

Symmetry-based efficient simulation of higher-order coherences in quantum many-body superradiance

Raphael Holzinger^{1,2,*}, Oriol Rubies-Bigorda^{3,1}, Susanne F. Yelin¹ and Helmut Ritsch²

¹Department of Physics, [Harvard University](#), Cambridge, Massachusetts 02138, USA

²Institute for Theoretical Physics, [University of Innsbruck](#), A-6020 Innsbruck, Austria

³Physics Department, [Massachusetts Institute of Technology](#), Cambridge, Massachusetts 02139, USA



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We propose an efficient method to numerically simulate the superradiant emission dynamics of large numbers of quantum emitters in ordered arrays in the presence of long-range dipole-dipole interactions mediated by the vacuum electromagnetic field. Using the spatial symmetries of the system, we rewrite the equations of motion in a collective spin basis and subsequently apply a higher-order cumulant expansion for the collective operators. By truncating the subradiant collective modes with a heavily suppressed decay rate and keeping only the effect from the radiating collective modes, we reduce the numerical complexity significantly. This allows to efficiently compute the dissipative dynamics of the observables of interest for linear and ring-shaped arrays of quantum emitters. In particular, we characterize the second-order intensity correlation function $g^{(2)}(\tau = 0)$, which is challenging to compute for extended systems with traditional cumulant expansion methods.

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I. INTRODUCTION

The accurate description of nonequilibrium dynamics such as collective radiative decay of interacting quantum emitters is one of the major challenges of open system quantum many-body theory. It is of central importance in many areas of physics, since an interacting array of emitters behaves quite differently from its individual constituents. When coupled via the electromagnetic field [1], excited emitters can trigger collective many-body quantum phenomena, resulting in short and intense bursts of light (generally termed superradiance or superfluorescence [2–6]) as well as directional emission of light [3,7], which has been observed in numerous experiments [8–16]. Understanding collective dissipation is crucial for enabling fundamental studies in quantum many-body physics [17,18], in the development of different light sources [19], or in quantum metrology with dense spin ensembles [20]. It is also vital to understand its implication on quantum error correction, where collective decay can directly impact fault tolerance in quantum systems involving many emitters [21–23]. Ordered atomic emitter arrays with subwavelength separations have emerged in recent years as a platform to study such dissipative many-body open quantum dynamics involving long-range interactions [10,24,25], illustrated in Fig. 1(a). Physical realizations include alkaline-earth atoms trapped in optical lattices [26–30] and in optical

tweezer arrays [31–35], which allow to trap thousands of atoms in ordered geometries [36,37]. The Hilbert space of an ensemble of N emitters grows exponentially with N . In the presence of collective dissipation or other decoherence processes, describing the dynamics of the system requires to solve for the $2^N \times 2^N$ density matrix over time. For smaller numbers of emitters, one can utilize a Monte Carlo wave function method, which converges to the exact expectation values based on the master equation when averaging over a sufficient number of trajectories [38]. Alternatively, for traditional cumulant expansion methods that include quantum correlations beyond a bare mean field treatment [39–45], the number of equations scales polynomially with the number of emitters, rendering the simulation of more than a few hundred emitters challenging. In particular, numerical simulations of higher-order correlations (such as the second-order intensity correlation function) are challenging but carry great interest for studying the quantum statistics and coherence properties of the emitted light [8,10]. Even though, in this work we will focus exclusively on the higher-order cumulant expansion method, we would like to note that phase space methods based on Wigner approximations are very successful in describing the dynamics of dissipative spin systems involving many emitters [46–50].

In this work, we present a different way to calculate the dynamics and the higher-order correlation functions in regular arrays by exploiting the symmetric nature of the collective super- and subradiant modes [51,52] that emerge at subwavelength separations between the emitters, as illustrated in Fig. 1(b). We demonstrate the effectiveness of this method by studying collective decay cascades in large ordered arrays of emitters, for which we show numerical simulations of systems containing up to 400 emitters in Fig. 1(c). Notably, this formalism can be leveraged to study the collective dissipative

*Contact author: raphael_holzinger@fas.harvard.edu

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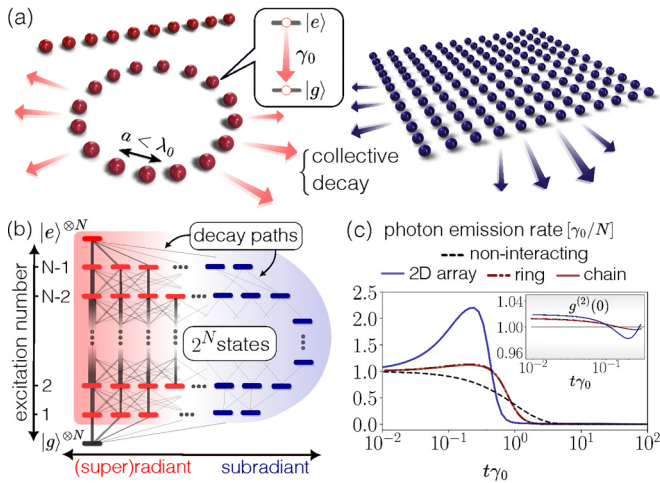


FIG. 1. (a) Ordered arrays of N collectively decaying two-level quantum emitters with subwavelength separation $a < \lambda_0 = 2\pi c/\omega_0$, where λ_0 is the transition wavelength. (b) Illustration of the decay dynamics for an initially fully excited ensemble with eigenstates organized according to their excitation number (up to down) and decay rates (left to right). Only a small number of eigenstates are significantly occupied during the time evolution (in red), while a majority of subradiant states (in blue) have a small occupation probability. In this work, we show how to simulate $\sim (a/\lambda_0)N$ (1D arrays) and $\sim (a/\lambda_0)^2 N$ (2D arrays) collective modes instead of N individual emitters, thus reducing the numerical complexity significantly. (c) The photon emission rate $p_{\text{out}}(t)$ in Eq. (8) and second-order correlation function $g^{(2)}(\tau = 0)$ in Eq. (9) at early times, computed by truncating (i.e., neglecting) the subradiant collective modes. We show the results for a chain and a ring of 100 two-level emitters, as well as for a square array of 400 atoms. We consider circularly polarized emitters with lattice spacing $a/\lambda_0 = 0.15$.

dynamics of quantum emitters (irrespective of their nature) coupled to a common radiation field. Apart from cold atoms, the formalism can also describe molecules [53], collective phenomena in waveguide quantum electrodynamics [54–57], magnons in solid-state environments [58], and quantum dots in self-assembled superlattices [59].

