

Spinor pairing order enforced by Berry phase

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We introduce a class of topological pairing orders characterized by a half-integer pair monopole charge, leading to Berry phase enforced half-integer partial wave symmetry. This exotic spinor pairing order can emerge from pairing between Fermi surfaces with Chern numbers differing by an odd integer. Using tight-binding models, we demonstrate spinor superconducting orders with monopole charges $\pm 1/2$, featuring a single gap node and nontrivial surface states. Additionally, the superfluid velocity follows a fractionalized Mermin-Ho relation in spatially inhomogeneous pairing orders. The concept extends to spinor density waves and excitons.

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Introduction. The discovery of a new pairing order is always accompanied by the establishment of a new paradigm to understand its physical properties and to probe the pairing symmetry. It has long been assumed that all pairing orders are completely classified by spherical harmonic symmetries and their lattice counterparts. Pioneering examples of unconventional superconductors and superfluids include p -wave superfluid ^3He [1–4], d -wave high- T_c cuprates [5–7], and $s_{\pm}$ -wave iron pnictides [8,9]. Distinct symmetry of pairing gap functions gives rise to characteristic properties in different superconducting states. On the other hand, significant progress has been made in the discovery of topological quantum materials characterized by nontrivial geometric phases of single-particle electron bands [10–14], which lead to the discovery of, for example, quantum anomalous Hall insulators [15–18] and Weyl semimetals [19–29].

Recently, monopole harmonic superconductivity [30–33] has been proposed based on the generalization of the single-particle Berry phase to “*pair* Berry phase”—the two-particle Berry phase for Cooper pairs in superconductors [34]. This topological class of 3D superconductors can possibly exist in, for example, magnetic Weyl semimetals under the proximity effect with an ordinary s -wave superconductor. Generally, when pairing occurs between two Fermi surfaces of opposite Chern numbers, the inter-Fermi-surface Cooper pair can nontrivially inherit band topology and acquire a nontrivial “*pair* Berry phase” in the weak-coupling regime. The “*pair* Berry phase,” as a type of topological obstruction, prevents the gap function from being well defined over an entire Fermi surface. Hence, the corresponding gap function is no longer describable by spherical harmonic functions or their lattice counterparts. Instead, it is characterized by monopole harmonic functions, which are eigenfunctions of angular momentum in the presence of a magnetic monopole

[30,35–38]. The “*pair* Berry phase” further enforces gap nodes and determines the total vorticity of gap nodes over a Fermi surface, independent of specific pairing mechanism. Therefore, monopole harmonic superconductivity fundamentally differs from previously known unconventional superconductivity. Furthermore, monopole harmonic superconductivity is only an example of topological many-particle order. In the particle-hole channel, the monopole harmonic charge-density-wave (CDW) order [39] has been proposed in a model of Weyl semimetal consisting of nested Fermi surfaces that surround Weyl points of the same chirality. This topological class of CDW states is also characterized by monopole harmonic symmetry and can host emergent Weyl nodes in the gap functions.

In this Letter, we first explain the difference between the familiar examples of topological superfluid/superconductors [2,4,40–47] and the monopole harmonic superconductors. Then, we investigate the spinor pairing as an exotic example of monopole harmonic pairing states. When electrons pair between two topological Fermi surfaces carrying Chern numbers with different even- and oddness, the Cooper pair acquires a half-integer monopole charge. The gap function can exhibit an odd number of nodes over a single Fermi surface, leading to a nontrivial Bogoliubov excitation spectrum. We demonstrate the simplest example of an angular momentum $j = 1/2$ spinor pairing order using tight-binding models in a cubic lattice, which exhibit a single Bogoliubov-de Gennes (BdG) gap node characterized by nontrivial chirality, providing a nontrivial example of a general result that the Nielsen-Ninomiya theorem [48,49] does not hold in lattice systems when $U(1)$ symmetry is broken. Furthermore, we show zero-energy Majorana surface states arising from momentum space phase winding of the low-energy spinor pairing order in the bulk. Finally, we show that the superfluid velocity obeys a fractional generalization of the Mermin-Ho relation [50] in the presence of spatial inhomogeneity of order parameters.

