

Vector-light interference with a multimode N00N state

Elvis Pillinen ^{*}, Tero Setälä, and Andreas Norrman Center for Photonics Sciences, [University of Eastern Finland](https://www.uef.fi), P.O. Box 111, FI-80101 Joensuu, Finland

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Multimode N00N states stand out as an excellent platform to explore higher-order quantum interference phenomena, with benefits in quantum-enhanced estimation of multiple parameters beyond classical limits. Here, we investigate higher-order polarization properties, geometric phase features, and associated wave-particle duality structures in double-slit vector-light interference involving a multimode N00N state. We particularly show that the interferometric polarization modulation and the accompanied geometric phase accumulation are N times faster for the multimode N00N state than for one-photon or classical light. We also formulate N th-order extensions of the three wave-particle duality relations established for a single vector-light photon: the two via distinguishabilities and visibilities [A. Norrman *et al.*, *Phys. Rev. Lett.* **119**, 040401 (2017)], and the one via the geometric phase and which-path information ratio [E. Pillinen *et al.*, *Phys. Rev. A* **111**, 063709 (2025)]. Our work contributes to the broader fundamental understanding of how vectorial quantum light acts in higher-order interference, with possible opportunities in precision metrology outside scalar-field regimes.

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I. INTRODUCTION

Interference lies at the heart of optical physics, offering a powerful tool to probe the classical and quantum natures of light. Quantum interference, in particular, has become a central concept in quantum optics and quantum information science, underpinning a wide range of nonclassical physical effects that are exploited in practical photonic applications [1,2]. Among the most striking examples of such interference are those provided by N00N states [3], superpositions of N photons that occupy one of two distinct modes: $|N, 0\rangle + |0, N\rangle$. Their capacity to achieve quantum-enhanced phase sensitivity [4,5], with interference fringes occurring at a frequency N times higher than those produced by classical states, makes them valuable resources for quantum lithography, metrology, and imaging [6–11]. When extended to encompass multiple modes, the so-called multimode N00N states offer an even richer landscape to explore complex interference effects and to perform multiparameter estimation in higher-dimensional quantum systems [12–16].

At the same time, optical fields that possess nonseparable combinations of spatial and polarization modes have opened pathways for manipulating photonic information and light-matter interactions across multiple degrees of freedom [17,18]. Importantly, such vector-light fields can exhibit interference not merely in terms of intensity fringes but also via polarization-state fringes, namely, periodic changes in the

polarization state [19]. This recognition has not only provided understanding of the coupling between polarization and coherence in interferometry and polarimetry [20], but also uncovered fundamental facets of wave-particle duality of the photon [21,22] that remain obscured when the interferometric polarization modulation is neglected [23–28]. A related key feature emerging in this context is the Pancharatnam-Berry phase [29–36]: a purely geometric phase contribution that the light field acquires as its polarization state evolves along a curved trajectory on the Poincaré sphere. In the classical domain, the geometric phase has been observed in the periodic polarization-state variation in double-slit interference [37,38], and at the single-photon level it has been interlinked to wave-particle duality [39]. Regarding N00N states, it has been demonstrated that they can gain a geometric phase that is N times larger than for classical or one-photon light [40–42]. But these studies concern the more usual case where the phase arises from the discrete changes of a two-mode polarization state along a single spatial mode by using additional optical elements. In the double-slit setup, on the other hand, the geometric phase occurs in the continuously varying polarization-state pattern of two interfering fields with different polarization states [43].

In this work, we study theoretically higher-order double-slit interference of vectorial light in a four-mode N00N state, with a focus on the higher-order links between polarization, geometric phase, and wave-particle duality stemming from such a setting. Here, the four-mode system contains two spatially separated slit modes, each of which has two orthogonal polarization modes. By introducing generalized N th-order Stokes parameters, we especially show that the interference of the multimode N00N-state light at the slits causes an N times faster polarization variation and geometric phase accumulation in the observation plane as compared to the corresponding classical and one-photon cases. Moreover, we establish

^{*}Contact author: elvis.pillinen@uef.fi

N th-order extensions of the two wave-particle duality relations for a single vector-light photon [21]: one between the Stokes (or polarization) distinguishability and intensity visibility, and another between the intensity distinguishability and Stokes (or total) visibility, reflecting two distinct wave-particle duality aspects of the multimode NOON-state light. The distinguishability measures can be viewed to characterize different types of particlelike behavior [which-path information (WPI)] at the slits, while the visibility measures can be interpreted to represent disparate wavelike features in the interference plane. Finally, we derive a single wave-particle duality relation for the multimode NOON-state light via the generally noncyclic geometric phase at the detection screen and the which-path information ratio between the intensity and Stokes distinguishabilities in the slit plane, which generalizes the link in the one-photon case [39].

II. HIGHER-ORDER STOKES FORMALISM

All information on the first-order correlations of a quantized vector-light field is, at two space-time points $x_1 = (\mathbf{r}_1, t_1)$ and $x_2 = (\mathbf{r}_2, t_2)$, included in the first-order coherence matrix [44,45]

$$\mathbf{G}^{(1)}(x_1, x_2) = \text{tr}[\hat{\rho} \hat{\mathbf{E}}^\dagger(x_1) \otimes \hat{\mathbf{E}}(x_2)]. \quad (1a)$$

Here, $\hat{\mathbf{E}}(x)$ is the positive frequency part of the generally three-component electric field operator, $\hat{\rho}$ is the density operator describing the quantum state, $\dagger$ denotes the Hermitian adjoint, $\otimes$ is the tensor product, and tr stands for the trace in Fock space. In other words, $\mathbf{G}^{(1)}(x_1, x_2)$ is the expectation of the first-order coherence matrix operator [46]

$$\hat{\mathbf{G}}^{(1)}(x_1, x_2) = \hat{\mathbf{E}}^\dagger(x_1) \otimes \hat{\mathbf{E}}(x_2). \quad (1b)$$