This work is organized as follows: First, we give a compact introduction to the open quantum system model used to describe the many-body dynamics involving two-level

systems that radiatively decay. We introduce the collective mode picture for interacting quantum emitters in ordered arrays and the higher-order cumulant expansion for the Heisenberg equations of motion. With these ingredients, we motivate and propose the collective mode truncation, which is the main result of this work. The full protocol of the method is illustrated in Fig. 2. Next, we benchmark our method with the exact numerics based on a master equation time evolution with a small number of emitters placed in a ring geometry. We then show the scaling of the total emission rate for large ensembles as a function of N . We make use of our truncation method to simulate system sizes that are computationally too costly for traditional cumulant expansion methods. Finally, we summarize the results and give an outlook to extend this method to a broader range of scenarios.

II. COLLECTIVE DECAY IN QUANTUM EMITTER ARRAYS

First, we introduce the theoretical framework used to describe the open quantum system dynamics of ordered array emitters interacting through the common free-space electromagnetic environment. We consider N atomic two-level emitters trapped in a periodic array with subwavelength nearest-neighbor separation a . Each emitter features a resonance frequency $\omega_0 = 2\pi/\lambda_0$ and spontaneous decay rate $\gamma_0 = k_0^3 |\mathbf{d}|^2 / (3\pi\epsilon_0\hbar)$, where $\mathbf{d}$ is the transition dipole moment and ϵ_0 the vacuum permittivity. For simplicity, we assume an angular momentum $J = 0$ ($|g\rangle$) to $J' = 1$ ($|e\rangle$) transition with a unique ground state ($|g\rangle$). After tracing out the vacuum electromagnetic field modes under the Born-Markov approximation [60,61], the dynamics of the system's density matrix $\hat{\rho} = |\psi\rangle\langle\psi|$ is governed by a master equation of the form ($\hbar = 1$)

$$\frac{d}{dt}\hat{\rho} = -i[\hat{H}, \hat{\rho}] + \sum_{n,m} \frac{\Gamma_{nm}}{2} (2\hat{\sigma}_n \hat{\rho} \hat{\sigma}_m^\dagger - \{\hat{\sigma}_n^\dagger \hat{\sigma}_m, \hat{\rho}\}), \quad (1)$$

where $\hat{H} = \sum_n \omega_0 \hat{\sigma}_n^{ee} + \sum_{n \neq m} J_{nm} \hat{\sigma}_n^\dagger \hat{\sigma}_m$ describes coherent dynamics between emitters. The operators $\hat{\sigma}_n = |g_n\rangle\langle e_n|$ act as the transition operators for emitters at positions $\{\mathbf{r}_n\}$ and $\hat{\sigma}_n^{ee} \equiv \hat{\sigma}_n^\dagger \hat{\sigma}_n$. For the sake of simplicity, we assume identical transition frequencies ω_0 for all emitters throughout this work. The coherent J_{nm} and dissipative Γ_{nm} couplings between

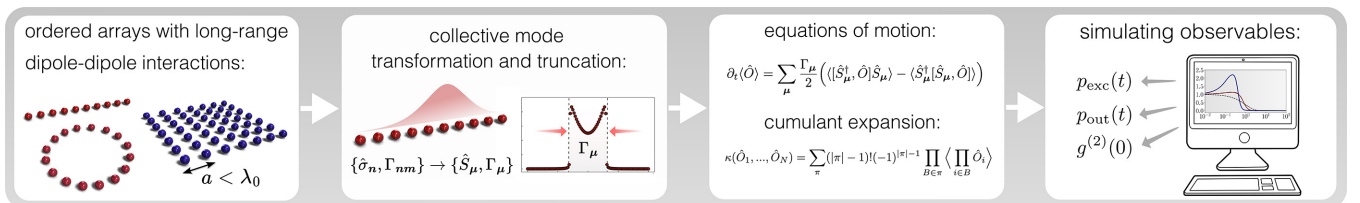


FIG. 2. The protocol for simulating dissipative quantum many-body systems using the collective mode truncation presented in this paper. Starting from ordered arrays of quantum emitters with subwavelength separations a , shown in Sec. II, we transform the system into the collective mode basis and apply the collective mode truncation in Sec. II. Next, we write down the equations of motion for the expectation values in the collective mode basis. For that, we apply a cumulant expansion to obtain a closed set of equations in Sec. III. Finally, the time evolution of the expectation values of the observables of interest are numerically simulated. In our case, these are the excited state population $p_{\text{ext}}(t)$ in Eq. (6), the total photon emission rate $p_{\text{out}}(t)$ in Eq. (8), and the second-order correlation function $g^{(2)}(0)$ in Eq. (9), presented in Sec. III.

emitters n and m are related to the Green's tensor in free-space $\mathbf{G}$ (detailed in the Supplemental Material [62]) via [52]

$$\frac{3\pi\gamma_0}{\omega_0} \mathbf{d}^\dagger \cdot \mathbf{G}(\mathbf{r}_{nm}, \omega_0) \cdot \mathbf{d} = -J_{nm} + \frac{i\Gamma_{nm}}{2}, \quad (2)$$

where the transition dipole moment $\mathbf{d}$ is assumed to be identical for all emitters and $\mathbf{r}_{nm} = \mathbf{r}_n - \mathbf{r}_m$ is the connecting vector between emitters n and m . At subwavelength separations, $a < \lambda_0 = 2\pi c/\omega_0$, the virtual photon exchange between emitters induces long-range dipole-dipole interactions Γ_{nm} . Then, the decay becomes collective and is governed by the jump operators obtained by diagonalizing the decay matrix $\mathbf{\Gamma}$ with elements Γ_{nm} . During a decay cascade, the state of the system can generally pass through a large fraction of the full Hilbert space (which has dimension 2^N), as illustrated in Fig. 1(b). Note that for an independent atomic ensemble only individual (diagonal) decay remains, namely, $\Gamma_{nm} = \gamma_0 \delta_{nm}$.