Monopole harmonic pairing and complex d vectors. We begin with examples of topological superconductivity, where the Fermi surface is topologically trivial, but the superconducting gap function exhibits nontrivial topology. For example,

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the spin-polarized $p_x + ip_y$ pairing is fully gapped in two dimensions (2D). It belongs to class D , which breaks both spin-rotation and time-reversal symmetry. The gap function $\Delta(\mathbf{k})$ is a complex function exhibiting a phase winding number $\nu = \frac{1}{2\pi} \oint d\mathbf{k} \partial_{\mathbf{k}} \theta(\mathbf{k})$ around the one-dimensional (1D) Fermi circle, where $\theta(\mathbf{k})$ is the $U(1)$ phase of the gap function. The three-dimensional (3D) time-reversal invariant pairing of the $^3\text{He-B}$ type belongs to class $DIII$, and the single-particle band structure exhibits twofold spin degeneracy. The corresponding gap function is no longer a scalar but is represented by a pairing matrix. It is proportional to a 2×2 unitary matrix, say, $\Delta(\mathbf{k}) = i\sigma_y \hat{\mathbf{d}}(\mathbf{k}) \cdot \boldsymbol{\sigma}$, where the real $\mathbf{d}$ vector exhibits a nontrivial texture over the Fermi surface characterized by a nontrivial Pontryagin index,

$$\nu = \frac{1}{8\pi} \oint_{\text{FS}} d\mathbf{k}^2 \epsilon_{\mu\nu} \hat{\mathbf{d}}(\mathbf{k}) \cdot \partial_{k_\mu} \hat{\mathbf{d}}(\mathbf{k}) \times \partial_{k_\nu} \hat{\mathbf{d}}(\mathbf{k}). \quad (1)$$

Under an open boundary condition, 2D class D topological superconductors exhibit 1D chiral Majorana modes on the edge, while 3D class $DIII$ time-reversal invariant topological superconductors exhibit helical 2D Majorana surface modes.

Recently, monopole superconductivity [30] has been proposed as a topological class of superconducting pairing order. The central idea is that when Cooper pairing occurs between two Fermi surfaces of different Chern numbers, the resulting Cooper pairs acquire a nontrivial two-particle pair Berry phase, which introduces topological obstruction in the $U(1)$ phase of the pairing order. For example, consider the simplest model of a Weyl semimetal state with a pair of Weyl points located at $\pm \mathbf{K}_0$. Upon doping, the Weyl points are enclosed by two separated Fermi surfaces, denoted by $\text{FS}_{c,\pm}$. Near these Fermi surfaces, the low-energy Hamiltonians take the form $H_{\pm}(\mathbf{k} \mp \mathbf{K}_0) = \pm v_F \mathbf{k} \cdot \boldsymbol{\sigma} - \mu$. The Fermi surfaces $\text{FS}_{c,\pm}$ exhibit nontrivial single-particle Berry phases, with single-particle monopole charges being $\pm q = \mp 1/2$. Alternatively, the corresponding Chern numbers are $\pm C$ with $C = 2q$. For the pairing between $\text{FS}_{c,+}$ and $\text{FS}_{c,-}$, the gap function can be represented by a 2×2 pairing matrix in the spin- $\uparrow, \downarrow$ basis. Monopole pairing is an example of *nonunitary* pairing, characterized by complex $\mathbf{d}$ vectors $(-d_x + id_y)/\sqrt{2} = u_k^2$, $(d_x + id_y)/\sqrt{2} = v_k^2$, $d_z = \sqrt{2}u_k v_k$, where $u_k = \cos \frac{\theta_k}{2}$ and $v_k = \sin \frac{\theta_k}{2} e^{i\phi_k}$. The pair Berry connection can be defined in terms of complex $\mathbf{d}$ vectors as

$$\mathbf{A}_{\text{pair}}(\mathbf{k}) = \hat{\mathbf{d}}^* \cdot i \nabla_{\mathbf{k}} \hat{\mathbf{d}} = 2q_{\text{pair}} \tan \frac{\theta_k}{2} \hat{\mathbf{e}}_{\phi_k}. \quad (2)$$

Correspondingly, the Berry curvature can be expressed as $\Omega_i(\mathbf{k}) = \epsilon_{ijl} \partial_j A_{\text{pair},l} = i \epsilon_{ijl} \partial_j \hat{\mathbf{d}}^* \cdot \partial_l \hat{\mathbf{d}}$. The total pair Berry flux through $\text{FS}_{c,+}$ is

$$\oint_{S_+} d\mathbf{k} \cdot \nabla_{\mathbf{k}} \times \mathbf{A}_{\text{pair}}(\mathbf{k}) = \oint_{S_+} d\mathbf{k} \cdot i \partial_j \hat{\mathbf{d}}^* \times \partial_k \hat{\mathbf{d}} = 4\pi q_{\text{pair}}, \quad (3)$$

where the pair monopole charges $q_{\text{pair}} = 2q$. Hence, the inter-Fermi-surface pairing inherits the Berry phases of electrons residing on different topological Fermi surfaces in a nontrivial way.

