

Quantum geometric bounds in spinful systems with trivial band topology

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We derive quantum geometric bounds in spinful systems with spin topology characterized by a single $\mathbb{Z}$ index protected by a spin gap. Our bounds provide geometric conditions on the spin topology, distinct from the known quantum geometric bounds associated with Wilson loops and nontrivial band topologies. As a result, we obtain broader bounds in time-reversal symmetric systems with a nontrivial $\mathbb{Z}_2$ index and also bounds in systems with a trivial $\mathbb{Z}_2$ index, where the quantum metric should be otherwise unbounded. We benchmark these findings with first-principles calculations in elemental bismuth realizing a nontrivial even spin-Chern number. Moreover, we connect these bounds to optical responses and show their robustness in the presence of disorder within a real-space marker formulation, demonstrating that spin-resolved quantum geometry is observable in realistic experimental settings of impure materials. Finally, we connect spin bounds to quantum Cramér-Rao bounds that are central to quantum metrology, showing that elemental Bi and other spin-topological phases hold promises for topological free fermion quantum sensors.

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Introduction. Topological materials [1–9] constitute intriguing new phases of matter, as they display robust windings in the phases of Bloch eigenvectors, which have direct physical consequences [10,11]. More recently, it has been shown that these windings, formally characterized by the Wilson loops, also have implications for the Riemannian geometry of the eigenvectors [12–18], placing robust lower bounds on the metric of the eigenvectors, integrated across the Brillouin zone (BZ). Intuitively, this is because the winding of the states cannot be trivialized without closing the energy gap; i.e., there must be some minimal momentum-space dependence in the eigenstates due to the nontrivial topology [18–25]. This has important implications for optics, as the optical responses of materials depend on the quantum metric [16,26]. Moreover, the geometry of states is of key importance for quantum metrology [27–30] and quantum sensing [31,32], as the metric is related to the quantum Fisher information [33], which places a bound on the sensitivity of sensors via the Cramér-Rao bound [27–30,33].

The paradigmatic example of a topological phase in time-reversal symmetric materials is the quantum spin Hall phase [34,35]. This phase is realized in real materials, perhaps most notably in (111) terminations of elemental bismuth (Bi) [36–38]. In general, the topological invariant associated with

this phase is $\mathbb{Z}_2$ -valued and given by the Kane-Mele index $\nu_{\mathbb{Z}_2} \in \{0, 1\}$ [35,39]. It was realized early on, however, that a $\mathbb{Z}$ -valued invariant C_s , the spin-Chern number, could also be defined for such phases, giving rise to the concept of spin topology. As long as the spin-orbit entanglement is moderately large, i.e., the spin gap is preserved [40], this remains well defined, but for large spin-orbit entanglement, i.e., when the spin gap is closed, only $\nu_{\mathbb{Z}_2} = C_s \bmod 2$ is robust. This has led to the realization that the topological $\nu_{\mathbb{Z}_2} = 1$ phase in elemental Bi can actually be characterized by a higher spin-Chern number $C_s = 3$ [41]. Most intriguingly, there also exists a phase of elemental Bi with $C_s = 2$, with a trivial $\nu_{\mathbb{Z}_2} = 0$ invariant (see Fig. 1). This phase shows interesting behavior, including weakly gapped edge states, but as there is generically no strict quantization of a bulk Wilson-loop invariant and no exact bulk-boundary correspondence, it is difficult to pin down a clear experimental signature of this phase. Here, we ask the question what implications this spin topology has for the quantum metric. We provide a lower bound on the metric by the spin-topological invariant, $C_s \in \mathbb{Z}$, the tightness of which depends on the degree of spin-orbit entanglement. For the highly entangled case, this bound generalizes the recently discovered bound for the $\nu_{\mathbb{Z}_2}$ invariant [25] and can be larger than it. Moreover, for the $\nu_{\mathbb{Z}_2} = 0$ case no metric bounds were previously identified. Thus, in this Letter, we describe the first lower bound for such phases. Our bound is a geometric bound purely due to the spin topology and associated winding of the projected spin operator (PS_zP) eigenvalues [42–44], which holds even in the presence of a trivial band topology characterized by a trivial Wilson loop winding. We further show that our result culminates in $\mathbb{Z}$ bounds on optical responses and band gaps in spin-topological materials. Having first detailed the spin topology (valid also in systems with

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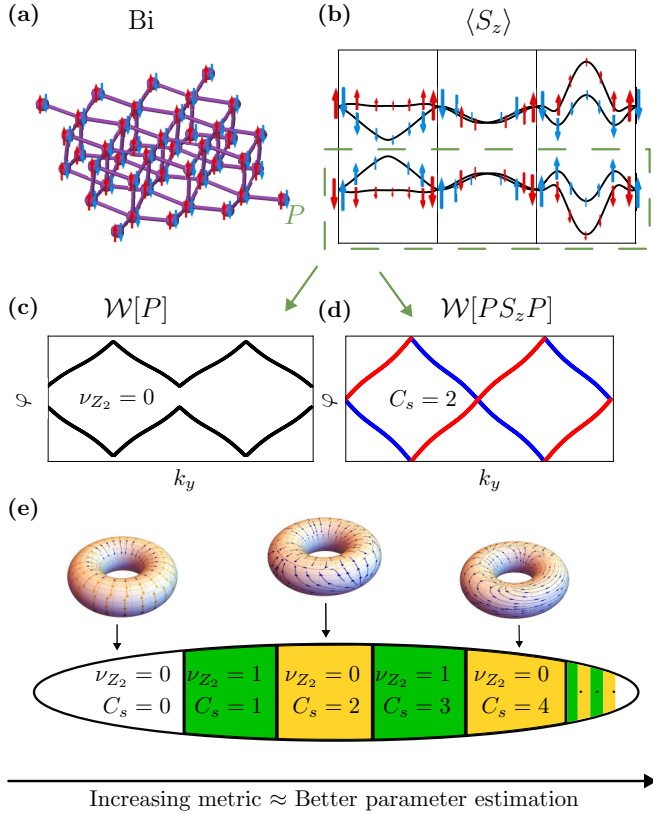


FIG. 1. Illustration of the main results of this work. (a) For a nonmagnetic material such as ultrathin bismuth, the energy bands may not have a definite spin value, due to spin-orbit entanglement, as illustrated schematically in panel (b), but the spin structure of energetically isolated bands (green dashed box) with projector P can still be defined via the projected spin operator (PS_zP). The electronic topology captured by the Kane-Mele invariant ν_{Z_2} is only protected for odd relative Wilson loop windings of the bands in P , $\mathcal{W}[P]$, while even relative Wilson loop windings of the bands can gap out (c). As long as the energy and spin gap (defined in Sec. I of the Supplemental Material [46]) stay open, however, the spin topology, captured by the spin-Chern number C_s , is well defined, and the spin-Wilson loop winding shown in panel (d) cannot gap. This has direct consequences for the lower bound of the integrated quantum metric, which is crucial for quantum metrology. Previously, only phases with $\nu_{Z_2} = 1$ [green in panel (e)] had a known lower bound. We formulate a new bound for phases with even C_s [orange in panel (e)], and extend the bound for phases with odd $C_s > 1$.

trivial Wilson loops) in the context of which we introduce a spin-resolved quantum geometry, we retrieve spin-topological bounds in ultrathin Bi [see Fig. 1(a)] with $\nu_{Z_2} = 0$ and in few-band effective Hamiltonians that can be engineered and used for spin-topological quantum sensing and optics.

