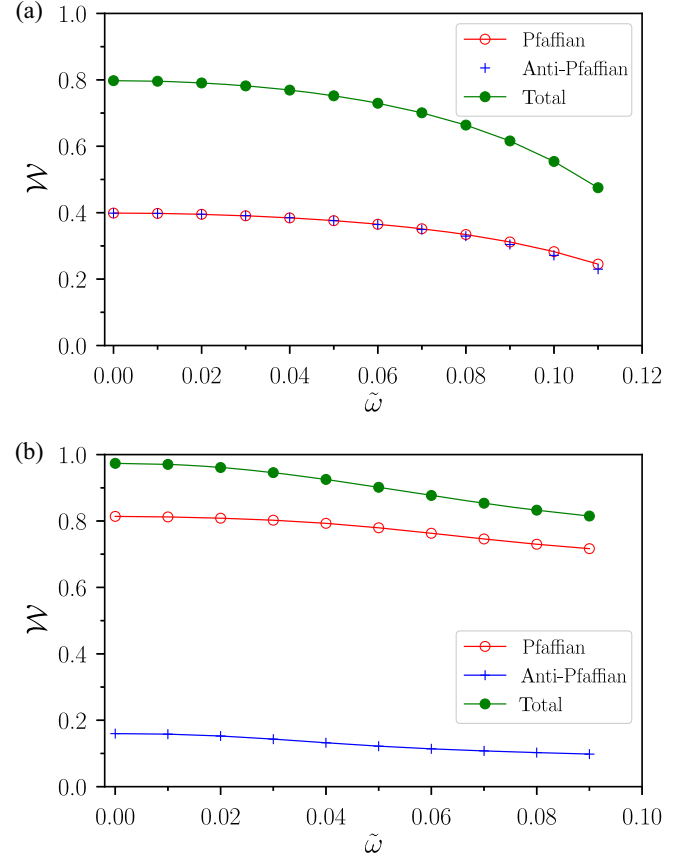


FIG. 11. (a) The weight of the Coulomb ground state in $|u_1\rangle$ on Pf and aPf model states as well as the total weight for $N_e = 14$ electrons. (b) The weight of the ground state on Pf and aPf model states as well as the total weight when a small amount of the three-body Pfaffian parent repulsion (projected to $|u_0\rangle$) is added to the Coulomb interaction (projected to $|u_1\rangle$). Here, the system size is $N_e = 8$ electrons. We show data up to the $\tilde{\omega}$ where the sixfold degeneracy is broken by level crossing with other states.

repulsion projected to $|u_0\rangle$, which can be obtained from numerical diagonalization. Notice $|u_0\rangle$ is the generalized lowest LL, or the standard ideal band; therefore, it has nontrivial quantum geometric background and the three-body interaction zero mode is a generalized Pfaffian wave function. The corresponding aPf state is obtained by applying the PH transformation to the Pf state. All weights have been averaged over the six ground states. The total weight of the Coulomb ground state on the Pf and aPf states is decent until the critical value of $\tilde{\omega} \sim 0.1$ where the sixfold ground-state degeneracy disappears. This is another strong evidence of the MR phase in addition to the sixfold ground-state degeneracy. However, we always find almost identical weights on the Pf and aPf states, even when the quantum geometric fluctuation is present. The absence of the expected quantum-geometry-induced selection of the Pf or aPf state in the system sizes studied by us is probably because the MR phase only exists for weak quantum geometric fluctuation, i.e., at relatively small $\tilde{\omega}$. In this

