

Protocol for purifying noisy preparation and measurements of qubits

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Noise affecting qubit preparation and measurements accounts for a significant fraction of errors in quantum information processing. This is especially critical in tasks like variational quantum algorithms, quantum error correction, and entanglement distribution through repeaters. In this work, we present a protocol to purify noisy state preparation and measurement (SPAM), effectively suppressing these errors to an arbitrarily low level. For instance, in a realistic scenario where qubits contain error rates around 0.05 in both preparation and measurement, the protocol can suppress error rates up to 10^{-3} with two ancillas and 10^{-6} with four ancillas. We show how to distill error-free SPAM by repeating noisy SPAMs. The protocol is also feasible with superconducting qubits. We envisage that our results can be used to realize quantum information tasks in computing and communication with negligible SPAM errors.

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I. INTRODUCTION

To address the problem of noise inherent in quantum devices, it is increasingly important to mitigate or suppress errors due to noisy state preparation and measurements (SPAMs). For example, hybrid quantum-classical computing approaches—such as variational quantum algorithms, which are promising for near-term quantum technologies—rely on repeated SPAM operations [1,2]. In doing so, SPAM errors may propagate through hybrid quantum-classical computing. Furthermore, SPAM is an essential element in quantum error-correcting codes [3]. Having sufficiently low SPAM error rates is also significant for fault-tolerant quantum computing.

Significant efforts have been devoted to improving the readout of noisy measurements. The purification of noisy measurements has been proposed by analyzing measurement statistics in an information-theoretic manner, e.g., using maximum-likelihood information principles [4]. Active readout-error mitigation has been proposed by following advantage distillations or quantum error correction schemes [5–7]. It is also worth mentioning a recent proposal that attempts to simulate or reproduce other measurements, such as those without noise [8]. We remark that all of the aforementioned proposals exploit ancillary qubits prepared in a noiseless manner. It remains unclear whether noisy preparations of ancillary qubits can be used to suppress readout errors. A cost-effective protocol with a minimal number of ancillary qubits is also desired.

SPAM errors are due to the inability to realize a fiducial setting. A noiseless scenario with n qubits initialized in a fiducial state is denoted by $|0\rangle \otimes \cdots \otimes |0\rangle = |0\rangle^{\otimes n}$. Similarly, a measurement of each qubit in the computational basis follows as $\{|0\rangle, |1\rangle\}$ [9]. Noisy SPAMs lead to a deviation from the noiseless setting. For instance, a noisy initialization of a single qubit can be described by a mixed state $\rho_0 = f|0\rangle\langle 0| + (1 - f)|1\rangle\langle 1|$ with $f \in (1/2, 1]$, and suppressing the initialization error can be quantified by the fraction f .

In this work, we present a practical and cost-effective protocol for purifying noisy SPAM using noisy SPAM and a set of auxiliary qubits. The protocol is practical in that it works when both preparation and measurements of qubits are noisy, in particular, for error rates of present-day technologies (e.g., Ref. [10]), and cost-effective since it may efficiently work with even a single ancilla qubit under realistic conditions. We show that, under realistic assumptions, a few qubits are immediately cost-effective to reach an error rate as low as 10^{-3} . The proposed protocol can suppress SPAM errors to arbitrarily low values. We propose an experimental implementation of the protocol for superconducting qubits, leveraging existing resources already available on state-of-the-art processors, and thus remaining resource-friendly. We then consider the purification of noisy SPAM for key tasks in a quantum network: One is two-way entanglement distillation protocols where local measurements can be noisy, and the other is a measurement for entanglement swapping in a quantum repeater. Thus, our results envisage an effectively noiseless quantum network with noisy SPAM.

II. SPAM ERRORS

Throughout this work, we define a noiseless scenario as one in which a quantum system is initialized in state $|0\rangle$,

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and measurements are performed in the computational basis $\{M_0, M_1\}$, where $M_k = |k\rangle\langle k|$ for $k = 0, 1$. We denote a noisy state preparation by the density matrix ρ , given by

$$\rho = \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix}. \quad (1)$$

The fidelity of the preparation is quantified by $f = \langle 0|\rho|0\rangle$. Similarly, we describe noisy measurements using positive operator-valued measure (POVM) elements $\tilde{M}_k$ corresponding to outcomes $k \in \{0, 1\}$. The measurement noise is characterized by a noise fraction $q = \langle \bar{k}|\tilde{M}_k|\bar{k}\rangle$, where $\bar{k} = k \oplus 1$ denotes bitwise addition. Throughout, we assume $q \in [0, 1/2]$, corresponding to a measurement that performs better than random guessing. A measurement with $q > 1/2$ can always be converted to one with $q < 1/2$ by relabeling outcomes.

Without loss of generality, we assume that noisy POVM elements are diagonal as follows:

$$\tilde{M}_k = (1 - q)M_k + qM_{k\oplus 1}, \quad \text{for } k = 0, 1. \quad (2)$$

This is because one can diagonalize a POVM element by incorporating random Pauli-Z gates as follows. For a qubit POVM element $\hat{M}_k$ having a noise fraction q , it holds that

$$\frac{1}{2}(\hat{M}_k + Z\hat{M}_kZ) = (1 - q)M_k + qM_{k\oplus 1}, \quad \text{for } k = 0, 1.$$

The left-hand side above is equivalent to cases where one randomly applies a Pauli-Z gate to a state before a measurement is performed. A POVM element resulting from random applications of a Pauli-Z gate is diagonalized while the noise fraction q remains unaltered. That is, a POVM element can be generally diagonalized, and it suffices to consider noisy POVMs in Eq. (2).

III. PROTOCOL FOR PURIFYING QUBIT-STATE PREPARATION

We propose a protocol for purifying noisy qubit-state preparations, with the goal of producing an initial state whose fidelity f approaches unity arbitrarily closely. The protocol requires n ancillary qubits and a collective CNOT gate on $(n + 1)$ qubits, defined as

$$V_{n+1}^{SA_1 \dots A_n} = |0\rangle\langle 0|^S \otimes \mathbb{1}^{\otimes n} + |1\rangle\langle 1|^S \otimes X^{\otimes n}, \quad (3)$$

where X denotes a Pauli-X gate. This collective operation V_n can be implemented as a sequence of standard two-qubit CNOT gates, $V_{n+1}^{SA_1 \dots A_n} = \prod_{i=1}^n V_2^{SA_i}$ (see Fig. 1), where $V_2^{SA_i}$ is a CNOT gate with the system qubit S as control and the auxiliary qubit A_i as target. Following the application of the collective CNOT gates, noisy measurements are performed on n target qubits [Eq. (2)]. The resulting state in the control register is accepted, denoted by $\rho^{(n)}$, only if all n measurement outcomes of the target qubits are zeros, meaning the output string is 0^n . It can be shown that the fidelity of the postselected state satisfies $f^{(n)} = \langle 0|\rho^{(n)}|0\rangle \rightarrow 1$, demonstrating that the protocol enables arbitrarily accurate state preparation. A detailed derivation is provided in Appendix A.

Proposition 1. For all $\epsilon > 0$, there exists $N \geq 1$ such that $|f^{(n)} - 1| < \epsilon$ whenever $n > N$.

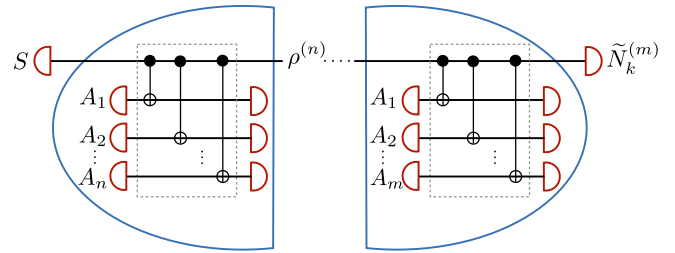


FIG. 1. Repeated noisy SPAM can purify noisy SPAM. The protocol starts with a collective CNOT applied to noisy qubits $SA_1, \dots, A_n$. The resulting state $\rho^{(n)}$ is accepted once the outcomes of the noisy qubit measurements $|A_1, \dots, A_n\rangle$ are all zeros 0^n . The fidelity converges to 1 as n increases. A noisy measurement on a system S is purified with noisy SPAM with m target qubits; a purified POVM element giving an outcome k is denoted by $\tilde{N}_k^{(m)}$, having a noise fraction converging to 0.

IV. PROTOCOL FOR PURIFYING NOISY MEASUREMENTS

The purification protocol of noisy qubit measurements aims to approximate the results of noiseless measurements in a computational basis. We show that the protocol can purify a noisy measurement with POVM elements in Eq. (2) by suppressing an error rate q .

The purification protocol, depicted in Fig. 1, comprises noisy qubit-state preparations ρ and noisy qubit-state measurements in Eq. (2). To purify a noisy measurement of state σ , m ancillary qubits are initialized in $\rho^{\otimes m}$. A collective CNOT V_m is applied to $\sigma \otimes \rho^{\otimes m}$. Subsequently, measurements are performed on each of the $(m + 1)$ qubits individually, as described in Eq. (2). The results are accepted only if the measurement results are identical, either 0^{m+1} or 1^{m+1} .

Suppose that the measurements are identical and accepted. We write that $\tilde{N}_k^{(m)}$ is a POVM element that yields k in the first register, since the results $m + 1$ are identical. The noisy fraction for the POVM element is given by $q^{(m)} = \langle \bar{k}|\tilde{N}_k^{(m)}|\bar{k}\rangle$ and it turns out that $q^{(m)} \rightarrow 0$ as m increases. The derivation is shown in Appendix B.