As a result, the topological obstruction in the wavefunction of Cooper pairs prevents its phase from being well defined over an entire Fermi surface, leading to generic nodal

structures in the pairing gap functions. The gap function $\Delta(\mathbf{k})$, when projected onto $\text{FS}_{c,\pm}$, exhibits a nodal structure with total vorticity $2q_{\text{pair}}$ in momentum space, which is independent of specific pairing mechanisms [30]. When $q_{\text{pair}} \neq 0$, $\Delta(\mathbf{k})$ cannot be a regular function throughout the $\text{FS}_{c,+}$. This nodal structure is fundamentally different from that of familiar pairing symmetries based on spherical harmonics $Y_{lm}(\hat{\mathbf{k}})$, which are well defined on the entire Fermi surface and correspond to $q_{\text{pair}} = 0$ with zero total vorticity. For example, in the $^3\text{He-A}$ type $p_x + ip_y$ pairing in 3D, the gap nodes appear at the north and south poles of the Fermi surface as a pair of momentum space vortex and antivortex, respectively.

Away from the Fermi surface, the nodes of $\Delta(\mathbf{k})$ extend into vortex lines in momentum space. The Weyl points of opposite chiralities are sources and drains for the fundamental vortex lines that intersect the Fermi surface, forming vortices and antivortices in momentum space with a total vorticity of $\pm 2q_{\text{pair}}$. Vortex lines that do not connect to Weyl points form closed loops and are nonfundamental—Their intersections with the Fermi surfaces create pairs of vortices and antivortices, which are not enforced by topology but enrich the symmetry of the pairing order. A remarkable feature of this system is that even though these band Weyl points are at high energy near the cutoff scale, far away from the Fermi energy after doping, the nonperturbative nature of topological properties governs the low-energy nodal excitations of $\Delta(\mathbf{k})$, which inherit band topology in a nontrivial way.

Spinor pairing from half-integer pair monopole charges. So far, we have only discussed monopole pairing with integer-valued monopole charges q_{pair} , where the pairing symmetry remains within integer partial wave channels. However, when pairing occurs between two Fermi surfaces with Chern numbers differing by an odd integer, the resulting pair monopole charge $q_{\text{pair}} = |C_1 - C_2|/2$ becomes a half-integer. A notable feature of this pairing is that the order parameter, typically assumed to be bosonic, actually forms a spinor representation of rotation symmetry.

Consider a system consisting of two different types of fermions. The first type is a spin-1/2 fermion, with annihilation operators denoted by c_α , $\alpha = \uparrow, \downarrow$ and mass m . This fermion has a 3D Weyl-type spin-orbit coupling as described by H_c^0 . The second type of fermion, described by H_d^0 , is a single-component fermion with annihilation operators denoted by d and mass M . It exhibits a simple parabolic dispersion:

$$H_c^0 = \sum_{\mathbf{k}} \sum_{\alpha, \beta = \uparrow, \downarrow} c_\alpha^\dagger(\mathbf{k}) \left(\frac{\hbar^2 k^2}{2m} - \lambda \mathbf{k} \cdot \boldsymbol{\sigma}_{\alpha\beta} - \mu_c \right) c_\beta(\mathbf{k}),$$

$$H_d^0 = \sum_{\mathbf{k}} d^\dagger(\mathbf{k}) \left(\frac{\hbar^2 k^2}{2M} - \mu_d \right) d(\mathbf{k}). \quad (4)$$

Here, the system breaks inversion symmetry, resulting in split Fermi surfaces, labeled as $\text{FS}_{c,\pm}$, which possess opposite monopole charges, $q_c = \pm 1/2$. The Fermi wave vectors, $k_{F;c,\pm}$ and $k_{F;d}$, corresponding to the three Fermi surfaces, satisfy $\hbar^2 k_{F;c,\pm}^2/2m \pm \lambda k_{F;c,\pm} = \mu_c$ and $\hbar^2 k_{F;d}^2/2M = \mu_d$. The synthetic spin-orbit coupling of the c fermion can be possibly realized in ultracold atom systems, which have progressed rapidly from 2D to 3D systems, from continuum to

lattice models, and from boson to fermion systems [51–57]. Particularly, a Weyl semimetal band was observed in ultracold bosons (^{87}Rb) in a 3D optical lattice, and the spin-orbit coupling can be achieved for ultracold fermions without obstacles in principle [57].