Similarly, the vector-field correlations between $2n$ space-time points $\mathbf{X} = \{x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}\}$ are fully characterized by the n th-order coherence matrix

$$\mathbf{G}^{(n)}(\mathbf{X}) = \text{tr}[\hat{\rho} \hat{\mathbf{G}}^{(n)}(\mathbf{X})], \quad (2a)$$

where we introduced the n th-order coherence matrix operator

$$\begin{aligned} \hat{\mathbf{G}}^{(n)}(\mathbf{X}) &= \left[\bigotimes_{m=1}^n \hat{\mathbf{E}}^\dagger(x_m) \right] \otimes \left[\bigotimes_{m=1}^n \hat{\mathbf{E}}(x_{n+m}) \right] \\ &=: \bigotimes_{m=1}^n \hat{\mathbf{G}}^{(1)}(x_m, x_{n+m}) :. \end{aligned} \quad (2b)$$

The second line above, stating that the n th-order coherence operator matrix is expressible as a normally ordered tensor product of first-order coherence matrix operators, can be shown by using a generalization of the tensor-product relation $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$ for matrices. In general, $\mathbf{G}^{(n)}(\mathbf{X})$ is a $3^n \times 3^n$ matrix, whose 9^n elements $G_{\mu_1 \dots \mu_n, \mu_{n+1} \dots \mu_{2n}}^{(n)}(x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}) = \text{tr}[\hat{\rho} \prod_{m=1}^n \hat{E}_{\mu_m}(x_m) \prod_{m=1}^n \hat{E}_{\mu_{n+m}}(x_{n+m})]$ amount to the $2n$ -point correlations between the three Cartesian field components labeled by $\mu_1, \dots, \mu_{2n} \in \{x, y, z\}$ [44,45].

Henceforth, we focus on a two-component field characterized by $\hat{\mathbf{E}}(x) = [\hat{E}_x(x), \hat{E}_y(x)]^T$, with T standing for the transpose, whereupon $\mathbf{G}^{(n)}(\mathbf{X})$ is a $2^n \times 2^n$ matrix. In the first-order case, instead of using the 2×2 coherence matrix

$\mathbf{G}^{(1)}(x_1, x_2)$, an alternative yet equivalent way to describe the correlations of the field is provided by the four coherence Stokes parameters [21]

$$S_j^{(1)}(x_1, x_2) = \text{tr}[\hat{\rho} \hat{\mathbf{E}}^\dagger(x_1) \sigma_j \hat{\mathbf{E}}(x_2)], \quad j \in \{0, \dots, 3\}, \quad (3a)$$

where σ_0 is the 2×2 identity matrix and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. These parameters are quantum versions of their classical counterparts [47,48] and expectations of the coherence Stokes operators [22,46]

$$\hat{S}_j^{(1)}(x_1, x_2) = \hat{\mathbf{E}}^\dagger(x_1) \sigma_j \hat{\mathbf{E}}(x_2), \quad j \in \{0, \dots, 3\}. \quad (3b)$$

Physically, $S_0^{(1)}(x_1, x_2)$ amounts to the sum of correlations of the x and y components at the two space-time points, while $S_1^{(1)}(x_1, x_2)$, $S_2^{(1)}(x_1, x_2)$, and $S_3^{(1)}(x_1, x_2)$ describe the correlation differences between the x - and y -polarized, $+\pi/4$ - and $-\pi/4$ -linearly polarized, and right- and left-circularly polarized components, respectively [49]. In view of Eqs. (1a) and (3a), the Stokes representation is linked to the matrix representation as

$$\begin{aligned} \mathbf{G}^{(1)}(x_1, x_2) &= \frac{1}{2} \sum_{j=0}^3 S_j^{(1)}(x_1, x_2) \sigma_j, \\ \hat{S}_j^{(1)}(x_1, x_2) &= \text{Tr}[\mathbf{G}^{(1)}(x_1, x_2) \sigma_j], \end{aligned} \quad (4)$$

where Tr denotes the matrix trace. Further, when $x_1 = x_2 = x$, the coherence (two-point) Stokes parameters/operators reduce to their polarization (single-point) counterparts [22,50], namely, $S_j^{(1)}(x, x) = S_j^{(1)}(x)$ and $\hat{S}_j^{(1)}(x, x) = \hat{S}_j^{(1)}(x)$.

Motivated by the connection between the coherence matrix and Stokes formalisms in the first-order case, we next introduce the n th-order coherence Stokes parameters

$$S_{\mathbf{J}}^{(n)}(\mathbf{X}) = \text{tr}[\hat{\rho} \hat{S}_{\mathbf{J}}^{(n)}(\mathbf{X})], \quad (5a)$$

in which $\mathbf{J} = \{j_1, \dots, j_n\}$ with $j_1, \dots, j_n \in \{0, \dots, 3\}$. In addition, $\hat{S}_{\mathbf{J}}^{(n)}(\mathbf{X})$ are the respective n th-order coherence Stokes operators,

$$\begin{aligned} \hat{S}_{\mathbf{J}}^{(n)}(\mathbf{X}) &= \bigotimes_{m=1}^n \hat{\mathbf{E}}^\dagger(x_m) \sigma_{\mathbf{J}} \bigotimes_{m=1}^n \hat{\mathbf{E}}(x_{n+m}) \\ &=: \prod_{m=1}^n \hat{S}_{j_m}^{(1)}(x_m, x_{n+m}) :, \end{aligned} \quad (5b)$$

where on the first line $\sigma_{\mathbf{J}} = \bigotimes_{m=1}^n \sigma_{j_m}$. When all the j_m indices are zero, $\sigma_{\mathbf{J}}$ is the $2^n \times 2^n$ unit matrix, otherwise it corresponds to generalized Pauli matrices [51]. The second line with the normally ordered product of first-order coherence Stokes operators can be proven similarly as in Eq. (2b). From Eqs. (2a), (2b), (5a), and (5b), we then find the n th-order generalization to the first-order correspondence between the coherence matrix and coherence Stokes parameters in Eq. (4):

$$\begin{aligned} \mathbf{G}^{(n)}(\mathbf{X}) &= \frac{1}{2^n} \sum_{\mathbf{J}} S_{\mathbf{J}}^{(n)}(\mathbf{X}) \sigma_{\mathbf{J}}, \\ \hat{S}_{\mathbf{J}}^{(n)}(\mathbf{X}) &= \text{Tr}[\mathbf{G}^{(n)}(\mathbf{X}) \sigma_{\mathbf{J}}]. \end{aligned} \quad (6)$$

