

How quantum fluctuations freeze a classical liquid and then melt it into a topological one

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(Received 2 May 2025; accepted 25 September 2025; published 22 October 2025)

Topologically ordered quantum liquids are highly sought-after quantum phases of matter, and recently, fractional Chern insulators (FCIs) joined the few experimental realizations of such phases. Here, we ask whether a gapped classical, highly degenerate liquid can be the birthplace of FCIs upon the addition of suitable quantum fluctuations. Two competing tendencies can be anticipated: (1) Following the quantum order-by-disorder paradigm, quantum fluctuations could induce symmetry-breaking (charge) order, or (2) the classical liquid builds up long-range entanglement and turns into a quantum liquid. We study spinless fermions on a honeycomb lattice subject to cluster-charging interactions and introduce quantumness through a Haldane kinetic term, featuring complex second-nearest-neighbor hopping. Based on extensive exact diagonalization calculations and high-order perturbation theory, we find that neither scenario (1) nor (2) prevails, but (1) and (2) manifest sequentially as the kinetic energy is increased. We demonstrate how the gradual lifting of kinematic constraints gives rise to this sequence of phases. Our results relate to the regime of intermediate-scale interactions present in moiré systems, where band projections are not suitable to model FCIs and competing charge-ordered phases have been identified.

DOI: [10.1103/ydxb-1tfx](https://doi.org/10.1103/ydxb-1tfx)

Introduction. The emergence of degenerate many-body ground states in topologically ordered systems stands as one of the most striking manifestations of interactions, as nature abhors degeneracy. In the fractional quantum Hall (FQH) effect at filling fraction $1/m$, the m -fold degenerate state is born out of a Landau level, a degenerate single-particle kinetic energy spectrum induced by the magnetic field. Weak interactions relative to the gap between the Landau levels separate the m states from the extensively degenerate *noninteracting* spectrum. In contrast, in a geometrically frustrated system, strong interactions can result in an extensively degenerate *many-body* spectrum in a classical limit. Kinetic energy introduces quantum fluctuations into such degenerate many-body states, which can lift the degeneracy via the so-called “order-by-disorder” mechanism and result in a crystalline state, such as a valence-bond solid. However, quantum fluctuations can also build up long-range entanglement and result in a quantum-liquid phase with topological degeneracy, such as a spin liquid. Fractional Chern insulators, lattice analogs of the FQH effect, were first proposed in flat Chern bands mimicking Landau levels [1–3], with their stability supported by extensive numerical studies of interacting topological bands [3–12] and recently in the context of moiré materials [13–20]. While the FQH analogy has guided much of our understand-

ing [21–24], the existence of a lattice necessarily opens the door to effects of lattice geometry. When an FCI emerges in a lattice with threefold rotational symmetry—the source of geometric frustration—a natural question arises: Can the FCI originate from a degenerate many-body spectrum driven by this geometric frustration?

Recent experimental advances in moiré materials discovered the emergence of FCIs in small or zero external magnetic fields [25–29], often accompanied by competing incompressible charge-ordered states with zero Chern number [28,29]. Since the moiré materials have (super)lattices with threefold rotation symmetry, the question about the role of geometric frustration in the emergence of an FCI and its competition with charge order is more timely than ever.

In the widely pursued weak-coupling approaches, partially filled isolated bands that resemble Landau levels go from a metal to an FCI upon adding interactions that are small compared to the band gap [15–24]. While this framework can also find a nesting-driven charge-density-wave state [17–19], such a charge-density-wave state is in general compressible, as only small portions of the Fermi surface acquire a gap. Most importantly, both the FCI and the charge-ordered state require fine tuning of the band structure. Alternatively, strongly interacting electrons, which partially fill a lattice, could go from a trivial compressible insulator [9–12] or a classical charge-ordered state [7–9] to an FCI upon adding complex hoppings. In these strong-coupling scenarios, the interaction can be on the order of the band gap and the FCI is insensitive to the details of the band structure. However, the insulating state in the limit of infinite interactions is either a degenerate compressible state or a classical charge-ordered state. Hence, the existing approaches from either a weak-coupling

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or strong-coupling angle fail to capture the competition between an incompressible *quantum* charge-ordered state and an FCI. Moreover, the potential role of geometric frustration has been overlooked.

