

Approaching the critical point through an exact ground-state evolution

Jun-Hao Lin,^{1,2} Ke-Xiong Yan,^{1,2} Jie Song,^{3,4} Ye-Hong Chen,^{1,2,4,5,*} and Yan Xia^{1,2,4,†}

¹*Fujian Key Laboratory of Quantum Information and Quantum Optics, Fuzhou University, Fuzhou 350116, China*

²*Department of Physics, Fuzhou University, Fuzhou 350116, China*

³*Department of Physics, Harbin Institute of Technology, Harbin 150001, China*

⁴*Institute of Quantum Science and Technology, Yanbian University, Yanji 133002, China*

⁵*Quantum Information Physics Theory Research Team, Center for Quantum Computing, RIKEN, Wako-shi, Saitama 351-0198, Japan*



(Received 31 March 2025; accepted 8 February 2026; published 11 March 2026)

We investigate critical quantum metrology in a system where a qubit is dispersively coupled to a parametrically driven bosonic field. By engineering a counterdiabatic driving Hamiltonian under realistic physical constraints, we demonstrate a high-fidelity preparation of quantum states near a quantum phase transition. Our protocol extends the application of criticality-enhanced metrology to the short-time regime, offering a significant advantage over nonassisted protocols. The metrological precision is quantified by the quantum Fisher information (QFI) and the signal-to-noise ratio (SNR). Even with a constrained driving strength, numerical simulations reveal a QFI enhancement factor exceeding 10^3 and an SNR enhancement of over 10^6 in this regime. This work opens a promising avenue for critical quantum metrology.

DOI: [10.1103/PhysRevResearch.8.013275](https://doi.org/10.1103/PhysRevResearch.8.013275)

I. INTRODUCTION

Quantum metrology [1–4] leverages quantum resources, such as superposition and entanglement, to enhance measurement precision beyond classical limits [4,5]. This capability is crucial for applications such as gravitational wave detection [6], atomic clocks [7], and magnetic field sensing [8], since even small improvements in precision can lead to significant scientific and technological advancements. Conventional quantum metrology often employs highly nonclassical states, particularly quantum entangled states, to achieve enhanced precision. However, these states are highly susceptible to noise and decoherence [9,10], limiting the practical utility of conventional quantum metrology.

To address these limitations, critical quantum metrology [11] offers a promising solution. It significantly enhances measurement precision, not by relying on entanglement [12], but by exploiting the quantum criticality [13–22] near the critical point, where even a tiny change in parameters can lead to dramatic shifts in the equilibrium properties. Despite numerous theoretical proposals for critical quantum metrology (e.g., in the quantum Rabi model [21], the anisotropic quantum Rabi model [23], and the anisotropic Dicke model [24]), practical applications remain challenging due to a fundamental obstacle: the precise preparation of initial states near the critical point. This obstacle arises from the symmetry breaking near the critical point, where the desired state becomes degenerate

with other states, making it inherently difficult to distinguish and stabilize the target state.

Adiabatic control [25–27] theoretically offers a path to prepare such states by leveraging the adiabatic theorem. However, its practical realization is fundamentally constrained by critical behavior [28–30]. Near the critical point, the vanishing energy gap requires infinitely slow variations in control parameters to maintain adiabaticity, thereby rendering the target state experimentally inaccessible [29,31,32]. This inherent limitation has driven the development of alternative strategies that can accelerate quantum protocols beyond adiabatic constraints [33].

A powerful framework for such strategies is shortcuts to adiabaticity (STA) [34]. Among the various STA techniques, counterdiabatic (CD) driving [35–37] has emerged as a particularly prominent and versatile method. This method has been widely reported and experimentally validated across diverse fields of quantum science [38,39]. For instance, it has been instrumental in accelerating the implementation of high-fidelity quantum gates for quantum computing [40], speeding up quantum optimization algorithms [41], and engineering nonclassical states of light in analogous physical systems [42,43].

In this work, we develop a critical-state preparation method based on CD driving. While CD driving is a well-established tool for accelerating quantum dynamics, its application to solve the fundamental state-preparation problem in critical quantum metrology remains largely unexplored. We demonstrate that this approach enables the high-fidelity preparation of critical states within experimentally feasible timescales, bypassing the need for the strict adiabatic condition [44,45]. By engineering a parameter-dependent driving term [46–48], our method suppresses nonadiabatic transitions in the critical region and forces the system to precisely follow the desired ground-state trajectory.

We validate this method in the squeezed dispersive coupling model (SDCM), which describes a qubit dispersively

*Contact author: yehong.chen@fzu.edu.cn

†Contact author: xia-208@163.com

coupled to a parametrically driven bosonic field and mimics the quantum Rabi model within a squeezed framework [49–51]. To quantify the performance, we employ the quantum Fisher information (QFI) [21,52] and the signal-to-noise ratio (SNR) [53] as metrics. The QFI characterizes the sensitivity of a quantum state to parameter changes, while the SNR reflects the practical measurement quality. Numerical simulations demonstrate that our method can significantly enhance the precision of parameter estimation. Specifically, compared to the nonassisted diabatic case, our method enhances the QFI by a factor of up to 10^3 and the SNR by over 10^6 in the short-time regime. In the long-time regime, both cases give similar results, but the implementation of our method more closely approaches the theoretical bounds. Our results establish CD driving as a practical method to prepare critical quantum states for enhanced metrological applications. The parameter-dependent design of the driving provides a universal framework that can be generalized to other quantum systems near criticality.

This manuscript is organized as follows. In Sec. II, we present the energy spectrum and critical behavior of the model. In Sec. III, we design the driving Hamiltonian and demonstrate its efficacy through numerical simulations. In Sec. IV, we characterize the quantum metrological potential of the system using the QFI and the SNR, and compare the results with and without the driving. In Sec. V, we investigate the experimental feasibility in circuit quantum electrodynamics (QED) systems. In Sec. VI, we examine the robustness of our method against the decay of the bosonic field. Finally, the conclusions are given in Sec. VII.