Proposition 2. For all $\epsilon > 0$, there exists $N \geq 1$ such that $q^{(m)} < \epsilon$ whenever $m > N$.

The results in Propositions 1 and 2 extend to cases where ancilla qubits are prepared with different values of preparation fidelity f_i and measurement noise q_i . The convergence for the purification, $f^{(n)} \rightarrow 1$ and $q^{(m)} \rightarrow 0$, holds whenever the noise parameters satisfy $f_i > 1/2$ and $q_i < 1/2$ for each i . The proof is detailed in Appendix B.

We note that the purification protocol is distinct from *quantum error mitigation* [11,12] in the following sense. The error-mitigation protocols aim to estimate noiseless expectation values of observables by repeating experiments. The proposed protocol attempts to purify noisy SPAM and aims to achieve noiseless statistics. As for shortcomings, the mitigation technique mentioned above may suffer from the costs of sampling [13], whereas the proposed purification protocol may have low success probabilities since it relies on postselection. In Table IV, we consider a realistic scenario where SPAM errors are about 5%–10%, and show that one or two ancillary qubits are sufficient to reach an error rate up to

TABLE I. Purification of noisy measurements with noisy SPAM is demonstrated. Note that $T^{(n)}$ [see Eq. (4)] is a trace distance between a noiseless POVM element and a purified one with n ancilla qubits, and $p_s^{(n)}$ denotes the success probability.

Error rate	$T^{(0)}$	$T^{(1)}$	$p_s^{(1)}$	$T^{(2)}$	$p_s^{(2)}$
$1 - f = q = 0.1$	0.1	0.024	0.756	0.005	0.608
$1 - f = q = 0.05$	0.05	0.005	0.864	0.001	0.779

10^{-3} . Note also that success probabilities are higher than 0.5. Further analysis with realistic parameters is provided in Appendix C. In fact, in what follows, we show that the resource requirement to suppress SPAM errors up to a desired level, for example, 10^{-3} or even less, can be achieved with a few ancilla qubits.

V. REALISTIC SCENARIOS

To quantify a purification of noisy measurements, we use the trace distance, $T(A, B) = \frac{1}{2} \|A - B\|_1$ for Hermitian operators A and B , where $\|X\|_1 = \text{tr} \sqrt{X^\dagger X}$.

For a purified POVM element $\tilde{N}_k^{(m)}$ with m additional qubits, the quantification may be addressed for $\delta > 0$ such that

$$T^{(m)} := T(\tilde{N}_i^{(m)}, M_i) \leq \delta, \quad \text{for } i = 0, 1. \quad (4)$$

Note also that the distance above corresponds to the noise fraction, that is, $q^{(m)} = T^{(m)}$. Equivalently, one can find the fidelity $F(\tilde{N}_i^{(m)}, M_i) = 1 - T^{(m)} \geq 1 - \delta$ since M_i to be rank 1. The other parameter is the probability of success that the measurement outcomes are identical, which can be computed as $p_{\text{succ}}^{(m)} = \alpha^m (1 - q) + (1 - \alpha)^m q$, where $\alpha = f(1 - q) + (1 - f)q$.

State-of-the-art quantum processors suffer from SPAM errors on the order of 1%–10% [10,14]. Assuming that error rates are balanced, that is, $1 - f = q \approx 5\%$, we find that an error rate is suppressed up to 10^{-3} , i.e., $1 - f^{(n)} = q^{(n)} < 10^{-3}$ for $n \geq 2$. The probability that three measurements are identical, that is, $n = 2$, is 0.7785, see Table I. Therefore, the protocol with a few ancillary qubits is immediately efficient and cost-effective. More instances of the resource requirements of the purification protocol are presented in Appendix C.

We note that the balanced error rates are assumed for the purpose of demonstrating the purification of noisy resources. We recall that the convergence in the purification is shown for general values of f and q in the propositions above. More instances when the parameters are not balanced, i.e., $1 - f \neq q$, are demonstrated in Appendix D.

VI. EXPERIMENTAL FEASIBILITY

To demonstrate the feasibility of the proposed protocol, we present a blueprint tailored to state-of-the-art superconducting quantum processors that is resource-efficient and readily implementable. Superconducting qubits are among the leading platforms for the realization of scalable quantum information processors [10,14,15]. Despite advances in superconducting qubit architectures, the fidelity of qubit-state preparation is

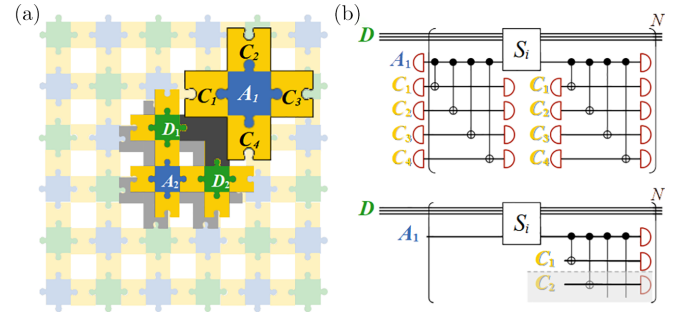


FIG. 2. SPAM purification protocol on a superconducting quantum processor encoded with surface code. (a) Data qubits (D, green), ancilla qubits (A, blue), and tunable couplers (C, yellow) are represented as puzzle pieces. (b) Top: Full protocol implementation using all four tunable couplers as ancilla qubits for each QEC cycle i (up to N cycles) and corresponding syndrome measurement S . Bottom: Resource-efficient implementation using one (or two) couplers, reducing SPAM error from 1.3% to 0.05% (0.002%).

often constrained by residual excitations, ground-state heating, and imperfect gate operations. More importantly, readout errors remain a significant challenge. These errors—caused by residual coupling, crosstalk, or limited signal-to-noise ratios—can misidentify the state of a qubit during measurement [16].

Modern superconducting architectures frequently incorporate tunable couplers between qubits, which serve to minimize crosstalk and facilitate interactions between two qubits on demand [10,17,18]. These features make superconducting qubits particularly well suited for implementing the proposed SPAM purification protocol without requiring additional hardware resources. Tunable couplers are qubits themselves and can thus act as auxiliary qubits needed to implement the proposed SPAM purification protocol, as illustrated in Fig. 2.

In superconducting qubit systems, the surface code—currently the most widely implemented quantum error correction scheme—relies critically on minimizing SPAM errors to preserve the integrity of logical qubits [10,19]. Each plaquette typically uses a dedicated ancilla qubit. Each ancilla qubit is surrounded by four tunable couplers enabling on-demand connection to respective data qubits.

We propose repurposing tunable couplers as ancillary qubits for the presented SPAM purification protocol. Each data or ancilla qubit can access up to four couplers. Utilizing couplers as qubits requires calibrating a two-qubit gate that enables conditional phase accumulation specific to the target qubit, while the phase on the untargeted qubit cancels out. This calibration involves optimizing the coupler's operating frequency, pulse duration, and shape. Control of the tunable coupler can be achieved either via an additional dedicated control line or through residual coupling to nearby control lines.

Readout can be performed via an additional resonator or through parasitic coupling to the resonator of the associated data or ancilla qubit. The coupler's tunability allows reducing the frequency gap between its operating point and the resonator, compensating for weaker coupling.

TABLE II. Superconducting quantum processor performance metrics.

Readout error	0.01
Readout duration	400 ns
Coupler readout error	0.02
CNOT-gate error	0.01
CNOT-gate duration	100 ns
T_1	70 μ s
$T_{2,\text{CPMG}}$	85 μ s
FPGA signal processing	400 ns
QNDness	0.001
Reset error	0.002

This protocol temporarily reassigns couplers as qubits to support SPAM purification. While it requires extra calibration for gate tuning and parallel readout, it avoids additional hardware and is compatible with current superconducting quantum processors.

To estimate the utility of the proposed protocol, we adopt performance metrics comparable to state-of-the-art quantum processors [10], as summarized in Table II.

Ancilla-qubit initialization achieves significantly higher fidelity than readout, making SPAM errors predominantly due to readout. Given the relatively long readout and signal processing times, the proposed implementation focuses on improving readout fidelity, with overhead limited to the number of CNOT gates between the tunable coupler and the ancilla.

The ancilla-SPAM error is approximately 1.3%. Using a tunable coupler during measurement—requiring one CNOT gate and incurring $\sim 0.1\%$ decoherence—this can be reduced to 0.05% with over 95% acceptance. With two couplers, the SPAM error drops to 0.002%, with $\sim 0.2\%$ decoherence and 92% acceptance. Given access to four couplers, we can select the most suitable one or two, achieving substantial SPAM suppression while maintaining high acceptance.

