

Programming stability and stiffness in two-dimensional multistable structures

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Multistable structures can reconfigure and passively maintain their geometry, enabling adaptive materials with tunable properties and performance tailored to changes in conditions. This work presents a theoretical and an experimental framework for designing and analyzing two-dimensional multistable structures composed of interconnected elastically buckled beams in different arrangements. First, we develop an analytical model and use numerical continuation to identify stable and unstable equilibrium configurations. Then, experimental samples validate the analytically predicted stable states, yielding structures with asymmetric energy potentials and reconfigurable effective stiffness. This approach provides a foundation for programmable, modular mechanical metamaterials assembled from bistable building blocks.

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I. INTRODUCTION

Multistable structures possess multiple stable static configurations and can reversibly transition between them, enabling mechanical systems with programmable responses and unusual effective properties [1]. Multistability arises from a nonconvex potential energy landscape containing multiple local minima separated by energy barriers. This feature has been obtained with a wide range of building blocks, including buckled beams [2–4], shell elements [5–7], origami-inspired architectures [8–11], and mechanism-based systems [12]. Networks assembled from individual multistable building blocks enable spatial patterning and increase robustness through collective effects. Appropriately designed couplings between units can yield emergent interactions, including tunable, reciprocal, and nonreciprocal responses, that are not present at the level of an isolated element [12].

To fully describe multistable structures, two complementary viewpoints can be considered: (1) a static viewpoint that characterizes the set of equilibrium configurations and their stability, and (2) a dynamic viewpoint that characterizes how the structure transitions between configurations. From the static viewpoint, the first question entails the equilibrium configurations and their stability. When multiple stable configurations are available, differences in geometry between states can be exploited for morphing structures that adapt their shape in response to external stimuli [13–15]. Interestingly, structures can be engineered to maximize the differences between states such that the structure is compact for transport and then deploys to a much larger footprint, for example, as

temporary shelters [9] or deployable space structures [16]. Structures with many stable states can also realize mechanical memory, in which each element functions as a mechanical “bit” and information can be written to and read from the structure [4,17,18]. Beyond geometry, different configurations can exhibit distinct mechanical properties, enabling reconfigurable stiffness [7,11,19,20] or adaptable compressive responses [21].

Access to multiple states is only valuable if the cost of switching between them is appropriate. The dynamic viewpoint focuses on the transition pathways between states, including the critical loads and energy barriers that govern switching [13–15]. These transition dynamics can also be leveraged for energy absorption and impact mitigation [22,23]. In other settings, the dynamical responses are the primary objective, as in transition waves [24] and mechanical avalanches [25], where local switching events trigger nonlinear wave propagation and enable reconfigurations of many elements from a single input. Overall, multistable structures are defined not only by their stable energy wells but also by the barriers and pathways between them, motivating the need to view them from both a static and a dynamic perspective.

In this work, we study multistable structures assembled from elastically buckled beams as the building block. Under axial compression, an initially flat slender beam with pinned-pinned boundary conditions buckles into a half-sine shape [Fig. 1(a)] through a pitchfork bifurcation [Fig. 1(b)]. The isolated beam is stable in both the buckled “up” state (positive midpoint displacement) and the buckled “down” state (negative midpoint displacement), and these two states are mechanically equivalent by symmetry. Elastically buckled beams have been studied extensively, including their static responses [26], vibration and modal characteristics [27,28], and dynamic snap-through behavior [2–4,29–32].

Elastically buckled beams have also been assembled into one-dimensional chains by connecting neighboring elements through shared pinned supports [24,33]. These studies show that a localized actuation can trigger a transition wave that

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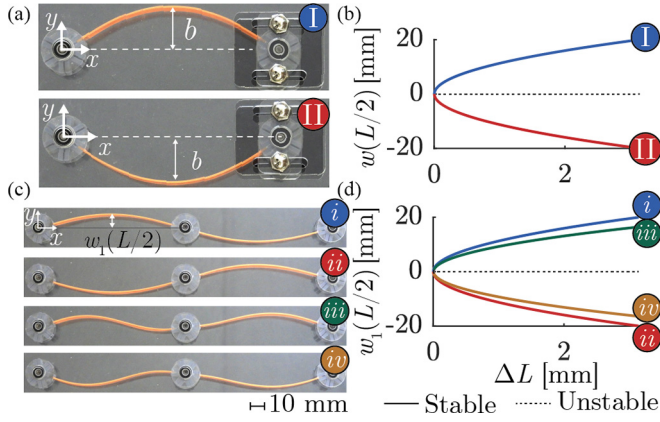


FIG. 1. Interconnected buckled beams' equilibrium states. (a) An elastically buckled beam supported by pins at both ends exhibits two stable configurations with equal mechanical properties. (b) The bifurcation diagram showing the pitchfork bifurcation of the elastically buckled beam. (c) Two connected elements of equal geometry exhibit four stable configurations observed in panel (d) as four stable branches in the bifurcation diagram ($L = 100$ mm, $b = 10$ mm, and $h = 0.76$ mm).