A. Collective quantum jumps

1. Ensemble with translational symmetry

Let us first consider ensembles of emitters that exhibit translational symmetry, such as rings with perpendicular or in-plane circular polarization and infinite ordered arrays. Then, the N eigenstates of the dissipative and coherent interactions given by the matrices $\mathbf{\Gamma}$ and $\mathbf{J}$ (with elements Γ_{nm} and J_{nm} , respectively) are spin wave or Bloch operators of the form

$$\hat{S}_\mu = \frac{1}{\sqrt{N}} \sum_{n=1}^N \exp\left(i \frac{2\pi}{N_{1D}a} \boldsymbol{\mu} \cdot \mathbf{r}_n\right) \hat{\sigma}_n. \quad (3)$$

The spin waves are labeled by their quasimomentum. For two-dimensional geometries such as squared arrays with spacing a , the quasimomentum in the xy -plane reads $\mathbf{k} = 2\pi \boldsymbol{\mu}/N_{1D}a$, where $\mu_i = 0, \pm 1, \pm 2, \dots, [\pm(N_{1D}-1)/2]$ [52]. Here, N_{1D} denotes the number of emitters along one direction, such that the total number of emitters is $N = N_{1D}^2$ for two-dimensional and $N = N_{1D}$ for one-dimensional arrays. For one-dimensional arrays, we simply set $\mu_y = 0$. All spin waves are fully delocalized over all sites, and their associated phase profile is dictated by the quasimomentum $\boldsymbol{\mu}$. The collective decay rates and energy shifts associated with each spin wave correspond to the eigenvalues of $\mathbf{\Gamma}$ and $\mathbf{J}$:

$$\Gamma_\mu = \frac{1}{N} \sum_{n,m} \exp\left(i \frac{2\pi}{N_{1D}a} \boldsymbol{\mu} \cdot (\mathbf{r}_n - \mathbf{r}_m)\right) \Gamma_{nm}, \quad (4a)$$

$$J_\mu = \frac{1}{N} \sum_{n \neq m} \exp\left(i \frac{2\pi}{N_{1D}a} \boldsymbol{\mu} \cdot (\mathbf{r}_n - \mathbf{r}_m)\right) J_{nm}, \quad (4b)$$

with $\sum_\mu \Gamma_\mu = N\gamma_0$. From Eq. (4), it can be seen that the eigenvalues are symmetric under the inversion $\boldsymbol{\mu} \leftrightarrow -\boldsymbol{\mu}$, meanwhile the collective mode with $\boldsymbol{\mu} = 0$ is always unique.

Notably, the master equation (1) becomes diagonal in the collective spin basis given by the operators $\hat{S}_\mu$:

$$\frac{d}{dt} \hat{\rho} = -i[\hat{H}, \hat{\rho}] + \sum_\mu \frac{\Gamma_\mu}{2} (2\hat{S}_\mu \hat{\rho} \hat{S}_\mu^\dagger - \{\hat{S}_\mu^\dagger \hat{S}_\mu, \hat{\rho}\}), \quad (5)$$

where the coherent interactions are given by the Hamiltonian $\hat{H} = \sum_\mu (\omega_0 + J_\mu) \hat{S}_\mu^\dagger \hat{S}_\mu$.

2. General ensembles

While the collective spin states $\hat{S}_\mu$ given in Eq. (3) are the eigenstates of the coherent and dissipative interactions $\mathbf{\Gamma}$ and $\mathbf{J}$ for translationally symmetric ensembles of emitters, this is no longer the case once this symmetry is broken [19,52,63]. For finite arrays, Eq. (3) is only approximately the eigenstates of the system, and the master equation (5) consequently acquires couplings between spin waves with different quasimomenta. As shown in the Supplemental Material [62] by comparing the decay rates Γ_ν given in Eq. (4a) with the decay rates obtained by diagonalizing the matrix $\mathbf{\Gamma}$ exactly, excellent agreement is found specially for large arrays. As a result, the master equation (5) derived for the translationally symmetric case can be used to simulate and study the dynamics of finite-sized arrays with high accuracy, as discussed in detail in Sec. III.

3. Hierarchy of timescales

In subwavelength configurations, the set of collective decay rates $\{\Gamma_\mu\}$ exhibits a high variability and range from superradiant ($\Gamma_\mu > \gamma_0$) to strongly subradiant ($\Gamma_\mu \ll \gamma_0$). The dissipative dynamics is governed by the set of collective jump operators and decay rates, $\{\hat{S}_\mu, \Gamma_\mu\}$, and results in a complex network of decay paths from the fully excited state, as sketched in Fig. 3(a). Notably, a majority of these paths involve subradiant modes at subwavelength separations. These paths occur with small probabilities proportional to their corresponding subradiant decay rates, and can therefore be neglected if one is only interested in the radiant dynamics. Intuitively, these occur due to a separation of timescales, whereby the radiating collective operators act at a much faster rate than their subradiant counterparts. For one-dimensional arrays, Fig. 3(b) shows that all radiant modes are confined within the lightcone, $|\mu_{x(y)}| \lesssim aN_{1D}$ [52,63], leading to a fraction of radiant modes equal to $2a/\lambda_0$. In Fig. 3(c), we numerically show that the number of modes that satisfy $\Gamma_\mu > \gamma_0/N$ approaches the above limit as N increases. In the case of two-dimensional square arrays, the light cone is a circle of radius aN_{1D} , and the radiant mode fraction for large arrays tends to $\pi(a/\lambda_0)^2$, as shown in Fig. 3(c) as well. As we further discuss in Secs. III and IV, neglecting the collective subradiant operators allows to drastically reduce the number of equations that needs to be solved to simulate the superradiant decay dynamics.

B. Collective photon emission

In this section, we focus on characterizing the collective photon emission properties of the atomic emitter array. We consider the excited-state population over time:

$$p_{\text{exc}}(t) = \sum_{n=1}^N \langle \hat{\sigma}_n^{ee} \rangle = N \langle \hat{S}_0^{ee} \rangle, \quad (6)$$

where $\hat{S}_0^{ee}$ is the collective excited state operator of the whole array with quasimomentum $\mathbf{0} = (0, 0)$ and reads

$$\hat{S}_0^{ee} = \frac{1}{N} \sum_{n=1}^N \exp\left(-i \frac{2\pi}{N_{1D}a} \boldsymbol{\mu} \cdot \mathbf{r}_n\right) \hat{\sigma}_n^{ee}. \quad (7)$$