II. THE QUANTUM PHASE TRANSITION IN THE SDCM

We start by considering a quantum system that consists of a qubit coupled to a parametrically driven bosonic field [54] (see Fig. 1). The Hamiltonian can be written as ($\hbar = 1$)

$$H = \omega_r a^\dagger a + \frac{\Omega}{2} \sigma_z + g(a^\dagger \sigma_- + a \sigma_+) + \frac{G}{2}(a^{\dagger 2} e^{i\omega_d t} + a^2 e^{-i\omega_d t}), \quad (1)$$

where $a^\dagger(a)$ is the creation (annihilation) operator for the bosonic field with frequency ω_r , g is the coupling strength between the qubit and bosonic field, and $\sigma_\pm$ are the Pauli operators for the qubit with transition frequency Ω . Moreover, G and ω_d represent the strength and frequency of the parametric drive [55–57], respectively.

By moving to a rotating frame at half the drive frequency $\omega_d/2$, the parametric drive term becomes time-independent. In this frame, the qubit and resonator frequencies are shifted to their effective detunings, $\delta = \omega_r - \omega_d/2$ and $\Omega' = \Omega - \omega_d/2$. Subsequently, in the dispersive limit, $|\Omega' - \delta| \gg g$, we can employ a Schrieffer-Wolff transformation [58,59] to transform the qubit-resonator interaction into a dispersive coupling. This procedure yields the squeezed dispersive coupling model [54]

$$H_r = \delta a^\dagger a + \frac{\Omega'}{2} \sigma_z + u a^\dagger a \sigma_z - \frac{G}{2}(a^{\dagger 2} + a^2), \quad (2)$$

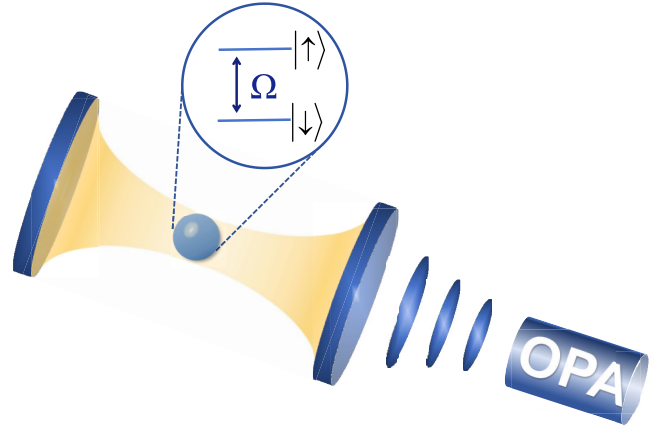


FIG. 1. Schematic illustration of the system, containing a single-mode bosonic field, a qubit, and an optical parametric amplifier (OPA). The qubit ($|\uparrow\rangle$ = excited state and $|\downarrow\rangle$ = ground state) is coupled to the bosonic field with the coupling strength Ω . The OPA is used to generate a squeezed-vacuum reservoir, which couples to the bosonic field mode.

where $u = g^2/(\Omega' - \delta)$ is the dispersive coupling strength. It is worth mentioning that H_r describes the SDCM, which features a qubit coupled to a parametrically driven bosonic field via dispersive interaction. The eigenstates of the free bosonic field are represented by the Fock states $|n\rangle$ ($n = 0, 1, 2, 3, \dots$), while the states of the qubit are denoted as $|\uparrow\rangle$ (excited) and $|\downarrow\rangle$ (ground).

To simplify the Hamiltonian H_r in Eq. (2), we apply a squeezing transformation $S(r) = \exp[r(a^{\dagger 2} - a^2)/2]$, with the squeezing parameter $r = \ln[(\delta + G)/(\delta - G)]/4$. The Hamiltonian H_r can be reexpressed in a more insightful form

$$H_{np} = \delta \text{sech}(2r) a^\dagger a - \frac{e^{2r}}{4} u (a^\dagger + a)^2 + E_c, \quad (3)$$

where $E_c = [\exp(2r)(\delta - G) - 2\Omega']/4$ is a constant energy shift. To study the low-energy behavior, we project H_{np} onto the spin-down subspace, yielding $H_{np} = \langle \downarrow | H_r | \downarrow \rangle$. The validity of this projection hinges on the qubit frequency Ω' being the dominant energy scale. This condition is met in the operational regime of interest where $G \approx \delta$, which ensures that all other relevant energy scales, such as the effective coupling ue^{2r} , are negligible compared to Ω' . We emphasize that Eq. (3) is similar to the low-energy effective Hamiltonian of the quantum Rabi model, suggesting that the SDCM also exhibits analogous phase transition characteristics.

Inspired by Ref. [60], Eq. (3) can be exactly diagonalized by introducing the squeezed operator $S(\varepsilon) = \exp[(\varepsilon/2)(a^{\dagger 2} - a^2)]$, where the squeezing parameter ε is chosen to eliminate the $(a^{\dagger 2} + a^2)$ term. The resulting diagonalized Hamiltonian is

$$H'_{np} = \epsilon_e a^\dagger a + E_g, \quad (4)$$

with the excitation energy ϵ_e ,

$$\epsilon_e = \sqrt{\Delta_r(\Delta_r - \tilde{u}^2)}, \quad (5)$$

and the ground-state energy E_g ,

$$E_g = -\frac{1}{4}\tilde{u}^2[2\sinh^2(\varepsilon) + \sinh(2\varepsilon) + 1] + \Delta_r \sinh^2(\varepsilon) + E_c. \quad (6)$$