We investigate the competition between trimer liquid (TL) [30], generalized Wigner crystal (gWC), and FCI in a honeycomb lattice with strong cluster-charging interactions at a filling of $1/3$ electrons per unit cell. The cluster-charging interaction is proposed to capture the leading-order repulsive interaction at this filling for twisted bilayer graphene [31–34]. These interactions introduce orbital geometric frustration, placing the system between the short-range- and the long-range-interaction limits. At filling $1/3$ and infinite cluster charging, the degeneracy is extensive [35,36], in contrast to the Coulomb interaction, where a gWC is the ground state. However, despite the extensive degeneracy, the state is incompressible, contrary to the nearest-neighbor-repulsion case [11,12].

The Haldane hopping terms introduce quantum fluctuations with two possible competing tendencies. On the one hand, quantum fluctuations can generate ordering, such as in the valence bond solids, via quantum order by disorder in dimer models on bipartite lattices [37,38]. On the other hand, the Haldane hopping terms imprint the band topology of Chern bands, which may favor an FCI. Using exact diagonalization (ED) and high-order perturbation theory in the strong-coupling limit, we show that quantum fluctuations play a dual role: (1) For small hopping, quantum fluctuations lift the degeneracy of the cluster-charging term and favor a gWC via fourth-order virtual hopping processes. (2) An FCI phase is realized in the intermediate hopping regime, where the bandwidth of the Chern band and the band gap are comparable to the interaction.

Model. We study the cluster-charging Hamiltonian with the Haldane-model hopping for spinless fermions,

$$\hat{H} = - \sum_{i,j} t_{ij} \hat{c}_i^\dagger \hat{c}_j + V_1 \sum_{\langle ij \rangle_1} \hat{n}_i \hat{n}_j + V_2 \sum_{\langle ij \rangle_2} \hat{n}_i \hat{n}_j + V_3 \sum_{\langle ij \rangle_3} \hat{n}_i \hat{n}_j, \quad (1)$$

where $\langle ij \rangle_1$, $\langle ij \rangle_2$, and $\langle ij \rangle_3$ denote nearest neighbors (NN), second nearest neighbors (2NN), and third nearest neighbors (3NN), respectively. The cluster-charging interaction is realized when $V_1 = 2V_2 = 2V_3$. Without the hopping terms, the ground state of the Hamiltonian is the so-called trimer-liquid phase of Ref. [30], with extensive degeneracy. Fermions can hop between NN, 2NN, and 3NN with hopping amplitudes $t_{\langle ij \rangle_1} = t_1$, $t_{\langle ij \rangle_2} = t_2 e^{\pm i\phi}$, and $t_{\langle ij \rangle_3} = t_3$, respectively. The sign convention of the acquired phase $\pm\phi$ is illustrated in Fig. 1(a). We perform an ED study on the model given in Eq. (1), with the following choice for the hopping parameters: $t_2/t_1 = 0.7$, $t_3/t_1 = -0.9$, and $\phi = 0.35\pi$ or $\phi = 0$ as specified below. Under these two choices, the noninteracting part of the Hamiltonian features two Chern bands with opposite (single-particle) Chern numbers $C = \pm 1$ for $\phi = 0.35\pi$ and is topologically trivial for $\phi = 0$.