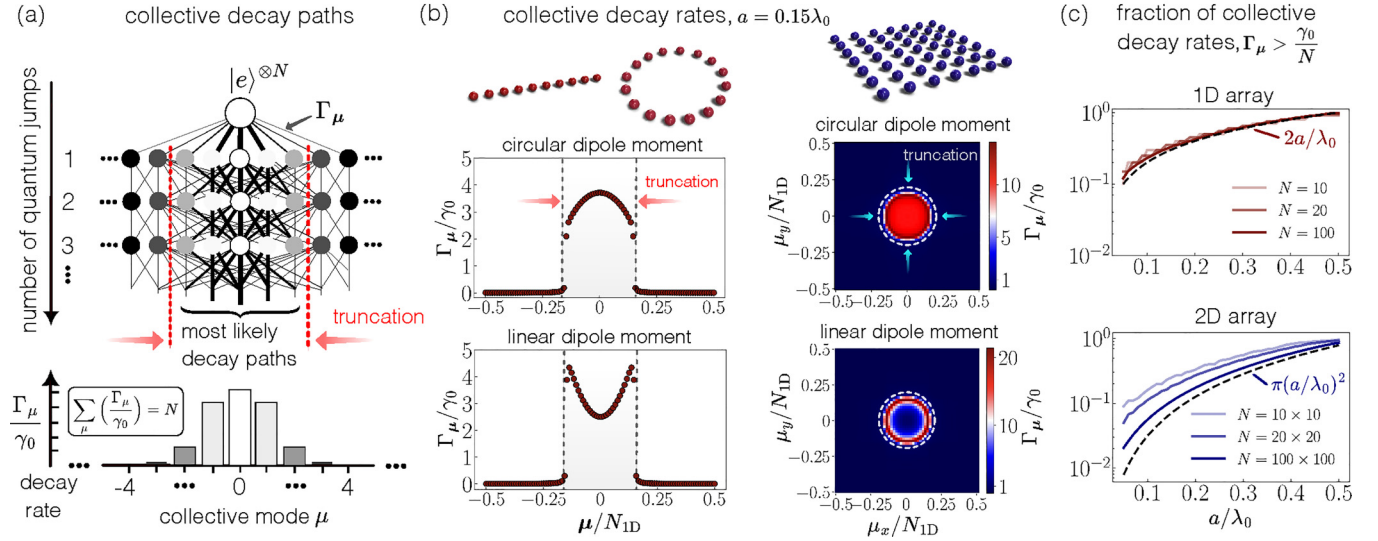


FIG. 3. (a) Repeated actions of collective jump operators $\hat{S}_\mu$ (circles) lead to collective decay paths (connecting lines). The most likely decay paths involve (super)radiant jumps with rates $\Gamma_\mu \gtrsim \gamma_0$, while paths generated by subradiant ($\Gamma_\mu \ll \gamma_0$) jump operators are unlikely. (b) Collective decay rates for one- and two-dimensional arrays with the dashed lines indicating the border between radiant and subradiant (dark) modes, and $a = 0.15\lambda_0$. The number of radiant decay channels is strongly reduced at subwavelength separation, comprising a tiny fraction of the total number of N decay channels. In the limit of large N , this fraction is given by $2a/\lambda_0$ for one-dimensional arrays and $\pi(a/\lambda_0)^2$ for two-dimensional arrays, as shown in panel (c). In the collective mode picture, we can truncate the system at the dashed lines. (c) Fraction of collective modes with decay rates $\Gamma_\mu > \gamma_0/N$ as a function of the array spacing a , obtained by numerical diagonalization of the symmetric $N \times N$ decay matrix with elements Γ_{mn} . For large enough emitter numbers, the fraction of radiant modes follows a linear scaling for one-dimensional arrays and a quadratic scaling for square arrays, leading to a substantial reduction in radiant modes for $a < \lambda_0$.

Next, the total photon emission rate of the array can be written in the individual and collective basis as

$$p_{\text{out}}(t) = \gamma_0 \sum_{n=1}^N \langle \hat{\sigma}_n^{ee} \rangle + \sum_{n \neq m}^N \Gamma_{nm} \langle \hat{\sigma}_n^\dagger \hat{\sigma}_m \rangle$$

$$= \sum_{\mu} \Gamma_{\mu} \langle \hat{S}_{\mu}^\dagger \hat{S}_{\mu} \rangle, \quad (8)$$

where we have used Eq. (3). In noninteracting emitters, only the first term in the first line of Eq. (8) is nonzero and exponential decay follows. At subwavelength separations, the light-induced dipole-dipole interactions can result in nonzero correlations $\langle \hat{\sigma}_n^\dagger \hat{\sigma}_m \rangle$ that extend over the whole array. When these correlations contribute positively, i.e., $\Gamma_{nm} \langle \hat{\sigma}_n^\dagger \hat{\sigma}_m \rangle > 0$, the total emission rate can attain values $p_{\text{out}}(t) > \gamma_0 N$ at $t > 0$, thereby giving rise to superradiant decay [5].

Second-order correlations between the atomic emitters allow to characterize two-photon emission statistics of the array during the dissipative dynamics. This is captured by the normalized second-order correlation function with zero time delay $g^{(2)}(\tau = 0)$ [3,42] and is expressed in both the individual and collective basis as

$$g^{(2)}(\tau = 0) = \frac{\sum_{k,l,m,n} \Gamma_{kn} \Gamma_{lm} \langle \hat{\sigma}_k^\dagger \hat{\sigma}_l^\dagger \hat{\sigma}_m \hat{\sigma}_n \rangle}{(\sum_{n,m} \Gamma_{nm} \langle \hat{\sigma}_m^\dagger \hat{\sigma}_n \rangle)^2}$$

$$= \frac{\sum_{\mu,\nu} \Gamma_{\mu} \Gamma_{\nu} \langle \hat{S}_{\mu}^\dagger \hat{S}_{\nu}^\dagger \hat{S}_{\nu} \hat{S}_{\mu} \rangle}{(\sum_{\mu} \Gamma_{\mu} \langle \hat{S}_{\mu}^\dagger \hat{S}_{\mu} \rangle)^2}. \quad (9)$$

Equation (9) is equivalent to the second-order intensity correlation of the total emitted light, detected in the far field, and as such quantifies the emitted photon statistics of the array

as a whole. The intensity correlations at any point in space can be calculated via the electric field operators [2,7] and are discussed in the Supplemental Material [62].

III. EQUATIONS OF MOTION

In this section, a closed set of equations of motion for expectation values of collective spin operators is derived. We analyze dissipative dynamics in the absence of external driving, with the system initially in the fully inverted product state [40,42,43],

$$|\psi_0\rangle = \left(\prod_{n=1}^N \hat{\sigma}_n^\dagger \right) |g\rangle^{\otimes N} = |e\rangle^{\otimes N}. \quad (10)$$