Here, $\tilde{u} = \sqrt{ue}$ is the rescaled dispersive coupling strength and $\Delta_r = \delta \operatorname{sech}(2r)$ is the rescaled detuning. The squeezing parameter is $\varepsilon = \ln[\Delta_r/(\Delta_r - \tilde{u}^2)]/4$, ensuring that the excitation energy ϵ_e is real only when

$$\tilde{u} \leq \tilde{u}_c = \sqrt{\Delta_r}. \quad (7)$$

Consequently, the excitation energy ϵ_e is a positive real number for $\tilde{u} \leq \tilde{u}_c$ and vanishes at $\tilde{u} = \tilde{u}_c$, indicating the occurrence of a quantum phase transition [61,62]. In this regime, the eigenstates and eigenenergies of H'_{np} are

$$|\phi_{\text{np}}\rangle = S(r)S(\varepsilon)|n\rangle|\downarrow\rangle, \\ E_{\text{np}} = n\epsilon_e + E_g. \quad (8)$$

In this work, we are particularly interested in the ground state $|\phi_{\text{np}}^0\rangle = S(r)S(\varepsilon)|0\rangle|\downarrow\rangle$. To characterize the critical behavior in the SDCM, we examine the energy spectrum with respect to the rescaled dispersive coupling $\tilde{u}$.

As illustrated in Fig. 2(a), the energy spectrum is degenerate at the critical point $\tilde{u} = \tilde{u}_c$, signaling the breaking of system symmetry. It is worth noting that an insufficient truncation dimension (N_c) in the numerical diagonalization can lead to numerical artifacts, such as an apparent nondegeneracy. The results presented here are obtained with a large truncation of $N_c = 20\,000$ to ensure convergence. Additionally, the second derivative of the ground-state energy, shown in Fig. 2(b), exhibits a sharp discontinuity at $\tilde{u} = \tilde{u}_c$, signifying that the quantum phase transition here is a second-order phase transition, similar to that in the Jaynes-Cummings model [63]. Notably, the derivation above imposes no restrictions on atomic transition frequencies. By using the condition $G \simeq \delta$, we replace the thermodynamic limit $\Omega \gg \omega_r$ in the quantum Rabi model [64], making our model more experimentally feasible.

III. THE CRITICAL GROUND-STATE PREPARATION

The quantum adiabatic theorem [65–67] describes the evolution of a quantum system when external parameters change slowly. When initialized in an eigenstate of the Hamiltonian $H[\zeta(0)]$, the system will follow the instantaneous eigenstate $|n[\zeta(t)]\rangle$ if the parameter $\zeta(t)$ changes adiabatically. This requires the condition

$$\mathcal{A}(t) \equiv \dot{\zeta}(t) \frac{|\langle m[\zeta(t)] | \partial_{\zeta} H[\zeta(t)] | n[\zeta(t)] \rangle|}{\Delta E[\zeta(t)]^2} \ll 1. \quad (9)$$

Here, $\Delta E[\zeta(t)] \equiv E_n[\zeta(t)] - E_m[\zeta(t)]$ denotes the instantaneous energy gap between eigenstates and $\dot{\zeta}(t)$ is the rate of change of the parameters. The numerator quantifies state-to-state transitions induced by parameter variation, while the denominator suppresses nonadiabatic excitations [66,67]. These excitations refer to unwanted quantum energy level transitions caused by the system's failure to adiabatically follow its instantaneous eigenstate. Such failures typically occur when the eigenstate tracking is hindered by vanishing energy

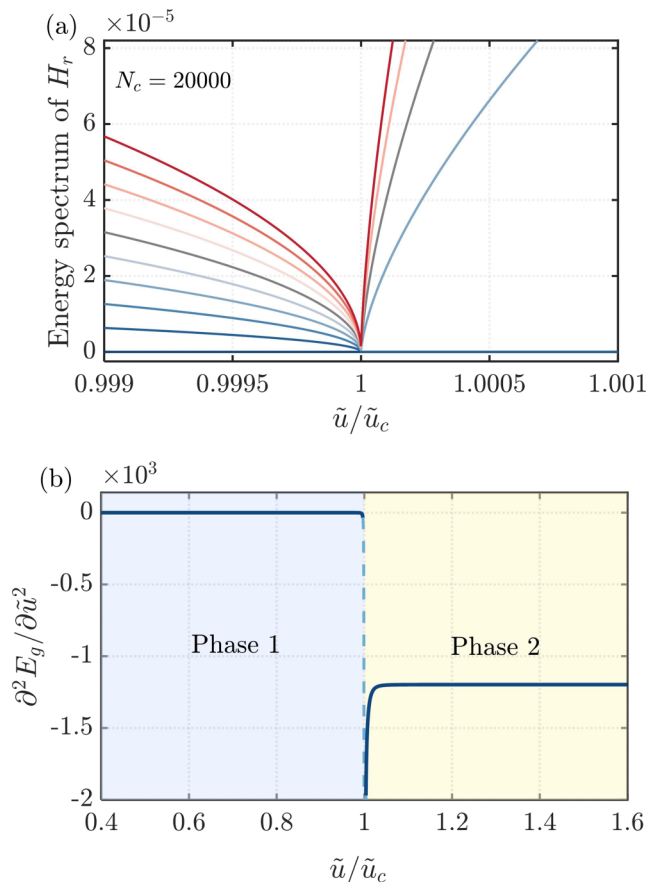


FIG. 2. (a) Low-energy excitation spectrum ($E_n - E_0$) of the SDCM near the critical point, calculated with a Hilbert space truncation dimension of $N_c = 20\,000$. The energy gap between the ground and first excited states closes at the critical point $\tilde{u}/\tilde{u}_c = 1$. (b) Second derivative of the ground-state energy with respect to $\tilde{u}$. The sharp discontinuity at the critical point confirms that the quantum phase transition is of second order.

gaps, as shown in Fig. 2(a). The vanishing energy gap near criticality causes divergences in $\mathcal{A}(t)$, necessitating exponentially slow parameter variation to maintain adiabaticity. Such an effect essentially freezes system dynamics before reaching the target critically point [29].