propagates through the chain, switching elements sequentially from one state to the other [34]. However, when a transition wave is unable to propagate through the entire structure because of damping or insufficient energy barrier conditions, additional stable configurations of a structure can exist. A chain of two connected elements with equal rise has two intuitive stable equilibrium configurations in which the elements have opposite midpoint displacement [Fig. 1(c)]. Remarkably, two additional stable configurations exist, where both arches have the same midpoint displacement [Fig. 1(c)]. Notably, between these two pairs of stable configurations the structure's mechanical properties vary. While the first two configurations (i) and (ii) have equivalent mechanical properties, the last two configurations (iii) and (iv) have mechanical properties that differ from the first pair. Differences also appear in relative stability. Configurations (iii) and (iv) require smaller perturbations to trigger switching (Supplementary Movie 1 [35]). One-dimensional chains limit each element to at most two neighbors, which limits the possible connection topologies and motivates extending these ideas to higher-dimensional networks.

Here, we show that two-dimensional (2D) networks provide a versatile platform for realizing multistable structures with reconfigurable effective mechanical properties. Prior work on 2D multistable architectures has often focused on a single choice of building block (e.g., square-based units [36], hexagonal arrays [37], or triangular and square tilings [18]) and then explored behavior within that single geometric design. Even within a single 2D geometry, there is rich static and dynamic behavior to study, including nonreciprocal energy transport [37], emergent memory and path-dependent states [10,18], and the influence of topological defects on mechanical response [38]. In contrast, here we focus on how the topology of connections between identical buckled-beam elements shapes the emergent stability and mechanics of the overall structure. We show that, even with a small number

of elements, changing only the number and arrangement of connections can program global behavior, including bi- and quad-stability, without redesigning the unit cell itself.

We present a generalized analytical model for arbitrary 2D networks of buckled beams and use numerical continuation to compute equilibrium branches, assess stability, and extract associated mechanical properties. Starting from the simplest 2D shape, a triangle, we investigate how small geometric imperfections bias the energy landscape and how differences between stable states can be leveraged to passively and reversibly reconfigure the effective mechanical response of a metastructure. Experimentally, we validate these predictions by measuring force-displacement curves and demonstrate a simple structure with reconfigurable effective stiffness across stable configurations. Together, these results highlight how simple buckled-beam units can yield robust and programmable behavior through the topology of connections alone.

II. METHODS

To gain analytical insights, we propose a general model for two-dimensional structures composed of buckled beams. The model consists of two parts. First, single buckled beams are modeled as independent, unconnected elements with unknown moments acting at both ends. Second, moment balance and slope continuity equations corresponding to the specific structure's geometry are imposed to fully close the system, solving for the unknown moments. Consider an arbitrary system of $\mathbb{E}$ buckled beams. Each element is modeled using Kirchhoff beam theory. The governing partial differential equation describing the deformed shape, w_i , of the i th buckled beam of a system can be written in terms of a local coordinate x_i [24]:

$$\rho A \frac{\partial^2 w_i}{\partial t^2} + \beta \frac{\partial w_i}{\partial t} + EI \frac{\partial^4 w_i}{\partial x_i^4} + P \frac{\partial^2 w_i}{\partial x_i^2} + M_{i,L}(t) \delta'(x_i) + M_{i,R}(t) \delta'(x_i - L) = 0, \quad (1)$$

where ρ is the volumetric density, A is the cross sectional area, t is time, β is the viscous damping, E is Young's modulus, I is the second moment of area, $L_{i,0}$ is the undeformed length of the i th beam, L is the deformed length of all beams, $M_{i,L}$ and $M_{i,R}$ are external moments acting on the left and right side of the beam, respectively, and δ' is the x_i derivative of the Dirac delta function. P is the midplane axial force given by

$$P = \frac{EA}{L_{i,0}}(L_{i,0} - L) - \frac{EA}{2L} \int_0^L \left(\frac{\partial w_i}{\partial x_i} \right)^2 dx_i. \quad (2)$$

Approximate solutions to Eq. (1) can be found via Galerkin's method [28,29]. The system is nondimensionalized using the following parameters:

$$\begin{aligned} \bar{x}_i &= \frac{x_i \pi}{L}, & \bar{w}_i &= \frac{w_i \pi}{L}, & \bar{t} &= t \frac{\pi \sqrt{E}}{L \sqrt{\rho}}, & \bar{A} &= A \frac{\pi^2}{L^2}, \\ \bar{L}_0 &= \frac{L_{0,0} \pi}{L}, & \bar{\beta} &= \beta \frac{\pi}{L \sqrt{\rho E}}, & \bar{I} &= I \frac{\pi^4}{L^4}, & \bar{M} &= M \frac{2\pi^2}{EL^3}, \\ \bar{\delta}(\cdot) &= \frac{L}{\pi} \delta(\cdot). \end{aligned} \quad (3)$$