In terms of density matrices, this product state can be written as $\hat{\rho}(0) = \otimes_{n=1}^N \hat{\rho}_n$, with $\hat{\rho}_n = |e_n\rangle\langle e_n|$. To numerically simulate the time evolution of operator expectation values $\langle \hat{O} \rangle$, one can use the master equation in Eq. (1) and evaluate $\frac{d}{dt} \langle \hat{O} \rangle = \text{Tr}(\hat{O} \frac{d}{dt} \hat{\rho})$. Equivalently, the Heisenberg equations of motion can be considered to express the time evolution for expectation values as [40,41,43]

$$\frac{d}{dt} \langle \hat{O} \rangle = i \langle [\hat{H}, \hat{O}] \rangle$$

$$- \sum_{\mu} \frac{\Gamma_{\mu}}{2} (\langle [\hat{S}_{\mu}^\dagger, \hat{O}] \hat{S}_{\mu} \rangle - \langle \hat{S}_{\mu}^\dagger [\hat{S}_{\mu}, \hat{O}] \rangle), \quad (11)$$

where we neglected quantum noise terms [40], as we are interested in operator averages. The operator $\hat{O}$ is generally a product involving r operators $\hat{O}_1 \hat{O}_2 \dots \hat{O}_r$. In most instances, the time derivative of $\langle \hat{O} \rangle$ will depend on expectation values of products of up to $r+1$ operators, on the right

hand side of Eq. (11). This will lead to a growing system of equations whose size scales polynomially with N . In order to obtain a closed set of equations, certain higher-order correlations between operators have to be neglected, which is achieved systematically through the cumulant expansion [39,43]. The Supplemental Material [62] provides a detailed discussion of the cumulant expansion and the resulting equations of motion, while we focus here on a compact description. Before proceeding, we would like to highlight that collective spin operators with different quasimomenta $\mu \neq \nu$ do not commute in general, $[\hat{S}_\mu^\dagger, \hat{S}_\nu] \neq 0$. This is in stark contrast to the local spin basis, for which operators associated with different spins always commute, e.g., $[\hat{\sigma}_n^\dagger, \hat{\sigma}_m] = (2\hat{\sigma}_n^{ee} - \hat{1})\delta_{nm}$ commutes for $n \neq m$. This fact leads to couplings among modes of different quasimomenta in the equations of motion and induces complex decay paths en route to the ground state $|g\rangle^{\otimes N}$ as is illustrated in Figs. 1(b) and 3(a). The set of commutation relations read

$$[\hat{S}_\mu^\dagger, \hat{S}_\nu] = 2\hat{S}_{\mu-\nu}^{ee} - \hat{1}\delta_{\mu\nu}, \quad (12a)$$

$$[\hat{S}_\mu^\dagger, \hat{S}_\nu^{ee}] = -\frac{1}{N}\hat{S}_{\mu+\nu}^\dagger, \quad (12b)$$

$$[\hat{S}_\mu, \hat{S}_\nu^{ee}] = \frac{1}{N}\hat{S}_{\mu-\nu}, \quad (12c)$$

where $\hat{1}$ is the identity operator. In Eq. (12a), we have used the orthogonality of the sum for periodic boundary conditions

$$\sum_{n=1}^N \exp\left(i\frac{2\pi}{N_{\text{ID}}} \mu \cdot \mathbf{r}_n\right) = N\delta_{\mu\mathbf{0}}, \quad (13)$$

with $N = N_{\text{2D}}^2$ for two-dimensional and $N = N_{\text{1D}}$ for one-dimensional arrays.

In order to obtain the equations of motion for the expectation values of the collective spin operators, we start from the set of operators $\{\hat{S}_\mu^\dagger, \hat{S}_\mu, \hat{S}_\mu^{ee}\}$ and apply Eq. (11) to derive their dynamics in terms of higher-order operators. In turn, we repeat this procedure for the higher-order operators too. One important consequence of the phase symmetry of the initial product state in Eq. (10) is the fact that expectation values such as $\langle \hat{S}_\mu \rangle$ and $\langle \hat{S}_\mu^{ee} \rangle$ are zero throughout the whole time evolution [40,42]. This also generates a constraint on the sum of quasimomenta for the expectation values. Additionally, the initial state in Eq. (10) also implies $\langle \hat{S}_\mu^\dagger \hat{S}_\nu \rangle = \delta_{\mu\nu}$, where we have used Eq. (13). Indeed, one can numerically check that only expectation values of the form $\langle \hat{S}_\mu^\dagger \hat{S}_\mu \rangle$ are nonzero throughout the time dynamics for ring geometries [for which Eq. (13) is exact]. This results in a significant reduction of the number of expectation values and the equations of motion that need to be simulated over time. Starting from the Heisenberg equations of motion in Eq. (11), we finally find the equations of motion for the following set of expectation values:

$$\{\langle \hat{S}_\mu^{ee} \rangle, \langle \hat{S}_\mu^\dagger \hat{S}_\mu \rangle, \langle \hat{S}_\mu^{ee} \hat{S}_{-\mu}^{ee} \rangle, \langle \hat{S}_\mu^\dagger \hat{S}_\nu \hat{S}_{\nu-\mu}^{ee} \rangle, \langle \hat{S}_\mu^\dagger \hat{S}_\nu^\dagger \hat{S}_\xi \hat{S}_{\nu-\mu-\xi} \rangle\}, \quad (14)$$

which in turn couple to expectation values such as $\langle \hat{S}_\mu^\dagger \hat{S}_\nu^\dagger \hat{S}_\xi \hat{S}_\sigma \hat{S}_{\xi+\sigma-\mu-\nu}^{ee} \rangle$ and $\langle \hat{S}_\mu^\dagger \hat{S}_\nu \hat{S}_\xi^{ee} \hat{S}_{\nu-\mu-\xi}^{ee} \rangle$. These terms are expanded in terms of lower-order expectation values using the

cumulant expansion method. The explicit expressions for the equations of motions as well as the cumulant expansions are provided in the Supplemental Material [62].

The values for the expectation values in Eq. (14) can be evaluated at $t = 0$ and we obtain

$$\langle \hat{S}_\mu^{ee} \rangle = \delta_{\mu\mathbf{0}}, \quad (15a)$$

$$\langle \hat{S}_\mu^\dagger \hat{S}_\mu \rangle = 1 \quad \text{for all } \mu, \quad (15b)$$

$$\langle \hat{S}_\mu^{ee} \hat{S}_{-\mu}^{ee} \rangle = \delta_{\mu\mathbf{0}}, \quad (15c)$$

$$\langle \hat{S}_\mu^\dagger \hat{S}_\nu \hat{S}_{\nu-\mu}^{ee} \rangle = \delta_{\mu\nu}, \quad (15d)$$

$$\langle \hat{S}_\mu^\dagger \hat{S}_\nu^\dagger \hat{S}_\xi \hat{S}_{\mu+\nu-\xi} \rangle = -2/N + \delta_{\mu\xi} + \delta_{\nu\xi}. \quad (15e)$$

Note that the expectation values in Eq. (15e) necessary to evaluate the second-order correlation $g^{(2)}(0)$ involve products of four operators, but only contain three momentum indices μ, ν, ξ . This results in $\mathcal{O}(N^3)$ terms. Conversely, the individual spin basis requires to solve for $\mathcal{O}(N^4)$ terms of the form $\langle \hat{\sigma}_n^\dagger \hat{\sigma}_m^\dagger \hat{\sigma}_k \hat{\sigma}_l \rangle$, making the evaluation of the $g^{(2)}(0)$ function numerically challenging for large N . We would like to note that generally the cumulant expansion is prone to errors particularly for studying subradiant dynamics in the long time limit, which is not considered in this work [40]. Furthermore, systematic comparisons with exact numerics are generally restricted to small emitter numbers, except in the Dicke limit [41] (see Refs. [64,65] for a compact analytic solution).