VII. PURIFYING SPAM ERRORS WITH NOISY GATES

We reiterate that the purification protocol needs noisy SPAM and noiseless CNOT gates. Here, we relax these conditions and consider the purification protocol with imperfect CNOT gates [20],

$$V_2(\cdot)V_2^\dagger \mapsto (1 - \epsilon)V_2(\cdot)V_2^\dagger + \epsilon \text{tr}[\cdot] \frac{\mathbb{1}}{2} \otimes \frac{\mathbb{1}}{2}, \quad (5)$$

where ϵ represents a noise fraction. It turns out that noisy SPAM cannot be completely purified with noisy CNOT gates. With three noise parameters, $1 - f$, ϵ , and q , the protocol can reach the fidelity in state preparation and a noise fraction in measurements as follows:

$$f_\epsilon^{(n)} \xrightarrow{n \rightarrow \infty} \frac{1}{1 - D + \sqrt{D^2 + 1}} < 1, \quad (6)$$

$$q_\epsilon^{(m)} \xrightarrow{m \rightarrow \infty} \frac{1}{1 + D + \sqrt{D^2 + 1}} > 0, \quad (7)$$

where $D = 2(2f - 1)(1 - 2q)(1 - \epsilon)/\epsilon$. The derivations are shown in Appendix E. The estimation of the parameters is also shown in Appendix F. It can be shown that the qubit-state-

TABLE III. Critical error rates ϵ_c of a CNOT gate for purifying noisy SPAM are shown [see Eq. (9)].

$1 - f$	0	0.01	0.03	0.05	0.07	0.1
ϵ_c	0	0.0374	0.0986	0.1460	0.1830	0.2236

preparation fidelity is strictly less than 1. Furthermore, noisy measurements can be purified up to an error $q_\epsilon^{(m)}$, which is strictly positive.

The purification protocol achieves a fidelity up to 0.984 with two target qubits for a realistic scenario with balanced error rates $\epsilon = 1 - f = q = 0.05$. The protocol purifies state preparation from 0.95 up to 0.984. The success probability with two target qubits is given by $p_{\text{succ}}^{(2)} = 0.7357$. For a noisy measurement, the protocol suppresses an error rate of 0.05 up to 0.016, as illustrated in Fig. 5.

VIII. THE PURIFICATION CONDITION

Equations (6) and (7) show that the convergence depends on the parameters f , q , and ϵ . This means that the purification protocol does not work if the first iteration does not improve fidelity, defining the condition for purifying noisy SPAM,

$$\text{the purification condition : } f < f_\epsilon^{(1)}. \quad (8)$$

As an instance, assuming that SPAM errors are balanced, that is, $1 - f = q$, an initial fidelity increases by the purification protocol whenever $\epsilon < \epsilon_c$, where the threshold ϵ_c is given by

$$\begin{aligned} \epsilon_c &= \frac{8f^3 - 12f^2 + 4f}{8f^3 - 12f^2 + 4f - 1} \\ &\approx 4(1 - f) - 28(1 - f)^2 + O((1 - f)^3). \end{aligned} \quad (9)$$

The expansion above shows that for error rates $1 - f = q$ around 10^{-2} , a threshold of an error rate ϵ_c is about four times more relaxed. In Table III, thresholds ϵ_c are computed for error rates relevant to current state-of-the-art architectures and applications. For example, for an error rate $1 - f = q = 0.01$, the critical error rate ϵ_c results as 0.0374, see Fig. 3.

IX. APPLICATION 1: DISTILLING ENTANGLEMENT

Entanglement is a general resource for quantum information applications [21]. Distilling an entangled bit (ebit),

$$|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \quad (10)$$

is a key task that enables various quantum information tasks, such as entanglement distillation [7]. Entanglement distillation begins when two parties share copies of an isotropic state,

$$\rho_F = F|\phi^+\rangle\langle\phi^+| + \frac{1 - F}{3}(\mathbb{1} \otimes \mathbb{1} - |\phi^+\rangle\langle\phi^+|),$$

which is entangled if and only if the fidelity satisfies $F > 1/2$. Two parties consider two copies and apply CNOT gates locally. Performing measurements in the second register, they accept a remaining state in the first register if the measurement outcomes are identical. Then, two parties may share a state with

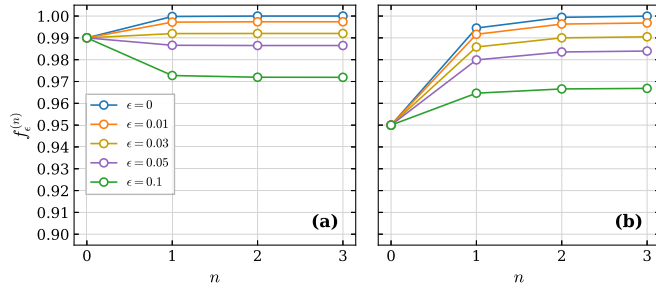


FIG. 3. The protocol purifies noisy qubit-state preparation with resources of noisy SPAM and imperfect CNOT gates on n additional qubits. For noiseless CNOT gates, one can purify noisy preparation up to a fidelity of 1. For an initial fidelity $f = 0.99$, one additional qubit suffices to achieve a noiseless preparation. If a CNOT gate is noisy with an error rate $\epsilon > 0.0374$, the purification is not fulfilled, and, thus, the fidelity decreases (purple and green plots). For $f = 0.95$, two or three additional noisy qubits can efficiently increase the fidelity.

fidelity F' , for which we have $F' > F$ whenever $F > 1/2$. Hence, all entangled two-qubit states are distillable.

In Ref. [22], it is shown that if the measurements in the second register are noisy, the distillation protocol cannot distill ebits from weakly entangled states. Here, we apply the purification protocol for distilling entanglement with noisy measurements and investigate the resources needed. We write by $\tilde{M}_k^{(n)}$ a POVM element purified with n ancillary qubits; it is obtained by noisy initialization ρ and noisy measurements $\tilde{M}_k$. We find a recurrence relation in POVM elements in the purification protocol,

$$\tilde{M}_k^{(n)} = \text{tr}_{A_n} [\rho^{A_n} V_2^\dagger \tilde{M}_k^{(n-1)} \otimes \tilde{M}_k^{A_n} V_2], \quad (11)$$

where noisy SPAMs are only exploited. We also write

$$r_0^{(n)} = \langle k | \tilde{M}_k^{(n)} | k \rangle \text{ and } r_1^{(n)} = \langle \bar{k} | \tilde{M}_k^{(n)} | \bar{k} \rangle. \quad (12)$$

Alice and Bob apply purified measurements to the distillation of entanglement and may find that

$$F' = \frac{F^2 + \left(\frac{1-F}{3}\right)^2 + g^{(n)}(F)}{F^2 + 2F\left(\frac{1-F}{3}\right) + 5\left(\frac{1-F}{3}\right)^2 + 4g^{(n)}(F)}, \quad (13)$$

where

$$g^{(n)}(F) = \left(\frac{F(1-F)}{3} + \left(\frac{1-F}{3} \right)^2 \right) \frac{r_{\text{odd}}^{(n)}}{r_{\text{even}}^{(n)}},$$

with $r_{\text{odd}}^{(n)} = 2r_0^{(n)}r_1^{(n)}$ and $r_{\text{even}}^{(n)} = r_0^{(n)}r_0^{(n)} + r_1^{(n)}r_1^{(n)}$.

Then, distilling entanglement works, i.e., $F' > F$, for $F > L^{(n)}$, where

$$L^{(n)} = \frac{1}{2} \left(\frac{r_0^{(n)} + r_1^{(n)}}{r_0^{(n)} - r_1^{(n)}} \right)^2.$$

In what follows, we investigate the lower bound $L^{(n)}$ for distillation. First, if the CNOT gates in the purification are noiseless, $\epsilon = 0$, we solve the recurrence relations in Eq. (12): $r_0^{(n)} = (1-q)\alpha^n$ and $r_1^{(n)} = q(1-\alpha)^n$, where $\alpha = f(1-q) + (1-f)q$. It follows that $L^{(n)} \xrightarrow{n \rightarrow \infty} \frac{1}{2}$, meaning that an ebit can be distilled from all entangled two-qubit states. Second, if

a CNOT gate is noisy, $\epsilon > 0$, we end up with $r_1^{(n)}/r_0^{(n)} \rightarrow d = \sqrt{D^2 + 1} - D$, where $D = 2(2f-1)(1-2q)(1-\epsilon)/\epsilon$. It turns out that

$$L^{(n)} \xrightarrow{n \rightarrow \infty} \frac{1}{2} \left(\frac{1+d}{1-d} \right)^2, \quad (14)$$

which is strictly greater than $1/2$. Hence, weakly entangled states cannot be distilled. Detailed derivations are shown in Appendix G.

X. APPLICATION 2: ENTANGLEMENT SWAPPING

We consider a repeater $R = R_1 R_2$ sharing ebits with Alice and Bob, $|\phi^+\rangle^{AR_1} |\phi^+\rangle^{R_2 B}$. A noiseless Bell measurement is desired, where a projection onto an ebit $|\phi^+\rangle^{R_1 R_2}$ enables two parties, Alice and Bob, to share an ebit. A successful projection for a state $\rho^{AR_1 R_2 B}$, denoted by a success probability p_R , can be realized as follows:

$$p_R \rho^{AB} = \text{tr}_{R_1 R_2} [\rho^{AR_1 R_2 B} V_2 H^{R_1} |00\rangle \langle 00|^{R_1 R_2} H^{R_1} V_2^\dagger] \quad (15)$$

since $|\phi^+\rangle = V_2 H^{R_1} |00\rangle$, where H^{R_1} denotes a Hadamard transform on R_1 and V_2 acts on $R_1 R_2$. The result 00 corresponds to a successful projection. Noisy measurements may fail in creating an ebit between two parties,

$$\tilde{\rho} \tilde{\rho}^{AB} = \text{tr}_{R_1 R_2} [\rho^{AR_1 R_2 B} V_2 H^{R_1} \tilde{M}_0 \otimes \tilde{M}_0 H^{R_1} V_2^\dagger], \quad (16)$$

the result of which leads to a mixture of four Bell states between Alice and Bob.

