Symmetries of anisotropic spin interaction models

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We show that anisotropic spin interactions do not merely break spin-space group (SSG) symmetries, but instead *twist* them through cohomology invariants, yielding symmetry classes beyond subgroups of $O(3) \times \mathrm{Isom}(\mathbb{R}^3)$. This requires redefining the spin-only group $\mathcal{S}_0$ in terms of proper spin rotations. Based on this unitary $\mathcal{S}_0$, we formulate a twisted SSG (tSSG) theory that captures the complete set of spin-space symmetries. We then study a spin-1 model with tSSG symmetry using linear flavor wave theory and find $\mathbb{Z}_2$ topological quadrupolar excitations defined on a spin Brillouin Klein bottle. Specifically, the quadrupolar excitations possess a momentum-space glide-reflection symmetry and the edge states exhibit a nonlocal momentum twist. These results establish the symmetry language required for interacting spin systems, whether realized in magnetic materials or on programmable quantum simulators, and open a route to unconventional magnetism.

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I. INTRODUCTION

Spin-space groups (SSGs) provide a unifying symmetry framework for magnetic materials, enabling systematic descriptions of magnetic topological electronic states [1–4], topological magnon bands [5–13], and the altermagnetism [14–21]. These systems are governed by isotropic Heisenberg exchange in which spin and spatial operations combine in a particularly transparent manner: Each symmetry operation is realized as a global spin rotation in O(3) accompanied by a lattice isometry, and the resulting SSGs form subgroups of $O(3) \times \mathrm{Isom}(\mathbb{R}^3)$ [22–27].

In realistic magnets, anisotropic spin interactions are ubiquitous and often essential [28–33]. Canonical examples include Dzyaloshinskii-Moriya (DM) exchange [34,35], compass-type bond interactions [36–39], and single-ion anisotropy originating from the crystal field [40–42]. Moreover, these terms are natural targets of the quantum simulation of magnetism: Programmable platforms based on trapped ions, optical lattices, and Rydberg arrays have realized a variety of interacting spin models [43–48], while polar molecules provide a direct route to long-range, tunable anisotropic exchange [49–51].

Spin anisotropies are commonly assumed to reduce or preserve the symmetry of isotropic magnets [52–54]. However, we propose the model [Eq. (1)] to show that anisotropy instead changes the *type* of symmetry realized: Exact symmetries of anisotropic models involve spatially varying spin rotations, so that the full symmetry group cannot embed into $O(3) \times \mathrm{Isom}(\mathbb{R}^3)$. We further find that these spin symmetries acquire a precise group-theoretic meaning within group extension theory only after removing effective time reversal from the spin-only group $\mathcal{S}_0$.

Motivated by these findings, we formulate a theory of twisted spin-space groups (tSSGs) that completes the symmetry description of interacting spin systems via group extension theory. The key step is to redefine $\mathcal{S}_0$ as the group of on-site *proper* spin rotations that are exact symmetries of the system. With this definition, we show that any spin-space operation g admits a natural decomposition into (1) a lattice operation l_g (unitary or antiunitary), (2) a global normalizer spin operation $R_g \in O(3)$ that implements the global conjugation action on $\mathcal{S}_0$, and (3) a site-dependent centralizer sector $\mathfrak{Z}_g(\vec{r})$ that commutes with $\mathcal{S}_0$ at every site. If $\mathfrak{Z}_g(\vec{r}) \equiv 1$, then $g \in O(3) \times \mathrm{Isom}(\mathbb{R}^3)$, and such operations necessarily form a subgroup of $O(3) \times \mathrm{Isom}(\mathbb{R}^3)$. In contrast, when there exists an element g with nontrivial centralizer sector $\mathfrak{Z}_g(\vec{r})$, the group structure, as will be shown below, is modified by a two-cocycle $[\omega_2] \in H^2_\phi(G_{\mathcal{L}}, \mathcal{Z}(\mathcal{S}_0))$. Here, $G_{\mathcal{L}}$ denotes the set of lattice operations $\{l_g, \forall g\}$ and $\phi : G_{\mathcal{L}} \to \mathrm{Aut}(\mathcal{Z}(\mathcal{S}_0))$ is the action induced by conjugation on the center of $\mathcal{S}_0$ via the normalizer sector R_g.

Remarkably, tSSGs give rise to a novel class of topological magnetic excitations on nonorientable manifolds. We study a two-dimensional anisotropic spin-1 model [Eq. (13)] on the wallpaper group *pm* using linear flavor wave theory. The Hamiltonian contains uniform *XYZ* anisotropy and DM interactions on x bonds, uniform off-diagonal symmetric anisotropy and DM interactions on y bonds, and single-ion anisotropy $D(S_i^z)^2$. In the large-D regime, the quadrupolar sector $\{Q_{xz}, Q_{yz}\}$ possesses a momentum-space glide-reflection symmetry; that is, the corresponding Hamiltonian obeys the sewing relation

$$U H_{\mathrm{BdG}}(k_x, k_y) U^{-1} = H_{\mathrm{BdG}}(-k_x, k_y + \pi).$$

Consequently, the corresponding topological excitations are characterized not on the conventional Brillouin torus, but on

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a Brillouin Klein bottle [55–58]. In particular, the two edge modes are related by the nonlocal twist $\omega_L(k_y) = \omega_R(k_y + \pi)$. These features sharply distinguish them from conventional topological magnetic excitations.