This deadlock can be resolved by applying the CD driving. By adding an auxiliary term $H_{\text{CD}}(t)$ to the original Hamiltonian, the nonadiabatic transitions can be actively suppressed. This engineered driving maintains adiabaticity even at finite $\dot{\zeta}(t)$, enabling precise control through critical points within experimentally accessible timeframes. The principle of this approach is schematically illustrated in Fig. 3. By precisely canceling the nonadiabatic excitations, the CD driving ensures that the trajectory of the system follows the ideal adiabatic path. This allows the protocol to bypass the slow evolution required by the adiabatic theorem and avoid errors accumulated during a rapid diabatic process. The target state (T) is strategically chosen to be near, but not exactly at, the critical point (C). This choice represents a crucial trade-off: The state is close enough to the critical point to harness significant criticality-enhanced sensitivity, yet remains sufficiently far to

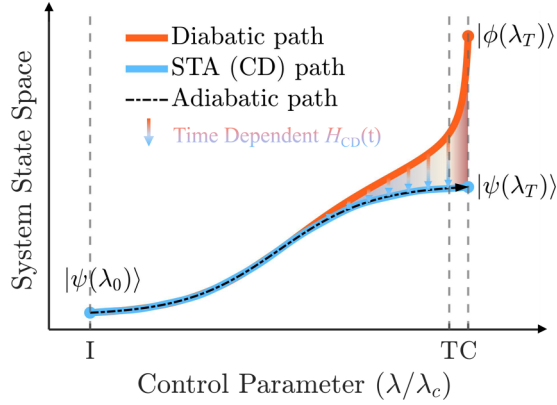


FIG. 3. Schematic of the CD driving principle. The protocol aims to evolve the system from an initial state (I) to a target state (T) near the critical point (C). The CD path (solid blue) perfectly tracks the ideal adiabatic path (dashed black) by adding a corrective drive $H_{CD}(t)$, in contrast to the error-prone diabatic path (solid orange).

avoid the divergent control resources that would be required to prepare the critical state itself.

To ensure a well-behaved system evolution and subsequent control fields, we employ a smooth pulse shape for the physical coupling strength $g(t)$. A key feature of this shape is a vanishing time derivative at the start and end of the protocol. This condition is met by the pulse shape

$$g(t) = g_f \sin^2\left(\frac{\pi t}{2t_f}\right), \quad (10)$$

where g_f is the final target coupling strength, chosen to be close to but less than the critical value, and t_f is the total evolution time.

Because $g(t)$ is time-dependent, a time-dependent Schrieffer-Wolff (SW) transformation [68,69] is required to derive an accurate low-energy effective Hamiltonian. As detailed in Appendix A, this transformation yields an effective Hamiltonian $H_{r,cor}(t)$ with an effective coupling strength $u_{cor}(t)$ given by

$$u_{cor}(t) = \frac{g(t)^2}{\Delta} - \frac{g(t)\ddot{g}(t) - \dot{g}(t)^2}{\Delta^3}. \quad (11)$$

This leads to the final effective Hamiltonian for the system:

$$H_{r,cor}(t) = \delta a^\dagger a + \frac{\Omega'}{2} \sigma_z + u_{cor}(t) a^\dagger a \sigma_z + \frac{G}{2} (a^{\dagger 2} + a^2). \quad (12)$$

The SW transformation also induces a small, time-dependent ac Stark shift to the qubit frequency. This shift is negligible compared to the effective frequency Ω' and is thus treated as a constant. The Hamiltonian in Eq. (12) provides an accurate description of the system's low-energy dynamics and serves as the basis for the CD protocol design.

The construction of the CD driving protocol is based on the effective Hamiltonian $H_{r,cor}(t)$ in Eq. (12), which governs the dynamics of the system including nonadiabatic corrections. The instantaneous eigenstates of this Hamiltonian are given by

$$|\phi_{np}(t)\rangle = S(r)S[\varepsilon(t)]|n\rangle|\downarrow\rangle, \quad (13)$$

where the squeezing parameter $\varepsilon(t)$ is now implicitly a function of the corrected coupling, $\varepsilon(t) \equiv \varepsilon[\tilde{u}_{cor}(t)]$, with $\tilde{u}_{cor}(t) = \sqrt{u_{cor}(t)}e^r$. The CD driving $H_{CD}(t)$ is constructed to compensate for nonadiabatic transitions. It is formally given by the expression for transitionless quantum driving [35,70]:

$$H_{CD}(t) = i\hbar \sum_n [|\partial_t \phi_{np}(t)\rangle \langle \phi_{np}(t)| - \langle \phi_{np}(t)| \partial_t \phi_{np}(t)\rangle |\phi_{np}(t)\rangle \langle \phi_{np}(t)|], \quad (14)$$

where $|\partial_t \phi_{np}(t)\rangle$ is the time derivative of the instantaneous eigenstate

$$|\partial_t \phi_{np}(t)\rangle = \frac{\dot{\varepsilon}(t)}{2} S(r)S[\varepsilon(t)](a^{\dagger 2} - a^2)|n\rangle|\downarrow\rangle. \quad (15)$$

In principle, when this Hamiltonian is well defined, the protocol is exact and should yield a final-state fidelity of unity. Focusing on the ground state and using the completeness relation of the instantaneous eigenbasis, the CD driving simplifies to

$$H_{CD}(t) = \frac{i\dot{\varepsilon}(t)}{2} (a^{\dagger 2} - a^2), \quad (16)$$

where $\dot{\varepsilon}(t)$ is given by

$$\dot{\varepsilon}(t) = \frac{\tilde{u}_{cor}(t)\dot{\tilde{u}}_{cor}(t)}{2[\tilde{u}_c^2 - \tilde{u}_{cor}^2(t)]}. \quad (17)$$