Collective mode truncation

The major step in order to reduce the computational cost in solving the time evolution for the set of expectation values in Eq. (14) consists in reducing the number of collective modes considered. We illustrate this for one-dimensional arrays ($N = N_{\text{1D}}$) with quasimomentum μ , although these concepts can be readily extended to two-dimensional arrays with quasimomenta $\mu = (\mu_x, \mu_y)$. In the limit of large N , as shown in Figs. 3(a) and 3(b), there are $\tilde{N}$ (super)radiant modes that satisfy $|\mu| \leq aN$. If one is interested in the radiating part of the dissipative time dynamics, one can keep these radiating modes while discarding the subradiant ones. For finite N , we numerically compute the collective decay rates in Eq. (4) and determine the number of modes $\tilde{N}$ satisfying $\Gamma_\mu > \gamma_0/N$. This leads to a reduction of the number of collective modes to $\mu = 0, \pm 1, \pm 2, \dots, \lceil \pm(\tilde{N} - 1)/2 \rceil$. The scaling of the fraction of modes satisfying this condition as a function of the array spacing a is shown in Fig. 3(c) for both one- and two-dimensional arrays.

As we show in Fig. 3(c), the number of radiant modes scales as $2a/\lambda_0 N$ and $\pi(a/\lambda_0)^2 N$ for one- and two-dimensional arrays, respectively, which leads to a substantial reduction of the relevant number of modes for arrays with subwavelength spacings between nearest neighbors. We list a comparison of the number of equations necessary to solve the time dynamics in Eqs. (1) and (11) between different numerical methods in Table I in terms. Moreover, the second-order correlation function $g^{(2)}(0)$ in the individual spin basis involves expectation values of products of four operators and thus $\mathcal{O}(N^4)$ equations need to be solved to obtain its time evolution. However, in the collective spin basis this reduces to $\mathcal{O}(N^3)$, as the necessary expectation values $\langle \hat{S}_\mu^\dagger \hat{S}_\nu^\dagger \hat{S}_\xi \hat{S}_{\mu+\nu-\xi} \rangle$

TABLE I. The numerical complexity, in terms of the number of equations for different methods that are required to solve the dissipative quantum dynamics in order to evaluate $g^{(2)}(\tau = 0)$ in Eq. (9). The cumulant expansion in the individual spin basis scales polynomially with the number of emitters [39]. For the collective mode truncation meanwhile, the numerical complexity is reduced by orders of magnitude for subwavelength spacings $a < \lambda_0$ between nearest neighbors.

Method	Number of equations
Cumulant expansion 4th order	$\sim N^4$
Collective mode truncation in 1D	$\sim N^3 \cdot (a/\lambda_0)^3$
Collective mode truncation in 2D	$\sim N^3 \cdot (a/\lambda_0)^6$

only involve three quasimomenta. This allows to evaluate the time evolution for the $g^{(2)}(0)$ function for large numbers of emitters, as will be shown in the next section.

IV. RESULTS AND BENCHMARKING

In this section, we present the application of the cumulant expansion in the collective mode basis and the collective mode truncation for specific geometries. We first consider emitters in a ring geometry, for which the collective mode transformation in Eq. (3) is exact independently of N . The ring geometry can therefore be used to benchmark the validity of the method with exact numerics based on the time evolution of the density matrix in Eq. (1) [66,67]. In the second part of this section, we apply the collective mode truncation to simulate one- and two-dimensional arrays with large numbers of emitters and show a comparison with the traditional cumulant expansion for individual spin operators [40,42,43]. In the Supplemental Material [62], we discuss the extensions to three-dimensional cubic arrays and arrays coupled to one-dimensional waveguides.

The numerical simulations are performed within the Julia programming language [68] and the time dynamics are computed using the DifferentialEquations.jl [69] package.

A. Comparison with exact numerics

Here, we consider emitters in a ring geometry and provide a comparison with exact numerics based on the master equation. In Fig. 4(a), we show the time evolution for the observables introduced in Sec. II for 12 emitters with circular in-plane polarization, $\mathbf{d} = (1, i, 0)^T / \sqrt{2}$, and for various spacings a . The collective mode basis shows excellent agreement with the master equation for the photon emission rate $p_{\text{out}}(t)$ as well as the excited state population $p_{\text{ex}}(t)$. In both cases, noninteracting emitters with exponential decay $\exp(-t\gamma_0)$ are shown as well. We plot the relative error $\Delta p_{\text{out}}^{\text{max}}$ (between the collective mode basis and the master equation) as a function of N in Fig. 4(b) for the photon emission rate at peak emission, and observe a reduction of the relative error with increasing N .

The second-order correlation function $g^{(2)}(0)$ is a key and very sensitive parameter characterizing photon statistics from nonclassical antibunching ($g^{(2)}(0) < 0$) via classical coherent states ($g^{(2)}(0) = 1$) to thermal emission ($g^{(2)}(0) \approx 2$) [3]. In