Thus, Eq. (6) offers a unified framework to quantify n th-order correlations of any two-component field in terms of either the $2^n \times 2^n$ coherence matrix or the 4^n coherence Stokes

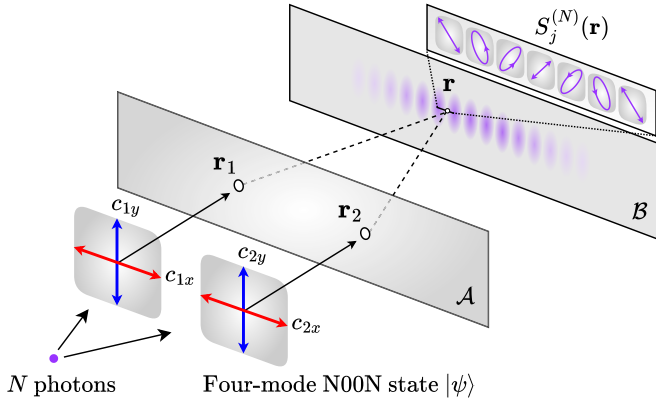


FIG. 1. Higher-order vector-light interference in the double-slit setup. An N -photon field in a four-mode NOON state $|\Psi\rangle = c_{1x}|N, 0, 0, 0\rangle + c_{1y}|0, N, 0, 0\rangle + c_{2x}|0, 0, N, 0\rangle + c_{2y}|0, 0, 0, N\rangle$ impinges on screen $\mathcal{A}$ with two small pinholes at points $\mathbf{r}_1$ and $\mathbf{r}_2$. The interfering light is observed at point $\mathbf{r}$ in plane $\mathcal{B}$ where periodic N th-order intensity and polarization-state fringes occur via the Stokes parameters $S_j^{(N)}(\mathbf{r})$.

parameters. As will become apparent in the next section, the Stokes formalism is particularly advantageous for characterizing higher-order correlations in double-slit interference (as in the first-order case [20,21]).

Before this we address a few basic properties of the n th-order coherence Stokes parameters, bearing similarities with those of the n th-order coherence matrix [44,45]. It is readily verified from Eqs. (5a) and (5b) that they fulfill the symmetry relation

$$S_{j_n \dots j_1}^{(n)}(x_{2n}, \dots, x_1) = S_{j_1 \dots j_n}^{(n)*}(x_1, \dots, x_{2n}), \quad (7)$$

with $*$ standing for complex conjugation. Further, when all the pairs of points are the same, the coherence Stokes parameters are real (and the coherence Stokes operators are Hermitian), namely,

$$\begin{aligned} S_{\mathbf{j}}^{(n)}(x_1, \dots, x_n, x_n, \dots, x_1) \\ = S_{\mathbf{j}}^{(n)*}(x_1, \dots, x_n, x_n, \dots, x_1). \end{aligned} \quad (8)$$

Eventually, if the density operator limits the number of photons to be smaller than or equal to some value M , the normal ordering of the field operators in Eq. (5b) leads to $S_{\mathbf{j}}^{(n)}(\mathbf{X}) = 0$ for $n > M$.

III. INTERFEROMETRIC POLARIZATION MODULATION

Let us turn our focus to the double-slit setup, as shown in Fig. 1. A monochromatic vector-light field hits two small slits (pinholes) located at $\mathbf{r}_1$ and $\mathbf{r}_2$ in an opaque screen $\mathcal{A}$, whereafter the light emanating from the slits is detected on another screen $\mathcal{B}$ far from $\mathcal{A}$ at position $\mathbf{r}$. In this scenario, the spatial part of electric field operator at $\mathcal{B}$ can be expressed as [21,52]

$$\hat{\mathbf{E}}(\mathbf{r}) = \frac{K}{r} [\hat{\mathbf{E}}(\mathbf{r}_1)e^{ikr_1} + \hat{\mathbf{E}}(\mathbf{r}_2)e^{ikr_2}], \quad (9)$$

where K is a constant, k is the wave number, $r_m = |\mathbf{r} - \mathbf{r}_m|$ is the distance from slit $m \in \{1, 2\}$ to the observation point, and

$r = r_1 \approx r_2$ in the denominator holds in the paraxial regime. Moreover, $\hat{\mathbf{E}}(\mathbf{r}_1) = C(\hat{a}_{1x}, \hat{a}_{1y})^T$ and $\hat{\mathbf{E}}(\mathbf{r}_2) = C(\hat{a}_{2x}, \hat{a}_{2y})^T$ are the field operators in the slit plane $\mathcal{A}$, in which C is another constant and $\hat{a}_{m\mu}$ is the annihilation operator of the $\mu \in \{x, y\}$ polarized mode in slit $m \in \{1, 2\}$. We are now specifically interested in the case where the light is in the multimode NOON state

$$\begin{aligned} |\Psi\rangle = c_{1x}|N, 0, 0, 0\rangle + c_{1y}|0, N, 0, 0\rangle \\ + c_{2x}|0, 0, N, 0\rangle + c_{2y}|0, 0, 0, N\rangle, \end{aligned} \quad (10)$$

in which $|n_{1x}, n_{1y}, n_{2x}, n_{2y}\rangle$ is a four-mode Fock state, with $n_{m\mu}$ specifying the number of photons in each mode, while $|c_{m\mu}|^2$ obeying the normalization $|c_{1x}|^2 + |c_{1y}|^2 + |c_{2x}|^2 + |c_{2y}|^2 = 1$ is the probability that the photons are in the respective mode.