Spin topology. We consider 2D spinful time-reversal symmetric ($\mathcal{T}^2 = -1$), periodic systems described by Bloch Hamiltonians $H(\mathbf{k})$. In a tight-binding basis with spinful orbitals $|\alpha\rangle \otimes |\sigma\rangle$, such Hamiltonians can be written as

$$H(\mathbf{k}) = \sum_{\alpha, \beta=1}^{N_{\text{orb}}} \sum_{\sigma, \sigma'=\uparrow, \downarrow} H_{\alpha\sigma, \beta\sigma'}(\mathbf{k}) [|\alpha\rangle \otimes |\sigma\rangle][\langle\beta| \otimes \langle\sigma'|]. \quad (1)$$

The spin operator along $\hat{n}$ associated with this basis is then given by $S_{\hat{n}} = 1_{\text{orb}} \otimes \hat{n} \cdot \sigma$. If $[H(\mathbf{k}), S_{\hat{n}}] = 0$, then the eigenstates of $H(\mathbf{k})$ are spin eigenstates, and can be labeled as $|\tilde{u}_{n\mathbf{k}}^\sigma\rangle$, where $n \in \{1, \dots, N_{\text{orb}}\}$. When this can be done for all $\mathbf{k}$ (i.e., there is no entanglement between spin and orbit degrees of freedom), one can define the multiband spin-Berry curvature:

$$\Omega_{ij, nm}^\sigma(\mathbf{k}) = i[\langle \partial_{k_i} \tilde{u}_{n\mathbf{k}}^\sigma | \partial_{k_j} \tilde{u}_{m\mathbf{k}}^\sigma \rangle - \text{c.c.}]. \quad (2)$$

Tracing over the occupied bands with spin σ , $N_{\text{occ}}^\sigma = N_{\text{occ}}/2$, this will be quantized when integrated over a 2D BZ, giving rise to the spin-resolved Chern numbers $C_s^\sigma = \frac{1}{2\pi} \int_{\text{BZ}} d\mathbf{k} \text{Tr} \Omega_{xy}^\sigma \in \mathbb{Z}$, with the overall spin-Chern number $C_s = \frac{1}{2}(C_s^\uparrow - C_s^\downarrow)$. Time-reversal symmetry ensures that the Chern numbers of the opposite spin bands satisfy $C_s^\uparrow = -C_s^\downarrow$ [40,45]. When spin is not conserved, one can instead formulate the equivalent quantities in terms of the eigenstates $|u_{n\mathbf{k}}^\sigma\rangle$ of the *projected* spin operator $S_P^\sigma = PS_zP$ [see Fig. 1(b)], where P is the projector onto the occupied bands. These eigenstates will not generically be energy eigenstates when spin is not a good quantum number, but $P = P_\uparrow + P_\downarrow$, where $P_\sigma = \sum_n^{N_{\text{occ}}^\sigma} |u_{n\mathbf{k}}^\sigma\rangle \langle u_{n\mathbf{k}}^\sigma|$ projects onto the top and bottom N_{occ}^σ states respectively; for a review of projected spin operators, see Ref. [44] and Sec. I of the Supplemental Material [46]. As we show in the following, formulating the spin-Berry curvature Ω_{xy}^σ in terms of projected spin eigenstates $|u_{n\mathbf{k}}^\sigma\rangle$ provides a local bound on the quantum metric, which holds at every point of momentum space, even when spin is not a good quantum number.

Spin-resolved quantum geometry. We now define the spin-resolved quantum geometry, which is fully characterized by a spin-resolved quantum geometric tensor (QGT), termed spin-QGT:

$$Q_{ij}^\sigma \equiv \sum_n^{N_{\text{occ}}^\sigma} \langle \partial_{k_i} u_{n\mathbf{k}}^\sigma | 1 - P_\sigma | \partial_{k_j} u_{n\mathbf{k}}^\sigma \rangle = g_{ij}^\sigma - \frac{i}{2} \Omega_{ij}^\sigma, \quad (3)$$

with $\sigma = \uparrow, \downarrow$ and spin-resolved quantum metric (spin metric) g_{ij}^σ . The spin-QGT definitionally satisfies a positive semidefiniteness condition $\det Q_{ij}^\sigma > 0$, and as a consequence, we derive the spin-geometric bound (see Sec. II of the Supplemental Material [46]):

$$\frac{1}{2} \text{Tr} g_{ij}^\sigma \geq \sqrt{\det g_{ij}^\sigma} \geq \frac{|\text{Tr} \Omega_{xy}^\sigma|}{2}. \quad (4)$$

When spin- z is a good quantum number, we further show that the quantum metric $g_{ij} = \frac{1}{2} \text{Tr} [\partial_{k_i} P \partial_{k_j} P] = g_{ij}^\uparrow + g_{ij}^\downarrow$ [12] (see Sec. III of the Supplemental Material [46]). On combining with the spin-resolved bounds and integrating, one obtains [47]

$$\frac{1}{2\pi} \int_{\text{BZ}} d^2\mathbf{k} (g_{xx} + g_{yy}) \geq |C_s^\uparrow| + |C_s^\downarrow| \equiv S_{\lambda=0} \in \mathbb{Z}. \quad (5)$$

We next focus on the case where spin- z is no longer a good quantum number, i.e., $[H, S_z] \neq 0$, but the spin gap is preserved [44,48]. In this case, utilizing the Cauchy-Schwarz inequality (see Sec. III of the Supplemental Material [46]) allows us to derive a fundamental bound with a suppression by the number of occupied bands, N_{occ} , due to the spin-orbit

entanglement (see Sec. IV of the Supplemental Material [46]),

$$\frac{1}{2\pi} \int_{\text{BZ}} d^2\mathbf{k} (g_{xx} + g_{yy}) \geq \frac{|C_s^\uparrow| + |C_s^\downarrow|}{N_{\text{occ}}} \equiv S_{\lambda \neq 0} \in \mathbb{Q}, \quad (6)$$

where on the right hand side we have a rational number $S_{\lambda \neq 0}$. Intuitively, the suppression by the factor N_{occ} arises due to the mixing of all occupied bands with spin-orbit coupling (SOC) while preserving the winding of the spin-Wilson loop realized by the PS_zP eigenstates. For odd spin-Chern numbers, our bound is consistent with the $\mathbb{Z}_2$ bound of Ref. [25], yet realizes a possible generalization to tighter inequalities for higher odd spin-Chern numbers, e.g., $|C_s| = 3$, depending on the ratio of $|C_s|$ and N_{occ} . Furthermore, our bound provides for a more general fundamental quantum geometric condition, even if the $\mathbb{Z}_2$ invariant is trivial, i.e., when $C_s^\sigma \equiv 0 \pmod{2}$. Namely, for even spin-Chern numbers, Eq. (6) is the first topological bound due to spin topology, yet accompanied by trivial band topology. We further detail direct consequences of the spin-geometric bounds, which we further demonstrate to apply to spin-topological materials such as ultrathin Bi.