Here, the overbar indicates a nondimensional quantity (Sec. 1 of the Supplemental Material [35]). A basis of sine functions is used for an N -mode approximation of the shape of each buckled beam:

$$\bar{w}_i(\bar{x}_i, \bar{t}) = \sum_{n=1}^N a_{i,n}(\bar{t}) \sin n\bar{x}_i. \quad (4)$$

Substituting Eq. (4) into Eq. (1) and minimizing the residual using Galerkin's scheme converts the system of $\mathbb{E}$ partial differential equations into a system of $\mathbb{E} \cdot N$ ordinary differential equations.

$$\begin{aligned} \ddot{a}_{i,n} + \bar{\beta}\dot{a}_{i,n} + \frac{1}{12}\bar{h}^2 n^4 a_{i,n} - \frac{n^2}{4} \left[(\bar{b}\delta_i)^2 - \left(\sum_{k=1}^N k^2 a_{i,k}^2 \right) \right] a_{i,n} \\ - n\bar{M}_{i,L} + (-1)^{n+1} n\bar{M}_{i,R} = 0, \quad \text{for } n = 1, 2, 3, \dots, N, \\ \text{for } i = 1, 2, 3, \dots, \mathbb{E}, \end{aligned} \quad (5)$$

where $\bar{h}$ is the nondimensional thickness assuming a rectangular cross section, $\bar{b}$ is the nominal nondimensional rise of the buckled beam due to end shortening, and δ_i represents the i th arch's geometric imperfection from the nominal value. After imposing static equilibrium conditions (i.e., $\ddot{a}_{i,n} = \dot{a}_{i,n} = 0$), Eq. (5) represents a system of $\mathbb{E}(N+2)$ unknowns and $\mathbb{E} \cdot N$ equations. To fully close the system, moment balance and slope continuity conditions are imposed. The pinned-pinned connections at the boundary of each element require that the sum of the moments at each support be zero. For a system with V total hinges connecting the $\mathbb{E}$ buckled beams, the moment balance conditions can be written as

$$\mathcal{B}\mathbf{M} = 0, \quad (6)$$

where $\mathbf{M}$ is a $2\mathbb{E} \times 1$ vector of the moments listed as $[M_{1,L}, M_{1,R}, M_{2,L}, \dots, M_{\mathbb{E},L}, M_{\mathbb{E},R}]^T$ and $\mathcal{B}$ is a $V \times 2\mathbb{E}$ matrix defined by

$$\mathcal{B}_{i,j} = \begin{cases} 1, & \text{if the } j\text{th moment acts on the } i\text{th hinge,} \\ 0, & \text{otherwise.} \end{cases} \quad (7)$$

The j th moment in Eq. (7) refers to the index of moment in the moment vector $\mathbf{M}$. Equation (6) represents additional V equations imposed by the pinned boundary conditions. The remaining $2\mathbb{E} - V$ equations required to fully close the system come from slope continuity boundary conditions (Sec. 2 of the Supplemental Material [35]).

At each hinge, all buckled beams connected to the hinge experience the same rotation. Hinges that connect multiple buckled beams are identified through rows of the moment balance coefficient matrix $\mathcal{B}$, whose row sums are greater than 1. For the i th row, r_i , of the matrix $\mathcal{B}$ in which $\sum_{j=1}^{2\mathbb{E}} \mathcal{B}_{ij} > 1$, the set of slope boundary conditions that apply at that connection can be written as

$$\begin{aligned} S(r_i) = \left\{ \frac{\partial \bar{w}_k}{\partial \bar{x}_k} \right\}_{\bar{x}_k = \frac{\pi}{2}(1+(-1)^j)} : 1 \leq j \leq 2\mathbb{E}, \mathcal{B}_{ij} = 1, \\ k = \left\lfloor \frac{j}{2} \right\rfloor + 1 \}. \end{aligned} \quad (8)$$

Equating the rotation of all buckled beams connected to the same hinge produces the final $2\mathbb{E} - V$ conditions required to fully close the system

$$\forall s \in S(r_i) : s = \theta_i. \quad (9)$$

Numerical methods can be used to find solutions to the fully defined system of $\mathbb{E}(N+2)$ unknowns and $\mathbb{E}(N+2)$ equations. Treating the nominal rise b as an additional free parameter, the numerical continuation software COCO [39] is used to track one-dimensional manifolds of static equilibrium solutions of the system. To do so, the system of second-order differential equations (5) is cast into first-order form defining a vector field of $2\mathbb{E}N$ equations. Naively applying the continuity equations (9) induces linear dependence on the modes, which causes pairs of eigenvalues of the Jacobian to be zero. To properly determine the stability of the system accounting for these zero eigenvalues, continuation is performed on a partial system where $2\mathbb{E} - V$ modes are solved for in terms of the remaining modes using the slope continuity equations (9) (Sec. 3 of the Supplemental Material [35]). In this manner, for $C = 2\mathbb{E} - V$ constraint equations, there are C modes removed from the full system, and continuation is performed on the partial system. Stability analysis performed on the partial system contains C fewer pairs of eigenvalues than the full system (Fig. 2). The corresponding eigenvectors of the omitted eigenvalues all contain mode shapes that violate the slope continuity conditions (Fig. 2, vii–ix), making these perturbations kinematically inadmissible and rightfully dismissed in the stability analysis of the system.