We apply a purified measurement with m noisy additional qubits on R_1 and R_2 , respectively, which constructs POVM elements $\tilde{M}_0^{(m)} \otimes \tilde{M}_0^{(m)}$. The resulting state of Alice and Bob after the projection is denoted by $\sigma^{(m)}$ and can be quantified by the fidelity,

$$\tilde{F}^{(m)} = \langle \phi^+ | \sigma^{(m)} | \phi^+ \rangle = \left[1 + \left(\frac{1-\alpha}{\alpha} \right)^m \left(\frac{q}{1-q} \right) \right]^{-2}.$$

Since $\alpha > 1/2$, one can find $m \geq 1$ such that $\tilde{F}^{(m)}$ is sufficiently close to 1.

XI. CONCLUSION

In conclusion, we have presented a protocol that purifies noisy SPAM. The protocol exploits noisy SPAM of auxiliary qubits as a resource. It efficiently supports quantum information tasks with noisy SPAM in general and is feasible with currently available quantum technologies, such as superconducting quantum processors. In particular, the protocol enables entanglement distillation with noisy SPAM [22] as well as entanglement swapping: Both are essential to build a quantum network. We have shown that under realistic assumptions, a few qubits are cost-effective in reaching a sufficiently low error rate in SPAM. The proposed protocol offers a general approach to mitigating SPAM errors, which have posed a significant obstacle to the reliable implementation of noisy-intermediate-scale-quantum algorithms, quantum communication protocols, quantum metrology (e.g., Ref. [23]), and the performance of quantum error correction codes.

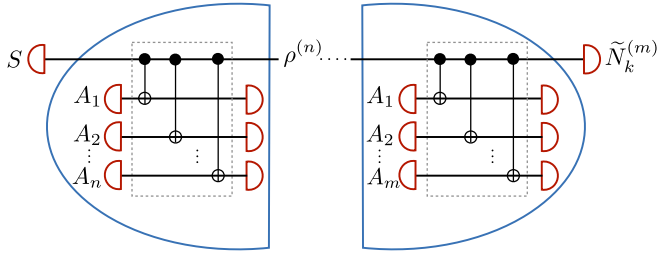


FIG. 4. Noisy SPAM can purify noisy SPAM. A collective CNOT in Eq. (A1) applies to noisy qubits $SA_1, \dots, A_n$. A resulting state $\rho^{(n)}$ accepted once the outcomes of noisy measurements on noisy qubits $A_1, \dots, A_n$ are all zeros 0^n has a fidelity converging to 1 as n increases. A noisy measurement on a system S is purified with noisy SPAM with m target qubits; a purified POVM element giving an outcome k is denoted by $\tilde{N}_k^{(m)}$ having a noise fraction converging to 0.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: PURIFICATION OF NOISY STATE PREPARATION

Summary of the protocol. The protocol for purifying noisy state preparation works with n ancillary qubits and a collective CNOT gate on $(n+1)$ qubits,

$$V_{n+1}^{SA_1 \dots A_n} = |0\rangle\langle 0|^S \otimes \mathbb{1}^{\otimes n} + |1\rangle\langle 1|^S \otimes X^{\otimes n}, \quad (\text{A1})$$

where X denotes a Pauli- X gate. Note also that V_{n+1} can be implemented as a sequence of CNOT gates, $V_{n+1}^{SA_1 \dots A_n} = \prod_{i=1}^n V_2^{SA_i}$ (see also Fig. 4), where $V_2^{SA_i}$ is a CNOT gate. After the collective CNOT gates, noisy measurements are performed on n target qubits. A resulting state in the control register is accepted, denoted by $\rho^{(n)}$, only when all n outcomes on target qubits are zeros, i.e., 0^n .

Proposition A1. For all $\epsilon > 0$, there exists $N \geq 1$ such that $|f^{(n)} - 1| < \epsilon$ whenever $n > N$.

Proof. We write by $R^{(n)}$ an unnormalized state accepted after a collective CNOT V_{n+1} over a control and n target qubits,

$$R^{(n)} = \begin{bmatrix} R_{00}^{(n)} & R_{01}^{(n)} \\ R_{10}^{(n)} & R_{11}^{(n)} \end{bmatrix}. \quad (\text{A2})$$

Note that the probability that a control qubit is accepted is given by $p_{\text{succ}}^{(n)} = \text{tr} R^{(n)}$, and the fidelity is thus as follows:

$$f^{(n)} = \frac{R_{00}^{(n)}}{R_{00}^{(n)} + R_{11}^{(n)}} = \left(1 + \frac{R_{11}^{(n)}}{R_{00}^{(n)}}\right)^{-1}. \quad (\text{A3})$$

A collective CNOT V_{n+1} can be decomposed as

$$V_{n+1}^{SA_1 \dots A_n} = V_2^{SA_n} V_n^{SA_1 \dots A_{n-1}}$$

from which a recurrence relation for a state $R^{(n)}$ is obtained,

$$R^{(n+1)} = \text{tr}_{A_{n+1}} [V_2(R^{(n)} \otimes \rho^{A_{n+1}}) V_2^\dagger \tilde{M}_0^{A_{n+1}}], \quad (\text{A4})$$

where V_2 acts on systems S and A_{n+1} . The elements also satisfy the following relations:

$$R_{00}^{(n+1)} = \alpha R_{00}^{(n)} \text{ and } R_{11}^{(n+1)} = (1 - \alpha) R_{11}^{(n)}, \quad (\text{A5})$$

where $\alpha = (1 - q)f + q(1 - f)$. It is straightforward to see that $1 - 2\alpha < 0$, from which $\alpha > 1/2$. It follows that

$$\frac{R_{11}^{(n+1)}}{R_{00}^{(n+1)}} = \left(\frac{1 - \alpha}{\alpha}\right) \frac{R_{11}^{(n)}}{R_{00}^{(n)}} = \dots = \left(\frac{1 - \alpha}{\alpha}\right)^{n+1} \frac{R_{11}^{(0)}}{R_{00}^{(0)}}. \quad (\text{A6})$$

Since $\alpha > 1/2$, for all $\epsilon > 0$ one can find $n' \geq 1$ such that

$$\frac{R_{11}^{(n')}}{R_{00}^{(n')}} < \epsilon.$$

The fidelity in Eq. (A3) converges to 1 as n increases. ■

Remark A1. The protocol above extends to a nonuniform noise model in which each ancilla qubit A_i has its own noisy state preparation ρ^{A_i} and noisy measurement $\tilde{M}_0^{A_i}$.

Proof. Define

$$f_i := \langle 0 | \rho^{A_i} | 0 \rangle, \quad q_i := \langle 1 | \tilde{M}_0^{A_i} | 1 \rangle, \\ \alpha_i := (1 - q_i)f_i + q_i(1 - f_i). \quad (\text{A7})$$

Then, the recurrence relation in Eq. (A4) becomes

$$R^{(n+1)} = \text{tr}_{A_{n+1}} [V_2(R^{(n)} \otimes \rho^{A_{n+1}}) V_2^\dagger \tilde{M}_0^{A_{n+1}}], \quad (\text{A8})$$

and the diagonal elements satisfy

$$R_{00}^{(n+1)} = \alpha_{n+1} R_{00}^{(n)}, \quad R_{11}^{(n+1)} = (1 - \alpha_{n+1}) R_{11}^{(n)}. \quad (\text{A9})$$

Hence,

$$\frac{R_{11}^{(n)}}{R_{00}^{(n)}} = \left(\prod_{i=1}^n \frac{1 - \alpha_i}{\alpha_i}\right) \frac{R_{11}^{(0)}}{R_{00}^{(0)}}. \quad (\text{A10})$$

Experimental parameters from calibrated devices satisfy that $f_i \geq 1/2 + a$ and $q_i \leq 1/2 - b$ for all i and for some $a, b > 0$. For $\alpha_i = f_i(1 - q_i) + q_i(1 - f_i)$, we have $\alpha_i - \frac{1}{2} = 2(f_i - \frac{1}{2})(\frac{1}{2} - q_i) \geq 2ab > 0$. Therefore, we choose $\delta = 2ab$ so that $\alpha_i \geq \frac{1}{2} + \delta$ for all i . Then, it holds that

$$\alpha_i \geq \frac{1}{2} + \delta, \quad \text{for all } i \geq 1, \quad (\text{A11})$$

then

$$\frac{1 - \alpha_i}{\alpha_i} \leq \frac{\frac{1}{2} - \delta}{\frac{1}{2} + \delta} < 1, \quad \text{for all } i \geq 1, \quad (\text{A12})$$

so that the product in Eq. (A10) converges to 0 exponentially fast. Consequently, the fidelity $f^{(n)}$ in Eq. (A3) still converges

to 1. In particular, this holds whenever all ancilla preparations and measurements are better than random: $f_i > 1/2$ and $q_i < 1/2$ for all i . ■

APPENDIX B: PURIFICATION OF NOISY MEASUREMENTS

Summary of the protocol. Outcomes of noiseless measurements can be distilled with noisy state preparations ρ and noisy measurements (see also Fig. 4). To purify a noisy measurement on a system state σ , m ancillary qubits are initialized in noisy preparations $\rho^{\otimes m}$. A collective CNOT V_m is applied to the states $\sigma \otimes \rho^{\otimes m}$, and then noisy measurements are performed on the $(m+1)$ qubits individually. The outcomes are accepted only when they are identical, either 0^{m+1} or 1^{m+1} .