Spin-resolved geometry in optics. We proceed to show that as a direct physical consequence, the spin-resolved quantum geometry has direct manifestations in optical responses of materials with nontrivial spin-resolved topology, such as bismuth central to this work. The integral of the quantum metric amounts to the integral of the optical weight due to the optical conductivities $\sigma_{xx}(\omega)$, $\sigma_{yy}(\omega)$ at frequency ω [26,47], which therefore satisfies the spin-topological bounds,

$$\int_0^\infty d\omega \frac{\text{Re}[\sigma_{xx}(\omega) + \sigma_{yy}(\omega)]}{\omega} \geq \frac{e^2}{2\hbar} S_\lambda. \quad (7)$$

We further retrieve an optical bound on the gap of spin-topological systems (see Sec. V of the Supplemental Material [46]),

$$E_g \leq \frac{2\pi n \hbar^2}{m S_\lambda}, \quad (8)$$

where n is electron charge density, m is electron mass, and E_g is the energy band gap. Our bounds hold also in the $\mathbb{Z}_2$ -trivial systems, i.e., where the Kane-Mele invariant is vanishing $\nu_{\mathbb{Z}_2} = 0$, placing our bound beyond the findings of Refs. [25,47]. We, moreover, validate our quantum geometric bounds in real material settings of ultrathin Bi, which we retrieve below.

Quantum geometry in spin-topological Bi. We now demonstrate the interplay of spin topologies with spin-resolved quantum geometries in the material context of ultrathin elemental bismuth. As a consequence, we further validate and confirm that the optical weights and the band gaps are topologically bounded. The associated first-principles calculations are detailed in Sec. VI of the Supplemental Material [46].

In Fig. 2, we consider ultrathin Bi in the puckered $Pmn2_1$ phase, as studied theoretically in Ref. [50] and experimentally in Ref. [49]. We find that it has spin-Chern numbers $|C_s| = 2$, as previously reported in Refs. [41,51]. As a consequence of the nontrivial spin topology, the spin-Wilson loop winds as shown in Fig. 2(d). Having confirmed the nontrivial spin topology, we observe that the spin-Berry curvatures

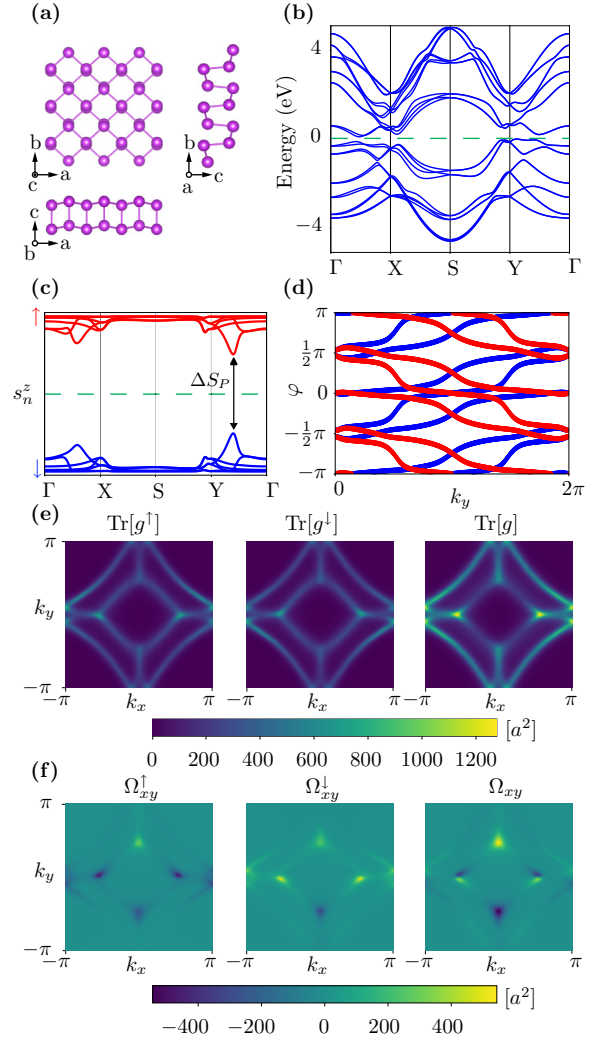


FIG. 2. Quantum (spin) geometry in ultrathin Bi. (a) Crystal structure corresponding to the puckered $Pmn2_1$ phase [49] of ultrathin Bi. (b) Band structure. (c) Spin-z band structure with manifestly large spin gap ΔS_P . (d) Topological winding of projected spin operator PS_zP eigenstates in a spin-Wilson loop. (e) Metrics where we observe that the quantum metric (g_{ij}) pattern is supported by the spin-resolved quantum metrics $g_{ij}^\uparrow$, $g_{ij}^\downarrow$. The spin-topological geometric bound is satisfied at every point within the BZ, with spin-Berry curvatures bounding the spin-resolved quantum metric from below (f). We find as expected that $(2\pi)^{-1} \int_{\text{BZ}} \Omega_{xy}^\uparrow = -(2\pi)^{-1} \int_{\text{BZ}} \Omega_{xy}^\downarrow = 2$, whereas $(2\pi)^{-1} \int_{\text{BZ}} \Omega_{xy} = 0.0$. In contrast, $(2\pi)^{-1} \int_{\text{BZ}} \text{Tr}[g^\uparrow] = (2\pi)^{-1} \int_{\text{BZ}} \text{Tr}[g^\downarrow] = 15.1$, whereas $(2\pi)^{-1} \int_{\text{BZ}} \text{Tr}[g] = 25.6$, demonstrating an enhanced quantum geometry in spin-topological Bi with $|C_s^\sigma| = 2$.