To balance the necessity of using a sufficient number of modes to accurately describe the behavior of the deformed shape with the increased computational complexity of considering excessive modes, modal convergence was performed to ensure that a sufficient number of modes was chosen for each shape. First, the deformed state of each structure was determined for varying choices of N [Fig. 3(a)]. Next, the error is computed as the area differences between the reference shape of each element, $w_i^{(R)}$, and the solution found with an N -mode approximation, $w_i^{(N)}$ [Fig. 3(b)]. The total error between the two shapes is calculated using

$$\% \text{Error} = \frac{\sum_{i=1}^{\mathbb{E}} \int_0^\pi ||w_i^{(N)}(x_i) - w_i^{(R)}(x_i)|| dx_i}{\sum_{k=1}^{\mathbb{E}} 2 \int_0^\pi ||w_k^{(R)}(x_k)|| dx_k} \times 100. \quad (10)$$

Modal convergence is defined when additional modes would reduce the error below 2% [Fig. 3(c)]. For all shapes presented in the text, convergence is found for $N = 3$, except the square, which requires $N = 5$.

Analytical results are validated with experimental samples manufactured with PETG (polyethylene terephthalate glycol) shim stock connected by three-dimensional printed hinges and supported with ball bearings to approximate pinned-pinned supports (Sec. 4 of the Supplemental Material [35]).

III. RESULTS

We begin with the simplest two-dimensional shape that can be created using the minimum number of elements. Consider a triangle with no variation in the undeformed side length (i.e., $L_{1,0} = L_{2,0} = L_{3,0}$) [Fig. 4(a)]. Each side of the structure

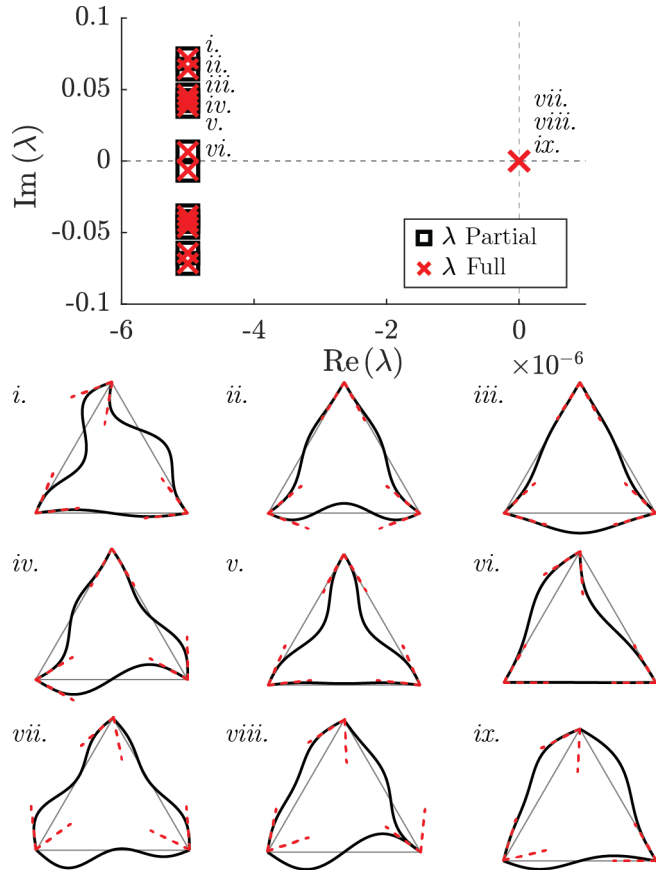


FIG. 2. Stability of solutions. Comparing the eigenvalues and eigenvectors of the partial system and full system. The partial system solves for the omitted C modes using the slope continuity equations. The partial system in turn has C fewer eigenvalues and eigenvectors. These omitted eigenvalues are all zero eigenvalues as a result of the linear dependence that is introduced from the constraint equations. The corresponding eigenvectors (i–ix) of the omitted values correspond to perturbations that violate the slope continuity condition. Because these perturbations are kinematically inadmissible, they are neglected during stability analysis.