We also present a general scheme for models realizing tSSGs based on the cocycle-free construction of group extensions and provide a complete classification of spin point groups with $\mathcal{S}_0$ valued in the 11 chiral point groups, summarized in the Supplemental Material [59].

II. A MINIMAL MODEL WITH TWISTED COLLINEAR SSG

The model is defined on a two-dimensional square lattice with spatial symmetry $p4mm$. It contains nearest-neighbor XY interactions along the $\vec{x}$ direction and DM interactions along the $\vec{y}$ direction. Specifically, for an undirected bond $\langle ij\rangle \parallel \vec{x}$, $H^{XY}_{\langle ij\rangle} = S^x_i S^x_j + S^y_i S^y_j$, whereas for a directed bond $\langle k \to l\rangle \parallel \vec{y}$, $H^{\mathrm{DM}}_{\langle k\to l\rangle} = S^x_k S^y_l - S^y_k S^x_l$. Both interactions have equal strength α, a fixed nonzero real constant. The coupling to the antiferromagnetic (AFM) background field is described by $H_a = \sum_l (-1)^l h S^z_l$, with h its strength. The total Hamiltonian then reads

$$H = \sum_{\langle ij\rangle} \left(\alpha\, H^{XY}_{\langle ij\rangle} + S^z_i S^z_j\right) + \sum_{\langle k\to l\rangle} \left(\alpha\, H^{\mathrm{DM}}_{\langle k\to l\rangle} + S^z_k S^z_l\right) + H_a. \tag{1}$$

We first analyze the unitary symmetries of the model. Since the space group $p4mm$ is symmorphic, it admits the decomposition $p4mm \equiv \mathcal{C}_{4v} \ltimes (\mathbb{T}_x \times \mathbb{T}_y)$, where $\mathcal{C}_{4v}$ is generated by $\{C_4, \mathcal{M}_x\}$. The Wyckoff position $1b = (1/2, 1/2)$ is fixed by $\mathcal{C}_{4v}$, and $\mathbb{T}_x$ and $\mathbb{T}_y$ denote translations along the x and y directions, respectively. With this notation, the Hamiltonian is invariant under five global unitary operations: spin rotations $\mathrm{SO}(2)_z = \{R_z(\theta)\}$, two effective translations $(R_x(\pi)\,||\,\mathbb{T}_{x,y})$, and two effective mirror reflections $(R_x(\pi)\,||\,\mathcal{M}_{x,y})$. These symmetries are all contained in collinear SSGs.

An exotic unitary symmetry arises when the lattice rotation C_4 is spin decorated to become a symmetry of the Hamiltonian. The resulting effective rotation necessarily exchanges the XY interaction on x bonds with the DM interaction on y bonds,

$$\widetilde{C}_4 : \ x\text{-bond } XY \longleftrightarrow y\text{-bond DM}. \tag{2}$$

This transformation cannot be realized by lattice C_4 followed by any global spin rotation. Nevertheless, owing to the local gauge equivalence between the XY and DM terms [60,61], it can be implemented by introducing a site-dependent spin operation $\mathfrak{Z}_{C_4}(\vec{r})$ (see Fig. 1). On a single square, it reads

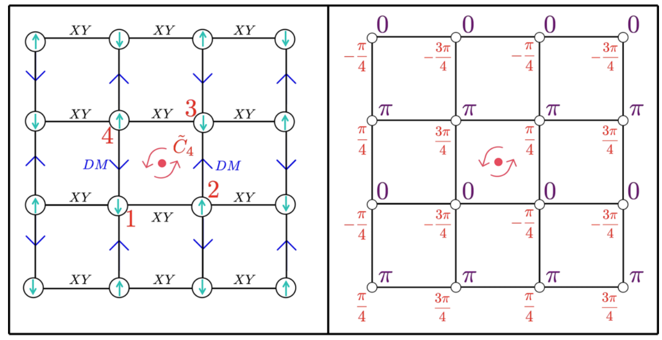
$$\mathfrak{Z}_{C_4}(r_1)\cdot \vec{S}_{r_1} = \vec{S}_{r_1} R_z\!\left(-\frac{3\pi}{4}\right), \quad \mathfrak{Z}_{C_4}(r_2)\cdot \vec{S}_{r_2} = \vec{S}_{r_2} R_z\!\left(-\frac{\pi}{4}\right),$$

$$\mathfrak{Z}_{C_4}(r_3)\cdot \vec{S}_{r_3} = \vec{S}_{r_3} R_z\!\left(\frac{\pi}{4}\right), \quad \mathfrak{Z}_{C_4}(r_4)\cdot \vec{S}_{r_4} = \vec{S}_{r_4} R_z\!\left(\frac{3\pi}{4}\right).$$

The combined operation $\widetilde{C}_4 \equiv (\mathfrak{Z}_{C_4}(\vec{r}) R_x(\pi)\,||\,C_4)$, which satisfies

$$\widetilde{C}_4 \mathrm{SO}(2)_z (\widetilde{C}_4)^{-1} = (\mathrm{SO}(2)_z)^{-1}, \quad (\widetilde{C}_4)^4 = R_z(\pi), \tag{3}$$

is a symmetry of the Hamiltonian (1).