manifestly provide a local, i.e., holding at every point of the momentum space, bound on the spin-resolved quantum metrics [Figs. 2(e) and 2(f)]. As a consequence, the total quantum metric realized by the electrons in the ground state of the material is lower-bounded by the $\mathbb{Z}$ index protected by the spin gap ΔS_P [Fig. 2(c)], and we, moreover, find that the band gap of Bi (E_g) satisfies the previously derived spin-topological quantum geometric bound. Intuitively, the large quantum met-

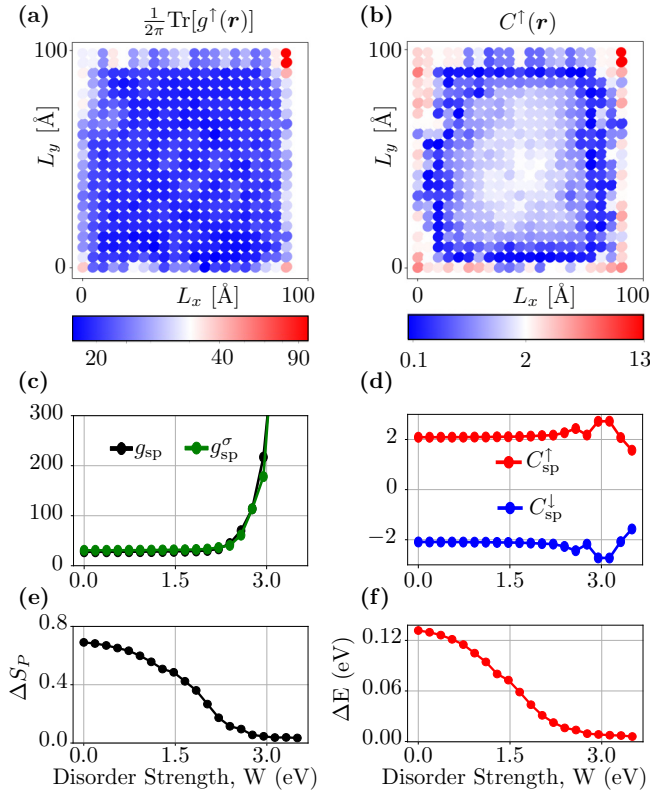


FIG. 3. Robustness of spin-topologically bounded quantum geometry in the presence of disorder. (a) Spin-resolved metric markers $\frac{1}{2\pi} \text{Tr}[g^\sigma(\mathbf{r})]$ and (b) spin-Chern markers $C^\sigma(\mathbf{r})$ in disordered ultrathin Bi flake system of 1746 atoms, with random potential disorder of strength $W = 1.5$ eV. Averaging over 120 atoms in the center yields: $\frac{1}{2\pi} \text{Tr}[g^\uparrow(\mathbf{r})] = 20.2$ and $C^\uparrow(\mathbf{r}) = 1.8$. (c) Scaling of single-point metrics $g_{\text{sp}}^{(\sigma)} \equiv \frac{1}{2\pi} \text{Tr}[g_{ij}^{(\sigma)}]_{\text{sp}}$, (d) single-point spin-Chern numbers C_{sp}^σ , (e) spin gap ΔS_P , and (f) energy gap ΔE against disorder strength W in ultrathin Bi system of 784 atoms, averaged over 20 disorder realizations.

ric in Bi, beyond the saturation of topological bound, likely arises in part due to positions of the orbitals within the unit cell [52].

Effects of disorder. We further validate the robustness of these results under the disorder present in realistic materials. In Fig. 3, we demonstrate that the bounds are stable up to disorder-induced delocalization transitions. We confirm the robustness of geometric bounds for model Bi nanoflakes with an evaluation of real-space spin-geometric and spin-Chern markers equivalent to the momentum-space invariants under the change of basis [53] [see Figs. 3(a) and 3(b)], and single-point spin-Chern numbers [54] and metrics equal to the momentum-space definitions in a supercell within a single-point ($\mathbf{k} = 0$) integral approximation [55] [see Figs. 3(c) and 3(d) and Appendixes A and B for technical details]. Remarkably, we find that no additional symmetries are needed to protect the bound against the naturally present disorder, as long as the spin and energy gaps are preserved [see Figs. 3(e) and 3(f)], which is a promising indication that the retrieved geometric bounds hold in real materials under experimental conditions. Further details

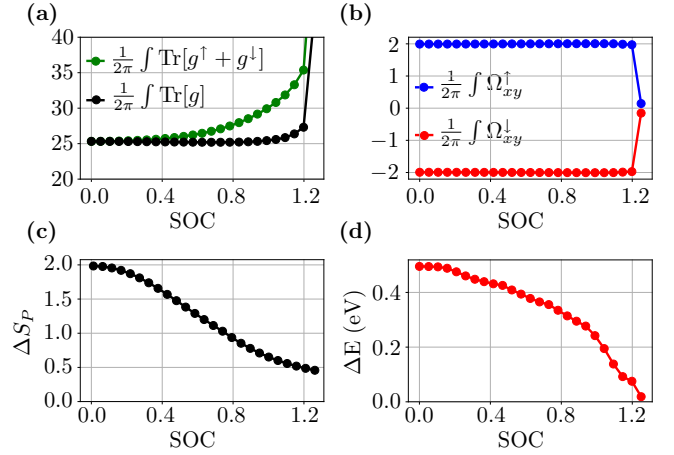


FIG. 4. Scaling behavior of the quantum metric in ultrathin bismuth with SOC, where SOC = 1 corresponds to the unperturbed Bi (for further information, see Sec. VI of the Supplemental Material [46]). We observe that (a) the metric diverges close to the critical value SOC = 1.2, where the spin-Chern numbers (b) become ill defined. In this transition, the spin gap ΔS_P (c) remains well defined up to the critical point, while the energy gap ΔE (d) closes. The analytical argument for the metric scaling is detailed in Sec. VII of the Supplemental Material [46].

on the spin-geometric markers and disorder calculations are provided in Secs. IX and X of the Supplemental Material [46], respectively.

Spin-resolved quantum metrology. We now discuss how the spin-resolved quantum geometry can be utilized for quantum metrology [33], providing for a set of applications within quantum sensing [31], and opening a new direction of creating topological quantum sensors with time-reversal symmetric spin-topological materials, given their topologically bounded quantum geometry. As such an implementation of a spin-topological Hamiltonian in a quantum, e.g., free fermion, sensor [32], offers for a topologically $\mathbb{Z}$ -bounded enhancement of measurement sensitivity in realistic $\mathcal{T}$ -symmetric, spinful systems. Quantum sensing relies on the uncertainty in the Cramér-Rao bounds [31], which by relation to the quantum Fisher information [33] can be improved with enhanced quantum metric. We demonstrate the scaling of the quantum sensing-adaptable metric against the SOC realized in the spin-topological Bi Hamiltonian (Fig. 4), which is controllable with external parameters, e.g., bending [56], thus reducing uncertainty in the Cramér-Rao bounds for quantum sensing (see also Appendix C).