To facilitate pairing between the c and d Fermi surfaces, consider the simplest scenario where the Fermi surface of d fermion, FS_d , matches with one of the helical Fermi surfaces of c fermion, say, $\text{FS}_{c,+}$. This can be achieved by tuning the chemical potential μ_d of d fermions such that $k_{F;d} = k_{F;c,+}$. In this case, Cooper pairing can occur between $\text{FS}_{c,+}$ and FS_d , which possess Chern numbers of -1 and 0 , respectively. The mean-field inter-Fermi-surface pairing Hamiltonian takes the form

$$H_\Delta = \sum_{\mathbf{k}; \alpha=\uparrow, \downarrow} \Delta_\alpha(\mathbf{k}) c_\alpha^\dagger(\mathbf{k}) d^\dagger(-\mathbf{k}) + \Delta_\alpha^*(\mathbf{k}) d(-\mathbf{k}) c_\alpha(\mathbf{k}). \quad (5)$$

Here, the gap function takes a two-component form $\Delta(\mathbf{k}) = (\Delta_\uparrow(\mathbf{k}), \Delta_\downarrow(\mathbf{k}))^T$. Since the kinetic Hamiltonian preserves spin-orbit-coupled rotational symmetry, the pairing gap function can be expanded in terms of partial wave channels of half-integer angular momentum quantum numbers j and m_j as $\Delta(\mathbf{k}) = \sum_{j, m_j} \Delta_{j, m_j}(\mathbf{k})$. Due to broken inversion symmetry, each partial wave pairing channel with angular momentum j and m_j is generally a superposition of two orbital angular momentum channels of opposite parity, $l = j - 1/2$ and $l + 1 = j + 1/2$: $\Delta_{j, m_j; \alpha}(\mathbf{k}) = \Delta_{j, m_j; l, \alpha}(\mathbf{k}) + \Delta_{j, m_j; l+1, \alpha}(\mathbf{k})$, with $\Delta_{j, m_j; l, \alpha}(\mathbf{k}) = \Delta_{j, m_j; l}(k) \phi_{j, m_j; l, \alpha}(\hat{\mathbf{k}})$. Here, $\Delta_{j, m_j; l}(k)$ is the pairing amplitude and $\phi_{j, m_j; l, \alpha}(\hat{\mathbf{k}}) = \langle j, m_j | l, m; \frac{1}{2}, \alpha \rangle Y_{lm}(\hat{\mathbf{k}}) \otimes |\alpha\rangle$ is the $\alpha = \uparrow, \downarrow$ component of spin spherical harmonic function [58].

To understand the pairing at low energy near Fermi surfaces, we note that states at $\text{FS}_{c, \pm}$ are, respectively, described by the helical band eigenstates $|\lambda_\pm(\mathbf{k})\rangle$ for c fermions. They satisfy $\boldsymbol{\sigma} \cdot \hat{\mathbf{k}} |\lambda_\pm(\mathbf{k})\rangle = \pm |\lambda_\pm(\mathbf{k})\rangle$. When FS_d matches with $\text{FS}_{c,+}$, at low energy, the spin spherical harmonics in the pairing order are projected to monopole harmonics as $P^{(+)}\phi_{j, m_j; l, \alpha}(\hat{\mathbf{k}}) = -P^{(+)}\phi_{j, m_j; l+1, \alpha}(\hat{\mathbf{k}}) = \frac{1}{\sqrt{2}} Y_{q_{\text{pair}}; j, m_j}(\hat{\mathbf{k}})$, where $q_{\text{pair}} = -1/2$, $j = l + 1/2$, and $Y_{q_{\text{pair}}; j, m_j}(\hat{\mathbf{k}})$ is the monopole harmonic function. It satisfies $J^2 Y_{q_{\text{pair}}; j, m_j} = j(j+1) Y_{q_{\text{pair}}; j, m_j}$ and $J_z Y_{q_{\text{pair}}; j, m_j} = m_j Y_{q_{\text{pair}}; j, m_j}$, with $j = |q_{\text{pair}}|, |q_{\text{pair}}| + 1, \dots$ and $|m_j| \leq j$ (see Sec. 1 in the Supplemental Material for details [59]). Consequently, the gap function $\Delta(\mathbf{k})$ becomes a function of half-charge monopole harmonics, $\Delta^{(+)}(\mathbf{k}) = \sum_{j, m_j} \Delta_{j, m_j} Y_{q_{\text{pair}}=-1/2; j, m_j}(\hat{\mathbf{k}})$, where $\Delta_{j, m_j} = (\Delta_{j, m_j; l=j-1/2}(k) - \Delta_{j, m_j; l=j+1/2}(k))/\sqrt{2}$. The projected pairing Hamiltonian then takes the form

$$H_\Delta^{(+)}(\mathbf{k}) = \sum_{j, m_j} \Delta_{j, m_j} Y_{q_{\text{pair}}=-1/2; j, m_j}(\hat{\mathbf{k}}) \chi_+^\dagger(\mathbf{k}) d^\dagger(-\mathbf{k}) + \text{H.c.}, \quad (6)$$

where $\chi_+^\dagger(\mathbf{k}) = \sum_{\alpha=\uparrow, \downarrow} \lambda_{+, \alpha}(\mathbf{k}) c_\alpha^\dagger(\mathbf{k})$ is the creation operator of a single-band eigenstate at $\mathbf{k}$ on $\text{FS}_{c,+}$.