Exact diagonalization. In ED, we consider finite clusters consisting of $N_1 \times N_2$ unit cells (with $N_s = 2N_1N_2$ the total number of sites) under periodic boundary conditions and exploit tilted geometries [6,39,40] to achieve near-unity aspect ratios. The cluster geometries used in this work and

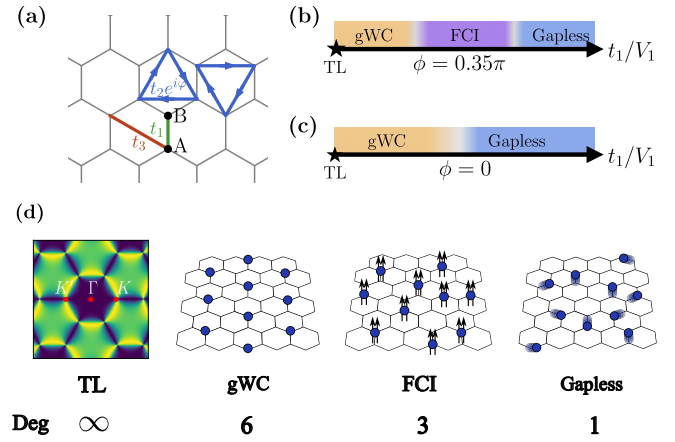


FIG. 1. (a) Haldane model on a honeycomb lattice. Possible hopping terms are depicted in different colors, with the arrows for 2NN hopping indicating the phase convention. Qualitative phase diagrams of the model as a function of t_1/V_1 for (b) $\phi = 0.35\pi$, and (c) $\phi = 0$, respectively. TL stands for the trimer liquid phase in the infinite-interaction limit $t_1/V_1 = 0$. gWC denotes generalized Wigner crystal and FCI denotes fractional Chern insulator. (d) Characteristics/cartoons and degeneracies of the phases in panels (b) and (c). TL features two twofold pinch points at $\mathbf{K}$ and $\mathbf{K}'$ of the static structure factor in momentum space, reflecting the constraint imposed by the cluster-charging interaction. In the FCI phase at filling fraction $1/3$, each of the fermions is attached with two Chern-Simons flux quanta to form a composite fermion.

the corresponding parameters are provided in the Supplemental Material [40]. We focus on the filling fraction $\nu = N_e/(N_1N_2) = 1/3$.

Figures 1(b) and 1(c) present the qualitative phase diagrams with $\phi = 0.35\pi$ and $\phi = 0$, respectively. The horizontal axis represents t_1/V_1 , which is gradually increased, along with the corresponding adjustments to t_2 and t_3 . In both cases, the model forms a critical trimer liquid at $t_1/V_1 = 0$ [30,35,41]. For small t_1/V_1 , a gWC phase with a $\sqrt{3} \times \sqrt{3}$ charge order emerges, featuring six degenerate ground states. The number six originates from the twofold degeneracy from the sublattice symmetry times threefold degeneracy from tripling of the unit cell. In the opposite limit, $t_1/V_1 \rightarrow \infty$, a gapless Fermi liquid phase forms. Unlike the trivial hopping phase $\phi = 0$, an FCI with a threefold degeneracy (on a torus) emerges for $\phi = 0.35\pi$.

Figure 2 presents numerical evidence for the three phases at $\phi = 0.35\pi$, using the 9×2 cluster as an example. The evolution of the low-lying spectrum when varying t_1/V_1 is shown in Fig. 2(a). We found two gapped phases separated by a gap-closing point. Both of them exhibit a threefold degeneracy on this cluster. The mismatch between the observed threefold degeneracy in the small t_1/V_1 regime on the finite cluster and the sixfold degeneracy of the $\sqrt{3} \times \sqrt{3}$ phase is due to finite-size effects: The threefold degeneracy observed on the finite cluster comes from the tripling of unit cells, but the finite-size effects lift the twofold degeneracy between two sublattices. The gap closing can be seen from the divergence of spacing/gap ratio s_3/Δ , where s_3 is the maximum spacing between adjacent levels in the threefold degenerate ground