According to Eqs. (5a), (5b), (9), and (10), the n th-order Stokes parameters at the point $\mathbf{r}$ on the observation screen for the four-mode NOON state are

$$S_{\mathbf{j}}^{(n)}(\mathbf{r}) = \langle \Psi | [\hat{\mathbf{E}}^\dagger(\mathbf{r})]^{\otimes n} \sigma_{\mathbf{j}} [\hat{\mathbf{E}}(\mathbf{r})]^{\otimes n} | \Psi \rangle. \quad (11)$$

Since the number of photons is bounded, we evidently have that

$$S_{\mathbf{j}}^{(n)}(\mathbf{r}) = 0, \quad \text{if } n > N. \quad (12)$$

For $n \leq N$, as shown in Sec. 1 of the Supplemental Material [53], the number of n th-order Stokes parameters given in Eq. (11) are reduced to only four that can be labeled by a single subindex $j \in \{0, \dots, 3\}$:

$$S_{\mathbf{j}}^{(n)}(\mathbf{r}) \rightarrow \hat{S}_{\mathbf{j}}^{(n)}(\mathbf{r}) = \langle \Psi | \hat{S}_{\mathbf{j}}^{(n)}(\mathbf{r}) | \Psi \rangle. \quad (13)$$

The respective operators are

$$\begin{aligned} \hat{S}_{\mathbf{j}}^{(n)}(\mathbf{r}) = \frac{|K|^{2n}}{r^{2n}} [\hat{S}_{\mathbf{j}}^{(n)}(\mathbf{r}_1) + \hat{S}_{\mathbf{j}}^{(n)}(\mathbf{r}_2) \\ + \hat{S}_{\mathbf{j}}^{(n)}(\mathbf{r}_1, \mathbf{r}_2)e^{ink\Delta r} + \hat{S}_{\mathbf{j}}^{(n)\dagger}(\mathbf{r}_1, \mathbf{r}_2)e^{-ink\Delta r}], \end{aligned} \quad (14a)$$

including the n th-order one-point Stokes operators in the slits $m \in \{1, 2\}$

$$\hat{S}_{\mathbf{j}}^{(n)}(\mathbf{r}_m) = |C|^{2n} [\hat{a}_{mx}^{\dagger n} \hat{a}_{my}^{\dagger n} \sigma_j [\hat{a}_{mx}^n \hat{a}_{my}^n]^T], \quad (14b)$$

the n th-order two-point Stokes operators between the slits

$$\hat{S}_{\mathbf{j}}^{(n)}(\mathbf{r}_1, \mathbf{r}_2) = |C|^{2n} [\hat{a}_{1x}^{\dagger n} \hat{a}_{1y}^{\dagger n} \sigma_j [\hat{a}_{2x}^n \hat{a}_{2y}^n]^T], \quad (14c)$$

and the distance separation $\Delta r = r_2 - r_1$.

The physical meaning of Eq. (13) becomes more clear by considering the expectations of the various terms in Eq. (14a). Regarding the one-point Stokes parameters in the slit plane $\mathcal{A}$, we find from Eq. (14b) that

$$S_0^{(n)}(\mathbf{r}_m) = |C|^{2n} \langle \Psi | \hat{a}_{mx}^{\dagger n} \hat{a}_{mx}^n + \hat{a}_{my}^{\dagger n} \hat{a}_{my}^n | \Psi \rangle, \quad (15a)$$

$$S_1^{(n)}(\mathbf{r}_m) = |C|^{2n} \langle \Psi | \hat{a}_{mx}^{\dagger n} \hat{a}_{mx}^n - \hat{a}_{my}^{\dagger n} \hat{a}_{my}^n | \Psi \rangle. \quad (15b)$$

Accordingly, $S_0^{(n)}(\mathbf{r}_m) [S_1^{(n)}(\mathbf{r}_m)]$ is proportional to the probability sum (difference) of detecting n number of x -polarized photons and n number of y -polarized photons in slit $m \in \{1, 2\}$. Moreover, on expressing the annihilation operators in the x and y polarization bases as $\hat{a}_{mx} = (\hat{a}_{ma} + \hat{a}_{mb})/\sqrt{2} = (\hat{a}_{mr} + \hat{a}_{ml})/\sqrt{2}$ and $\hat{a}_{my} = (\hat{a}_{ma} - \hat{a}_{mb})/\sqrt{2} =$

$i(\hat{a}_{mr} - \hat{a}_{ml})/\sqrt{2}$, where $\hat{a}_{ma}$ and $\hat{a}_{mb}$ are the annihilation operators in the $+\pi/4$ - and $-\pi/4$ -linear polarization bases and $\hat{a}_{mr}$ and $\hat{a}_{ml}$ are the annihilation operators in the right- and left-circular polarization bases, respectively, one verifies from Eq. (14b) that

$$S_2^{(n)}(\mathbf{r}_m) = 2^{1-n} |C|^{2n} \langle \Psi | \hat{a}_{ma}^{\dagger n} \hat{a}_{ma}^n - \hat{a}_{mb}^{\dagger n} \hat{a}_{mb}^n | \Psi \rangle, \quad (15c)$$

$$S_3^{(n)}(\mathbf{r}_m) = 2^{1-n} |C|^{2n} \langle \Psi | \hat{a}_{mr}^{\dagger n} \hat{a}_{mr}^n - \hat{a}_{ml}^{\dagger n} \hat{a}_{ml}^n | \Psi \rangle. \quad (15d)$$

Hence, akin to $S_1^{(n)}(\mathbf{r}_m)$, we conclude that $S_2^{(n)}(\mathbf{r}_m)$ [$S_3^{(n)}(\mathbf{r}_m)$] is proportional to the probability difference of recording n number of $+\pi/4$ -linearly (right-circularly) polarized photons and n number of $-\pi/4$ -linearly (left-circularly) polarized photons in slit $m \in \{1, 2\}$. The factor 2^{1-n} appearing in Eqs. (15c) and (15d) but not in Eq. (15b) is a consequence of our chosen xy polarization basis for the multimode N00N state in Eq. (10). For example, the term $|N, 0, 0, 0\rangle$ describes a state with N number of x -polarized photons in slit 1, which corresponds to an N -photon state where each photon is in a $\pm\pi/4$ -linearly and right/left-circularly polarized superposition. Similarly, the two-point Stokes parameters at the slits can be cast into the forms

$$S_0^{(n)}(\mathbf{r}_1, \mathbf{r}_2) = |C|^{2n} \langle \Psi | \hat{a}_{1x}^{\dagger n} \hat{a}_{2x}^n + \hat{a}_{1y}^{\dagger n} \hat{a}_{2y}^n | \Psi \rangle, \quad (16a)$$