We emphasize that, regardless of specific interactions in the pairing mechanism, the pairing order always exhibits half-integer charged monopole pairing, following singular spinor representations. For instance, consider the simplest case of

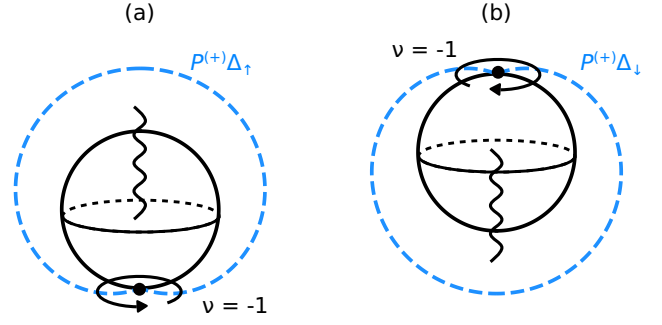


FIG. 1. Spinor pairing gap functions (a) $P^{(+)}\Delta_\uparrow \propto Y_{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}(\hat{\mathbf{k}})$ and (b) $P^{(+)}\Delta_\downarrow \propto Y_{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}}(\hat{\mathbf{k}})$ (blue, dashed) overlaid onto Fermi surface $\text{FS}_{c,+}$ (black, solid). They exhibit local chiral p -wave symmetry near the gap node under the choice of gauge with Dirac string (wavy line) going through the antipodal point in $\text{FS}_{c,+}$.

an attractive contact interaction between these two types of fermions,

$$H_{\text{int}} = -g \int d\mathbf{r} c_\alpha^\dagger(\mathbf{r}) d^\dagger(\mathbf{r}) d(\mathbf{r}) c_\alpha(\mathbf{r}), \quad \alpha = \uparrow, \downarrow, \quad (7)$$

which would only give rise to conventional s -wave pairing when Fermi surfaces are topologically trivial. In this case, before projection, $\Delta_{\alpha=\uparrow, \downarrow} = -\frac{g}{V} \int d\mathbf{r} \langle G | d(\mathbf{r}) c_\alpha(\mathbf{r}) | G \rangle$ is a constant independent of $\hat{\mathbf{k}}$. Nevertheless, after the projection, the gap functions $\Delta_{\alpha=\uparrow, \downarrow}$ become

$$P^{(+)}\Delta_\uparrow = \sqrt{2\pi} \Delta Y_{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}(\hat{\mathbf{k}}) = \Delta \cos \frac{\theta_{\mathbf{k}}}{2} e^{i\phi_{\mathbf{k}}},$$

$$P^{(+)}\Delta_\downarrow = \sqrt{2\pi} \Delta Y_{-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}}(\hat{\mathbf{k}}) = \Delta \sin \frac{\theta_{\mathbf{k}}}{2} e^{-i\phi_{\mathbf{k}}}, \quad (8)$$

which are time-reversal partners with each other. As shown in Figs. 1(a) and 1(b), the projected gap function $P^{(+)}\Delta_\uparrow$ ($P^{(+)}\Delta_\downarrow$) exhibits a single point node at the south (north) pole on $\text{FS}_{c,+}$. Locally, we describe the monopole spinor pairing order as familiar chiral p wave by choosing a gauge in which the pairing order is well defined at the respective nodal point. The corresponding single gap node contributes to nonvanishing total vorticity in momentum space, $\nu = \frac{1}{2\pi} \oint_{\mathcal{C}} d\mathbf{k} \cdot \mathbf{v} = 2q_{\text{pair}} = -1$. Here, $\mathbf{v}(\mathbf{k}) = \nabla_{\mathbf{k}} \phi(\mathbf{k}) - \mathbf{A}_p(\mathbf{k})$ is the gauge-invariant “velocity” in momentum space. Furthermore, we show the energetic stability of the spinor pairing order in Secs. 3 and 4 of the Supplemental Material [59].

Analogous to the concept of composite fermions [64], this monopole spinor pairing order can be described as a composite order characterized by a local chiral p -wave order under a specific gauge choice combined with a nonlocal Dirac string. The Dirac string introduces a Berry flux that shifts the angular momentum of the composite pairing order, leading to a spinor representation with gauge-invariant half-integer pairing angular momentum. Moreover, this monopole spinor pairing order can be extended to describe spinor density waves and spinor excitons in the particle-hole channel.