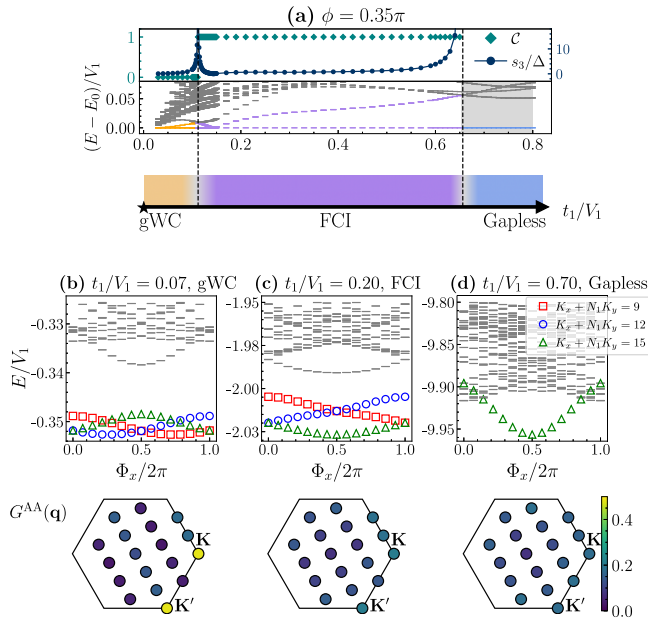


FIG. 2. Characterizations of the phases of Eq. (1) on the 9×2 cluster. (a) Evolution of the low-lying spectrum as t_1/V_1 increases. In the upper plot, we show the many-body Chern number $\mathcal{C}$ and the spacing/gap ratio s_3/Δ for the ground states of the gapped phases. The spectral flows and ground state structure factors $G^{AA}(\mathbf{q})$ of the three phases are shown in panels (b)–(d). The structure factor is calculated for the ground state with momentum number $K_x + N_1 K_y = 15$. In the gWC and FCI phases, the ground states occupy the same momentum sectors, marked in nongray colors. Both phases exhibit similar spectral flow patterns: The three ground states flow into each other upon flux insertion, maintaining a persistent gap from the excited states. This spectral flow pattern is to be compared with the level crossings shown in the gapless phase [panel (c)]. The structure factor of the gWC phase is peaked at $\mathbf{K}$ and $\mathbf{K}'$, indicating a translational-symmetry-breaking charge order with a tripled unit cell.

states and $\Delta = E_4 - E_3$ is the difference between the fourth-lowest energy E_4 and the third-lowest energy E_3 . Further increasing t_1/V_1 drives the system into a gapless state (shaded area). In determining whether the ground state is potentially gapless, we introduce twisted boundary conditions along the two periodic vectors $(\mathbf{T}_1, \mathbf{T}_2)$ of the cluster. Specifically, we enforce particles to acquire an additional phase $\Phi_x(\Phi_y)$ when translated by $\mathbf{T}_1(\mathbf{T}_2)$. We consider a ground state as gapped only if the many-body gap persists for all possible flux configurations $(\Phi_x, \Phi_y) \in [0, 2\pi)^{\times 2}$. Note that Fig. 2(a) only shows the spectrum evolution at a specific flux configuration $(\Phi_x, \Phi_y) = (\pi, 0)$, for which the gap-closing point between the two gapped phases can be directly observed. The seemingly non-zero gap in the gapless region is due to the flux-configuration choice. In Figs. 2(b)–2(d), we present the spectral flow upon varying Φ_x for the three phases. The level crossings in Fig. 2(d) clearly demonstrate the gapless nature in the large- t_1/V_1 regime, while a persistent gap is found for the two gapped phases. However, the spectral flows provide limited insight into distinguishing the two gapped phases in our case, as the ground states of both phases lie in the same set of

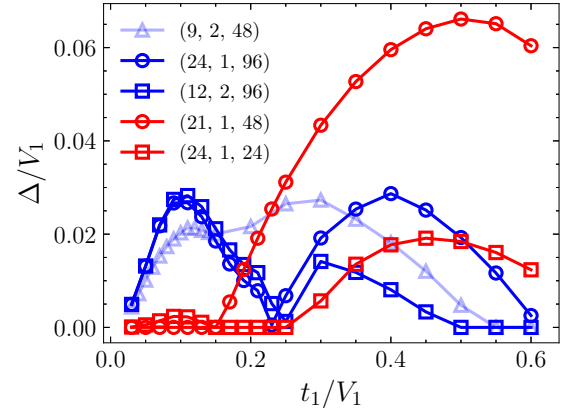


FIG. 3. Many-body gap $\Delta = E_4 - E_3$ as a function of t_1/V_1 for finite clusters. Clusters are labeled by (N_1, N_2, Deg) , with Deg the classical ground state degeneracy at $t_1/V_1 = 0$.

momentum sectors and exhibit similar spectral flows for all clusters studied in this work, as illustrated in Figs. 2(b) and 2(c).