$$S_1^{(n)}(\mathbf{r}_1, \mathbf{r}_2) = |C|^{2n} \langle \Psi | \hat{a}_{1x}^{\dagger n} \hat{a}_{2x}^n - \hat{a}_{1y}^{\dagger n} \hat{a}_{2y}^n | \Psi \rangle, \quad (16b)$$

$$S_2^{(n)}(\mathbf{r}_1, \mathbf{r}_2) = 2^{1-n} |C|^{2n} \langle \Psi | \hat{a}_{1a}^{\dagger n} \hat{a}_{2a}^n - \hat{a}_{1b}^{\dagger n} \hat{a}_{2b}^n | \Psi \rangle, \quad (16c)$$

$$S_3^{(n)}(\mathbf{r}_1, \mathbf{r}_2) = 2^{1-n} |C|^{2n} \langle \Psi | \hat{a}_{1r}^{\dagger n} \hat{a}_{2r}^n - \hat{a}_{1l}^{\dagger n} \hat{a}_{2l}^n | \Psi \rangle, \quad (16d)$$

which are seen to characterize linear combinations of n th-order correlations in the different polarization bases.

It is now readily verified from Eqs. (10) and (16a)–(16d) that for $n < N$ the two-point Stokes parameters in the slit plane $\mathcal{A}$ vanish, i.e., $S_j^{(n)}(\mathbf{r}_1, \mathbf{r}_2) = 0$ for all $j \in \{0, \dots, 3\}$. Consequently, from Eqs. (13) and (14a) we obtain

$$S_j^{(n)}(\mathbf{r}) = \frac{|K|^{2n}}{r^{2n}} [S_j^{(n)}(\mathbf{r}_1) + S_j^{(n)}(\mathbf{r}_2)], \quad \text{if } n < N, \quad (17)$$

which implies that the n th-order Stokes parameters in the observation plane are uniform and do not show any spatial variations. Yet, and most importantly, for $n = N$ we find from Eqs. (10), (13), (14a), and (15a)–(16d) that the N th-order Stokes parameters at $\mathcal{B}$ are accompanied by interferometric terms, namely,

$$S_j^{(N)}(\mathbf{r}) = \frac{|K|^{2N}}{r^{2N}} \{ S_j^{(N)}(\mathbf{r}_1) + S_j^{(N)}(\mathbf{r}_2) + 2|S_j^{(N)}(\mathbf{r}_1, \mathbf{r}_2)| \cos [Nk\Delta r + \theta_j^{(N)}(\mathbf{r}_1, \mathbf{r}_2)] \}. \quad (18a)$$

Here, the N th-order Stokes parameters in slits $m \in \{1, 2\}$ explicitly read

$$S_j^{(N)}(\mathbf{r}_m) = |C|^{2N} N! [c_{mx}^* \ c_{my}^*] \sigma_j [c_{mx} \ c_{my}]^T, \quad (18b)$$

whereas $|S_j^{(N)}(\mathbf{r}_1, \mathbf{r}_2)|$ and $\theta_j^{(N)}(\mathbf{r}_1, \mathbf{r}_2)$ are the magnitudes and arguments of the N th-order Stokes parameters between the

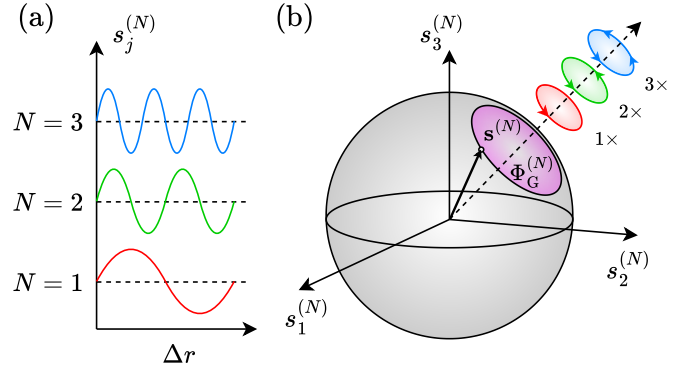


FIG. 2. Schematic illustration of higher-order polarization modulation and geometric phase accumulation of the four-mode N00N-state light in double-slit interference. (a) Modulation of the normalized N th-order Stokes parameters $s_j^{(N)} = S_j^{(N)}/S_0^{(N)}$ as a function of the path difference Δr for three different photon numbers N . (b) Associated circular polarization paths traced by $\mathbf{s}^{(N)}$ and geometric phases $\Phi_G^{(N)}$ on the Poincaré-type sphere.

slits

$$S_j^{(N)}(\mathbf{r}_1, \mathbf{r}_2) = |C|^{2N} N! [c_{1x}^* \ c_{1y}^*] \sigma_j [c_{2x} \ c_{2y}]^T. \quad (18c)$$

Equation (18a) forms the first main result of this work. It is the multimode N00N-state version of the first-order expressions for the one-photon state [21] and classical light [20]. We see from Eq. (18a) that the N th-order Stokes parameters at $\mathcal{B}$ oscillate sinusoidally when moving the detection point $\mathbf{r}$ transversally along the screen and, in particular, N times faster than in the one-photon and classical cases ($N = 1$), as illustrated in Fig. 2(a).

On introducing the three-component vector

$$\mathbf{S}^{(N)}(\mathbf{r}) = [S_1^{(N)}(\mathbf{r}), S_2^{(N)}(\mathbf{r}), S_3^{(N)}(\mathbf{r})]^T, \quad (19)$$

it is verified from Eqs. (18a)–(18c) that

$$|\mathbf{S}^{(N)}(\mathbf{r})| = S_0^{(N)}(\mathbf{r}). \quad (20)$$

The normalized vector $\mathbf{s}^{(N)}(\mathbf{r}) = \mathbf{S}^{(N)}(\mathbf{r})/S_0^{(N)}(\mathbf{r})$ can hence be represented on a Poincaré-type sphere, as depicted in Fig. 2(b), and interpreted as characterizing the N th-order polarization state of the multimode N00N-state field at the observation screen. Owing to the factor N in the cosine function of Eq. (18a), the rate of the cyclic polarization-state modulation increases with the number of photons, having no correspondence in the classical domain [19,20]. Furthermore, in the first-order scenario, it has been shown that $\mathbf{s}^{(1)}(\mathbf{r})$ traces a circular polarization-evolution loop on the usual Poincaré sphere [43], giving rise to a geometric phase [37–39]. Analogously, also $\mathbf{s}^{(N)}(\mathbf{r})$ outlines a cyclic circular path on the N th-order Poincaré-type sphere, but with an N times faster rate. This correspondence advocates the existence of an N th-order interferometric geometric phase for the multimode N00N-state light, as will be discussed in Sec. V.