(a) The Hamiltonian (b) The operations $\mathfrak{Z}_{C_4}$, $\mathfrak{Z}_{\mathcal{T}}$

FIG. 1. Spin model with twisted collinear SSG symmetries $\widetilde{C}_4$ and $\widetilde{\mathcal{T}}$. Panel (a) illustrates the Hamiltonian and the AFM background field (teal), while panel (b) shows the site-dependent spin operations associated with C_4 (red) and $\mathcal{T}$ (purple), respectively. Here, each $\theta \in [0, 2\pi)$ in the panel (b) stands for a local spin rotation $R_z(\theta) \in \mathrm{SO}(2)_z$.

We now demonstrate the nontrivial nature of $\widetilde{C}_4$ in comparison with conventional spin point groups (SPGs). The key result is as follows: There exist *exactly three* distinct groups $\mathcal{G}$ satisfying the quotient relation $\mathcal{G}/\mathrm{SO}(2)_z \cong C_4$ [62]. Among these, only two correspond to conventional SPGs, while the third is intrinsically distinct and is realized by the operation $\widetilde{C}_4$ obeying Eq. (3).

The two conventional SPGs correspond to the direct product $\mathcal{G}_1 \equiv \mathrm{SO}(2)_z \times C_4$ and the semidirect product $\mathcal{G}_2 \equiv \mathrm{SO}(2)_z \rtimes \mathscr{C}_4$, where $\mathscr{C}_4$ is generated by $\{(R_x(\pi)\,||\,C_4)\}$. To see this, we recall several basic facts about collinear SSGs [24–26]. From the classification theory, a collinear SSG G_{cl} is characterized by the decomposition $G_{cl} = G'_{cl} \times (\mathrm{SO}(2)_z \rtimes \mathbb{Z}_2^{\mathcal{T}'})$, where $\mathbb{Z}_2^{\mathcal{T}'}$ is generated by $\mathcal{T}' = (-R_x(\pi)\,||\,\mathcal{T})$ with $\mathcal{T}$ the time reversal. Here, $\mathrm{O}(2)_z \equiv \mathrm{SO}(2)_z \rtimes \mathbb{Z}_2^{\mathcal{T}'}$ is referred to as the spin-only group. Each element $g' \in G'_{cl}$ takes the form $(E\,||\,b_{g'})$ or $(-E\,||\,b_{g'}\mathcal{T})$, with $b_{g'}$ a spatial operation. Let $G_{\mathcal{L}}$ denote the space group generated by all such $b_{g'}$. Collinear SSGs are then in one-to-one correspondence with one-dimensional real representations of $G_{\mathcal{L}}$.

For our purposes, we assume that the lattice symmetry group $G_{\mathcal{L}}$ contains the cyclic subgroup $\mathcal{C}_4$. Restricting $G_{\mathcal{L}}$ to $\mathcal{C}_4$, there are exactly two SPG solutions, corresponding to the two one-dimensional real representations of $\mathcal{C}_4$. In the first case, G'_{cl} is generated by the unitary operation $(E\,||\,C_4)$; in the second, it is generated by the antiunitary operation $(-E\,||\,C_4\mathcal{T})$. These two SPG solutions can be rewritten as $G^{(+)}_{cl} = \mathrm{SO}(2)_z \rtimes (C_4 \times \mathbb{Z}_2)$, with C_4 generated by $(E\,||\,C_4)$, and $G^{(-)}_{cl} = \mathrm{SO}(2)_z \rtimes (\mathscr{C}_4 \times \mathbb{Z}_2)$, with $\mathscr{C}_4$ generated by the unitary operation $(R_x(\pi)\,||\,C_4)$. Restricting $G^{(+)}_{cl}$ and $G^{(-)}_{cl}$ to their unitary subgroups yields the direct product group $\mathcal{G}_1$ and the semidirect product group $\mathcal{G}_2$, respectively.

We now turn to the group $\mathcal{G}_3$ generated by $\mathrm{SO}(2)_z$ and $\widetilde{C}_4$, which satisfies Eq. (3). Owing to the presence of the spin flipping $R_x(\pi)$ in $\widetilde{C}_4$, $\mathcal{G}_3$ appears superficially similar to $\mathcal{G}_2$. Indeed, the semidirect product $\mathcal{G}_2$ can be expressed as

$$\mathscr{C}_4\, \mathrm{SO}(2)_z\, \mathscr{C}_4^{-1} = (\mathrm{SO}(2)_z)^{-1}, \quad (\mathscr{C}_4)^4 = R_z(0) = E, \tag{4}$$

where we denote $\{R_x(\pi)\,||\,C_4\}$ by $\mathscr{C}_4$. The crucial distinction between Eqs. (4) and (3) lies in the fourth power of the corresponding effective C_4 operation. Importantly, this distinction cannot be removed: $(\widetilde{C}_4)^4 = R_z(\pi)$ is in fact a cohomology invariant. In other words, $\mathcal{G}_2$ cannot be transformed into $\mathcal{G}_3$ by redefining $\mathscr{C}_4$ within $\mathcal{G}_2$. Mathematically, such a redefinition corresponds to a different choice of section and hence induces a coboundary transformation, which leaves the invariant unchanged. We further note that $\mathcal{G}_2$ and $\mathcal{G}_3$ are not only distinct in this mathematical sense, but are also physically inequivalent, as they are not conjugate to each other.