Next, we study the Bogoliubov quasiparticle excitations. As an example, we consider the case of $q_{\text{pair}} = -1/2$, $j = m_j = 1/2$, which occurs when inter-Fermi-surface pairing between $\text{FS}_{c,+}$ and FS_d takes the form $\Delta = (\Delta_0, 0)^T$, with Δ_0

being the pairing amplitude. In the Nambu basis $\psi(\mathbf{k}) = (\chi_+(\mathbf{k}), d^\dagger(-\mathbf{k}))^T$, we have

$$\mathcal{H}(\mathbf{k}) = \epsilon_{0,k} \tau_0 + \epsilon_k \tau_3 + \Delta_0 \sqrt{2\pi} Y_{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}(\hat{\mathbf{k}}) \tau_+ + \Delta_0 \sqrt{2\pi} Y_{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}^*(\hat{\mathbf{k}}) \tau_- \quad (9)$$

where $\tau_\pm = (\tau_x \pm i\tau_y)/2$. Here, $\epsilon_{0,k} = (\epsilon_{k;c,+} - \epsilon_{k;d})/2$ and $\epsilon_k = (\epsilon_{k;c,+} + \epsilon_{k;d})/2$, with $\epsilon_{k;c,+} = \hbar^2 k^2/(2m) - \lambda k - \mu_c$ and $\epsilon_{k;d} = \hbar^2 k^2/(2M) - \mu_d$ being the dispersions of the band eigenstates at the Fermi level. The Bogoliubov quasiparticle excitations are $\gamma_1^\dagger(\mathbf{k}) = \cos \frac{\beta_k}{2} \chi^\dagger(\mathbf{k}) + \sin \frac{\beta_k}{2} d(-\mathbf{k})$, $\gamma_2^\dagger(\mathbf{k}) = \sin \frac{\beta_k}{2} d(\mathbf{k}) - \cos \frac{\beta_k}{2} \chi^\dagger(-\mathbf{k})$, where $\tan \beta_k = \sqrt{2\pi} \Delta_0 Y_{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}(\hat{\mathbf{k}})/\epsilon_k$. The corresponding excitation spectrum is $E(\mathbf{k}) = \epsilon_{0,k} \pm \sqrt{\epsilon_k^2 + \Delta_0^2 \cos^2(\theta_k/2)}$. When pairing is instead $\Delta = (0, \Delta_0)^T$, corresponding to $m_j = -1/2$ channel, we have $E(\mathbf{k}) = \epsilon_{0,k} \pm \sqrt{\epsilon_k^2 + \Delta_0^2 \sin^2(\theta_k/2)}$. The two pairings exhibit a single BdG node at the south or north pole, respectively.

Tight-binding models. To further demonstrate the exotic spinor pairing and its surface states, we consider a tight-binding Hamiltonian with the kinetic part $\mathcal{H}_{c;t.b.}^0(\mathbf{k}) = \sum_i (2t_c \cos k_i \mathbb{1} - \lambda \sin k_i \sigma_i) - \mu_c \mathbb{1}$ and $\mathcal{H}_{d;t.b.}^0(\mathbf{k}) = \sum_i 2t_d \cos k_i - \mu_d$, and on-site pairing between c and d fermions. It takes the following form:

$$H_{t.b.} = \sum_{\mathbf{n}, \mathbf{n}'} \Psi_{\mathbf{n}}^\dagger \left(\begin{array}{c|c} [\mathcal{H}_c^0]_{\mathbf{n}, \mathbf{n}'} & \begin{array}{c} \Delta_\uparrow \delta_{\mathbf{n}, \mathbf{n}'} \\ \Delta_\downarrow \delta_{\mathbf{n}, \mathbf{n}'} \end{array} \\ \hline \begin{array}{c} \Delta_\uparrow^* \delta_{\mathbf{n}, \mathbf{n}'} \\ \Delta_\downarrow^* \delta_{\mathbf{n}, \mathbf{n}'} \end{array} & -[\mathcal{H}_d^0]_{\mathbf{n}, \mathbf{n}'} \end{array} \right) \Psi_{\mathbf{n}'}. \quad (10)$$

Here, $\Psi_{\mathbf{n}} = (c_{\mathbf{n},\uparrow}, c_{\mathbf{n},\downarrow}, d_{\mathbf{n}}^\dagger)^T$, with $c_{\mathbf{n},\sigma}$ ($d_{\mathbf{n}}$) being the annihilation operator of c fermion with spin σ (spinless d fermion) at site $\mathbf{n}$ in a cubic lattice. Nonzero matrix elements of $\mathcal{H}_c^0$ and $\mathcal{H}_d^0$ are $[\mathcal{H}_c^0]_{\mathbf{n}, \mathbf{n}} = -\mu_c \mathbb{1}_{2 \times 2}$, $[\mathcal{H}_c^0]_{\mathbf{n}, \mathbf{n}+\delta_i} = t_c \mathbb{1}_{2 \times 2} - \frac{i}{2} \lambda [\sigma_i]_{2 \times 2}$, $[\mathcal{H}_d^0]_{\mathbf{n}, \mathbf{n}} = -\mu_d$, and $[\mathcal{H}_d^0]_{\mathbf{n}, \mathbf{n}+\delta_i} = t_d$. Here, δ_i denotes the direct lattice vector of unit length between nearest neighboring sites along $i = x, y, z$. t_c and t_d are spin-independent hopping amplitudes for c and d fermions, respectively, and λ is the spin-orbit coupling strength. For $\mathcal{H}_c^0$, we take a convention in the tensor product representation that the two-dimensional spin- $\uparrow, \downarrow$ basis is nested under lattice site basis. To satisfy the Fermi surface matching condition to pair $\text{FS}_{c,+}$ with FS_d , we take $\mu_d = 2t_d(\cos k_F + 2)$ and $\mu_c = 2t_c(\cos k_F + 2) + \lambda |\sin k_F|$, with $k_F = k_{F;c,+} = k_{F;d}$ being the Fermi wave vector at the overlap of $\text{FS}_{c,+}$ and FS_d , as shown in Fig. 2(b).