Discussion. Below we discuss our results and their applicability. We first stress that the derivation of the here introduced spin-topological bounds was possible because of the presence of a spin gap, which defines the $P_\uparrow, P_\downarrow$ eigenstate manifolds. Closing the spin gap amounts to the trivialization of the here-derived geometric bounds, as well as of the spin-Chern numbers C_s^σ [40,45]. Moreover, the derivation of quantum geometric bounds is possible without a direct use of energy eigenstates; previously, a similar strategy was implemented for deriving geometric bounds in topological Euler bands [18,22,23], where auxiliary states rather than eigenstates were utilized for the implementation of a

geometric positive semidefiniteness condition, before ultimately connecting to the energy eigenstates via a unitary transformation. It should be noted that our bounds, rather than being of $\mathbb{Z}_2$ type, i.e., given by topological $\mathbb{Z}_2$ indices, as previously retrieved in electronic [25,57] and excitonic contexts [58,59], fundamentally belong to the class of $\mathbb{Z}$ bounds [18–20,22,23]. Having derived new geometric optical bounds due to spin topology, a question of their experimental validation naturally arises. In that regard, quantum geometry was shown to be measurable with inelastic X-ray scattering [60] and photoemission [61], which also applies to the measurements of the quantum metric induced by spin topology studied in this work. A further experimental study of an interplay of the spin-resolved quantum geometry, spin-density response, and entanglement provides for an interesting future pursuit, in particular, given the robustness of our findings in the presence of ubiquitous disorders. Finally, we stress that while preserving a $\mathbb{Z}$ character, our bounds applicable to quantum metrology and quantum sensing remain nontrivial in the presence of time-reversal symmetry, unlike the previous bounds for topological magnetic systems [28,32]. Hence, our bounds, adaptable even for four-state systems as discussed in the Supplemental Material [46], could enable a class of topological quantum sensors operating without magnetic environments.

Conclusions and outlook. We propose an avenue for probing a range of spin-topological systems from the perspective of quantum geometric relations, which notably, for a vanishing $\mathbb{Z}_2$ index, were previously not targetable with observables. We derive a set of spin-topological $\mathbb{Z}$ bounds, even for eigenstates that are trivial from a Wilson loop perspective, which provide optical constraints on optical weights and band gaps. We show that the bounds are valid in the presence of disorder and up to moderate spin-orbit entanglements, i.e., as long as the spin gap is preserved. We validated these quantum geometric conditions in bismuth and in an effective Hamiltonian with high spin-Chern number. By deriving quantum geometric relations central to quantum metrology, we show that quantum geometry bounded by the spin topology could enable spin-topological quantum sensors.

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Data availability. The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

Appendix A: Spin-geometric and topological markers in disordered systems. We here provide minimal technical details on the spin-geometric real-space markers and their interplay with the spin-Chern markers. For full technical details, see Sec. IX of the Supplemental Material [46].

The local spin-metric marker, which we define in real space under open boundary conditions for disordered spin-

topological systems studied in this work, reads

$$g_{ij}^{\sigma}(\mathbf{r}) = \frac{1}{2A_{\text{cell}}} \sum_{\alpha} \langle r_{\alpha} | P_{\sigma} \{ P_{\sigma} \hat{x}_i P_{\sigma}, P_{\sigma} \hat{x}_j P_{\sigma} \} | r_{\alpha} \rangle. \quad (\text{A1})$$

$|r_{\alpha}\rangle$ is a real-space physical orbital α at a unit cell centered at the position vector $\mathbf{r}$, A_{cell} is the area of a unit cell, and $\{\dots, \dots\}$ denotes an anticommutator of spin-projected operators $P_{\sigma} \hat{x}_i P_{\sigma}$, with spin-projector P_{σ} in real-space basis and position operator components $\hat{x}_i$. The spin-Chern marker is defined as [53]

$$C^{\sigma}(\mathbf{r}) = \frac{4\pi}{A_{\text{cell}}} \Im \sum_{\alpha} \langle r_{\alpha} | P_{\sigma} [\hat{x}, P_{\sigma}], [\hat{y}, P_{\sigma}] | r_{\alpha} \rangle. \quad (\text{A2})$$

On defining $\text{Tr}[g^{\sigma}(\mathbf{r})] \equiv g_{xx}^{\sigma}(\mathbf{r}) + g_{yy}^{\sigma}(\mathbf{r})$, the spin-metric marker and spin-Chern marker satisfy a topological bound consistent with the results presented in Fig. 3,

$$\sum_{i \in \text{cells}} \frac{1}{2\pi} \text{Tr}[g^{\sigma}(\mathbf{r}_i)] \geq \sum_{i \in \text{cells}} |C^{\sigma}(\mathbf{r}_i)|, \quad (\text{A3})$$

which we rigorously derive in Sec. IX of the Supplemental Material. Supplementary calculations with spin-metric and spin-Chern markers are presented in Sec. X of the Supplemental Material.

Appendix B: Single-point spin-geometric and spin-topological invariants in disordered systems. In the following, we outline minimal technical details on the single-point spin metric and its interplay with the single-point spin-Chern invariant [54]. For full technical details, see Sec. IX of the Supplemental Material.

The single-point formulation relies on approximating the momentum-space invariants, as in Eq. (5), with a single k -point sampling at a supercell BZ center ($\mathbf{k} = \mathbf{0}$), i.e., the Γ -point, assuming that the finite-size corrections of the shrunk BZ can be neglected [55]. In the limit of infinite (disordered) supercell, the supercell BZ shrinks to the Γ -point, justifying the approximation.

In this limit, the single-point spin-Chern numbers C_{sp}^{σ} ($C_s^{\sigma} = \frac{1}{2\pi} \int_{\text{BZ}} d\mathbf{k} \text{Tr} \Omega_{xy}^{\sigma} \rightarrow C_{\text{sp}}^{\sigma}$) are defined as [54]

$$C_{\text{sp}}^{\sigma} \equiv -\frac{|\mathbf{b}_1||\mathbf{b}_2|}{\pi} \sum_n^{N_{\text{occ}}} \Im \langle \partial_{k_x} u_{n\mathbf{k}=\Gamma}^{\sigma} | \partial_{k_y} u_{n\mathbf{k}=\Gamma}^{\sigma} \rangle, \quad (\text{B1})$$

where $\mathbf{b}_1, \mathbf{b}_2$ are reciprocal lattice vectors of a two-dimensional supercell. Moreover, we here define the single-point (spin) metric

$$[g_{ij}^{(\sigma)}]_{\text{sp}} \equiv 2|\mathbf{b}_1||\mathbf{b}_2| \sum_n^{N_{\text{occ}}} \Re \langle \partial_{k_i} u_{n\mathbf{k}=\Gamma}^{(\sigma)} | \partial_{k_j} u_{n\mathbf{k}=\Gamma}^{(\sigma)} \rangle, \quad (\text{B2})$$

with $\frac{1}{2\pi} \text{Tr}[g_{ij}^{(\sigma)}]_{\text{sp}} \equiv \frac{1}{2\pi} ([g_{xx}^{(\sigma)}]_{\text{sp}} + [g_{yy}^{(\sigma)}]_{\text{sp}})$. We retrieve a bound inherited from the local momentum-space inequality, Eq. (4):

$$\frac{1}{2\pi} \text{Tr}[g_{ij}^{\sigma}]_{\text{sp}} \geq |C_{\text{sp}}^{\sigma}|. \quad (\text{B3})$$