We next analyze the antiunitary symmetries of the model. The effective time-reversal symmetry $\mathcal{T}' = (-R_x(\pi)\,||\,\mathcal{T})$, present in all collinear SSGs, is explicitly broken. Nevertheless, it can be restored by combining it with a site-dependent spin operation $\mathfrak{Z}_{\mathcal{T}}(\vec{r})$. On the square in Fig. 1, we define $\mathfrak{Z}_{\mathcal{T}}(\vec{r}_{1,2}) = 1$, $\mathfrak{Z}_{\mathcal{T}}(\vec{r}_{3,4}) = R_z(\pi)$, under which the Hamiltonian is invariant with respect to $\widetilde{\mathcal{T}} = (-\mathfrak{Z}_{\mathcal{T}}(\vec{r})\,R_x(\pi)\,||\,\mathcal{T})$.

The site-dependent spin operations $\mathfrak{Z}_g(\vec{r})$ in $\widetilde{C}_4$ and $\widetilde{\mathcal{T}}$ modify the cohomology invariants and generate symmetry groups absent in the conventional formulation. This mechanism originates from the nontrivial conjugation of $SO(2)_z$ by $R_x(\pi)$, which necessitates decomposing the full symmetry group with respect to the unitary subgroup $SO(2)_z$, rather than the spin-only group $O(2)_z$. A decomposition based on $O(2)_z$ instead obscures this conjugation structure and the associated cohomology invariants. This choice is further enforced by the fact that the conventional antiunitary operation $\mathcal{T}' \in O(2)_z$ is broken and replaced by the site-dependent symmetry $\widetilde{\mathcal{T}}$.

III. BASIC SETUP

Below, we redefine the spin-only group and formulate a basic conceptual framework for symmetries in spin systems. Detailed constructions of tSSGs are presented in the following section.

A spin-only group is defined as a subgroup of SO(3) and is denoted by $\mathcal{S}_0$. Representative examples of $\mathcal{S}_0$ include the 11 chiral point groups consisting solely of proper rotations: 6 abelian groups $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_6$, and $\mathcal{D}_2$, and 5 nonabelian groups $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{D}_6$, $\mathcal{T}$, and $\mathcal{O}$. Continuous groups such as $SO(2)_z$, SO(3) also fall into this category. In the following, we denote by $\mathcal{Z}(\mathcal{S}_0)$ the center of $\mathcal{S}_0$, and by $\mathfrak{N}_{\mathcal{S}}(\mathcal{S}_0)$ and $\mathfrak{Z}_{\mathcal{S}}(\mathcal{S}_0)$ its normalizer and centralizer in a group $\mathcal{S} \subset O(3)$, respectively.

For a given spin Hamiltonian, the spin-only group $\mathcal{S}_0$ is defined as the set of all on-site global proper rotations that leave the Hamiltonian invariant. By definition, $\mathcal{S}_0$ is a normal subgroup of the full symmetry group $\mathcal{G}$. Then, $\mathcal{G}$ admits a coset decomposition,

$$\mathcal{G} = \mathcal{S}_0 \sqcup u_1\mathcal{S}_0 \sqcup u_2\mathcal{S}_0 \sqcup \cdots \sqcup T_1\mathcal{S}_0 \sqcup T_2\mathcal{S}_0 \sqcup \cdots, \tag{5}$$

where $u_1, u_2, \ldots$ are unitary elements and $T_1, T_2, \ldots$ are antiunitary elements. These cosets themselves form a group isomorphic to a magnetic space group (MSG) $G_{\mathcal{L}}$, equivalently $\mathcal{G}/\mathcal{S}_0 \cong G_{\mathcal{L}}$. Conversely, the complete class $\mathscr{G}$ of symmetry groups for spin systems with a given $\mathcal{S}_0$ is naturally defined as

$$\mathscr{G} = \{\mathcal{G}\,|\,\mathcal{G}/\mathcal{S}_0 \cong \text{MSG}\}. \tag{6}$$

The set $\mathscr{G}$ splits into two classes. One class consists of groups that can be embedded as subgroups of the direct product $O(3) \times \text{Isom}(\mathbb{R}^3)$; the other does not admit such an embedding. All elements of $\mathscr{G}$ are referred to as tSSGs, including cases with nontrivial cohomology twists such as $\mathcal{G}_3$ defined in Eq. (3). Groups belonging to the first class are, by convention, also referred to as SSGs. Two tSSGs $\mathcal{G}, \mathcal{G}' \in \mathscr{G}$ are defined to be physically equivalent if there exists a spin transformation $s \in O(3)$ such that $\mathcal{G} = s\,\mathcal{G}'\,s^{-1}$. Equivalently, the two groups are related by a global change of spin coordinates.