The form of Eq. (17) highlights a critical challenge inherent to transitionless driving near a quantum phase transition. As the system approaches the critical point [$\tilde{u}_{cor}(t) \rightarrow \tilde{u}_c$], the energy gap vanishes and the denominator in Eq. (17) approaches zero. Consequently, a divergent driving strength $\dot{\varepsilon}(t)$ would be required to suppress all nonadiabatic excitations. However, this potential divergence is elegantly circumvented by the pulse design in Eq. (10). By engineering the numerator [proportional to $\tilde{u}_{cor}(t)$] to vanish at the start ($t = 0$) and end ($t = t_f$) of the protocol, the driving strength $\dot{\varepsilon}(t)$ is ensured to be zero at these boundaries. This design guarantees that the control field is not only finite throughout the evolution but also smoothly ramped on and off, making it physically realizable.

The numerical simulations summarized in Fig. 4 confirm the substantial advantage of this protocol. Here, the fidelity is defined as $F = |\langle \psi(T) | \phi_{np}(T) \rangle|^2$, where $|\psi(T)\rangle$ is the prepared state and $|\phi_{np}(T)\rangle$ is the target state. The global performance is presented in Fig. 4(a), where the CD protocol maintains near-unity fidelity for rapid state preparation, even as the target coupling strength $\tilde{u}_f$ approaches the critical value $\tilde{u}_c$. Conversely, the nonassisted protocol exhibits a significant performance degradation in the critical regime. Its fidelity is substantially reduced as $\tilde{u}/\tilde{u}_c \rightarrow 1$, a trend that persists even for long evolution times. This contrast is further quantified in Figs. 4(b) and 4(c). For a fixed short evolution time, Fig. 4(b) shows that the fidelity of the nonassisted protocol drops sharply, whereas the CD protocol maintains near-unity fidelity. The challenge for the nonassisted approach is clearly demonstrated in Fig. 4(c), where achieving high fidelity for a state at $\tilde{u}_f = 0.99\tilde{u}_c$ and $\tilde{u}_f = 0.999\tilde{u}_c$ necessitates an evolution time on the order of $10^6/\omega_r$ and $10^7/\omega_r$, respectively. In contrast, the CD protocol prepares these states with high fidelity across all considered evolution times.

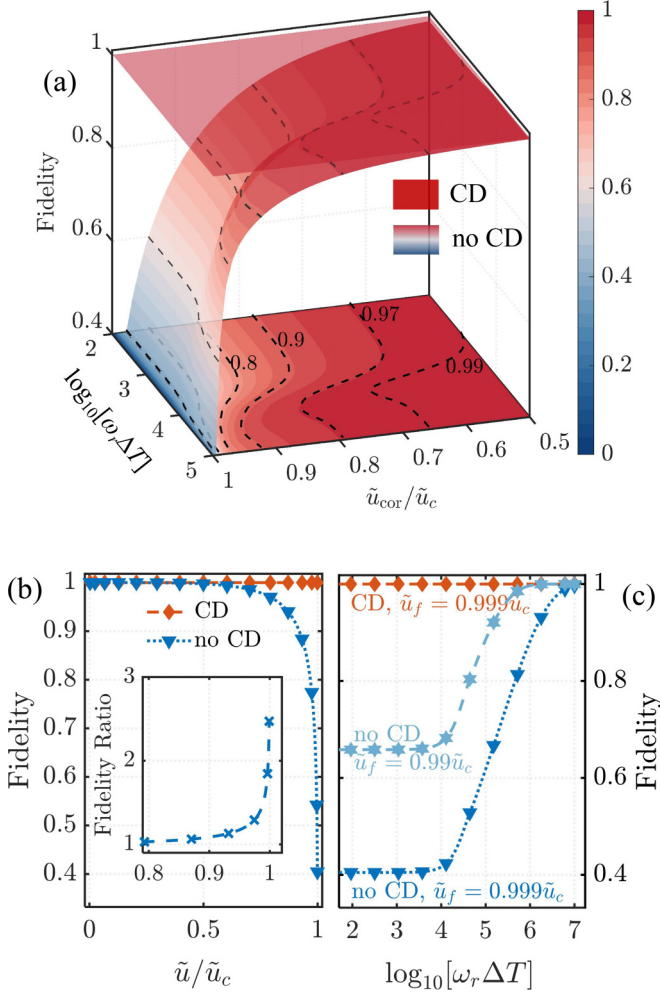


FIG. 4. Fidelity of critical target state preparation. (a) 3D surface plot comparing the fidelities of the CD-assisted (red surface) and nonassisted (gradient surface) protocols. The dashed contours project the fidelity of the nonassisted case onto the bottom plane. (b) Fidelity vs target coupling strength for a fixed short evolution time. (c) Fidelity vs evolution time, illustrating the substantial time cost for the nonassisted protocol to achieve high fidelity near criticality.

IV. CRITICAL QUANTUM METROLOGY ENHANCEMENT

Building on the effective Hamiltonian derived in Sec. II, we demonstrate the feasibility of achieving high-precision quantum metrology. To quantify the precision enhancement, we employ two key metrics: the QFI and the SNR.