IV. VISIBILITY AND DISTINGUISHABILITY

Due to the interferometric modulations of the N th-order Stokes parameters on $\mathcal{B}$ for the multimode N00N-state light, we may define for each $j \in \{0, \dots, 3\}$ a visibility parameter

$$V_j^{(N)} = \frac{\max[S_j^{(N)}(\mathbf{r})] - \min[S_j^{(N)}(\mathbf{r})]}{\max[S_0^{(N)}(\mathbf{r})] + \min[S_0^{(N)}(\mathbf{r})]}, \quad (21a)$$

where $\max$ ($\min$) stands for the local maximum (minimum). From Eqs. (18a), (18c), and (21a), we then obtain

$$V_0^{(N)} = 2|c_{1x}^*c_{2x} + c_{1y}^*c_{2y}|, \quad (21b)$$

$$V_1^{(N)} = 2|c_{1x}^*c_{2x} - c_{1y}^*c_{2y}|, \quad (21c)$$

$$V_2^{(N)} = 2|c_{1x}^*c_{2y} + c_{1y}^*c_{2x}|, \quad (21d)$$

$$V_3^{(N)} = 2|c_{1x}^*c_{2y} - c_{1y}^*c_{2x}|, \quad (21e)$$

which are similar to those met in the single-photon case [21] and satisfy $0 \leq V_j^{(N)} \leq 1$ for all $j \in \{0, \dots, 3\}$. In this regard, the quantity $V_0^{(N)}$ can be interpreted as an N th-order intensity visibility and $V_1^{(N)}, V_2^{(N)}, V_3^{(N)}$ as N th-order polarization visibilities. Furthermore, motivated by single-photon studies [21,22], we also introduce the N th-order Stokes (or total) visibility

$$V_S^{(N)} = \left\{ \frac{1}{2} \sum_{j=0}^3 [V_j^{(N)}]^2 \right\}^{1/2}, \quad (22a)$$

which includes both the intensity and polarization visibilities and hence characterizes the modulations of all four N th-order Stokes parameters on $\mathcal{B}$. According to Eqs. (21b)–(22a),

$$V_S^{(N)} = 2\sqrt{p_1 p_2}, \quad (22b)$$

where $p_1 = |c_{1x}|^2 + |c_{1y}|^2$ and $p_2 = |c_{2x}|^2 + |c_{2y}|^2$ are the path probabilities that all N photons traverse slit 1 or 2, respectively. Now it is evident from Eq. (22b) that $0 \leq V_S^{(N)} \leq 1$, with the lower limit taking place when $p_1 = 0$ or $p_2 = 0$, while the upper limit is met for $p_1 = p_2 = 1/2$. In particular, when both p_1 and p_2 are nonzero, then $V_S^{(N)} > 0$ and at least one $S_j^{(N)}(\mathbf{r})$ is modulated.

We further characterize the light in the slit plane $\mathcal{A}$ through

$$D_0^{(N)} = \frac{|S_0^{(N)}(\mathbf{r}_1) - S_0^{(N)}(\mathbf{r}_2)|}{S_0^{(N)}(\mathbf{r}_1) + S_0^{(N)}(\mathbf{r}_2)} = |p_1 - p_2|, \quad (23)$$

where Eq. (18b) was used for the second equality. The quantity $0 \leq D_0^{(N)} \leq 1$ can be viewed as the multimode N00N-state version of the so-called intensity distinguishability (or path predictability) in the single-photon scenario [21]. It quantifies the chance to correctly guess which of the slits N photons traverse based on their path probabilities. The lower bound $D_0^{(N)} = 0$ occurs when there is an equal possibility to detect N photons in both slits ($p_1 = p_2 = 1/2$), while the upper bound $D_0^{(N)} = 1$ is saturated if N photons pass via one slit only ($p_1 = 0$ or $p_2 = 0$). Another measure that we use to

differentiate the light at $\mathcal{A}$ is

$$D_S^{(N)} = \frac{|\mathbf{S}^{(N)}(\mathbf{r}_1) - \mathbf{S}^{(N)}(\mathbf{r}_2)|}{S_0^{(N)}(\mathbf{r}_1) + S_0^{(N)}(\mathbf{r}_2)} = \sqrt{1 - 4|c_{1x}^*c_{2x} + c_{1y}^*c_{2y}|^2}, \quad (24)$$

in which $\mathbf{S}^{(N)}(\mathbf{r}_m) = [S_1^{(N)}(\mathbf{r}_m), S_2^{(N)}(\mathbf{r}_m), S_3^{(N)}(\mathbf{r}_m)]$ is an N th-order Poincaré-type vector in slit $m \in \{1, 2\}$, and the second equality follows from Eq. (18b). Similarly to $D_0^{(N)}$, the quantity $0 \leq D_S^{(N)} \leq 1$ is to be interpreted as the multimode N00N-state analog of the so-called Stokes/polarization distinguishability (or path distinguishability) for the single-photon state [21], describing one's capability to discriminate the path of N photons with respect to the N th-order polarization at the two slits. The minimum $D_S^{(N)} = 0$ is encountered if $\mathbf{S}^{(N)}(\mathbf{r}_1) = \mathbf{S}^{(N)}(\mathbf{r}_2)$, corresponding to full N th-order intensity visibility [see Eq. (25)]. The maximum $D_S^{(N)} = 1$ is reached whenever the multimode N00N-state light is N th-order orthogonally polarized in the slit plane, for instance, $c_{1y} = c_{2x} = 0$ or $c_{1x} = c_{2y} = 0$.