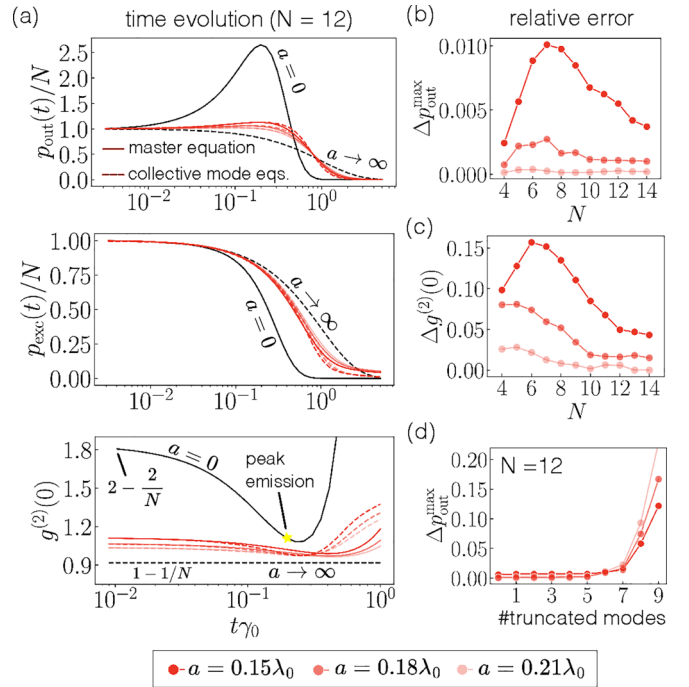


FIG. 4. Benchmarking the time evolution of the expectation values in Eq. (14) (dashed lines) against exact numerics using the master equation in Eq. (1) (continuous lines) for emitters in a ring geometry. (a) The total photon emission $p_{\text{out}}(t)$ [Eq. (8)], the excited state population $p_{\text{ex}}(t)$ [Eq. (6)], and the second-order correlation function $g^{(2)}(\tau = 0)$ [Eq. (9)] as a function of time. The black continuous and dashed curves show the limiting cases $a = 0$ in the absence of coherent couplings J_{nm} (Dicke superradiance) and noninteracting emitters with $a \rightarrow \infty$, respectively. (b) Relative error $\Delta p_{\text{out}}^{\text{max}}$ of the peak photon emission rate as a function of the emitter number N . (c) Relative error $\Delta g^{(2)}(0)$ of the second-order correlation function at peak photon emission. In panels (a), (b), and (c), no modes are truncated when solving the cumulant expansion in the collective mode basis. (d) Relative error of the peak photon emission as a function of the number of truncated modes. For $N = 12$ emitters, up to seven modes ($\sim 60\%$) can be neglected before the relative error increases significantly. We consider $\mathbf{d} = (1, i, 0)^T / \sqrt{2}$ in all plots.

Fig. 4(a), we see good quantitative agreement between master equation and the collective mode equations at early times, $t \lesssim 1/\gamma_0$, where it decreases to below one around the peak intensity of photon emission. After the peak in emission, $g^{(2)}$ increases again when the remaining excitation is low and is predominantly stored in subradiant states, which have a small photon emission rate. This creates a small denominator in Eq. (9) that results in an increase of $g^{(2)}(0)$. As expected, this feature cannot be very well represented in our truncated model and errors increase significantly at late times. For comparison, we also show the two special cases of noninteracting independent emitters ($a \rightarrow \infty$) and Dicke collective superradiance ($a \rightarrow 0$ in the absence of coherent dipole-dipole interactions, $J_{nm} = 0$). The noninteracting case leads to an exponential decay of both the photon emission and excited state population and to a constant $g^{(2)}(0) = 1 - 1/N$. The Dicke limit, on the other hand, shows the highest possible peak photon emission and super-exponential decay of the excited state population

in Fig. 4(a). The second-order correlation can be analytically found at the initial time ($t = 0$), $g^{(2)}(0) = 2 - 2/N$. As occurred for the case of ring geometries, the $g^{(2)}(0)$ function in the Dicke limit decrease at initial times until a minimum is reached. This feature is discussed below in more detail.

The large deviation of the $g^{(2)}(0)$ at late times $t \gtrsim \gamma_0/10$ shown by the cumulant expansion in the collective mode basis as opposed to the master equation require further investigation. A first analysis indicates that these growing errors are connected to the relevance of very subradiant states in the dynamics. One possible improvement consists on the extension of the equations in Eq. (14) to include the time evolution of $\langle \hat{S}_0^\dagger \hat{S}_0^\dagger \hat{S}_0 \hat{S}_0 \rangle$. We show in the Supplemental Material [62] that the cumulant expansion of this expectation is smaller than the exact value. This discrepancy could explain in turn the underestimation observed in Fig. 4(a). The relative errors of the peak photon emission $\Delta p_{\text{out}}^{\text{max}}$ and the second-order correlation function $\Delta g^{(2)}(0)$ at peak photon emission both tend to decrease toward as a function of the emitter number N . Note that in Figs. 4(a)–4(c) no modes are truncated, thus any error stems from the cumulant expansion performed in Eq. (14).

Finally, we show $\Delta p_{\text{out}}^{\text{max}}$ as a function of the number of truncated collective modes for $N = 12$ emitters in Fig. 4(d). Notably, the relative error remains close to zero if at most 7 out of the 12 total modes are truncated. In other words, the magnitude of the superradiant peak only diverges from the exact value when eight or more modes are truncated. This constitutes a 50% reduction of the number of modes that need to be considered.

B. Second-order correlation function $g^{(2)}(\tau = 0)$

The second-order correlation with zero time delay $g^{(2)}(\tau = 0)$ in Eq. (9) gives a measure of emitting two photons simultaneously from an ensemble of emitters. In other words, it is a measure of how much photons bunch, i.e., how much more likely is it, if we measure one photon, to have a second one arrive at the same time. It has been shown previously [3] that the value of $g^{(2)}(0)$ at $t = 0$ predicts whether the system will exhibit a superradiant outburst of radiation or not. The minimal condition for this to happen is $g^{(2)}(0)|_{t=0} > 1$ [3,42,70]. For subwavelength rings, Fig. 4(a) shows that the $g^{(2)}(0)$ function is above unity at $t = 0$, but then decreases at later times until a local minimum is reached after peak photon emission.

Utilizing the collective equations of motion for Eq. (14) and provided in the Supplemental Material [62], we can evaluate the time derivate at $t = 0$ and obtain

$$\begin{aligned} \frac{d}{dt} g^{(2)}(0) \Big|_{t=0} &= \frac{2}{N^2} \sum_{\mu} \Gamma_{\mu}^3 - \frac{4}{N^2} \sum_{\mu} \Gamma_{\mu}^2 \\ &\quad - \frac{2}{N^3} \left(\sum_{\mu} \Gamma_{\mu}^2 \right)^2 + \frac{4}{N} < 0. \end{aligned} \quad (16)$$

The derivative is confined to the interval $[-4 + 4/N, 0)$, thus confirming that the $g^{(2)}(0)$ will always show an initial decrease. In the Dicke limit ($a = 0$) [51], where $\Gamma_{\mu} = \delta_{0\mu} N$, it equals $-4(1 - 1/N)$. In the opposite limit of independent