However, the definition in Eq. (6) is overly general and may include groups $\mathcal{G}$ that are not physically meaningful for spin systems. For example, when $\mathcal{S}_0 = SO(2)_z$, the nontrivial conjugation action on $\mathcal{S}_0$ is not unique to the spin flipping $R_x(\pi)$, but can also be realized by other unitary matrices, such as $U = \text{diag}(i\sigma_x, 1) \in SU(3)$. Allowing such conjugations would lead to an SU(3) theory, which lies outside the scope of spin physics. We therefore impose the following restriction: The conjugation actions of any operations in $\mathcal{G} \in \mathscr{G}$ on $\mathcal{S}_0$ must be realized by elements of O(3). Under this condition, $\mathscr{G}$ defines a complete class of tSSGs relevant to spin systems. In the following, $\mathscr{G}$ refers exclusively to this restricted class.

IV. TWISTED SPIN-SPACE GROUPS

We now construct tSSGs. The construction is abstract, and the representative example introduced above will be used to clarify the key ideas when appropriate. In what follows, $\mathcal{G}$ denotes a tSSG with spin-only group $\mathcal{S}_0$, satisfying $\mathcal{G}/\mathcal{S}_0 \cong G_{\mathcal{L}}$, where $G_{\mathcal{L}}$ is an MSG.

We show that the structure underlying the operation $\widetilde{C}_4 = (\mathfrak{Z}_{C_4}R_x(\pi)\,||\,C_4)$ is generic to all tSSGs. Owing to the global nature of $\mathcal{S}_0$, each element $g \in \mathcal{G}$ admits a systematic decomposition into three sectors. Let $s_g(\vec{r}_i) \in O(3)$ denote the spin operation implementing the conjugation action of g on $\mathcal{S}_0$ at site $\vec{r}_i$. Since elements of $\mathcal{S}_0$ act globally, the conjugation action of g preserves $\mathcal{S}_0$ at all sites, implying

$$s_g(\vec{r}_i)\,\mathcal{S}_0\,s_g(\vec{r}_i)^{-1} \equiv s_g(\vec{r}_j)\,\mathcal{S}_0\,s_g(\vec{r}_j)^{-1} = \mathcal{S}_0. \tag{7}$$

Thus, $s_g(\vec{r}_i)$ and $s_g(\vec{r}_j)$ normalize $\mathcal{S}_0$, and their difference $s_g(\vec{r}_i)s_g(\vec{r}_j)^{-1}$ lies in the centralizer of $\mathcal{S}_0$. Fixing a reference site $\vec{r}_0$, each element $g \in \mathcal{G}$ can therefore be written as

$$g = \big(\mathfrak{Z}_g(\vec{r})\,R_g^0\big|\big|l_g\big),$$

where $l_g \in G_{\mathcal{L}}$ is the lattice operation of g, R_g^0 is the global operation valued in the normalizer $\mathfrak{N}_{O(3)}(\mathcal{S}_0)$ at $\vec{r}_0$, and $\mathfrak{Z}_g(\vec{r})$ is a site-dependent operation valued in the centralizer $\mathfrak{Z}_{SO(3)}(\mathcal{S}_0)$.

We now show that, analogous to the fact that the two spin operations $R_x(0) = E$ and $R_x(\pi)$ fully determine conventional SPGs, the sector R_g^0 determines all subgroups $\mathcal{G} \subset O(3) \times \text{Isom}(\mathbb{R}^3)$ satisfying $\mathcal{G}/\mathcal{S}_0 \cong G_{\mathcal{L}}$. Since any two elements $g, \widetilde{g} \in \mathcal{G}$ with the same lattice sector $l_g = l_{\widetilde{g}}$ belong to the same coset of $\mathcal{G}/\mathcal{S}_0$, we have $\widetilde{g} = s_{\widetilde{g}}g$ with $s_{\widetilde{g}} \in \mathcal{S}_0$. According to the above argument, we decompose g and $\widetilde{g}$ as $g = (\mathfrak{Z}_g(\vec{r})\,R_g^0\,||\,l_g)$ and $\widetilde{g} = (\mathfrak{Z}_{\widetilde{g}}(\vec{r})\,R_{\widetilde{g}}^0\,||\,l_{\widetilde{g}})$. Since $\mathfrak{Z}_g(\vec{r})$ commutes with $\mathcal{S}_0$, we obtain

$$\widetilde{g} = \big(\mathfrak{Z}_{\widetilde{g}}(\vec{r})\,R_{\widetilde{g}}^0\big|\big|l_{\widetilde{g}}\big) = \big(\mathfrak{Z}_g(\vec{r})\,s_{\widetilde{g}}R_g^0\big|\big|l_g\big). \tag{8}$$