Further details on the evaluation of numerical derivatives $|\partial_{k_i} u_{nk=\Gamma}^\sigma\rangle$ are provided in Ref. [54], and the further simplified expressions are detailed in Sec. IX of the Supplemental Material [46].

Appendix C: Quantum Cramér-Rao bound and quantum sensing. The quantum metric, through its relations to the quantum Fisher information, sets a lower bound on multiparameter estimation with pure states, as captured by the so-called quantum Cramér-Rao bound (QCRB), that is of central interest for quantum metrology and can be used in quantum sensing [31]. In the following, we consider pure states $|\phi_{\mathbf{k}}\rangle$ with density matrix $\rho = |\phi_{\mathbf{k}}\rangle\langle\phi_{\mathbf{k}}|$ constituted by the coherent superpositions of occupied single-particle eigenstates (see Sec. VII of the Supplemental Material [46] for more details). We furthermore provide a four-band model for $\mathbb{Z}$ -indexed spin topology in Sec. VIII of the Supplemental Material [46], which can be adapted for qudits or coupled qubits, and where the role of Bloch momenta can be taken by two angles, i.e., $(k_x, k_y) \rightarrow (\phi, \theta)$. The QCRB, given for a pure state by the quantum

metric, reads [33]

$$\sqrt{\det \Sigma_{ij}} \geq \frac{1}{4M\sqrt{\det g_{ij}}}, \quad (\text{C1})$$

where M is the number of measurements and we introduce the covariance matrix Σ_{ij} for an unbiased two-parameter estimator $\mathbf{k}$. Here, the estimator that was identified with the angular parameters takes values in the two-parameter family over the two-torus $[\mathbf{k} = (k_x, k_y) \in T^2]$ under a set of positive operator-valued measurements, Π_p yielding the expected measurement values of individual parameters $\bar{k}_i$, with $i = x, y$. Explicitly, the covariance matrix reads [33]

$$\Sigma_{ij} = \langle \delta k_i \delta k_j \rangle \equiv \sum_p (k_i - \bar{k}_i)(k_j - \bar{k}_j) \text{Tr}[\rho \Pi_p], \quad (\text{C2})$$

which quantifies the uncertainty in two measured parameters k_i, k_j . As we show numerically in spin-topological Hamiltonians (Fig. 4), Σ_{ij} can be topologically suppressed by the metric values provided by our $\mathbb{Z}$ bound, for spin-topological quantum sensors with topologically reduced uncertainties.

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- [1] X.-L. Qi and S.-C. Zhang, Topological insulators and superconductors, *Rev. Mod. Phys.* **83**, 1057 (2011).
 - [2] M. Z. Hasan and C. L. Kane, Colloquium: Topological insulators, *Rev. Mod. Phys.* **82**, 3045 (2010).
 - [3] N. P. Armitage, E. J. Mele, and A. Vishwanath, Weyl and Dirac semimetals in three-dimensional solids, *Rev. Mod. Phys.* **90**, 015001 (2018).
 - [4] R.-J. Slager, A. Mesaros, V. Juričić, and J. Zaanen, The space group classification of topological band-insulators, *Nat. Phys.* **9**, 98 (2013).
 - [5] J. Kruthoff, J. de Boer, J. van Wezel, C. L. Kane, and R.-J. Slager, Topological classification of crystalline insulators through band structure combinatorics, *Phys. Rev. X* **7**, 041069 (2017).
 - [6] H. C. Po, A. Vishwanath, and H. Watanabe, Symmetry-based indicators of band topology in the 230 space groups, *Nat. Commun.* **8**, 50 (2017).
 - [7] R.-J. Slager, The translational side of topological band insulators, *J. Phys. Chem. Solids* **128**, 24 (2019).
 - [8] B. Bradlyn, L. Elcoro, J. Cano, M. G. Vergniory, Z. Wang, C. Felser, M. I. Aroyo, and B. A. Bernevig, Topological quantum chemistry, *Nature (London)* **547**, 298 (2017).
 - [9] L. Fu, Topological crystalline insulators, *Phys. Rev. Lett.* **106**, 106802 (2011).
 - [10] Q. Niu, D. J. Thouless, and Y.-S. Wu, Quantized Hall conductance as a topological invariant, *Phys. Rev. B* **31**, 3372 (1985).
 - [11] F. D. M. Haldane, Model for a quantum Hall effect without Landau levels: Condensed-matter realization of the “parity anomaly”, *Phys. Rev. Lett.* **61**, 2015 (1988).
 - [12] J. Provost and G. Vallee, Riemannian structure on manifolds of quantum states, *Commun. Math. Phys.* **76**, 289 (1980).
 - [13] N. Marzari and D. Vanderbilt, Maximally localized generalized Wannier functions for composite energy bands, *Phys. Rev. B* **56**, 12847 (1997).
 - [14] R. Resta, The insulating state of matter: A geometrical theory, *Eur. Phys. J. B* **79**, 121 (2011).
 - [15] J. Ahn, G.-Y. Guo, and N. Nagaosa, Low-frequency divergence and quantum geometry of the bulk photovoltaic effect in topological semimetals, *Phys. Rev. X* **10**, 041041 (2020).
 - [16] J. Ahn, G.-Y. Guo, N. Nagaosa, and A. Vishwanath, Riemannian geometry of resonant optical responses, *Nat. Phys.* **18**, 290 (2022).
 - [17] P. Törmä, Essay: Where can quantum geometry lead us? *Phys. Rev. Lett.* **131**, 240001 (2023).
 - [18] A. Bouhon, A. Timmel, and R.-J. Slager, Quantum geometry beyond projective single bands, *arXiv:2303.02180*.
 - [19] R. Roy, Band geometry of fractional topological insulators, *Phys. Rev. B* **90**, 165139 (2014).
 - [20] F. Xie, Z. Song, B. Lian, and B. A. Bernevig, Topology-bounded superfluid weight in twisted bilayer graphene, *Phys. Rev. Lett.* **124**, 167002 (2020).
 - [21] Y.-Q. Zhu, W. Zheng, S.-L. Zhu, and G. Palumbo, Band topology of pseudo-Hermitian phases through tensor Berry connections and quantum metric, *Phys. Rev. B* **104**, 205103 (2021).
 - [22] W. J. Jankowski, A. S. Morris, A. Bouhon, F. N. Ünal, and R.-J. Slager, Optical manifestations and bounds of topological Euler class, *Phys. Rev. B* **111**, L081103 (2025).
 - [23] S. Kwon and B.-J. Yang, Quantum geometric bound and ideal condition for Euler band topology, *Phys. Rev. B* **109**, L161111 (2024).
 - [24] W. J. Jankowski, A. S. Morris, Z. Davoyan, A. Bouhon, F. N. Ünal, and R.-J. Slager, Non-Abelian Hopf-Euler insulators, *Phys. Rev. B* **110**, 075135 (2024).
 - [25] J. Yu, J. Herzog-Arbeitman, and B. A. Bernevig, Universal Wilson loop bound of quantum geometry, *Phys. Rev. Lett.* **135**, 086401 (2025).
 - [26] I. Souza, T. Wilkens, and R. M. Martin, Polarization and localization in insulators: Generating function approach, *Phys. Rev. B* **62**, 1666 (2000).