In quantum metrology, the QFI is a fundamental concept that characterizes the lower bound on the variance of a parameter estimate [52,71,72], defining the maximum achievable precision. Since the protocol under study can be framed as a frequency estimation task, its performance is subject to fundamental bounds, often referred to as the Heisenberg limit (HL) [73]. For a closed system with a maximum photon number N_{max} and evolution time T , a relevant bound is given by the HL, which scales as $\text{QFI} \sim N_{\text{max}}^2 T^2$ [74–76]. For a pure state $|\phi_{\tilde{u}}\rangle$, the QFI with respect to a parameter $\tilde{u}$ is given by

$$\mathcal{I}_{\tilde{u}} = 4[(\langle \partial_{\tilde{u}} \phi_{\tilde{u}} | \partial_{\tilde{u}} \phi_{\tilde{u}} \rangle + (\langle \partial_{\tilde{u}} \phi_{\tilde{u}} | \phi_{\tilde{u}} \rangle)^2]. \quad (18)$$

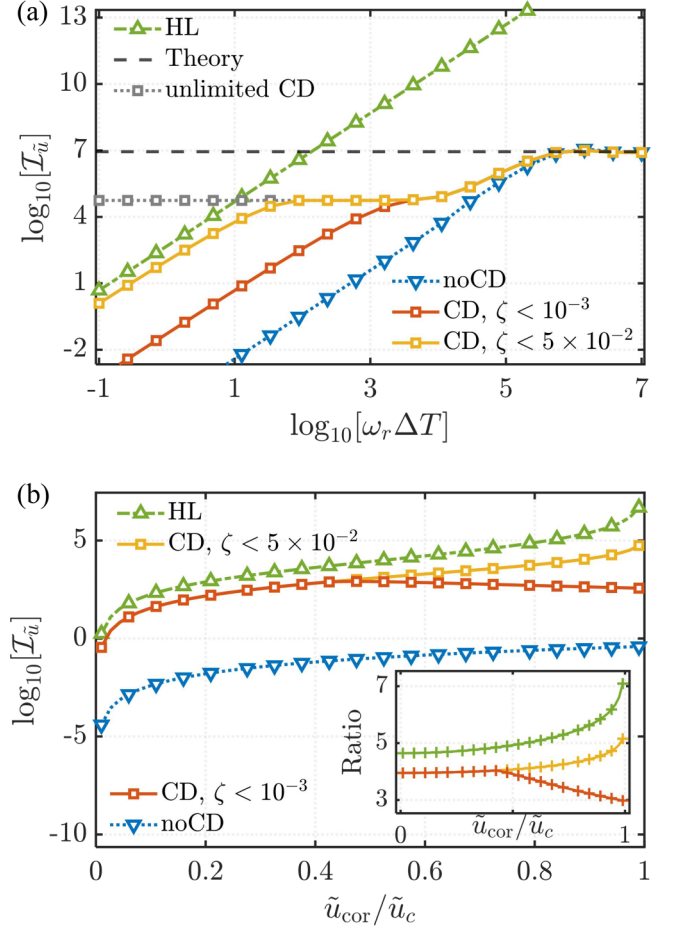


FIG. 5. Physically constrained quantum Fisher information. (a) Final-state QFI as a function of total evolution time ΔT . The unlimited CD protocol (gray dotted squares), while reaching a high plateau, unphysically exceeds the HL (green dashed triangles) at short times. By imposing a physical limit on the maximum driving strength, defined by $\zeta = |\dot{\epsilon}(t)|_{\text{m}}/\omega_r$, the protocol becomes physically consistent. The constrained CD protocols ($\zeta < 10^{-3}$ and $\zeta < 5 \times 10^{-2}$) remain below the HL for all times. As the constraint becomes stricter, the performance gracefully degrades toward the nonassisted (no CD) case. (b) Final-state QFI as a function of the target coupling strength for a fixed evolution time $\Delta T = 100/\omega_r$. The constrained CD protocols still offer a significant advantage over the no CD case, while respecting the HL. The inset shows the QFI ratio relative to the no CD baseline, confirming a substantial performance enhancement.

For the ground state $|\phi_{\text{np}}\rangle = S(r)S(\epsilon)|0\rangle|\downarrow\rangle$ in the SDCM, the QFI near the critical point can be approximated as

$$\mathcal{I}_{\tilde{u}} \simeq \frac{1}{8} \left(\frac{\tilde{u}}{\Delta_r - \tilde{u}^2} \right)^2. \quad (19)$$

The idealized CD protocol, while showing a high performance plateau, exhibits unphysical behavior at very short evolution times. As shown by the “unlimited CD” curve in Fig. 5(a), the calculated QFI unphysically surpasses the HL. This inconsistency arises because preparing a high-fidelity state in a vanishingly short time necessitates an infinitely strong CD drive. Such a dominant field would dictate the

system dynamics, leading to the unphysical constant QFI observed at these timescales.

To address this and construct a physically meaningful protocol, we introduce a crucial constraint: a hard limit on the maximum allowed strength of the CD driving field. We quantify this limit using the dimensionless parameter $\zeta = |\dot{\epsilon}(t)|_m/\omega_r$. The results under this constraint are also shown in Fig. 5(a).

A clear and physically consistent picture emerges. When the driving strength is limited (e.g., $\zeta < 5 \times 10^{-2}$ or $\zeta < 10^{-3}$), the QFI remains strictly bounded by the HL across all evolution times, resolving the previous inconsistency. This constraint introduces a natural trade-off between speed and precision: Stronger available driving fields (larger ζ) allow for reaching the QFI plateau at shorter times. As the constraint becomes more stringent (smaller ζ), the performance of the CD protocol gracefully degrades, eventually converging toward the nonassisted (no CD) case.

Figure 5(b) further explores the performance under these constraints for a fixed, short evolution time of $\Delta T = 100/\omega_r$. It demonstrates that even with significant physical limitations (e.g., $\zeta < 10^{-3}$), the CD protocol still provides a substantial, multi-order-of-magnitude enhancement in QFI compared to the nonassisted protocol. The inset, showing the QFI ratio, confirms that this significant advantage is maintained across the entire critical region, while fully respecting the fundamental bounds of quantum mechanics. This analysis solidifies the protocol as a robust and physically realizable pathway toward rapid, high-precision critical quantum sensing.