is compressed by the same amount, simultaneously buckling each side. Allowing the nominal nondimensional rise, $\bar{b}$, to vary, the bifurcation diagram of the system is found using numerical continuation. The full bifurcation diagram exists in an $(\mathbb{E}(N+2)+1)$ -dimensional space; however, in order to plot the results, the first two modes of the first arch ($a_{1,1}, a_{1,2}$) are plotted against the nondimensional rise, showing stable and unstable solutions of the system [Fig. 4(b)]. Six stable branches are found corresponding to the sides buckled into a mirror symmetric shape observed elsewhere in the literature [18,40]. One pair of the stable chiral configurations is shown; leveraging the 120° symmetry of the triangle, the remaining four branches can be found via rotations (Sec. 5 of the Supplemental Material [35]). All other solutions are unstable; notably, six of the unstable solutions correspond to the “chiral” configuration, where two of the elements are predominantly in the first mode (i.e., a “C” shape) and one arch is in the second mode (i.e., an “S” shape). While it has been shown that the stability of the chiral and mirror

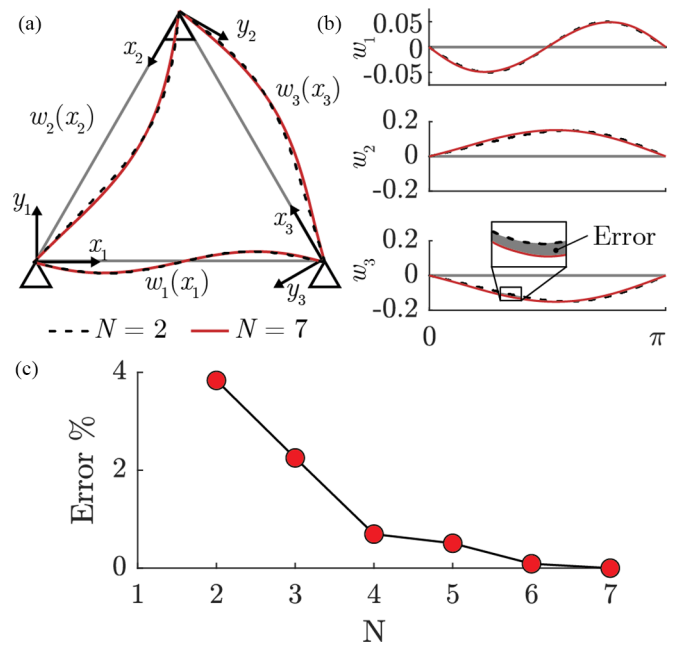


FIG. 3. Modal convergence analysis. (a) Comparing the solved deformed shape for a two-mode approximation and a seven-mode approximation. (b) Error is defined as the area difference between a deformed shape and a reference shape using Eq. (10). (c) Convergence analysis of a triangular system showing the effect of considering additional modes.

symmetric configurations of 2D patterned triangular lattices can be driven by varying the thickness of the buckled beam [40], small imperfections introduced among the sides of the elements can also control the solution’s stability.

In practice, small side length mismatches are unavoidable due to manufacturing tolerances and can have a significant effect on postbuckling stability. Here, we use a controlled length detuning as a representative imperfection to show how such asymmetries can arise and be leveraged to select a desired stable solution. If one side of the triangular structure is slightly shorter than the other two ($L_{1,0} < L_{2,0} \approx L_{3,0}$), the strain energy is minimized when the shortest beam deforms into the second mode and the two longer arches deform primarily into the first mode configurations. Experimentally, this is observed in samples with one side intentionally made 3% shorter ($L_{1,0} = 102$ mm, $L_{2,0} = L_{3,0} = 105$ mm) and compressed to $L = 100$ mm [Fig. 4(c)]. The bifurcation diagram including the corresponding variance ($\delta_1 = 1, \delta_2 = \delta_3 = 1.5$) [Fig. 4(d)] predicts that only the chiral solution with the first arch in the “S” shape is stable (Sec. 6 of the Supplemental Material [35]). This theoretical finding is also confirmed by the experimental samples (Supplementary Movie 2 [35]). Quantitatively, the error between the predicted shape and the experimental shape is computed by reconstructing the experimental curve via curve fitting and computing the normalized area difference between the two curves using Eq. (10) (Sec. 7 of the Supplemental Material [35]). Excellent agreement is found between the predicted and experimental shapes for each of the two stable configurations.

Fabrication-induced or intentional imperfections between side lengths can occur in either direction; a side may be