- [27] M. Yu, Y. Liu, P. Yang, M. Gong, Q. Cao, S. Zhang, H. Liu, M. Heyl, T. Ozawa, N. Goldman, and J. Cai, Quantum Fisher information measurement and verification of the quantum Cramér–Rao bound in a solid-state qubit, *npj Quantum Inf.* **8**, 56 (2022).
- [28] B. Mera, A. Zhang, and N. Goldman, Relating the topology of Dirac Hamiltonians to quantum geometry: When the quantum metric dictates Chern numbers and winding numbers, *SciPost Phys.* **12**, 018 (2022).
- [29] M. Yu, X. Li, Y. Chu, B. Mera, F. N. Ünal, P. Yang, Y. Liu, N. Goldman, and J. Cai, Experimental demonstration of topological bounds in quantum metrology, *Nat. Sci. Rev.* **11** nwae065 (2024).
- [30] T. B. Wahl, W. J. Jankowski, A. Bouhon, G. Chaudhary, and R.-J. Slager, Exact projected entangled pair ground states with topological Euler invariant, *Nat. Commun.* **16**, 284 (2025).
- [31] C. L. Degen, F. Reinhard, and P. Cappellaro, Quantum sensing, *Rev. Mod. Phys.* **89**, 035002 (2017).
- [32] S. Sarkar, C. Mukhopadhyay, A. Alase, and A. Bayat, Free-fermionic topological quantum sensors, *Phys. Rev. Lett.* **129**, 090503 (2022).
- [33] J. Liu, H. Yuan, X.-M. Lu, and X. Wang, Quantum Fisher information matrix and multiparameter estimation, *J. Phys. A: Math. Theor.* **53**, 023001 (2020).
- [34] C. L. Kane and E. J. Mele, Quantum spin Hall effect in graphene, *Phys. Rev. Lett.* **95**, 226801 (2005).
- [35] C. L. Kane and E. J. Mele, Z_2 topological order and the quantum spin Hall effect, *Phys. Rev. Lett.* **95**, 146802 (2005).
- [36] S. Murakami, Quantum spin Hall effect and enhanced magnetic response by spin-orbit coupling, *Phys. Rev. Lett.* **97**, 236805 (2006).
- [37] I. K. Drozdov, A. Alexandradinata, S. Jeon, S. Nadj-Perge, H. Ji, R. J. Cava, B. Andrei Bernevig, and A. Yazdani, One-dimensional topological edge states of bismuth bilayers, *Nat. Phys.* **10**, 664 (2014).
- [38] S. Ito, B. Feng, M. Arita, A. Takayama, R.-Y. Liu, T. Someya, W.-C. Chen, T. Iimori, H. Namatame, M. Taniguchi, C.-M. Cheng, S.-J. Tang, F. Komori, K. Kobayashi, T.-C. Chiang, and I. Matsuda, Proving nontrivial topology of pure bismuth by quantum confinement, *Phys. Rev. Lett.* **117**, 236402 (2016).
- [39] L. Fu and C. L. Kane, Time reversal polarization and a Z_2 adiabatic spin pump, *Phys. Rev. B* **74**, 195312 (2006).
- [40] E. Prodan, Robustness of the spin-Chern number, *Phys. Rev. B* **80**, 125327 (2009).
- [41] B. Peng, G. F. Lange, D. Bennett, K. Wang, R.-J. Slager, and B. Monserrat, Photoinduced electronic and spin topological phase transitions in monolayer bismuth, *Phys. Rev. Lett.* **132**, 116601 (2024).
- [42] G. F. Lange, A. Bouhon, and R.-J. Slager, Spin texture as a bulk indicator of fragile topology, *Phys. Rev. Res.* **5**, 033013 (2023).
- [43] R. González-Hernández and B. Uribe, Average spin Chern number, *Phys. Rev. B* **110**, 125129 (2024).
- [44] K.-S. Lin, G. Palumbo, Z. Guo, Y. Hwang, J. Blackburn, D. P. Shoemaker, F. Mahmood, Z. Wang, G. A. Fiete, B. J. Wieder, and B. Bradlyn, Spin-resolved topology and partial axion angles in three-dimensional insulators, *Nat. Commun.* **15**, 550 (2024).
- [45] D. N. Sheng, Z. Y. Weng, L. Sheng, and F. D. M. Haldane, Quantum spin-Hall effect and topologically invariant Chern numbers, *Phys. Rev. Lett.* **97**, 036808 (2006).
- [46] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/zlxq-fxgc> for details on the properties of the projected spin operator (Sec. I), derivation of spin-resolved quantum geometric bounds (Sec. II), connections of spin-resolved geometry to quantum geometry (Sec. III), effects of spin-orbit coupling and spin-orbit entanglement (Sec. IV), optical responses due to spin-resolved quantum geometry (Sec. V), details of first-principles calculations (Sec. VI), quantum metrology with spin-resolved geometry (Sec. VII), spin-resolved geometry in minimal models realizing spin topology (Sec. VIII), real-space formulation of spin-resolved geometry and spin-topological bounds in the presence of disorder (Sec. IX), and numerical details on the disorder calculations (Sec. X). The Supplemental Material also contains Refs. [62–76].
- [47] Y. Onishi and L. Fu, Fundamental bound on topological gap, *Phys. Rev. X* **14**, 011052 (2024).
- [48] A. Bouhon, A. M. Black-Schaffer, and R.-J. Slager, Wilson loop approach to fragile topology of split elementary band representations and topological crystalline insulators with time-reversal symmetry, *Phys. Rev. B* **100**, 195135 (2019).
- [49] T. Nagao, J. T. Sadowski, M. Saito, S. Yaginuma, Y. Fujikawa, T. Kogure, T. Ohno, Y. Hasegawa, S. Hasegawa, and T. Sakurai, Nanofilm allotrope and phase transformation of ultrathin Bi film on Si(111)– 7×7 , *Phys. Rev. Lett.* **93**, 105501 (2004).
- [50] S. Singh, Z. Zanolli, M. Amsler, B. Belhadji, J. O. Sofo, M. J. Verstraete, and A. H. Romero, Low-energy phases of Bi monolayer predicted by structure search in two dimensions, *J. Phys. Chem. Lett.* **10**, 7324 (2019).
- [51] Y. Bai, L. Cai, N. Mao, R. Li, Y. Dai, B. Huang, and C. Niu, Doubled quantum spin Hall effect with high-spin Chern number in α -antimonene and α -bismuthene, *Phys. Rev. B* **105**, 195142 (2022).
- [52] S. H. Simon and M. S. Rudner, Contrasting lattice geometry dependent versus independent quantities: Ramifications for Berry curvature, energy gaps, and dynamics, *Phys. Rev. B* **102**, 165148 (2020).
- [53] N. Baù and A. Marrazzo, Theory of local Z_2 topological markers for finite and periodic two-dimensional systems, *Phys. Rev. B* **110**, 054203 (2024).
- [54] R. Favata and A. Marrazzo, Single-point spin Chern number in a supercell framework, *Electron. Struct.* **5**, 014005 (2023).
- [55] D. Ceresoli and R. Resta, Orbital magnetization and Chern number in a supercell framework: Single $\mathbf{k}$ -point formula, *Phys. Rev. B* **76**, 012405 (2007).
- [56] A. Johan Skarpeid, H. G. Hugdal, T. Salamone, M. Amundsen, and S. H. Jacobsen, Non-constant geometric curvature for tailored spin-orbit coupling and chirality in superconductor-magnet heterostructures, *J. Phys.: Condens. Matter* **36**, 235302 (2024).
- [57] J. Herzog-Arbeitman, V. Peri, F. Schindler, S. D. Huber, and B. A. Bernevig, Superfluid weight bounds from symmetry and quantum geometry in flat bands, *Phys. Rev. Lett.* **128**, 087002 (2022).
- [58] W. J. Jankowski, J. J. P. Thompson, B. Monserrat, and R.-J. Slager, Excitonic topology and quantum geometry in organic semiconductors, *Nat. Commun.* **16**, 4661 (2025).
- [59] J. J. P. Thompson, W. J. Jankowski, R.-J. Slager, and B. Monserrat, Topologically-enhanced exciton transport, [arXiv:2410.00967](https://arxiv.org/abs/2410.00967).