To quantify the practical advantage of the protocol, Fig. 6 shows the performance enhancement ratios for both QFI and SNR, using the nonassisted (no CD) protocol as the baseline (ratio of one). The results highlight the importance of imposing a physical constraint on the CD drive. Without a constraint, the protocol exhibits unphysical behavior at short times. The enhancement ratio for the “unlimited CD” case, shown in Fig. 6, diverges and even surpasses the benchmark set by the HL. In contrast, the constrained protocols establish large and stable enhancement plateaus in the experimentally crucial short-time regime. For instance, even under a strict constraint of $\zeta < 10^{-4}$, the protocol enhances the QFI by over an order of magnitude [Fig. 6(a)] and the SNR by approximately 2 orders of magnitude [Fig. 6(b)] in the short-time regime.

This demonstrates that the protocol can provide a substantial metrological advantage within realistic physical constraints. The enhancement is bounded by the HL ratio, which represents the theoretical maximum improvement possible given the quantum resources. Our constrained protocol, particularly for larger allowed driving strengths (e.g., $\zeta < 5 \times 10^{-2}$), can closely approach this fundamental limit, offering a highly efficient route to leveraging quantum criticality.

It is worth noting, however, that the QFI achieved by the CD protocol at short times remains below the theoretical limit of the ideal target state, as seen in Fig. 5(a). This phenomenon arises from the nature of the QFI itself. For an unknown parameter θ , the QFI can be expressed as [77]

$$\mathcal{I}_\theta = \lim_{\delta\theta \rightarrow 0} \frac{4}{(\delta\theta)^2} (1 - |\langle \psi(T, \theta) | \psi(T, \theta + \delta\theta) \rangle|^2), \quad (20)$$

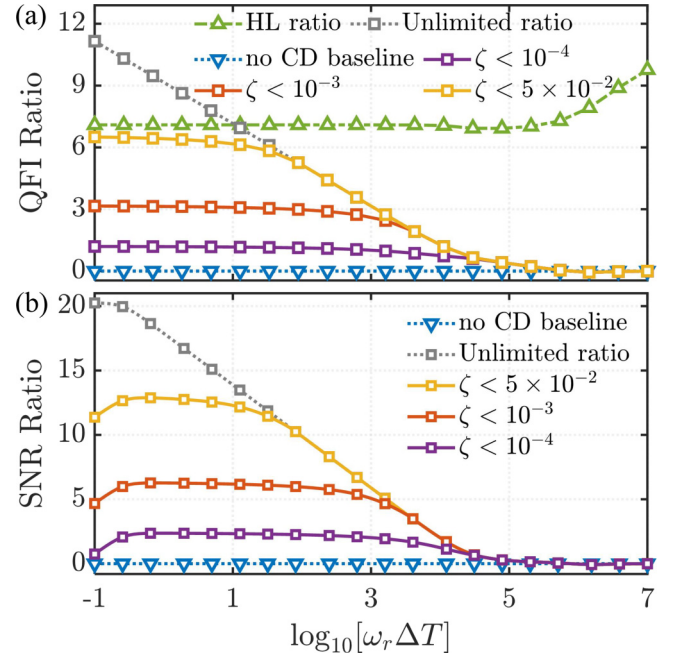


FIG. 6. Performance enhancement ratios for physically constrained protocols. Both panels show the enhancement relative to the nonassisted (no CD) protocol, which serves as the baseline (blue dotted triangles at unity). (a) QFI ratio. (b) SNR ratio. The unlimited CD protocol (gray dotted squares) shows a diverging ratio at short times, reflecting its unphysical nature. In contrast, the physically constrained protocols (e.g., $\zeta < 10^{-3}$, 10^{-4}) exhibit large, stable enhancement plateaus in the short-time regime. The HL ratio (green dashed triangles) provides a benchmark for the maximum possible enhancement based on the available resources.

where the overlap $|\langle \psi(T, \theta) | \psi(T, \theta + \delta\theta) \rangle|^2$ quantifies the distinguishability of final states. The QFI is maximized when this overlap is minimized. While a slow adiabatic process allows the final states to evolve into distinct eigenstates, the CD driving forces the system to perfectly track a predefined trajectory, suppressing the distinguishability between states evolved under θ and $\theta + \delta\theta$. This increases their overlap and consequently reduces the QFI below the absolute theoretical limit, even as it vastly outperforms the nonassisted diabatic protocol.

Likewise, the SNR is another crucial metric for evaluating parameter estimation methods, as it directly relates to the mean-square error of the estimator [78,79]. For a given observable O , the SNR used to estimate the parameter A can be expressed as [80]

$$S_A[O] = \frac{|\partial_A \langle O \rangle_A|^2}{\Delta O_A^2}, \quad (21)$$

where $\Delta O_A^2 = \langle O^2 \rangle_A - \langle O \rangle_A^2$ is the variance of O . In this work, we consider the SNR of heterodyne detection, which uses the quadrature operator $O = X^2 + P^2$ of the bosonic field as the detection observable. Using the creation ($a^\dagger$) and annihilation (a) operators, this observable can be rewritten as $O = aa^\dagger$. The SNR for this observable is expressed as [53]

$$S_u^{\text{Het}} = \frac{|\partial_u \langle aa^\dagger \rangle_u|^2}{\langle a^2 (a^\dagger)^2 \rangle_u - \langle aa^\dagger \rangle_u^2}. \quad (22)$$