This implies that any two elements $g, \widetilde{g} \in \mathcal{G}$ with the same lattice sector $l_g = l_{\widetilde{g}}$ have the same normalizer sector modulo $\mathcal{S}_0$. Consequently, there exists a map $\hat{R} : l_g \mapsto R_g^0 \mathcal{S}_0$. We therefore can rewrite the coset representative R_g^0 as $R_{l_g}^0$ and $\mathfrak{Z}_{\widetilde{g}}$ as $\mathfrak{Z}_{l_g}$, emphasizing that they depend only on the lattice sector l_g. Notably, since the centralizer group $\mathfrak{Z}_{\mathrm{SO}(3)}(\mathcal{S}_0)$ is a normal subgroup of the normalizer group $\mathfrak{N}_{\mathrm{O}(3)}(\mathcal{S}_0)$, the map $\hat{R}$ is a group homomorphism

$$\hat{R} : G_{\mathcal{L}} \to \mathfrak{N}_{\mathrm{O}(3)}(\mathcal{S}_0)/\mathcal{S}_0. \tag{9}$$

This homomorphism maps unitary (antiunitary) elements of $G_{\mathcal{L}}$ to cosets with determinant $+1$ (-1), and is well defined since $\mathcal{S}_0 \subset \mathrm{SO}(3)$.

Conversely, any such group homomorphism $\hat{R}$ uniquely specifies a subgroup of $\mathrm{O}(3) \times \mathrm{Isom}(\mathbb{R}^3)$. This subgroup is constructed by assembling the cosets $(\hat{R}(l_g) \,||\, l_g)$ for all $l_g \in G_{\mathcal{L}}$, with multiplication

$$\left(s_m R_{l_{g_1}}^0 \,||\, l_{g_1}\right)\left(s_n R_{l_{g_2}}^0 \,||\, l_{g_2}\right) = \left(s_m R_{l_{g_1}}^0 \, s_n R_{l_{g_2}}^0 \,||\, l_{g_1} l_{g_2}\right), \tag{10}$$

where $s_{m,n} \in \mathcal{S}_0$, and $R_{l_g}^0$ denotes a fixed coset representative of $\hat{R}(l_g)$. A reformulation of conventional SSGs in this language is presented in the Supplemental Material [59].

Each tSSG is formally similar to a subgroup of $\mathrm{O}(3) \times \mathrm{Isom}(\mathbb{R}^3)$, as illustrated by the relation between $\mathcal{G}_3$ and $\mathcal{G}_2$ in Eqs. (4) and (3). These "similar" groups share the same group action $\phi : G_{\mathcal{L}} \to \mathrm{Aut}(\mathcal{Z}(\mathcal{S}_0))$, induced by conjugation of the normalizer sector on the center of $\mathcal{S}_0$, and are distinguished by elements of the second group cohomology $H_\phi^2(G_{\mathcal{L}}, \mathcal{Z}(\mathcal{S}_0))$. Indeed, the centralizer sector $\mathfrak{Z}_{l_g}(\vec{r})$ encodes these cohomology invariants and gives rise to tSSGs that cannot be embedded into $\mathrm{O}(3) \times \mathrm{Isom}(\mathbb{R}^3)$. When $\mathfrak{Z}_{l_g}(\vec{r})$ is present, the multiplication between elements $\widetilde{g}_1 = (\mathfrak{Z}_{l_{g_1}}(\vec{r})\, s_{\widetilde{g}_1} R_{l_{g_1}}^0 \,||\, l_{g_1})$ and $\widetilde{g}_2 = (\mathfrak{Z}_{l_{g_2}}(\vec{r})\, s_{\widetilde{g}_2} R_{l_{g_2}}^0 \,||\, l_{g_2})$ takes the form (see the Supplemental Material [59])

$$\widetilde{g}_1 \cdot \widetilde{g}_2 = \omega_2(l_{g_1}, l_{g_2})\, \widetilde{g_1 g_2}, \tag{11}$$

where $\widetilde{g_1 g_2} = (\mathfrak{Z}_{l_{g_1} l_{g_2}}(\vec{r})\, s_{\widetilde{g}_1} R_{l_{g_1}}^0\, s_{\widetilde{g}_2} R_{l_{g_2}}^0 \,||\, l_{g_1} l_{g_2})$, and $\omega_2(l_{g_1}, l_{g_2}) \in \mathcal{Z}(\mathcal{S}_0)$ is a two-cocycle. For $\mathfrak{Z}_{l_g}(\vec{r}) \equiv E$, the product (11) is reduced to the law (10).

We now present a universal construction of models realizing tSSGs. This mechanism is rigorously supported by the cocycle-free construction of group extensions [63]. In general, tSSGs with any given $\mathcal{S}_0$ can arise from the coexistence of on-site single-ion anisotropy Hamiltonians H_{site}, with spin-only group $\mathcal{S}_0$, and bond spin interactions H_{bond}, whose spin-only group is $\mathcal{Z}(\mathcal{S}_0)$ (see Fig. 2). In this setting, the group homomorphism $\hat{R}$ is fixed by the symmetries of H_{site}, while H_{bond} admits site-dependent sector valued in $\mathfrak{Z}_{\mathrm{SO}(3)}(\mathcal{S}_0) \subseteq \mathfrak{Z}_{\mathrm{SO}(3)}(\mathcal{Z}(\mathcal{S}_0))$. A Hamiltonian with such tSSG symmetry therefore takes the generic form

$$H = \lambda_1 H_{\mathrm{site}} + \lambda_2 H_{\mathrm{bond}}, \quad \lambda_1, \lambda_2 \in \mathbb{R}. \tag{12}$$

In the limit $\lambda_2 \to 0$, the model reduces to one whose symmetries form an SSG with spin-only group $\mathcal{S}_0$ and no centralizer sector. In contrast, for $\lambda_1 \to 0$, it reduces to a model whose symmetries form a tSSG with spin-only group $\mathcal{Z}(\mathcal{S}_0)$, with the centralizer sector taking values in $\mathfrak{Z}_{\mathrm{SO}(3)}(\mathcal{S}_0)$. Rigorous proofs are provided in the Supplemental Material [59].