To further characterize the two gapped phases, we measure the ground state structure factor $G_i^{AA}(\mathbf{q})$ between the A-A sublattice sites, defined as

$$G_i^{AA}(\mathbf{q}) = \frac{1}{N_1 N_2} \left(\langle g_i | \hat{n}_{\mathbf{q}}^A \hat{n}_{-\mathbf{q}}^A | g_i \rangle - \langle g_i | \hat{n}_{\mathbf{q}=0}^A | g_i \rangle^2 \delta_{\mathbf{q},0} \right), \quad (2)$$

where $|g_i\rangle$ is one of the ground states and $\hat{n}_{\mathbf{q}}^A = \sum_{\mathbf{r}_j} \hat{n}_{\mathbf{r}_j}^A e^{i\mathbf{q} \cdot \mathbf{r}_j}$. The structure factors $G^{AA}(\mathbf{q})$ in Figs. 2(b) and 2(c) reveal distinct characteristics: The small- t_1/V_1 phase displays a charge order with ordering momenta $\mathbf{K}$ and $\mathbf{K}'$, whereas the second phase exhibits a roughly uniform $G^{AA}(\mathbf{q})$, indicating no translational symmetry breaking. Furthermore, we compute the many-body Chern number $\mathcal{C}$ for the threefold degenerate ground states, defined as a topological invariant obtained through integration over the twisted boundary conditions (as detailed in the Supplemental Material [40]). The Hall conductivity σ_{xy} is related to the many-body Chern number by $\sigma_{xy} = \frac{1}{m} \frac{\mathcal{C} e^2}{h}$, where $m = 3$ is the degeneracy of the ground states. The Chern-number calculations demonstrate that the small- t_1/V_1 phase is a symmetry-breaking phase with $\mathcal{C} = 0$, whereas the second phase is identified as an FCI state with $\mathcal{C} = 1$ (see the Supplemental Material [40]), as shown in the upper plot of Fig. 2(a).

In Fig. 3, we plot the many-body gap Δ as a function of t_1/V_1 for all clusters studied in ED. Here, we use the notation (N_1, N_2, Deg) to distinguish tilted clusters with the same (N_1, N_2) , where Deg refers to the classical ground state degeneracy at $t_1/V_1 = 0$. For the $(9, 2, 48)$ cluster, Δ is determined by the minimum gap among all possible flux configurations (Φ_x, Φ_y) . For larger clusters, only Δ at the fixed $(\Phi_x, \Phi_y) = (0, 0)$ is plotted, which is sufficient to indicate the gap closing at the phase transition. We expect the variation of Δ with (Φ_x, Φ_y) to be suppressed for larger system sizes, as the flux-induced vector potential $A_i \sim \Phi_i/L_i \rightarrow 0$ vanishes in the thermodynamic limit. Among the five clusters, three (colored blue) exhibit the same phase structure as previously observed: a transition from a gWC phase to an FCI phase and finally to a gapless phase, as confirmed by Chern-number

calculations. However, the other two clusters (colored red) show a gapless spectrum in the small- t_1/V_1 regime, despite the geometries being compatible with the tripled unit cell structure expected for the gWC phase. Irrespectively, as t_1 increases, all clusters invariably enter into an FCI phase, which suggests that the FCI phase is robust and likely to persist in the thermodynamic limit. Motivated by the inconsistency in the small- t_1/V_1 regime, in the following section we develop a perturbative analysis to address finite-size effects.