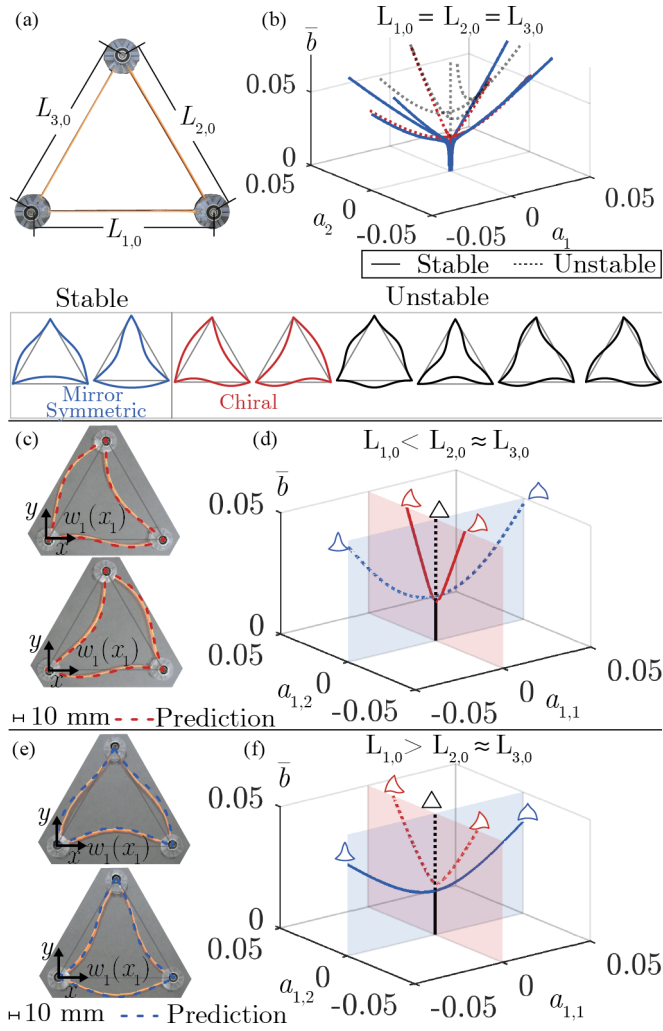


FIG. 4. Triangular structure. (a) Triangular structure with sides of undeformed lengths $L_{1,0}$, $L_{2,0}$, and $L_{3,0}$, shown before deformation. (b) Continuation diagram of a symmetric triangle ($\bar{h} = 0.01\pi$). (c), (d) Experimental sample and continuation diagram of a sample with an undeformed length of one side slightly shorter than the other two (3.2% and 5.9% errors). (e), (f) Experimental sample and continuation diagram of a sample with an undeformed length of one side slightly longer than the other two (5.1% and 2.4% errors).

slightly shorter or slightly longer. If, instead, one side is longer than the other two by 3% ($L_{1,0} = 105$ mm, $L_{2,0} = L_{3,0} = 102$ mm), the stability between the chiral and mirror symmetric configuration is exchanged. It becomes energetically favorable for the longer arch to be primarily in the first mode and the remaining two elements resolve into the mirror symmetric shape [Fig. 4(e)]. Including the corresponding variance δ_i in the analytical model, the numerical continuation is in agreement with the experimental results [Fig. 4(f)] quantitatively and qualitatively (Supplementary Movie 2 [35]) for each of the two configurations. This small variation between undeformed side lengths can be used to control the deformed structure's stability. The variation in undeformed side lengths due to manufacturing processes must be accounted for, either by minimizing it or by purposely adding it to guide the structure toward the desired stable configurations.

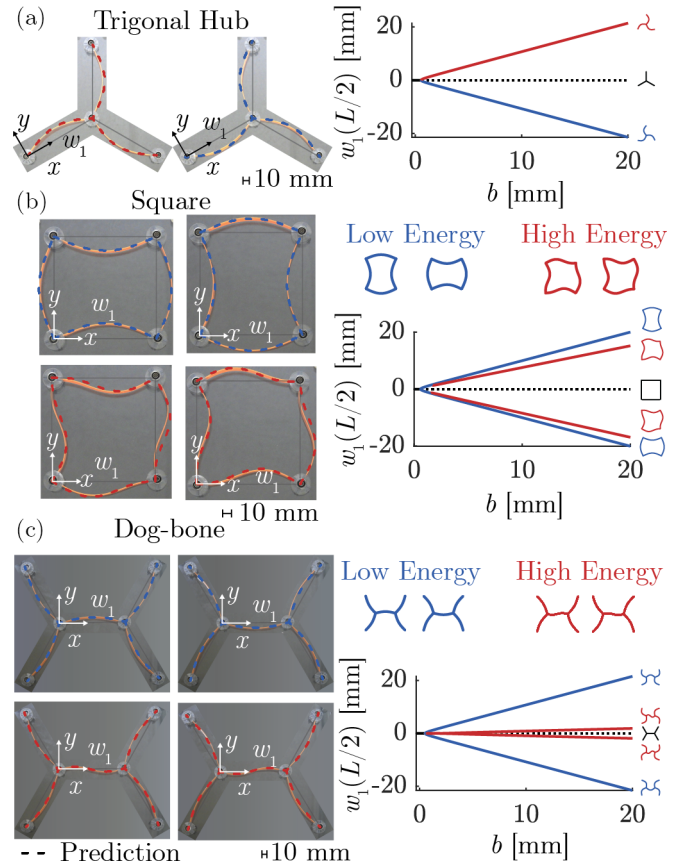


FIG. 5. Multiple 2D arrangements. Experimental structures ($\bar{b} = 0.15\pi$, $\bar{h} = 0.01\pi$) and bifurcation diagrams of (a) trigonal hub (structure of three elements connected at a shared hinge) with two stable states at symmetric potential energy (2.5% and 2.8% errors), (b) square structure with four stable states, with pairs of configurations sharing the same potential energy, but a low-potential-energy pair and a high-potential-energy pair (2.4%, 4.3%, 6.2%, and 7.8% errors), and (c) dog-bone structure that like the square has four stable states that are paired into two low-energy states and two high-energy states (3.3%, 4.9%, 4.7%, and 5.5% errors).