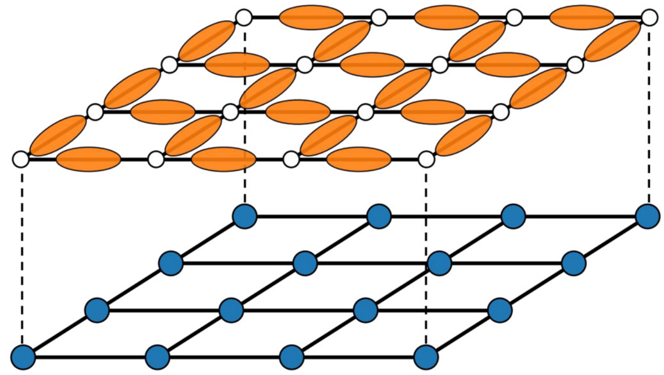


FIG. 2. The Hamiltonian (12) realizing tSSGs with arbitrary $\mathcal{S}_0$. The upper layer represents the spin-exchange Hamiltonian H_{bond}, while the lower layer represents the on-site single-ion anisotropy Hamiltonian H_{site}.

V. TOPOLOGICAL QUADRUPOLAR BANDS ON SPIN BRILLOUIN KLEIN BOTTLE

As an application of the tSSG theory, we show that spin models protected by tSSGs can yield quadrupolar excitations on a spin Brillouin zone whose fundamental domain is topologically a Klein bottle, arising from the presence of a free action, namely, a glide reflection in the spin Brillouin zone [55]. This Klein-bottle topology reshapes the topological classification of quadrupolar excitation bands [64,65]: Every topological band protected by the glide-reflection symmetry is topologically equivalent to a conventional topological band defined on the Klein bottle. We refer to these excitations as topological quadrupolars on spin Brillouin Klein bottle. Following Ref. [55], they are classified by a $\mathbb{Z}_2$ invariant, rather than the $\mathbb{Z}$-valued Chern invariant on the Brillouin torus.

Consider the following anisotropic spin-1 Hamiltonian with $\mathcal{S}_0 = \mathcal{C}_2$ on the lattice $\vec{r} = (r_x, r_y) \in \mathbb{Z} \times \mathbb{Z}$ with wallpaper group pm:

$$H = \sum_{\vec{r}} \left[\boldsymbol{S}_{\vec{r}}^{\mathsf{T}} \boldsymbol{J}_1 \boldsymbol{S}_{\vec{r}+\hat{x}} + \boldsymbol{S}_{\vec{r}}^{\mathsf{T}} \boldsymbol{J}_2 \boldsymbol{S}_{\vec{r}+\hat{y}} \right] + D \sum_{\vec{r}} \left(S_{\vec{r}}^z\right)^2, \tag{13}$$

where $\hat{x} = (1, 0)$, $\hat{y} = (0, 1)$, $\boldsymbol{S}_{\vec{r}} = (S_{\vec{r}}^x, S_{\vec{r}}^y, S_{\vec{r}}^z)^T$, and

$$\boldsymbol{J}_1 = \begin{pmatrix} a+b & \lambda & 0 \\ -\lambda & a-b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad \boldsymbol{J}_2 = \begin{pmatrix} 0 & \beta+\gamma & 0 \\ \beta-\gamma & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

with $a, b, c, \lambda, \alpha, \beta, \gamma, D$ all arbitrary real parameters. The tSSG symmetries of the Hamiltonian (13) are the following: $\widetilde{T}_x = (E \,||\, \mathbb{T}_x)$, $\widetilde{T}_y = (E \,||\, \mathbb{T}_y)$, $\widetilde{\mathcal{M}}_x = (\mathfrak{Z}_{\mathcal{M}_x}(\vec{r}) R_{[100]}(\pi) \,||\, \mathcal{M}_x)$, with $\mathfrak{Z}_{\mathcal{M}_x}(\vec{r}) = (R_{[001]}(\pi))^{r_y}$. Notice that $\widetilde{M}_x \widetilde{T}_y \widetilde{M}_x^{-1} \widetilde{T}_y^{-1} = R_{[001]}(\pi) \in \mathcal{S}_0$. The model is analyzed in detail in the Supplemental Material [59] using linear flavor wave theory, and we present the main results below.