Perturbative analysis. To better understand the charge ordering in the limit of $t_1/V_1 \rightarrow 0$, we perform a perturbative analysis. We write the Hamiltonian of Eq. (1) as $\hat{H} = \hat{H}_0 + \hat{H}_1$, with $\hat{H}_0$ the cluster-charging interaction and $\hat{H}_1 = -\sum_{i,j} t_{ij} \hat{c}_i^\dagger \hat{c}_j$ as the perturbation. The eigenstates of H_0 form degenerate flat bands with energies $E_n^{(0)} = nV_1/2$, and we use $\hat{P}$ to denote the projector onto the ground state manifold of $\hat{H}_0$. By applying a Schrieffer-Wolff (SW) transformation [40,42,43], we obtain the effective Hamiltonian defined on the ground state manifold of $\hat{H}_0$:

$$\hat{H}_{\text{eff}} = \sum_{n=1}^{\infty} \hat{P}(\hat{H}_1 \hat{D})^{n-1} \hat{H}_1 \hat{P} + \dots, \quad (3)$$

where $\hat{D} = (\mathbb{1} - \hat{P})/(\hat{H}_0 - E_0^{(0)})$, which projects the intermediate states onto the high-energy space of $\hat{H}_0$ and $\dots$ represents the “disconnected” terms such as $\hat{P}\hat{H}_1\hat{D}\hat{H}_1\hat{P}\hat{H}_1\hat{D}^2\hat{H}_1\hat{P}$. When evaluated in the occupation number basis $|n_1, \dots, n_{N_s}\rangle$, the effective Hamiltonian can be interpreted as a set of virtual hopping processes, which connect one classical ground state to another. Importantly, under periodic boundary conditions, there exist virtual processes in which the hopping electrons wrap around the torus.

On finite clusters, the order of these virtual hopping processes is proportional to the linear system size L , and they vanish as $(t_1/V_1)^{\alpha L}$ with $\alpha \sim O(1)$ when $L \rightarrow \infty$. However, on small clusters they can dominate over the true leading-order processes that survive in the thermodynamic limit, explaining the inconsistency in the small- t_1/V_1 regime. To avoid the dynamics dominated by such spurious terms, we directly calculate the effective Hamiltonian in the thermodynamic limit. For the off-diagonal part of $\hat{H}_{\text{eff}}$, it has been shown that the lowest-order nontrivial process is at the eighth order, giving rise to the lemniscate operator [41]. The diagonal part of $\hat{H}_{\text{eff}}$, in contrast, can have lower-order contributions, which have the form

$$\hat{H}_{\text{eff}}^{\text{diag}} = \sum_{ij} V_{ij} \hat{n}_i \hat{n}_j + \sum_{ijk} V_{ijk} \hat{n}_i \hat{n}_j \hat{n}_k + \dots \quad (4)$$

Up to third order, all ground-state configurations are renormalized by the same energy, while nontrivial splittings among the ground states appear for fourth-order diagonal terms. As listed in the Supplemental Material [40], the fourth-order diagonal terms only involve a (two-body) density-density interaction, ranging from 4NN to 13NN. Therefore, in the limit of $t_1/V_1 \ll 1$, the off-diagonal lemniscate operator is subleading compared to the diagonal terms. With $\hat{H}_{\text{eff}}$ obtained in the thermodynamic limit, we then calculate its energy spectrum on finite clusters much larger than those accessible with full ED. Since the spurious processes responsible for finite-size effects have been excluded in the calculation of $\hat{H}_{\text{eff}}$, the