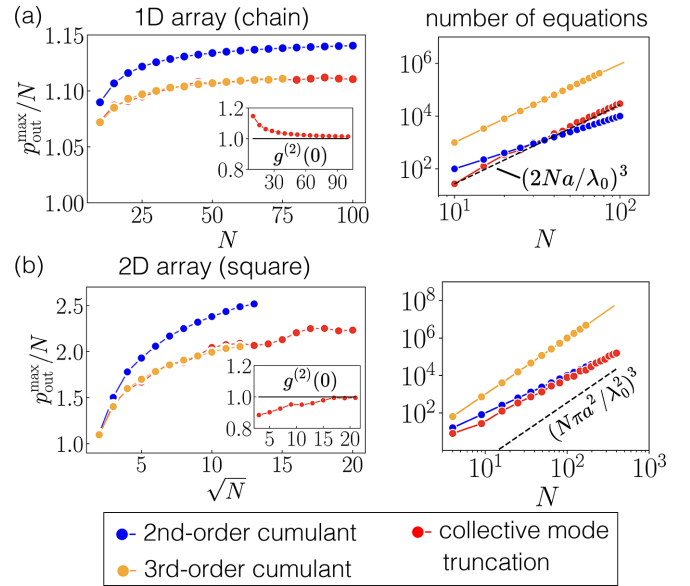


FIG. 5. Comparison of the peak photon emission and numerical complexity as a function of N using a second- and third-order cumulant expansion in the individual spin basis and the cumulant expansion in the collective spin basis with collective mode truncation. The linear chain in panel (a) and the square array in panel (b) show excellent agreement between the third-order cumulant expansion and the collective mode equations in Eq. (14). Here, all subradiant modes are truncated and the scaling of the numerical complexity is shown on the right-hand side (red dots). The two-dimensional square array exhibits small oscillations, stemming from the collective mode truncation. The second-order correlation $g^{(2)}(0)$ at peak photon emission, evaluated with the collective mode truncation shows a convergence toward unity at peak photon emission for increasing N . The second-order cumulant expansion clearly overestimates the peak photon emission rate both for chain and square geometries [40,42]. We consider $a = 0.15\lambda_0$ and $\mathbf{d} = (1, i, 0)^T/\sqrt{2}$ in all plots.

decay ($a \rightarrow \infty$), $\Gamma_{\mu} \rightarrow \gamma_0$ for all μ and Eq. (16) approaches zero. Then, the second-order correlation remains constant and equals $1 - 1/N$ throughout the time evolution, as shown in Fig. 4(a). In the presence of interactions, however, $g^{(2)}(\tau = 0)$ always decreases at $t = 0$, and for the cases shown in Fig. 4(a), will reach a local minimum after peak photon emission. In a physical picture, this means that the simultaneous emission of two photons becomes less and less likely after $t = 0$ and until after peak photon emission, even though the photon emission itself keeps increasing. At late times, $g^{(2)}(\tau = 0)$ increases due to the rapid decrease of the emitted intensity in the denominator of Eq. (9). We believe that this warrants further investigation and discuss possible directions in Sec. V.

C. Large-scale ordered arrays

We apply the collective mode truncation to ordered arrays involving large numbers of emitters and provide a comparison with second- and third-order cumulant expansions in the individual spin basis as a function of N . In Fig. 5, we truncate the number of collective modes used to solve Eq. (15) to a total of $\tilde{N}$ modes satisfying $\Gamma_{\mu} > \gamma_0/N$. In Fig. 5(a), the peak photon emission $p_{\text{out}}^{\text{max}}$ for the chain array saturates at $\approx 1.1N$

for the third-order cumulant expansion and collective mode truncation. The second-order cumulant expansion meanwhile shows an overestimation of approximately 5%. The second-order correlation $g^{(2)}(0)$ is calculated with the collective mode truncation and shows convergence to unity at peak photon emission. The number of equations, i.e., numerical complexity for the collective mode truncation is well described by $(2Na/\lambda_0)^3$, as previously obtained in Fig. 3(c). As a result, there are approximately two orders of magnitude less equations to be solved when using the collective spin basis with mode truncation than when using the third-order cumulant expansion in the individual spin basis for arrays with spacing $a = 0.15\lambda_0$.

The peak photon emission for the two-dimensional square array in Fig. 5(b) shows again excellent agreement between the third-order cumulant expansion and the collective mode truncation. The second-order cumulant expansion meanwhile shows now an overestimation of approximately 10%–20%. The numerical complexity as a function of N for the collective mode truncation is now slowly converging to the analytical value obtained earlier, $(N\pi a^2/\lambda_0^2)^3$. As a consequence, the number of equations that have to be solved as compared to the third-order cumulant expansion in the individual spin basis are reduced by a factor $\sim 10^3$ at $a = 0.15\lambda_0$ and for large N .

V. DISCUSSION AND OUTLOOK

We have presented an efficient numerical technique for simulating large numbers of dipole-coupled quantum emitters in ordered arrays in the presence of collective long-range decay. For subwavelength array spacings, we have demonstrated the emergence of collective super- and subradiant modes by transforming the individual spin operators into a collective spin basis. Subsequently, we have applied a cumulant expansion to the Heisenberg equations of motion to derive a closed set of equations that approximately describes the collective decay dynamics.

Crucially, a majority of collective modes for subwavelength spacings are highly subradiant and therefore do not participate significantly in the radiant dissipative dynamics. By neglecting or truncating these modes, the numerical complexity of the system of equations can be strongly reduced while ensuring an accurate description of the dissipative time evolution of the quantum many-body system.

We have illustrated and benchmarked this collective mode truncation method for various geometries by comparing the resulting dynamics with that obtained by solving small systems of up to 14 emitters exactly or by applying a cumulant expansion based on the individual spin basis for larger numbers of emitters. The method presented in this work allows to make accurate predictions about excited state populations,

photon emission rates, and second-order correlations, highly relevant for many quantum many-body experiments and dipolar spin ensemble experiments [8,71]. We have also studied the photon statistics of the emitted light by analyzing the second-order intensity correlation function with zero time delay, $g^{(2)}(\tau = 0)$, thereby characterizing two-photon processes. Unexpectedly, in the presence of dipole-dipole interactions, this quantity showed a decrease until after peak photon emission. Future studies should further elucidate the behavior of the second- and higher-order correlation functions at $t \gtrsim 0$, as it provides key insights into the internal correlations of the ensemble of emitters [2,3].

Future implementations of the collective mode truncation method might include incoherent driving of the individual emitters [19], emitters with multiple energy levels [42], initially coherent states [2], and coherent driving [8,10] to study, for instance, the generalization of the driven Dicke model in free space [8,72]. This method can also be leveraged to simulate the collective dissipative dynamics of emitter arrays coupled to electromagnetic environments different than free space, such as waveguides [54,55,73] (see the Supplemental Material [62]).

Even though the method presented in this work relies on ordered atomic emitter arrays, we expect that it will be valid approximately for arrays with small positional disorder. This can, for instance, arise in arrays where imperfections in the creation of the optical trap array can lead to small positional disorders of the trapped emitters [18].

Finally, we would like to point out that the collective basis presented in this work also opens the possibility to explore subradiant dynamics in a perturbative fashion. Here, subradiant modes with a specific quasimomentum can be selectively added to or removed from the existing system of equations, thereby allowing to study their role in the dissipative dynamics.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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