Interestingly, connections between elements do not always increase the number of stable configurations of a structure. When multiple elements are connected with a single common shared hinge such as in a trigonal hub of beams meeting at a joint, the structure is bistable [Fig. 5(a)]. The two stable configurations correspond to all buckled beams experiencing a positive rotation about the shared central hinge or negative rotation about the shared central hinge. Thus, the only structural stable states correspond to each buckled beam being in the first mode in a direction that satisfies the slope continuity condition.

The number of stable solutions available to a given system depends on both the connection topology and the geometry and bending rigidity of the buckled beams. An unconnected system of $\mathbb{E}$ bistable elements has $2^{\mathbb{E}}$ stable states. When elements are connected, however, the imposed slope-continuity constraints couple their deformations, reshape the energy landscape, and can reduce the number of accessible stable equilibria. This reduction is more pronounced for beams with higher bending rigidity. Buckled beams with higher bending

rigidity resist local slope changes; if the rigidity is sufficiently high, solutions that require regions of high curvature to satisfy slope-continuity conditions can lose stability. Conversely, when the bending rigidity is very low, configurations involving higher curvature can become stable, increasing the number of stable solutions. For the chain of two connected elements in Fig. 1, the last two configurations, (iii) and (iv), exhibit higher curvature regions than the first two configurations, (i) and (ii). Increasing the bending rigidity, for example, by increasing beam thickness, causes configurations (iii) and (iv) to lose stability beyond a critical value, leaving the structure bistable (Sec. 8 of the Supplemental Material [35]). The number of stable solutions depends on the topology of connections, the curvature demanded by solutions satisfying the slope continuity conditions, and the bending rigidity of the buckled beams. This has been observed in the literature where independent binary behavior of each element emerges as stiffness at the hinges decreases [18].

Notably, for the stable configurations of both the triangle and the trigonal hub, the stable configurations have identical potential energy, as the states are related by the reflection symmetry associated with a pitchfork bifurcation. This is not always the case as a structure can have multiple stable configurations with unequal potential energies. Consider a square configuration [Fig. 5(b)]; two stable configurations arise from patterning the first mode along each side of the square in alternating directions. While these two states are associated with the lowest potential energy, the structure can also be placed into a pair of stable higher-potential-energy states. In a perfectly symmetric sample, there would be four higher-energy stable states due to the symmetry of the square structure, however, with a slight variation between the side lengths due to manufacturing tolerances ($L_{1,0} = 105.6$ mm, $L_{2,0} = 105.1$ mm, $L_{3,0} = 105.3$ mm, $L_{4,0} = 104.5$ mm) in the experimental samples, only two of the higher-energy stable states appear. Within the bifurcation diagram, four stable branches are observed, two corresponding to a lower-energy state and two corresponding to a higher-energy state. The potential-energy landscape of the square can be described as “asymmetric” because it has stable configurations with unequal associated potential energy. The asymmetric energy potential is the result of the connectedness between the elements and it indicates unequal mechanical properties between stable configurations.

Another structure with asymmetric energy potentials is a single element connected at both hinges with two additional elements in a “dog-bone” configuration [Fig. 5(c)]. This structure has two stable states, where each element is in the first mode, satisfying the geometric constraints. However, two stable higher-energy configurations also exist when the middle element is in the second mode and the pair of elements on both ends are in the first mode [Fig. 5(c)]. In all configurations, the outer elements either constrain the middle element in the first mode configuration or hold the middle element in the second mode configuration. The difference in potential energy of the states is easily observed in the middle arch. When the middle arch is held in the first mode, the potential energy of the structure is lower than when the middle arch is held in the second mode. This difference in potential energy affects each configuration’s effective stiffness.