A noisy measurement can be generally described as follows:

$$\tilde{M}_k = (1-q)M_k + qM_{k\oplus 1}, \quad \text{for } k = 0, 1, \quad (\text{B1})$$

where q denotes a noise fraction. Otherwise, one can admix two instances, outcomes from a noisy measurement and the other obtained after applying a Pauli Z so that a POVM element is given as above, i.e.,

$$\frac{1}{2}(\tilde{M}_k + Z\tilde{M}_kZ) = (1-q)M_k + qM_{k\oplus 1}, \quad \text{for } k = 0, 1.$$

One can achieve the diagonalization by randomly admixing two probabilities, one from a noisy measurement and the other from a measurement after a Pauli gate Z.

Once outcomes are identical and thus accepted, we write that $\tilde{N}_k^{(m)}$ is a POVM element giving outcome k in the first register, given that $m+1$ outcomes are identical. The noisy fraction for the POVM element is given by $q^{(m)} = \langle \bar{k} | \tilde{N}_k^{(m)} | \bar{k} \rangle$, where $\bar{k} = k \oplus 1$.

Proposition B1. For all $\epsilon > 0$, there exists $N \geq 1$ such that $q^{(m)} < \epsilon$ whenever $m > N$.

Proof. A noisy POVM element can be purified with noisy SPAM as follows. The probability of having identical outcomes k^{m+1} for $k \in \{0, 1\}$ is given by, for a state σ on a system S ,

$$p(k^{m+1}) = \text{tr}[V_{m+1} \sigma^S \otimes \rho^{\otimes m} V_{m+1}^\dagger \tilde{M}_k^{\otimes m+1}].$$

From above, we write by $p^{(m)}(k)$ a probability of having an outcome k on a system once noisy measurements on a system and m target qubits give identical outcomes,

$$p^{(m)}(k) = \frac{1}{p_s^{(m+1)}} \text{tr}[\sigma \tilde{M}_k^{(m)}], \quad (\text{B2})$$

where

$$\tilde{M}_k^{(m)} = \text{tr}_{A_1 \dots A_m} \left[\rho^{\otimes m} V_m^\dagger M_k^S \bigotimes_{i=1}^m \tilde{M}_k^{A_i} V_m \right],$$

$$p_s^{(m+1)} = p(0^{m+1}) + p(1^{m+1}). \quad (\text{B3})$$

The decomposition of a collective CNOT gate applies to derive a recurrence relation for a POVM element $\tilde{M}_k^{(m)}$:

$$\tilde{M}_k^{(m+1)} = \text{tr}_{A_{m+1}} [\rho^{A_{m+1}} V_2^\dagger \tilde{M}_k^{(m)} \otimes \tilde{M}_k^{A_{m+1}} V_2], \quad (\text{B4})$$

where V_2 acts on systems S and A_{m+1} . We solve the recurrence relation with the initial condition in Eq. (B1) and have

$$\tilde{M}_k^{(m)} = \alpha^m (1-q) |k\rangle \langle k| + (1-\alpha)^m q |\bar{k}\rangle \langle \bar{k}|,$$

where $\alpha = f(1-q) + (1-f)q > 1/2$. The success probability is also computed, $p_s^{(m+1)} = \alpha^m (1-q) + (1-\alpha)^m q$. A purified POVM element in Eq. (B2) corresponds to

$$\tilde{N}_k^{(m)} = \frac{1}{p_s^{(m+1)}} \tilde{M}_k^{(m)}. \quad (\text{B5})$$

The noise fraction is obtained as

$$q^{(m)} = \left[1 + \left(\frac{\alpha}{1-\alpha} \right)^m \left(\frac{1-q}{q} \right) \right]^{-1}.$$

Since $\alpha > 1/2$, for all $\epsilon > 0$ there exists $m \geq 1$ such that $q^{(m)} < \epsilon$. Hence, it is shown that a noise fraction can be arbitrarily suppressed. ■

Remark B1. The proof above also extends to a nonuniform noise model in which each ancilla qubit A_i has its own noisy state preparation ρ^{A_i} and noisy measurement $\tilde{M}_k^{A_i}$.

Proof. Define

$$f_i := \langle 0 | \rho^{A_i} | 0 \rangle, \quad q_i := \langle \bar{k} | \tilde{M}_k^{A_i} | \bar{k} \rangle,$$

$$\alpha_i := (1-q_i)f_i + q_i(1-f_i). \quad (\text{B6})$$

Then, the recurrence relation in Eq. (B4) becomes

$$\tilde{M}_k^{(m+1)} = \text{tr}_{A_{m+1}} [\rho^{A_{m+1}} V_2^\dagger (\tilde{M}_k^{(m)} \otimes \tilde{M}_k^{A_{m+1}}) V_2], \quad (\text{B7})$$

where V_2 acts on systems S and A_{m+1} . Writing

$$r_0^{(m)} := \langle k | \tilde{M}_k^{(m)} | k \rangle, \quad r_1^{(m)} := \langle \bar{k} | \tilde{M}_k^{(m)} | \bar{k} \rangle, \quad (\text{B8})$$

one finds

$$r_0^{(m+1)} = \alpha_{m+1} r_0^{(m)}, \quad r_1^{(m+1)} = (1-\alpha_{m+1}) r_1^{(m)}. \quad (\text{B9})$$

Hence,

$$\frac{r_1^{(m)}}{r_0^{(m)}} = \left(\prod_{i=1}^m \frac{1-\alpha_i}{\alpha_i} \right) \frac{r_1^{(0)}}{r_0^{(0)}}. \quad (\text{B10})$$

Therefore, if there exists $\delta > 0$ such that

$$\alpha_i \geq \frac{1}{2} + \delta, \quad \text{for all } i \geq 1, \quad (\text{B11})$$

then the ratio in Eq. (B10) converges to 0 exponentially fast. Since the noise fraction of the purified POVM element $\tilde{N}_k^{(m)}$ is determined by the ratio $r_1^{(m)}/r_0^{(m)}$, it follows that $q^{(m)} \rightarrow 0$ as m increases. ■

APPENDIX C: RESOURCES FOR PURIFYING NOISY SPAM

In Table IV, the purification with n additional qubits and the success probability are computed in realistic scenarios, $q \in [0.05, 0.25]$, assuming errors are balanced, $1-f=q$. For instance, for $n=0$, without the purification, no postselection is performed, $p_s^{(0)}=1$. For SPAM errors less than 10%, in which $1-f=q=0.05$, it holds that $T^{(1)} < 10^{-2}$ with one additional qubit. The success probability is given as $p_s^{(1)}=0.86$. For a higher error rate $1-f=q=0.1$, two additional qubits can be used to suppress an error up to 5×10^{-3} with the success probability $p_s^{(2)}=0.6$. The purification with

TABLE IV. The purification of noisy measurements with noisy SPAM is demonstrated.

Error rate (q)/ purification (n)	$T^{(0)}$	$p_s^{(0)}$	$T^{(1)}$	$p_s^{(1)}$	$T^{(2)}$	$p_s^{(2)}$
$1 - f = q = 0.25$	0.25	1	0.167	0.562	0.107	0.328
$1 - f = q = 0.2$	0.2	1	0.105	0.608	0.052	0.390
$1 - f = q = 0.15$	0.15	1	0.057	0.672	0.020	0.482
$1 - f = q = 0.1$	0.1	1	0.024	0.756	0.005	0.608
$1 - f = q = 0.05$	0.05	1	0.005	0.864	0.001	0.779

one additional qubit shows a success probability larger than $1/2$ in a realistic scenario.

APPENDIX D: PURIFICATION OF SPAM WITH NONUNIFORM NOISE

Tables V and VI demonstrate the purification of noisy SPAM when noise parameters vary across qubits, as established in Remarks A1 and B1. The convergence requires only that each ancilla satisfies $\alpha_i = (1 - q_i)f_i + q_i(1 - f_i) > 1/2$. Table V considers moderate noise; the results are comparable to the uniform case in Table IV. Table VI considers a more demanding scenario in which the first ancilla has a preparation fidelity of $f_1 = 0.85$ and a measurement error of $q_1 = 0.12$. The protocol still converges, with lower success probabilities reflecting the reduced filtering power of the weaker ancilla.

APPENDIX E: PURIFYING SPAM ERRORS WITH NOISY GATES

For the full generality, we consider the purification protocol when CNOT gates are noisy [20], for which a noise model is

$$V_2(\cdot) V_2^\dagger \mapsto (1 - \epsilon) V_2(\cdot) V_2^\dagger + \epsilon \text{tr}[\cdot] \frac{\mathbb{1}}{2} \otimes \frac{\mathbb{1}}{2}, \quad (\text{E1})$$

where ϵ denotes a noise fraction.