To demonstrate spinor pairing at $m_j = 1/2$ channel, we take $\Delta_\uparrow = \Delta_0$ and $\Delta_\downarrow = 0$. The bulk BdG spectrum of the system is shown in Fig. 2(a), which exhibits a single point node at the south pole of the overlapped Fermi surfaces $\text{FS}_{c,+}$ and FS_d . Near the south pole $(0, 0, -k_F)$, the low-energy BdG Hamiltonian in the $(\chi_+(\mathbf{k}), d^\dagger(-\mathbf{k}))^T$ Nambu basis takes the form $\mathcal{H}(\mathbf{k}) \equiv \mathcal{H}(\tilde{\mathbf{k}} - k_F \hat{z}) \approx \begin{pmatrix} -v_\parallel \tilde{k}_z & v_\parallel (\tilde{k}_x + i\tilde{k}_y) \\ v_\parallel (\tilde{k}_x - i\tilde{k}_y) & v_2 \tilde{k}_z \end{pmatrix}$, with $v_\parallel = \Delta_0/(2|\sin k_F|)$, $v_1 = -2t_c \sin k_F + \lambda \cos k_F > 0$, and $v_2 = -2t_d \sin k_F > 0$. It exhibits an emergent BdG Weyl node with chirality -1 . This is a nontrivial example of a general statement that the Nielsen-Ninomiya theorem does not hold in lattice systems when $U(1)$ symmetry is broken [49].

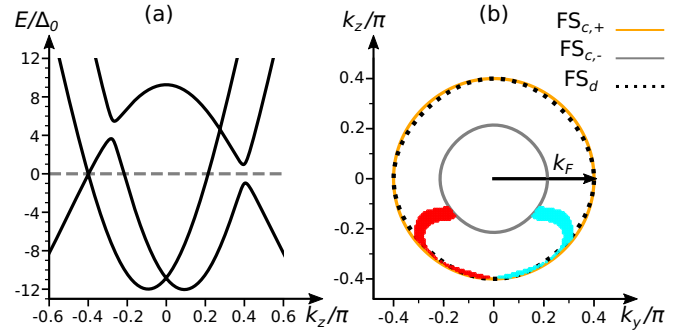


FIG. 2. (a) Bulk BdG spectrum of the three-band tight-binding model of spinor superconductor in Eq. (10) along the k_z axis. It exhibits a single BdG Weyl node at $k_z = -0.4\pi$. (b) Zero-energy surface arcs (colored) within 2D surface Brillouin zone. States localized at open boundary $x = 0$ ($x = L_x$) are shown in red (cyan). Overlaid are the $k_x = 0$ cross sections of bulk Fermi surfaces, FS_d (dashed black line), $\text{FS}_{c,+}$ (solid orange line), and $\text{FS}_{c,-}$ (solid gray line). Parameters are $t = -1$, $t_d/t = 1$, $t_c/t = 2$, $\mu_d/t = 4.62$, $\mu_c/t = 10.38$, $\lambda/|t| = 1.20$, $k_F = 0.4\pi$, and $L_x = 300$. The pairing is given by $\Delta = (\Delta_0, 0)^T$, with $\Delta_0/|t| = 0.15$.

Furthermore, the nontrivial pairing phase winding near the gap node exhibits local chiral p -wave symmetry and gives rise to zero-energy surface modes localized at opposite sides of the system, as shown in Fig. 2(b). These nontrivial surface states originate from the single pairing gap node before merging into the unpaired $\text{FS}_{c,-}$ bulk states, as shown in Figs. 3(a)–3(c). This is a unique characterization of this spinor pairing order and contrasts, for example, a three-dimensional $p_x + ip_y$ superconductor, where the zero-energy surface arc connects between the vortex and antivortex at the two poles of the Fermi surface. As shown in Figs. 3(a) and 3(b), there are two distinct surface states localized at $x = 0$ (L_x) that have negative (positive) group velocity along y and manifest the local $p_x + ip_y$ symmetry of the pairing order. When pairing occurs between $\text{FS}_{c,-}$ and FS_d , we have similar results for the spinor pairing with opposite pair monopole charge, as shown in Sec. 2 of the Supplemental Material [59].