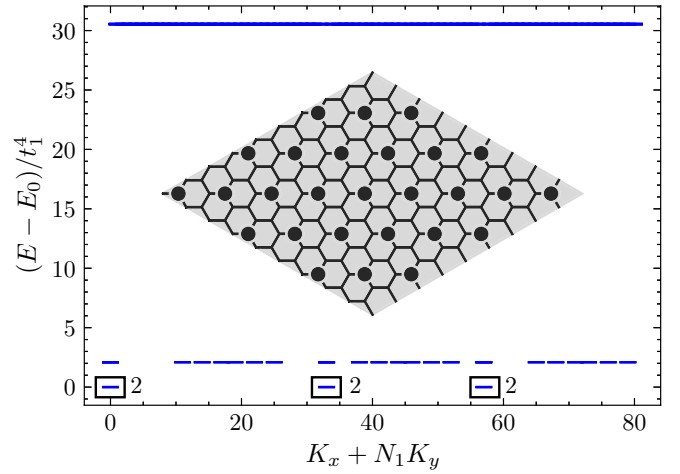


FIG. 4. Low-lying spectrum of the effective Hamiltonian $\hat{H}_{\text{eff}}^{(4)}$ up to fourth order on a 9×9 cluster. Within each box the energy level is doubly degenerate. There are in total six degenerate ground states. The inset illustrates the fermion number distribution for one of the ground states, where the fermions are colored in black.

resulting ground states are expected to correspond to those in the thermodynamic limit. Figure 4 shows the low-lying spectrum of $\hat{H}_{\text{eff}}^{(4)} = \sum_{i,j} V_{ij} \hat{n}_i \hat{n}_j$ on a 9×9 cluster, where a sixfold degeneracy is found. Figure 4 also illustrates the particle distribution in one of the six ground states, which demonstrates that the ground state has a $\sqrt{3} \times \sqrt{3}$ structure. Those configurations maximize the number of 5NN interactions per fermion. However, the competition between the 4NN, 5NN, and 6NN interactions can give rise to a stripe order when the phase ϕ of the NNN hopping is tuned (see the Supplemental Material [40]).

FCI from constrained Hilbert space. The FCI phase of the Hamiltonian given in Eq. (1) is in a distinct regime from the FCI observed in partially filled Chern band with negligible band mixing [15–19,21,23,24]. In the latter scenarios, the single-particle band gap is considered to be much larger than the interaction strength, allowing the physics to be fully captured by the lower Chern band, which plays the role of the lowest Landau level in the fractional quantum Hall effect. In our case, the system enters the FCI phase when the band gap of the single-particle Hamiltonian $\Delta_{\text{single}} \approx 0.6t_1$ is comparable to the interaction strength, making the lower band projection picture unjustified. Additionally, the bandwidth of the lower Chern band is larger than the band gap $\delta \approx 0.86t_1 > \Delta_{\text{single}}$, deviating from the ideal flat-band limit. FCIs in comparable regimes were previously studied in Refs. [8,11]. To better understand how the FCI arises in this intermediate regime, we adopt a complementary approach that starts from the strong-coupling limit of the cluster-charging Hamiltonian $\hat{H}_0$ and study the quantum fluctuations induced by the hopping terms. As the hopping amplitudes increase, more low-lying eigenstates of $\hat{H}_0$ are activated and contribute to the ground states. In this picture, the hierarchy of interaction levels provides a way to truncate the Hilbert space, with the hopping amplitudes determining how many energy levels are included in the description. Motivated by this observation, we propose the following model. First, we introduce an energy cutoff

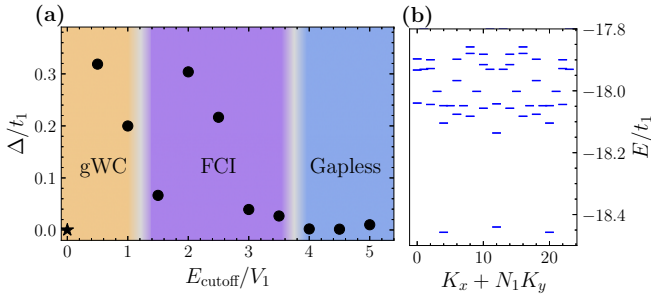


FIG. 5. (a) Phase diagram of $\hat{H}_c$ on the $(N_1, N_2, \text{deg}) = (24, 1, 96)$ cluster, with black data points representing the gap. Results are shown for $E_{\text{cutoff}} \leq 5V_1$, but note that there are states with cluster-charging energies above $5V_1$. (b) The low-lying spectrum of the FCI phase at $E_{\text{cutoff}}/V_1 = 2$.