In particular, from Eqs. (21b), (22b), (23), and (24) we find that the multimode N00N-state light obeys two strict identities:

$$[D_0^{(N)}]^2 + [V_S^{(N)}]^2 = 1, \quad [D_S^{(N)}]^2 + [V_0^{(N)}]^2 = 1. \quad (25)$$

These complementarity relations, constituting the second major result of our work, are N th-order generalizations of the two wave-particle relations for a single-photon vector-light field in double-slit interference [21]. Equation (25) indicates that the intensity distinguishability $D_0^{(N)}$ and the Stokes visibility $V_S^{(N)}$, as well as the Stokes distinguishability $D_S^{(N)}$ and the intensity visibility $V_0^{(N)}$, are strictly mutually complementary quantities. On this basis, Eq. (25) can be taken as reflecting two different aspects of wave-particle duality of the multimode N00N-state light, such that $D_0^{(N)}$ and $D_S^{(N)}$ reflect its particlelike properties at the slits, while $V_0^{(N)}$ and $V_S^{(N)}$ represent its wavelike features in the interference plane. If the pure state in Eq. (10) is replaced with a mixed state, it is straightforward to show that the equalities in Eq. (25) are converted into inequalities, consistent with the behavior in first-order wave-particle duality with vectorial light [21,22].

V. GEOMETRIC PHASE

For first-order interference, following Pancharatnam's original definition [29], the total phase of the light between two transverse points $\mathbf{r}'$ and $\mathbf{r}''$ on $\mathcal{B}$ is $\Phi_T^{(1)} = \arg[S_0^{(1)}(\mathbf{r}', \mathbf{r}'')]$ [37,43], with $S_0^{(1)}(\mathbf{r}', \mathbf{r}'')$ being the related coherence Stokes parameter. This total phase difference consists of two parts: the dynamical phase $\Phi_D^{(1)}$ and the geometric phase $\Phi_G^{(1)}$. The dynamical phase is build up from the incremental phase differences when moving from $\mathbf{r}'$ to $\mathbf{r}''$, and the geometric phase is found by subtracting it from the total phase: $\Phi_G^{(1)} = \Phi_T^{(1)} - \Phi_D^{(1)}$.

In view of these facts, for the multimode N00N-state light we can regard the total phase between $\mathbf{r}'$ and $\mathbf{r}''$ to be

$$\Phi_T^{(N)} = \arg[S_0^{(N)}(\mathbf{r}', \mathbf{r}'')], \quad (26)$$

where $S_0^{(N)}(\mathbf{r}', \mathbf{r}'')$ is the N th-order coherence Stokes parameter on $\mathcal{B}$. The associated dynamic phase is obtained by integrating over infinitesimal phase changes from $\mathbf{r}'$ to $\mathbf{r}''$ as [54]

$$\Phi_D^{(N)} = \int_{\mathbf{r}'}^{\mathbf{r}''} \arg [S_0^{(N)}(\mathbf{r}, \mathbf{r} + d\mathbf{r})]. \quad (27)$$

The geometric phase is then the difference between these total and dynamic phases:

$$\Phi_G^{(N)} = \Phi_T^{(N)} - \Phi_D^{(N)}. \quad (28)$$

As we show in Sec. 2 of the Supplemental Material [53], the total phase of the multimode N00N-state light on $\mathcal{B}$ between $\mathbf{r}'$ and $\mathbf{r}''$ can be expressed as

$$\Phi_T^{(N)} = \arg \{ 1 \pm D_0^{(N)} + (1 \mp D_0^{(N)})e^{iN\Delta\gamma} + 2V_0^{(N)}e^{iN\Delta\gamma/2} \cos [N\bar{\gamma} + \theta_0^{(N)}(\mathbf{r}_1, \mathbf{r}_2)] \}. \quad (29)$$

Here, we defined $\Delta\gamma = k(\Delta r'' - \Delta r')$ and $\bar{\gamma} = k(\Delta r' + \Delta r'')/2$, with the distance separations $\Delta r' = r'_2 - r'_1$ and $\Delta r'' = r''_2 - r''_1$ corresponding to the observation points $\mathbf{r}'$ and $\mathbf{r}''$, respectively. Furthermore, the upper (lower) sign in Eq. (29) refers to the case $p_1 > p_2$ ($p_1 < p_2$). The dynamic phase, whose derivation is outlined in Sec. 3 of the Supplemental Material [53], has the form

$$\Phi_D^{(N)} = \frac{N\Delta\gamma}{2}(1 \mp \Upsilon^{(N)}) \pm \Upsilon^{(N)}F^{(N)}(\mathbf{r}', \mathbf{r}''). \quad (30)$$

In the expression above,

$$F^{(N)}(\mathbf{r}', \mathbf{r}'') = \arg \{ 1 + D_S^{(N)} + (1 - D_S^{(N)})e^{iN\Delta\gamma} + 2V_0^{(N)}e^{iN\Delta\gamma/2} \cos [N\bar{\gamma} + \theta_0^{(N)}(\mathbf{r}_1, \mathbf{r}_2)] \}, \quad (31)$$

which is seen to bear similarity with Eq. (29), and

$$\Upsilon^{(N)} = \frac{D_0^{(N)}}{D_S^{(N)}}. \quad (32)$$

The quantity $\Upsilon^{(N)}$ is analogous to the WPI ratio for a single photon [39]. It is readily verified from Eqs. (23) and (24) that $D_0^{(N)} \leq D_S^{(N)}$ and therefore $0 \leq \Upsilon^{(N)} \leq 1$. The minimum $\Upsilon^{(N)} = 0$ takes place when the N th-order path predictability is zero, $D_0^{(N)} = 0$, while the maximum $\Upsilon^{(N)} = 1$ occurs if the N th-order polarization states in the slits are the same, $D_0^{(N)} = D_S^{(N)}$. From Eqs. (28)–(30), as elucidated in Sec. 4 of the Supplemental Material [53], we eventually find that the geometric phase in the observation plane $\mathcal{B}$ is

$$\begin{aligned} \Phi_G^{(N)} = & \pm \arg \{ 1 + D_0^{(N)} + (1 - D_0^{(N)})e^{iN\Delta\gamma} \\ & + 2V_0^{(N)}e^{iN\Delta\gamma/2} \cos [N\bar{\gamma} + \theta_0^{(N)}(\mathbf{r}_1, \mathbf{r}_2)] \} \\ & \mp \left[\frac{N\Delta\gamma}{2}(1 - \Upsilon^{(N)}) + \Upsilon^{(N)}F^{(N)}(\mathbf{r}', \mathbf{r}'') \right]. \end{aligned} \quad (33)$$

The upper and lower signs refer to the situations $p_1 > p_2$ and $p_1 < p_2$ for which the rotation direction of the enclosed N th-order polarization-state path on the Poincaré-type sphere is counterclockwise and clockwise, respectively, as $\Delta\gamma$ increases and when viewed toward the origin. For $p_1 = p_2$, the geometric phase is either 0 or $\pm\pi$, depending on $\Delta\gamma$, and the

rotation direction is undefined (see the discussion at the end of Sec. 4 in the Supplemental Material [53]).