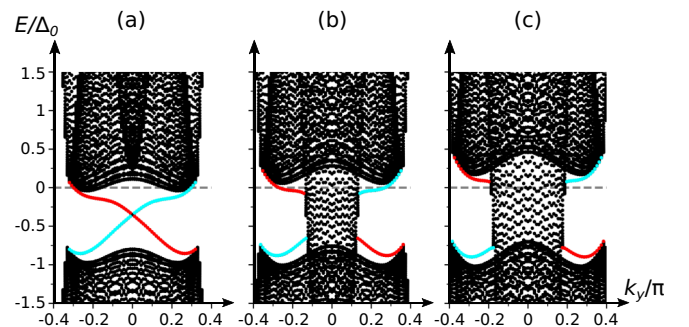


FIG. 3. Energy dispersion of surface modes in Fig. 2(b) at (a) $k_z = -0.22\pi$, (b) $k_z = -0.16\pi$ where surface modes merge into bulk spectrum (black) for $|k_y| < k_{F;c,-}$, and (c) $k_z = -0.10\pi$ where only bulk states of unpaired $\text{FS}_{c,-}$ reside at zero energy. The color scheme for surface states and parameters are the same as those in Fig. 2(b).

Fractional Mermin-Ho relation. Finally, we explore the situation of inhomogeneous spatial distribution of the spinor pairing order. Consider the simplest case where, before projection, the pairing gap function reads $\Delta(\mathbf{r}) = |\Delta(\mathbf{r})|e^{i\phi(\mathbf{r})}\eta(\mathbf{r})$, where $\eta(\mathbf{r})$ is a fundamental spin-1/2 spinor. In momentum space, η is projected to the monopole harmonic function $P^{(+)}\eta = \eta_\alpha Y_{-\frac{1}{2}, \frac{1}{2}, \alpha}(\hat{\mathbf{k}})$. Through Hopf map, the spinor gap function is mapped to a unit three-vector as $\hat{\mathbf{n}}(\mathbf{r}) = \eta^\dagger \boldsymbol{\sigma} \eta$. In $^3\text{He-A}$ p -wave superfluid, the direction of Cooper pairing orbital angular momentum is denoted by the one-vector, and the curl of the superfluid velocity is determined by the spatial variation of one-vector via $(\nabla \times \mathbf{v}_s)_i = (\hbar/2m^*)\epsilon_{ijk}\hat{\mathbf{l}} \cdot \partial_j \hat{\mathbf{l}} \times \partial_k \hat{\mathbf{l}}$, with m^* the mass of the Cooper pair, which is the celebrated Mermin-Ho relation [50]. Here, for the spinor pairing order, the corresponding Mermin-Ho relation is fractionalized. The spatial variation of $\hat{\mathbf{n}}$ leads to the following nontrivial circulation of the superfluid velocity (see Sec. 5 in the Supplemental Material for details [59]):

$$(\nabla \times \mathbf{v}_s)_i = \frac{\hbar}{4m^*}\epsilon_{ijk}\hat{\mathbf{n}} \cdot \partial_j \hat{\mathbf{n}} \times \partial_k \hat{\mathbf{n}}, \quad (11)$$

where the 1/2 factor arises due to the spinor order. In a spherical harmonic trap, there will appear a single vortex on the boundary induced by geometric curvature.

In summary, we have studied a class of topological nodal superconducting states characterized by a nontrivial two-particle pairing Berry phase. The low-energy gap func-

tion is described by monopole harmonic functions, not only in the integer monopole charge channels but also in more exotic half-integer monopole charge channels. The half-integer monopole pairing order follows spinor representation, even though it arises from Cooper pairs that are bosonic. The Berry phase enforced spinor pairing order can exist in, for example, ultracold atom systems with synthetic spin-orbit coupling, when Cooper pairing occurs between two Fermi surfaces with Chern numbers differing by an odd integer. Furthermore, we have demonstrated the 1/2 monopole charge spinor superconducting order in a lattice. The system exhibits a single BdG Weyl node in the bulk spectrum and exotic surface states arising from nontrivial phase winding of spinor pairing order about the gap node. Finally, when the spinor pairing order parameter has a spatial gradient, the superfluid velocity ceases to be irrotational, and a fractional version of the Mermin-Ho relation has been derived.

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Data availability. The code and data for the tight-binding model is available at [65].

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