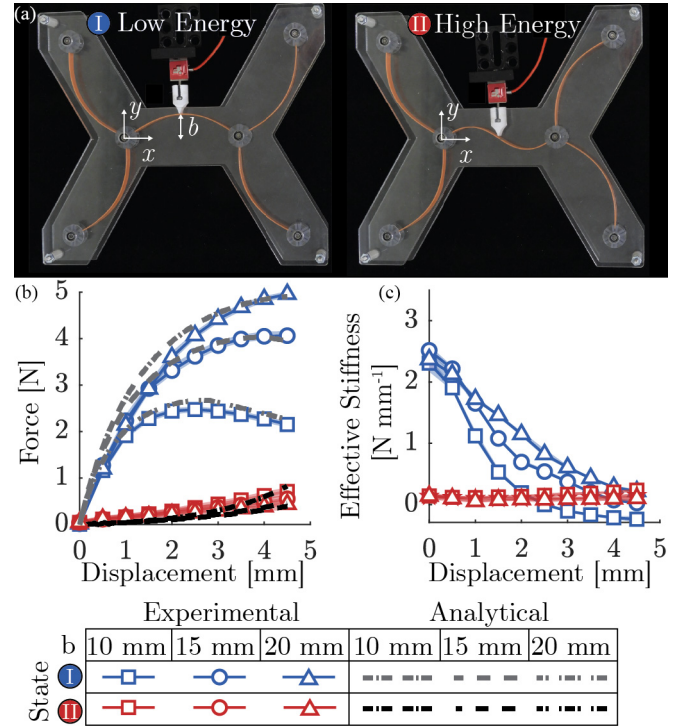


FIG. 6. Variable effective stiffness. Comparison of force-displacement data and analytically predicted values of the dog-bone structure for elements of three different rises ($\bar{b} = 0.1\pi, 0.15\pi, 0.20\pi$, and $\bar{h} = 0.01\pi$) from (a) the low- and high-energy configurations. (b) Force-displacement data when the structure is in the low-energy configuration (blue) and high-energy configuration (red) for each rise against the analytically predicted response (dashed). From the force-displacement, the instantaneous slope yields (c) the effective stiffness of the structure. For all rises, the initial effective stiffness of the high-energy state is much lower (0.15 N mm^{-1}) compared to the initial stiffness of the lower-energy state (2.3 N mm^{-1}).

To measure the effective stiffness of the structure, force-displacement data were gathered for three samples of varying rise ($b = 10, 15$, and 20 mm) in both the low-energy and high-energy configurations [Fig. 6(a)]. An indenter with a load cell mounted to a linear stage was aligned at the center of the buckled beam, initially out of contact with the structure. The stage advanced the indenter a small distance (4.5 mm), so as not to induce snap-through or reconfiguration, at a constant velocity (0.08 mm/s) while measuring the displacement and reaction force [Fig. 6(b)] [41]. The effective stiffness of the structure is taken as the instantaneous slope of the force-displacement curve [Fig. 6(c)]. For all samples, the initial effective stiffness of the low-energy state ($\approx 2.3 \text{ N mm}^{-1}$) was significantly higher than the initial effective stiffness of the high-energy state ($\approx 0.15 \text{ N mm}^{-1}$). Varying the rise of all elements in the structure does not strongly affect the initial effective stiffness, but increasing the rise delays the onset of nonlinear behavior. Beyond the initial linear regime, the force-displacement curves of the low-energy state exhibit a negative slope; if the trials were run to larger displacements, this trend would eventually cause the structure to reconfigure once the reaction force crosses zero. Over this larger displacement

range ($\approx 3\text{--}5$ mm), the effective stiffness of the high-energy state exceeds that of the low-energy state, as the low-energy state approaches a snap-through reconfiguration, evidenced by a decrease in reaction force. Shown here for the dog-bone geometry, the same principles apply to other structures that satisfy the condition of having multiple stable configurations with asymmetric energy potentials such as the square (Sec. 9 of the Supplemental Material [35]).

To keep measurements within the linear regime and avoid triggering snap-through, the indenter engaged the structure only over a small displacement. However, if perturbed sufficiently, the structure can reconfigure from the high-energy state to the low-energy state. Separating the stable configurations is an energy barrier, U_B , an unstable higher-potential-energy state that the structure must pass through before reconfiguring. Determining the energy barrier and the potential energy of each configuration can be used to calculate ΔU_B , the amount of energy that must be input to reconfigure the structure (Sec. 10 of the Supplemental Material [35]). As the rise of the structure increases, the corresponding energy barriers ΔU_B increase. For a given sample, the critical force required to reconfigure varies with the load location, but the energy barrier is constant.

IV. CONCLUSION

We presented a general analytical and experimental framework for multistable two-dimensional structures composed of interconnected buckled beams. Although demonstrated on relatively small systems, the approach naturally extends to assemblies of any size. Of particular interest is applying these methods to large structures by using periodic boundary

conditions to model the behavior and properties of metamaterials (Sec. 11 of the Supplemental Material [35]). Periodic structures with asymmetric potential-energy landscapes could support the transmission of nonlinear waves, such as transition waves and sequential reconfigurations. Using numerical continuation, we identified multiple equilibrium configurations and verified them experimentally. By connecting elements with symmetric energy potentials, we showed that asymmetric global energy landscapes and tunable effective stiffness can emerge solely from geometric coupling. This framework offers a systematic foundation for designing programmable mechanical metamaterials with reconfigurable stability and tunable effective stiffness.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [41].

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