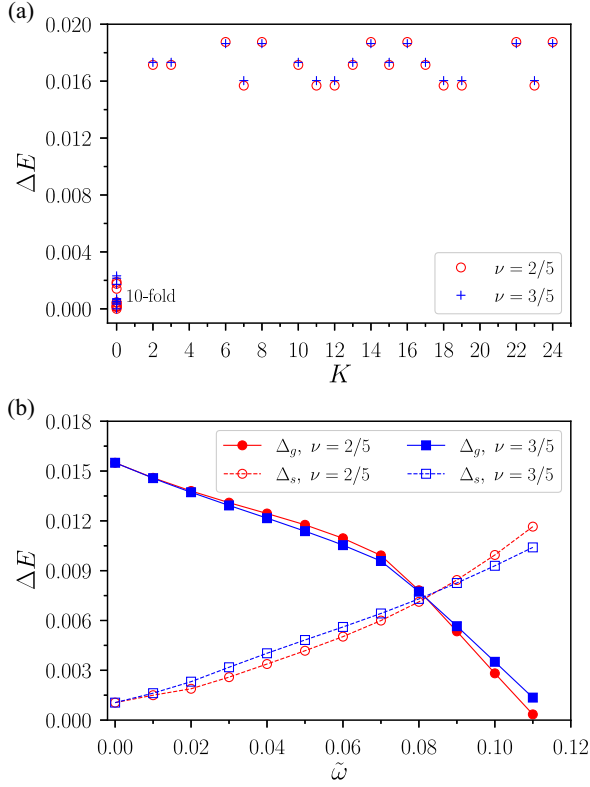


FIG. 12. (a) The energy spectrum of Coulomb interaction in a partially filled first generalized Landau level $|u_1\rangle$ at the filling fraction for Read-Rezayi state. Screening is added to the interaction to improve the tenfold degeneracy. We choose $\tilde{\omega} = 0.02$. (b) The neutral gap Δ_g and splitting Δ_s as a function of the Berry curvature modulation parameter $\tilde{\omega}$. The system size is $N_e = 10$ electrons at $\nu = 2/5$ and $N_e = 15$ electrons at $\nu = 3/5$.

case, ϵ_k and $\langle n_k \rangle_{\text{Pf}}$ in Eq. (F2) only weakly vary with k , leading to small Δ for small system sizes, such that Eq. (F5) is dominated by the off-diagonal elements and the ground state is still approximately a cat state of Pf and aPf. We hence expect that much larger system sizes beyond the limit of exact diagonalization are necessary to suppress $\Delta_{1,2}$ and enhance the effects of Δ .

Another mechanism to break the PH symmetry in a single LL is the three-body interaction (being included either in purpose of model study or originating from LL mixing) [110,111]. When this three-body interaction is the Pfaffian parent repulsion, we expect that the Pf state is favored and can be selected out even in small systems. We test this in the presence of quantum geometric fluctuation, as shown in Fig. 11(b). Once we add a small amount of Pfaffian parent repulsion in the Hamiltonian, the ground-state weight on the Pf state dominates over that on the aPf state.

4. The Read-Rezayi parafermion state

In addition to the MR family, other non-Abelian fractional quantum Hall states, such as the Read-Rezayi

parafermion state at $\nu = 3/5$ and its particle-hole conjugate at $\nu = 2/5$, have also been proposed to be realizable in the $n = 1$ LL [61,166,167]. One of the motivations for realizing the RR states is the topological quantum computation [168]: The MR states support Majorana-type excitations, which are not enough for universal quantum computation; instead, the RR states have Fibonacci anyons, which, in theory, is sufficient for universal topological quantum computation.

We briefly discuss the stability of the RR states against quantum geometric fluctuations. We project the Coulomb interaction to $|u_1\rangle$. Remarkably, we observe tenfold ground-state degeneracy at both $\nu = 2/5$ and $\nu = 3/5$ within a range of modulation parameter $\tilde{\omega}$ (Fig. 12). This degeneracy, and the momentum sectors in which the ground states appear, are consistent with the RR states. The RR states at $\nu = 2/5$ and $\nu = 3/5$ are both destroyed at roughly the same strength of quantum geometric fluctuation as the MR states. As the ground-state gap Δ_g and splitting Δ_s of both the $\nu = 2/5$ and $\nu = 3/5$ states show similar evolution with increasing $\tilde{\omega}$ [Fig. 12(b)], the PH symmetry breaking by quantum geometric fluctuation is quite weak in system sizes tractable by exact diagonalization, which is consistent with our observation for the MR states in Sec. F2.

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