1. Purifying noisy SPAM

Applying the purification protocol when CNOT gates are noisy, one can find a recurrence relation for a state in Eq. (A2),

$$\begin{aligned} R_{00}^{(n+1)} &= (1 - \epsilon) \alpha R_{00}^{(n)} + \frac{\epsilon}{4} (R_{00}^{(n)} + R_{11}^{(n)}), \\ R_{11}^{(n+1)} &= (1 - \epsilon) (1 - \alpha) R_{11}^{(n)} + \frac{\epsilon}{4} (R_{00}^{(n)} + R_{11}^{(n)}). \end{aligned} \quad (\text{E2})$$

TABLE V. Purification of noisy SPAM with nonuniform moderate noise. The system and ancilla qubits have parameters $(f_s, q_s) = (0.97, 0.05)$, $(f_1, q_1) = (0.94, 0.02)$, $(f_2, q_2) = (0.98, 0.08)$, $(f_3, q_3) = (0.96, 0.07)$.

n (m)	State preparation		Measurement	
	$f^{(n)}$	$p_{\text{succ}}^{(n)}$	$q^{(m)}$	$p_s^{(m+1)}$
0	0.9700	1	0.0500	1
1	0.9974	0.897	4.408×10^{-3}	0.880
2	0.9997	0.808	4.743×10^{-4}	0.792
3	0.99997	0.724	5.532×10^{-5}	0.709

Note also that noisy SPAM are applied in the protocol. To see the convergence of a resulting state, let us compute the ratio from Eq. (E2),

$$g = \lim_{n \rightarrow \infty} \frac{R_{11}^{(n)}}{R_{00}^{(n)}} = \sqrt{D^2 + 1} - D,$$

$$\text{where } D = 2(2f - 1)(1 - 2q) \frac{(1 - \epsilon)}{\epsilon}. \quad (\text{E3})$$

Then, the purification with n target qubits leads to a convergent state up to a fidelity in the following:

$$f_\epsilon^{(n)} = \frac{R_{00}^{(n)}}{\text{tr} R^{(n)}} \xrightarrow{n \rightarrow \infty} \frac{1}{1 - D + \sqrt{D^2 + 1}} < 1. \quad (\text{E4})$$

Thus, noiseless state preparation can be approximated with an error $1 - f_\epsilon^{(n)}$.

Similarly, for the purification of noisy measurements, the recurrence relation for a POVM element in Eq. (B4) is obtained,

$$\begin{aligned} \langle k | \tilde{M}_k^{(m+1)} | k \rangle &= (1 - \epsilon) \alpha \langle k | \tilde{M}_k^{(m)} | k \rangle + \frac{\epsilon}{4} \text{tr}[\tilde{M}_k^{(m)}], \\ \langle \bar{k} | \tilde{M}_k^{(m+1)} | \bar{k} \rangle &= (1 - \epsilon) (1 - \alpha) \langle \bar{k} | \tilde{M}_k^{(m)} | \bar{k} \rangle + \frac{\epsilon}{4} \text{tr}[\tilde{M}_k^{(m)}]. \end{aligned}$$

Then, a noise fraction of a purified POVM element depends on both the number of target qubits m and a noise fraction ϵ ; the noise fraction is also convergent,

$$q_\epsilon^{(m)} = \langle \bar{k} | \tilde{N}_k^{(m, \epsilon)} | \bar{k} \rangle \xrightarrow{m \rightarrow \infty} \frac{1}{1 + D + \sqrt{D^2 + 1}}, \quad (\text{E5})$$

which is strictly positive, thus achieving a noiseless POVM element up to an error $q_\epsilon^{(m)}$. One can notice that a noise fraction above is equal to $1 - f_\epsilon^{(m)}$ in Eq. (E4).

TABLE VI. Purification of noisy SPAM with nonuniform strong noise. The system and ancilla qubits have parameters $(f_s, q_s) = (0.92, 0.07)$, $(f_1, q_1) = (0.85, 0.12)$, $(f_2, q_2) = (0.93, 0.11)$, $(f_3, q_3) = (0.94, 0.09)$.

n (m)	State preparation		Measurement	
	$f^{(n)}$	$p_{\text{succ}}^{(n)}$	$q^{(m)}$	$p_s^{(m+1)}$
0	0.9200	1	0.0700	1
1	0.9741	0.723	2.248×10^{-2}	0.7288
2	0.9948	0.592	4.510×10^{-3}	0.598
3	0.9998	0.579	7.321×10^{-4}	0.513

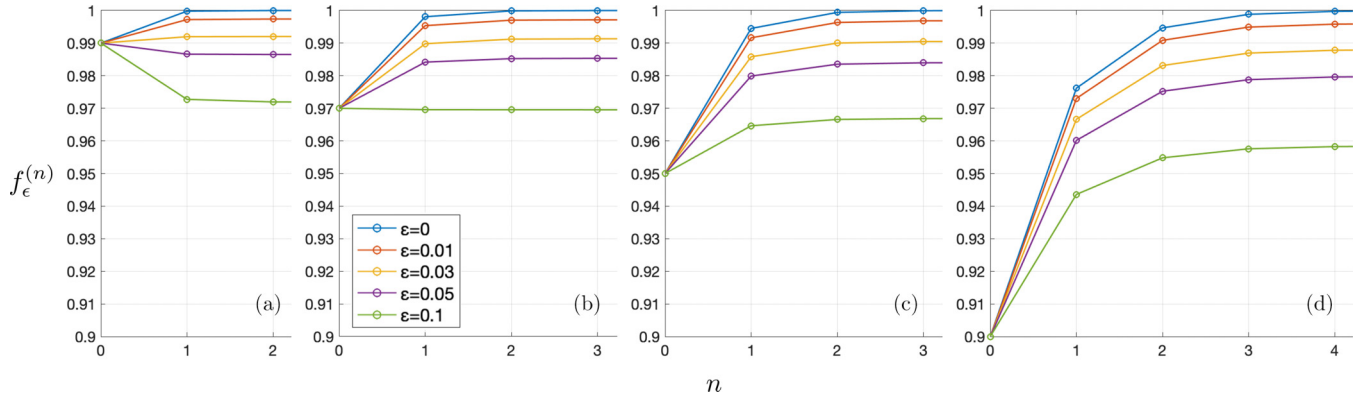


FIG. 5. The purification of noisy initialization is demonstrated when a CNOT gate contains an error rate ϵ in Eq. (E1) and errors are balanced, $1 - f = q$. An initial fidelity $f_\epsilon^{(0)}$ of state preparation may increase according to n noisy target qubits. The convergence is $f_\epsilon^{(\infty)}$ in Eq. (E4). For an initial fidelity $f = 0.99$ in panel (a), a single target qubit suffices to achieve noiseless preparation. A noisy CNOT gate with an error rate $\epsilon > 0.0374$ cannot be used to increase the fidelity (see Table VII). For $f = 0.97$ and $f = 0.95$ in panels (b) and (c), respectively, two or three noisy target qubits suffice to reach noiseless state preparation. Four target qubits are needed for $f = 0.9$ in panel (d). For all cases, the purification protocol is efficient, and a few noisy target qubits are cost-effective.

2. Feasibility: error suppression in realistic scenarios

From realistic constraints, let us consider balanced error rates $\epsilon = 1 - f = q = 0.05$. The purification achieves a fidelity up to 0.984 with two target qubits; i.e., the protocol can purify noisy state preparation from 0.95 up to 0.984. The success probability with two target qubits is given by $p_{\text{succ}}^{(2)} = 0.7357$. For a noisy measurement, the protocol suppresses an error rate from 0.05 up to 0.016 [see Fig. 5(c)].

APPENDIX F: VERIFICATION OF SPAM ERRORS

The purification condition in Eq. (8) tells that it is necessary to learn error rates in SPAM, $1 - f$ and q , to find whether a purification works when a CNOT gate is noisy, $\epsilon > 0$. These error rates can be verified from measurements on two qubits (see Fig. 6). For noisy preparations with a fidelity f , a noisy CNOT gate in Eq. (E1) with an error rate ϵ is applied and noisy measurements with a noise fraction q are performed. Probabilities $p(ij)$ for outcomes $i, j \in \{0, 1\}$ are given according to the error rates,

$$\begin{aligned} p(00) &= (1 - \epsilon)[f^2(1 - q)^2 + (1 - f)^2q^2 \\ &\quad + f(1 - f)(1 - q)q + (1 - f)^2q(1 - q) \\ &\quad + (1 - f)f q^2] + \epsilon/4, \\ p(01) &= (1 - \epsilon)[f^2(1 - q)q + (1 - f)^2q^2 \\ &\quad + f(1 - f)(1 - q)^2 + (1 - f)^2q^2 \\ &\quad + (1 - f)f q(1 - q)] + \epsilon/4, \\ p(10) &= (1 - \epsilon)[f^2q(1 - q) + (1 - f)^2(1 - q)q \\ &\quad + f(1 - f)q^2 + (1 - f)^2(1 - q)^2 \\ &\quad + (1 - f)f(1 - q)q] + \epsilon/4, \\ p(11) &= (1 - \epsilon)[f^2q^2 + (1 - f)^2(1 - q)^2 \\ &\quad + f(1 - f)q(1 - q) + (1 - f)^2(1 - q)q \\ &\quad + (1 - f)f(1 - q)^2] + \epsilon/4. \end{aligned}$$

TABLE VII. Critical error rates ϵ_c of a CNOT gate for purifying noisy SPAM are shown.

$1 - f$	0	0.01	0.03	0.05	0.07	0.1
ϵ_c	0	0.0374	0.0986	0.1460	0.1830	0.2236

$$\begin{aligned} &+ f(1 - f)q^2 + (1 - f)^2(1 - q)^2 \\ &+ (1 - f)f(1 - q)q] + \epsilon/4, \\ p(11) &= (1 - \epsilon)[f^2q^2 + (1 - f)^2(1 - q)^2 \\ &\quad + f(1 - f)q(1 - q) + (1 - f)^2(1 - q)q \\ &\quad + (1 - f)f(1 - q)^2] + \epsilon/4. \end{aligned}$$

Hence, all noise parameters of SPAM and a noisy CNOT gate can be verified by solving equations above.