Equation (33) constitutes the third main result of our work and provides a fully analytic formula for the generally noncyclic geometric phase of the multimode N00N-state light in double-slit interference. Several important conclusions can be drawn from Eq. (33). Most importantly, we see that the spatial dynamics of $\Phi_G^{(N)}$ are scaled by the photon number N . The geometric phase accumulation is therefore N times faster for the multimode N00N-state light than for one-photon light [39] or classical light [37,38,43] as illustrated in Fig. 2(b) (yet the accumulation rate is varying over the polarization path). In contrast to standard two-mode (scalar) N00N interferometry, where the N -fold phase enhancement originates from path-dependent dynamical phase accumulation, the N -fold enhancement considered here arises from polarization-dependent geometric phase accumulation enabled by the vectorial multimode structure of the state. Under cyclic evolution, the geometric phase reduces to the form

$$\Phi_G^{(N)} = \mp\pi(1 - \Upsilon^{(N)}), \quad \Delta\gamma = 2\pi/N. \quad (34)$$

In this situation $\Phi_G^{(N)}$ is determined solely by the N th-order WPI ratio $\Upsilon^{(N)}$. We further conclude from Eq. (33) that if $\Upsilon^{(N)} = 1$, in which case the N th-order polarization states in the slits are the same and no polarization modulation occurs on $\mathcal{B}$ [as explained below, Eq. (32)], then $\Phi_G^{(N)} = 0$.

Eventually, by introducing the π -normalized geometric phase $\tilde{\Phi}_G^{(N)} = \Phi_G^{(N)}/\pi$ obeying $0 \leq |\tilde{\Phi}_G^{(N)}| \leq 1$, we discover from Eq. (33) the complementarity relation

$$|\tilde{\Phi}_G^{(N)}| + \Upsilon^{(N)} \leq 1. \quad (35)$$

Equation (35), forming the fourth major result of our work, can be verified as follows. In the cyclic case, we immediately find from Eq. (34) that the upper bound in Eq. (35) is always saturated, i.e., $|\tilde{\Phi}_G^{(N)}| + \Upsilon^{(N)} = 1$. In the noncyclic case, we rely on the so-called geodesic rule [57–60]: The start and end points in any noncyclic evolution should be connected with a geodesic, and the geometric phase is specified by half the enclosed solid angle (surface area). Hence, since this rule implies that for any given $\Upsilon^{(N)}$ the geometric phase is necessarily smaller in the noncyclic case than in the cyclic case, we conclude that Eq. (35) also holds under noncyclic evolution. Furthermore, besides the cyclic case, the upper bound in Eq. (35) is reached in only two other situations. The first is the trivial scalar-light scenario of $\Upsilon^{(N)} = 1$ with no polarization modulation, for which Eq. (33) gives $|\tilde{\Phi}_G^{(N)}| = 0$ [as explained below, Eq. (34)]. The second is the special case of $\Upsilon^{(N)} = 0$ that is met solely if the path predictability in Eq. (23) is zero. In this case, Eq. (33) dictates that $|\tilde{\Phi}_G^{(N)}|$ can only assume the value of 0 or 1 depending on $\Delta\gamma$ (see the discussion at the end of Sec. 4 in the Supplemental Material [53]), with the latter case saturating the upper bound in Eq. (35).

Physically, we can view Eq. (35) as a quantifier of wave-particle duality of the multimode N00N-state light in terms of the geometric phase in the observation plane $\mathcal{B}$ (wave aspect) and the WPI ratio in the slit plane $\mathcal{A}$ (particle aspect). It is the multimode N00N-state version of the first-order relation for the one-photon state [39]. In cyclic evolution, the two qualities

are strictly mutually exclusive, i.e., reducing (increasing) $\Upsilon^{(N)}$ increases (reduces) $|\tilde{\Phi}_G^{(N)}|$ such that their sum is exactly one. We finally note that Eq. (35) is valid only for the pure state in Eq. (10). For a mixed state, the constraint $D_0^{(N)} \leq D_S^{(N)}$ may be violated, and consequently the bound $0 \leq \Upsilon^{(N)} \leq 1$, as required in Eq. (35), does not necessarily hold.

VI. CONCLUSIONS

In summary, we have established a theoretical framework based on higher-order Stokes parameters to characterize double-slit interference of vectorial quantum light in a multimode N00N state. Particular attention was paid to the higher-order connections between polarization, geometric phase, and wave-particle duality. We found that the interferometric polarization-state evolution and the consequent accumulation of the generally noncyclic geometric phase occur at an N times higher rate for the multimode N00N state than in the single-photon and classical scenarios. Furthermore, we derived N th-order complementarity identities between the intensity distinguishability and Stokes visibility as well as the Stokes distinguishability and intensity visibility. The identities are N th-order extensions of the ones for a single vector-light photon and they can be seen to reflect two distinct wave-particle duality facets of the multimode N00N-state light: The distinguishabilities represent its two particlelike

qualities at the slits, while the visibilities describe its two wavelike features in the interference plane. As a final result, we formulated a single N th-order wave-particle duality relation that links the geometric phase at the detection screen and the which-path information ratio in the slit plane. This connection extends the corresponding complementarity relation for a one-photon vector-light field to the multimode N00N-state light. Our work thus offers fundamental insights into the polarization characteristics, geometric phase properties, and dual wave-particle structures of vectorial light in higher-order quantum interference, and suggests possible applications in quantum-enhanced interferometric and polarimetric schemes that leverage polarization-dependent geometric phases with multimode quantum states beyond scalar-field regimes.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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