E_{cutoff} and truncate the Hilbert space by keeping states with energy $E \leq E_{\text{cutoff}}$ in the classical cluster-charging model. In the truncated Hilbert space, we then discard the interactions and study the hopping Hamiltonian on this restricted Hilbert space. The final Hamiltonian is written as

$$\hat{H}_c(E_{\text{cutoff}}) = \hat{P}(E_{\text{cutoff}}) \left[- \sum_{i,j} t_{ij} \hat{c}_i^\dagger \hat{c}_j \right] \hat{P}(E_{\text{cutoff}}), \quad (5)$$

where $\hat{P}(E_{\text{cutoff}})$ denotes the projector onto the space with $E \leq E_{\text{cutoff}}$ in the classical model.

Using ED, we find that the constrained model $\hat{H}_c$ reproduces the same phase diagram as the full Hamiltonian in Eq. (1) on the finite clusters analyzed in the previous sections (see the Supplemental Material [40]), with E_{cutoff} replacing t_1 . As an example, we plot the gap $\Delta = E_4 - E_3$ of $\hat{H}_c$ as a function of E_{cutoff} on the $(N_1, N_2, \text{deg}) = (24, 1, 96)$ cluster in Fig. 5(a). Figure 5(b) shows the energy spectrum of the FCI phase at $E_{\text{cutoff}} = 2V_1$, where the threefold-degenerate ground-state space carries a nonzero Chern number $\mathcal{C} = 1$, obtained from numerical calculations. The correspondence between $\hat{H}_c$ and $\hat{H}$ indicates that the FCI phase emerges from the first several levels of the classical model. In this picture, the interaction provides a natural way to energetically truncate the Hilbert space, with the states within the sub-Hilbert space satisfying the constraint that the number of frustrated plaquettes is bounded by some fixed number. The single-particle kinetic terms introduce coupling between the states within the subspace, resulting in a topologically ordered FCI state. It contrasts with the Landau level picture, where the sub-Hilbert space is formed by the lower Chern band, and the interaction projected to the lower band leads to an FCI state.

Conclusion. In summary, we studied the effect of quantum fluctuations in the geometrically frustrated setting of a strongly interacting cluster-charging model at fractional filling of the honeycomb lattice on the torus. We found the time-reversal-breaking hopping in the form of the Haldane model to gradually reduce the exponential degeneracy of the novel TL phase first into an incompressible gWC phase with sixfold degeneracy, and then to an FCI with threefold degeneracy, and eventually into a gapless metallic state upon increasing the strength of quantum fluctuations. This model provides an intriguing example of competing gWC and FCI ordering, both driven by quantum fluctuations. Notably, both phases are reached without fine-tuning of the band structure, as the interaction strength is on par with the band gap. The richness of the phase diagram we obtained showcases how the frustrated intermediate range interaction of cluster charging and the resulting gapless incompressible TL phase serve as a fertile birthplace for correlated states of matter. Notably, the FCI phase we found is an example of the “quantum charge liquid” [44], as analogous to the quantum spin liquid. Our results invite the questions such as whether the FCI can be reached directly from the TL through spontaneous time-reversal-symmetry breaking as in a chiral spin liquid state, and what the nature of the transition between gWC and FCI is. Additionally, a natural question to ask is whether our model can provide insights into the competition among different phases observed in recent experiments on moiré systems. In moiré materials such as twisted MoTe₂, the tunable twist angle and displacement field change the kinetic energy in a complicated way. However, the mechanisms for stabilizing the gWC and the FCI seem generic and mostly dependent on the overall bandwidth versus interaction strength. In particular, the exchange terms that favor the gWC would also be present for a generic screened Coulomb interaction, a topic that we leave for future studies.

Acknowledgments. This project was supported by the Swiss National Science Foundation through a Consolidator Grant (iTQC, TMC2-213805). G.W. was supported by the Swiss National Science Foundation (SNSF) via Ambizione Grant No. PZ00P2-216183. D.M. and E.-A.K. were partially supported by the Gordon and Betty Moore Foundation’s EPIQS Initiative, Grant No. GBMF10436 to E.-A.K. E.-A.K. acknowledges support by the NSF through Grant No. OAC-2118310.

Data availability. The data that support the findings of this Letter are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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