In Table VIII, the verification of parameters $f^{(0)}$, q , and ϵ from probabilities $p(ij)$ for $i, j = 0, 1$ is demonstrated. Once error rates are verified from probabilities of measurement outcomes, one can find whether the purification condition, $f < f_\epsilon^{(1)}$, is fulfilled and may estimate the number of target qubits required for the purification.

In case 1, a CNOT gate is noiseless; an error rate of 10^{-3} is achieved with three additional target qubits. In Cases 2, 3, and 4, a CNOT gate is noisy and the state preparation is purified up to $f_\epsilon^{(\infty)}$; a few qubits are cost effective. In Case 5, a CNOT gate is too noisy to purify noisy state preparation: The condition $f < f_\epsilon^{(1)}$ is not fulfilled.

APPENDIX G: APPLICATIONS TO QUANTUM NETWORKS

Distilling and swapping entanglement are essential to realize a quantum network. Entanglement distillation enables two parties to share an entangled bit (ebit) over a distance L , $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ from noisy entangled states. Once

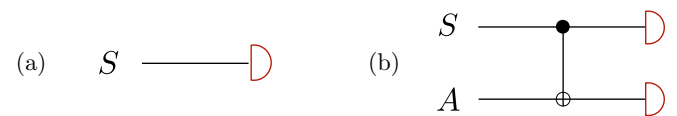


FIG. 6. (a) Single-copy level SPAM cannot verify errors of preparation and measurements. (b) Measurement outcomes can verify error rates in preparation, measurement, and a CNOT gate.

TABLE VIII. The purification of state preparation is demonstrated with noisy SPAM verified from measurement outcomes.

Cases	Measurement outcomes			Verification			Purification with n qubits			
	$p(01)$	$p(10)$	$p(11)$	$f^{(0)}$	q	ϵ	$f^{(1)}$	$f^{(2)}$	$f^{(3)}$	$f^{(\infty)}$
1	0.154	0.09	0.09	0.9	0.1	0	0.976	0.995	0.999	1
2	0.1252	0.0540	0.0896	0.9	0.05	0.01	0.979	0.994	0.996	0.996
3	0.0777	0.0530	0.0366	0.97	0.05	0.03	0.989	0.991	0.991	0.991
4	0.0961	0.0576	0.0576	0.95	0.05	0.05	0.980	0.983	0.984	0.984
5	0.0754	0.0674	0.0357	0.99	0.05	0.1	0.99	0.971	0.970	0.970

two parties Alice and Bob share ebits with a repeater, $|\phi^+\rangle^{AR_1}$ and $|\phi^+\rangle^{R_2B}$, a joint measurement on R_1R_2 in a repeater creates an ebit $|\phi^+\rangle^{AB}$ between Alice and Bob. Then, entanglement swapping allows two parties to share an ebit over a distance $2L$.

To see how measurements in entanglement distillation can be noisy, we recall that two parties first apply twirling to share copies of one-parameter states, called Werner states,

$$\rho_F = F|\phi^+\rangle\langle\phi^+| + \frac{1-F}{3}(\mathbb{1} \otimes \mathbb{1} - |\phi^+\rangle\langle\phi^+|),$$

which is entangled if and only if the fidelity satisfies $F > 1/2$. After applying bilateral CNOT gates on two copies of a Werner state, two parties perform measurements in the second register; when measurement outcomes are identical, a resulting state in the first register is accepted. Let F' denote a resulting fidelity and we have $F' > F$ whenever $F > L$, where $L = 1/2$; i.e., an entangled Werner state can be distilled. If measurements in the second register are noisy, we have some threshold $L > 1/2$ [22].

With the purification protocol for noisy SPAM, we now show that an entangled network can be achieved through entanglement distillation and entanglement swapping by purifying noisy resources for SPAM (see Fig. 7).

1. Distilling entanglement with noisy SPAM

In this Appendix, we purify noisy measurements and show that entanglement can be distilled with noisy SPAM. The result resolves the problem pointed out in Ref. [22] that noisy measurements in entanglement distillation [24] prevent parties from distilling entanglement from weakly entangled states.

Proposition G1. A larger set of entangled states can be distilled by purifying noisy SPAM.

Proof. Consider two nodes in a network with noisy resources for SPAM. Applying the protocol for purifying noisy measurements, we have a recurrence relation,

$$\tilde{M}_k^{(n)} = \text{tr}_{A_n}[\rho^{A_n} V_2^\dagger \tilde{M}_k^{(n-1)} \otimes \tilde{M}_k^{A_n} V_2], \quad (\text{G1})$$

where noisy SPAMs are only exploited. We also write by $r_0^{(n)} = \langle k | \tilde{M}_k^{(n)} | k \rangle$ and $r_1^{(n)} = \langle \bar{k} | \tilde{M}_k^{(n)} | \bar{k} \rangle$. From the recurrence relation, we have

$$\begin{aligned} r_0^{(n+1)} &= (1-\epsilon)\alpha r_0^{(n)} + \frac{\epsilon}{4}(r_0^{(n)} + r_1^{(n)}), \\ r_1^{(n+1)} &= (1-\epsilon)(1-\alpha)r_1^{(n)} + \frac{\epsilon}{4}(r_0^{(n)} + r_1^{(n)}). \end{aligned} \quad (\text{G2})$$

With the purification on noisy measurements, Alice and Bob apply the entanglement distillation protocol in Ref. [24]. We

have a resulting fidelity F' ,

$$F' = \frac{F^2 + \left(\frac{1-F}{3}\right)^2 + g^{(n)}(F)}{F^2 + 2F\left(\frac{1-F}{3}\right) + 5\left(\frac{1-F}{3}\right)^2 + 4g^{(n)}(F)}, \quad (\text{G3})$$

where

$$g^{(n)}(F) = \left(\frac{F(1-F)}{3} + \left(\frac{1-F}{3} \right)^2 \right) \frac{r_{\text{odd}}^{(n)}}{r_{\text{even}}^{(n)}},$$

with $r_{\text{odd}}^{(n)} = 2r_0^{(n)}r_1^{(n)}$ and $r_{\text{even}}^{(n)} = r_0^{(n)}r_0^{(n)} + r_1^{(n)}r_1^{(n)}$. From the relations, we compute

$$F' - F = 8\left(F - \frac{1}{4}\right)(F - L^{(n)})(1 - F)$$

and derive the distillability condition: $F > L^{(n)}$, where

$$L^{(n)} = \frac{1}{2} \left(\frac{r_0^{(n)} + r_1^{(n)}}{r_0^{(n)} - r_1^{(n)}} \right)^2.$$

On the one hand, suppose that CNOT gates in the purification are noiseless, $\epsilon = 0$. We can solve recurrence relations in Eq. (G2), $r_0^{(n)} = (1-q)\alpha^n$ and $r_1^{(n)} = q(1-\alpha)^n$. It follows that

$$L^{(n)} \xrightarrow{n \rightarrow \infty} \frac{1}{2}, \quad (\text{G4})$$

meaning that an ebit can be distilled from all entangled two-qubit states.

On the other hand, let us consider a case where a CNOT gate is noisy, $\epsilon > 0$, for which we derive a relation

$$\begin{aligned} \frac{r_1^{(n+1)}}{r_0^{(n+1)}} &= \frac{(1-\epsilon)(1-\alpha)r_1^{(n)} + \frac{\epsilon}{4}(r_0^{(n)} + r_1^{(n)})}{(1-\epsilon)\alpha r_0^{(n)} + \frac{\epsilon}{4}(r_0^{(n)} + r_1^{(n)})} \\ &= \frac{(1-\epsilon)(1-\alpha)\left(\frac{r_1^{(n)}}{r_0^{(n)}}\right) + \frac{\epsilon}{4}\left(1 + \left(\frac{r_1^{(n)}}{r_0^{(n)}}\right)\right)}{(1-\epsilon)\alpha + \frac{\epsilon}{4}\left(1 + \left(\frac{r_1^{(n)}}{r_0^{(n)}}\right)\right)}. \end{aligned} \quad (\text{G5})$$

From the above, we compute the convergence,

$$\frac{r_1^{(n)}}{r_0^{(n)}} \rightarrow d = \sqrt{D^2 + 1} - D, \quad (\text{G6})$$

where

$$d = \sqrt{D^2 + 1} - D, \quad D = 2\left(\frac{1-\epsilon}{\epsilon}\right)(2f-1)(1-2q).$$

Thus, the lower bound is convergent as follows:

$$L^{(n)} \xrightarrow{n \rightarrow \infty} \frac{1}{2} \left(\frac{1+d}{1-d} \right)^2, \quad (\text{G7})$$