In the Cartesian spin-1 basis $\{|x\rangle, |y\rangle, |z\rangle\}$, the large D single-ion anisotropy selects the ferroquadrupolar vacuum $|z\rangle$ at every site. Condensing the $|z\rangle$ flavor and keeping harmonic terms gives $Q_{xz} \sim -(b_x + b_x^\dagger)$ and $Q_{yz} \sim -(b_y + b_y^\dagger)$, so the $\{Q_{xz}, Q_{yz}\}$ sector is a genuine one-particle bosonic problem. Thus, one can write down a four-band bosonic

Bogoliubov–de Gennes (BdG) Hamiltonian in the basis $(b_{x,\boldsymbol{k}}, b_{y,\boldsymbol{k}}, b^\dagger_{x,-\boldsymbol{k}}, b^\dagger_{y,-\boldsymbol{k}})^{\mathrm{T}}$. Meanwhile, it is shown that the harmonic kernel equivalently reduces to a 2×2 Hermitian matrix,

$$M_1(\boldsymbol{k}) = D\,\mathbb{I}_2 + 4a\cos k_x\,\mathbb{I}_2 - 4\beta\cos k_y\,\sigma_x + 4(\lambda\sin k_x + \gamma\sin k_y)\sigma_y - 4b\cos k_x\,\sigma_z, \quad (14)$$

whose two positive bosonic branches are $\omega_\pm(\boldsymbol{k}) = \sqrt{D[D + 4a\cos k_x \pm 4\rho(\boldsymbol{k})]}$, with $\rho^2 = b^2\cos^2 k_x + \beta^2\cos^2 k_y + (\lambda\sin k_x + \gamma\sin k_y)^2$. The crucial point is

$$\widetilde{\mathcal{M}}_x:\ \boldsymbol{b}_{(k_x,k_y)} \mapsto e^{-\mathrm{i}k_x}(-\sigma_z)\boldsymbol{b}_{(-k_x,\,k_y+\pi)}, \quad (15)$$

which is a momentum glide reflection. Indeed, the kernel itself also obeys the glide-reflection sewing relation:

$$M_1(-k_x, k_y+\pi) = (-\sigma_z)M_1(k_x,k_y)(-\sigma_z)^{-1}. \quad (16)$$

The bosonic spectrum therefore satisfies $\omega_\pm(k_x,k_y) = \omega_\pm(-k_x, k_y+\pi)$, and the fundamental domain of the sector is a Klein bottle rather than a torus [55].

This bulk glide structure has a sharp boundary consequence. For open boundaries along x, the exactly solvable line $a=0$, $b=\lambda$ admits closed-form edge modes with energies $\omega_{L,R}(k_y) = \sqrt{D(D\pm 4\beta\cos k_y)}$ and decay factor $r_{L,R}(k_y) = \mp\mathrm{i}\gamma\sin k_y/\lambda$. The left and right edge branches are finally sewn by nonlocal twist $\omega_L(k_y) = \omega_R(k_y+\pi)$. Their existence is controlled by the winding of the off-diagonal polynomial $q(k_x;k_y) = -4(\lambda e^{\mathrm{i}k_x} + \mathrm{i}\gamma\sin k_y)$, namely,

$$\nu(k_y) = \frac{1}{2\pi\mathrm{i}}\int_{-\pi}^{\pi}\mathrm{d}k_x\,\partial_{k_x}\ln q = \begin{cases}1, & |\gamma\sin k_y| < |\lambda|,\\ 0, & |\gamma\sin k_y| > |\lambda|,\end{cases}$$

which counts the number of roots of $q(z;k_y)$ inside the unit disk. This $\mathbb{Z}_2$ quantity is the slice invariant governing edge localization: Any symmetry-preserving perturbation that leaves the glide sewing intact and keeps q nonzero on the unit circle cannot change $\nu(k_y)$ and hence cannot remove the corresponding boundary mode. For the lower positive-frequency band, the bulk Klein-bottle invariant satisfies $\nu_{\mathrm{K}} = \nu(-\pi/2) \pmod 2$. For instance, setting $\lambda = 1$ and $\gamma = 0.9$ gives $\nu(k_y) = 1$ for every k_y, implying both $\nu_{\mathrm{K}} = 1$ and a nonlocally sewn edge pair spanning the entire spin Brillouin zone rather than fragmentary edge arcs.

VI. DISCUSSION AND OUTLOOK

Physically, since the homomorphism $\hat{R}$ defined in Eq. (9) characterizes all subgroups of the symmetry group of the Heisenberg model, it encodes every symmetry-breaking pattern induced by anisotropic perturbations in realistic materials. On the other hand, since DM vectors act as a gauge field for spin currents [66], the site-dependent centralizer sector $\mathfrak{Z}_g(\vec{r})$ in Fig. 1 admits a natural interpretation as a *spin gauge*. From this perspective, tSSGs with nontrivial $\mathfrak{Z}_g(\vec{r})$ arise from gauging the centralizer of $\mathcal{S}_0$.

Both $\hat{R}$ and gauge-dressed spin symmetries can break the homomorphic relation between lattice and spin operations (see Sec. 3 of the Supplemental Material [59]), thereby generating magnetic excitations protected by *projective* crystalline symmetries. These quasiparticles can exhibit Brillouin platycosm topology [57,58]. To our knowledge, this provides the first interacting spin model supporting topological magnetic excitations on spin Brillouin Klein bottle. This work motivates the theoretical and experimental search for magnetic excitations on Brillouin platycosms, in real magnetic materials and on programmable platforms for the quantum simulation of magnetism alike.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the author.

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