- [60] D. Balut, B. Bradlyn, and P. Abbamonte, Quantum entanglement and quantum geometry measured with inelastic X-ray scattering, *Phys. Rev. B* **111**, 125161 (2025).
- [61] M. Kang, S. Kim, Y. Qian, P. M. Neves, L. Ye, J. Jung, D. Puntel, F. Mazzola, S. Fang, C. Jozwiak, A. Bostwick, E. Rotenberg, J. Fuji, I. Vobornik, J.-H. Park, J. G. Checkelsky, B.-J. Yang, and R. Comin, Measurements of the quantum geometric tensor in solids, *Nat. Phys.* **21**, 110 (2025).
- [62] M. Lysne, M. Schüler, and P. Werner, Quantum optics measurement scheme for quantum geometry and topological invariants, *Phys. Rev. Lett.* **131**, 156901 (2023).
- [63] R. Bianco and R. Resta, Mapping topological order in coordinate space, *Phys. Rev. B* **84**, 241106 (2011).
- [64] T. Ozawa and N. Goldman, Extracting the quantum metric tensor through periodic driving, *Phys. Rev. B* **97**, 201117 (2018).
- [65] S. Kivelson, Wannier functions in one-dimensional disordered systems: Application to fractionally charged solitons, *Phys. Rev. B* **26**, 4269 (1982).
- [66] P. Giannozzi *et al.*, QUANTUM ESPRESSO: A modular and open-source software project for quantum simulations of materials, *J. Phys.: Condens. Matter* **21**, 395502 (2009).
- [67] P. Giannozzi, O. Baseggio, P. Bonfà, D. Brunato, R. Car, I. Carnimeo, C. Cavazzoni, S. de Gironcoli, P. Delugas, F. Ferrari Ruffino, A. Ferretti, N. Marzari, I. Timrov, A. Urru, and S. Baroni, QUANTUM ESPRESSO toward the exascale, *J. Chem. Phys.* **152**, 154105 (2020).
- [68] T. Sohler, M. Calandra, and F. Mauri, Density functional perturbation theory for gated two-dimensional heterostructures: Theoretical developments and application to flexural phonons in graphene, *Phys. Rev. B* **96**, 075448 (2017).
- [69] A. A. Mostofi, J. R. Yates, Y.-S. Lee, I. Souza, D. Vanderbilt, and N. Marzari, WANNIER90.: A tool for obtaining maximally-localised Wannier functions, *Comput. Phys. Commun.* **178**, 685 (2008).
- [70] A. A. Mostofi, J. R. Yates, G. Pizzi, Y.-S. Lee, I. Souza, D. Vanderbilt, and N. Marzari, An updated version of WANNIER90: A tool for obtaining maximally-localised Wannier functions, *Comput. Phys. Commun.* **185**, 2309 (2014).
- [71] G. Pizzi *et al.*, WANNIER90 as a community code: New features and applications, *J. Phys.: Condens. Matter* **32**, 165902 (2020).
- [72] J. Henke, M. Kurttutan, J. Kruthoff, and J. van Wezel, Topological invariants of rotationally symmetric crystals, *Phys. Rev. B* **104**, L201110 (2021).
- [73] X.-L. Qi, Y.-S. Wu, and S.-C. Zhang, Topological quantization of the spin Hall effect in two-dimensional paramagnetic semiconductors, *Phys. Rev. B* **74**, 085308 (2006).
- [74] A. Marrazzo and R. Resta, Local theory of the insulating state, *Phys. Rev. Lett.* **122**, 166602 (2019).
- [75] M. S. M. de Sousa, A. L. Cruz, and W. Chen, Mapping quantum geometry and quantum phase transitions to real space by a fidelity marker, *Phys. Rev. B* **107**, 205133 (2023).
- [76] N. Baù and A. Marrazzo, Local Chern marker for periodic systems, *Phys. Rev. B* **109**, 014206 (2024).