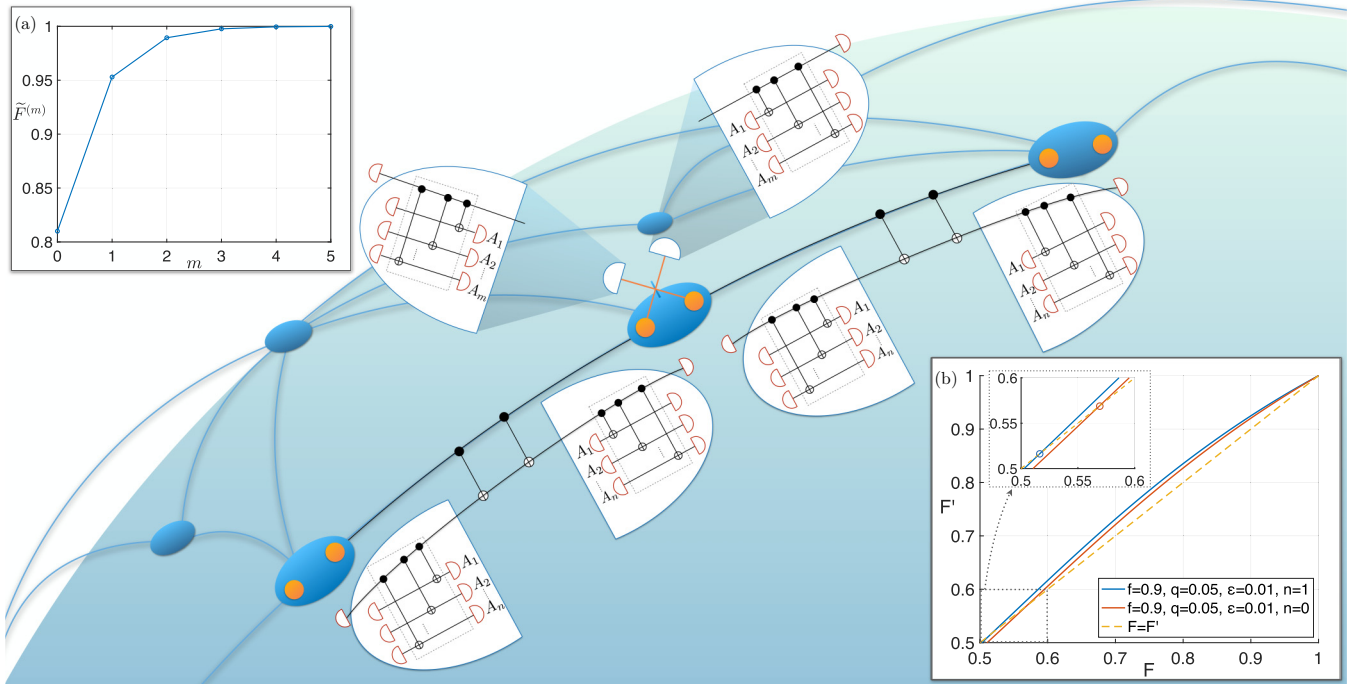


FIG. 7. Noisy measurements in a quantum network may appear in entanglement distillation and entanglement swapping in repeaters. In entanglement swapping in panel (a), a network results in a pure state with a sufficiently high fidelity by purifying noisy measurements, for which three noisy qubits are cost-effective. In panel (b), noisy measurements in entanglement distillation can be purified by noisy SPAM; a lower bound for distillation improves from 0.569 (red) to 0.517 (blue) where the purification works with one noisy target qubit having a fidelity 0.9, a measurement with an error rate 0.05, and a noisy CNOT gate with noise fraction 0.05.

which is higher than the ultimate lower bound $1/2$. ■

We note that the success probability in the distillation protocol, i.e., the probability that the copy in the first register is accepted, is given by

$$p_{\text{succ}}^{(n)}(F) = \left(F^2 + 2F \left(\frac{1-F}{3} \right) + 5 \left(\frac{1-F}{3} \right)^2 \right) r_{\text{even}}^{(n)} + \left(4F \left(\frac{1-F}{3} \right) + 4F \left(\frac{1-F}{3} \right)^2 \right) r_{\text{odd}}^{(n)}. \quad (\text{G8})$$

The success probability in Eq. (G8) takes two steps into account: One is the probability that the purification protocol with n qubits is successful, and the other is the probability that the first copy is accepted in the distillation protocol.

2. Optimizing resources: single target qubit is optimal

Although the convergence is shown in Eqs. (G4) and (G7), it is left to answer how large n is needed for distilling ebits from a larger set of entangled states. With the constraint on a fidelity with an ebit higher than 0.999, we estimate the number of additional qubits for distilling 1 ebit in what follows.

Note that each round of the distillation protocol makes a postselection of $N p_{\text{succ}}^{(n)}/2$ copies out of N shared copies. Let F_j denote a singlet fidelity after j rounds: An initial fidelity is thus denoted by F_0 and a final one by $F_j > 0.999$.

Then, the critical number of copies N_c having an initial fidelity F_0 for distilling one ebit is given by

$$N_c = \prod_{j=0}^{J-1} \frac{2}{p_{\text{succ}}^{(n)}(F_j)}, \quad (\text{G9})$$

with the purification of noisy measurements with n qubits.

The results for realistic scenarios where preparation and measurement errors are 5% are shown in Table IX. An initial fidelity F_0 is given for $[0.6, 0.9]$.

(1) N_c is minimal when one additional qubits is exploited on both sides. Thus, one additional qubit on both sides is optimal for distilling entanglement by purifying noisy SPAM.

(2) The success probabilities when additional qubits are exploited for the purification are comparable; see, e.g., $p_{\text{succ}}^{(1)}(F_0)$ and $p_{\text{succ}}^{(0)}(F_0)$.

Notably, N_c is not monotonic in n : The minimum occurs at $n = 1$ for all initial fidelities considered. This reflects a trade-off between measurement quality and success probability. Increasing n further suppresses the noise fraction $q^{(n)}$, which lowers the distillation threshold $L^{(n)}$ and may reduce the number of distillation rounds J required to reach the target fidelity. However, each additional ancilla also reduces the per-round success probability $p_{\text{succ}}^{(n)}(F_j)$ by imposing an additional postselection condition. At $n = 1$, the dominant measurement errors are already substantially suppressed (from $q = 0.05$ to $q^{(1)} \approx 0.005$), and the marginal improvement from further ancillae does not compensate for the reduced success probability. Thus, a single ancilla per party is optimal for distilling

TABLE IX. The number of copies of shared states N_c for distilling an ebit is computed when a measurement error is 5% [see Eq. (G9)].

	$N_c p_{\text{succ}}^{(0)}(F_0)$	$N_c p_{\text{succ}}^{(1)}(F_0)$	$N_c p_{\text{succ}}^{(2)}(F_0)$	$N_c p_{\text{succ}}^{(3)}(F_0)$	$N_c p_{\text{succ}}^{(4)}(F_0)$
$F_0 = 0.6, 1 - f = q = 0.05$	Undistillable	4.904×10^{11}	1.184×10^{13}	9.371×10^{14}	7.557×10^{16}
		0.453	0.369	0.302	0.247
$F_0 = 0.7, 1 - f = q = 0.05$	1.432×10^{12}	1.323×10^9	1.657×10^{10}	5.985×10^{11}	2.175×10^{13}
	0.646	0.505	0.412	0.337	0.276
$F_0 = 0.8, 1 - f = q = 0.05$	1.937×10^9	2.494×10^7	5.914×10^8	1.438×10^{10}	3.507×10^{11}
	0.718	0.570	0.466	0.381	0.312
$F_0 = 0.9, 1 - f = q = 0.05$	1.429×10^7	6.627×10^5	8.798×10^6	1.178×10^8	1.578×10^9
	0.804	0.648	0.530	0.434	0.356

entanglement by purifying noisy SPAM under chosen parameters.

3. Entanglement swapping

For ebits shared with a repeater $R = R_1 R_2$, $|\phi^+\rangle^{AR_1} |\phi^+\rangle^{R_2 B}$, a noiseless Bell measurement is desired, where a projection onto an ebit $|\phi^+\rangle^{R_1 R_2}$ enables two parties Alice and Bob to share an ebit. A successful projection for a state $\rho^{AR_1 R_2 B}$, denoted by a success probability p_R , can be realized in practice as follows:

$$p_R \rho^{AB} = \text{tr}_{R_1 R_2} [\rho^{AR_1 R_2 B} V_2 H^{R_1} |00\rangle\langle 00|^{R_1 R_2} H^{R_1} V_2^\dagger] \quad (\text{G10})$$

since $|\phi^+\rangle = V_2 H^{R_1} |00\rangle$, where H^{R_1} denotes a Hadamard transform on R_1 and V_2 acts on $R_1 R_2$. Then, outcomes 00 correspond to a successful projection in Eq. (G10).

Noisy measurements lead to

$$\tilde{p}_R \tilde{\rho}^{AB} = \text{tr}_{R_1 R_2} [\rho^{AR_1 R_2 B} V_2 H^{R_1} \tilde{M}_0 \otimes \tilde{M}_0 H^{R_1} V_2^\dagger], \quad (\text{G11})$$

the result of which leads to a mixture of four Bell states between Alice and Bob.

Proposition G2. Entanglement swapping can be realized by noisy SPAM with the purification protocol.

Proof. A purified measurement with m noisy target qubits on R_1 and R_2 , respectively, constructs POVM elements $\tilde{N}_0^{(m)} \otimes \tilde{N}_0^{(m)}$. The resulting state of Alice and Bob after applying the measurements on two ebits may be denoted by $\sigma^{(m)}$ and can be quantified by the fidelity

$$\tilde{F}^{(m)} = \langle \phi^+ | \sigma^{(m)} | \phi^+ \rangle = \left[1 + \left(\frac{1 - \alpha}{\alpha} \right)^m \left(\frac{q}{1 - q} \right) \right]^{-2}.$$

Since $\alpha > 1/2$, one can find $m \geq 1$ such that $\tilde{F}^{(m)}$ is sufficiently close to 1. ■

Thus, noisy measurements distributing mixed states over a network are purified, which then enables a noiseless and large-size entangled network (see Fig. 7).

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