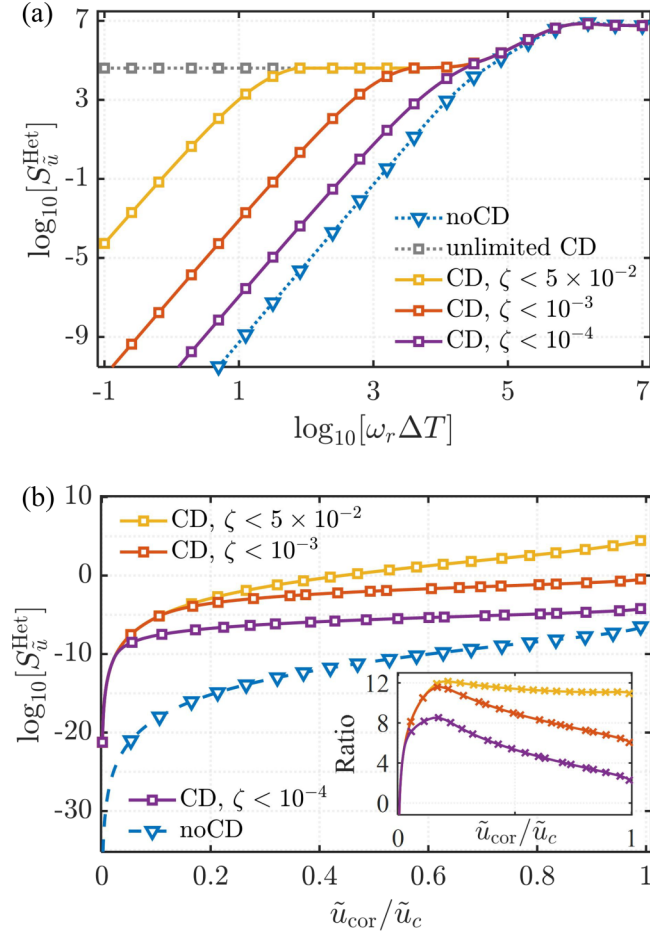


FIG. 7. Metrological performance quantified by the SNR under physical constraints. (a) Final-state SNR as a function of total evolution time ΔT . The unlimited CD protocol shows the best performance but is unphysical. The physically constrained CD protocols ($\zeta < 5 \times 10^{-2}$, etc.) still offer a substantial advantage over the nonassisted (no CD) case, especially at short times. (b) Final-state SNR as a function of the target coupling strength for a fixed evolution time $\Delta T = 100/\omega_r$. The inset displays the SNR enhancement ratio relative to the no CD baseline, which can reach several orders of magnitude.

As shown in Fig. 7, the SNR exhibits a diverging trend near the critical point, mirroring the behavior of the QFI. This similarity arises because the signal term $\partial_{\tilde{u}} \langle aa^\dagger \rangle_{\tilde{u}}$ introduces

$$\partial_{\tilde{u}} \langle aa^\dagger \rangle_{\tilde{u}} \propto (\tilde{u}_c^2 - \tilde{u}^2)^{-1}, \quad (23)$$

which diverges at the critical point, analogous to the QFI. However, the divergence of the SNR is tempered by the noise variance in the denominator, preventing it from reaching the same magnitude as the QFI. Beyond the fundamental precision limit described by the QFI, the SNR offers a more practical benchmark for metrological performance. As shown in Fig. 7, the SNR exhibits a behavior that closely mirrors that of the QFI, confirming that the quantum advantage translates into a tangible gain for realistic measurement schemes.

Figure 7(a) presents the final-state SNR as a function of the total evolution time, incorporating the same physical

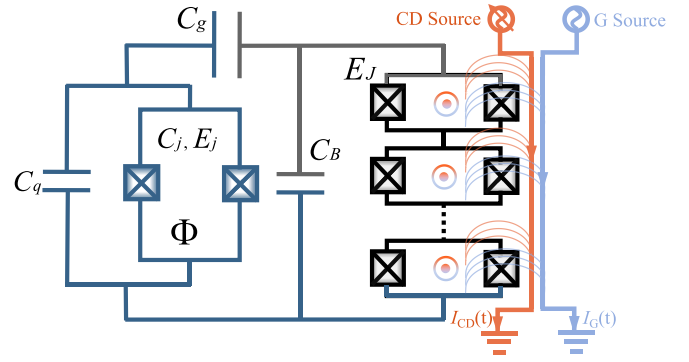


FIG. 8. Diagram of the setup for a transmon qubit with capacitance and Josephson energy C_j and E_j coupled to an array of Josephson junctions with capacitance and Josephson energy C_j and E_j , which is shunted by an additional large capacitance C_B that realizes parametric oscillators. The oscillators are driven by a total time-dependent magnetic flux, $\Phi(t)$, which is composed of two components. A fixed-amplitude stabilizing flux, $\Phi_G(t)$, is generated by the current $I_G(t)$. Concurrently, a variable-amplitude counterdiabatic pulse, $\Phi_{\text{CD}}(t)$, is generated by the current $I_{\text{CD}}(t)$. The total flux is thus given by their superposition: $\Phi(t) = \Phi_G(t) + \Phi_{\text{CD}}(t)$.

constraints on the driving strength ζ as in the QFI analysis. The results are qualitatively identical: The physically constrained CD protocols significantly outperform the nonassisted case, particularly in the short-time regime. For instance, with a constraint of $\zeta < 10^{-3}$, the SNR is enhanced by more than 5 orders of magnitude for $\Delta T \approx 1/\omega_r$. This demonstrates that our protocol can dramatically reduce the time required to reach a target SNR.

The performance at a fixed short evolution time is detailed in Fig. 7(b). Even under stringent constraints, the CD protocol yields a much higher SNR across the entire critical region. The inset quantifies this advantage, showing that the enhancement ratio can exceed 10^{11} near the critical point for moderately strong driving fields. This confirms that the rapid preparation of critical states via the protocol translates into a dramatic improvement in the practical signal-to-noise ratio, solidifying its potential for real-world quantum sensing applications.

V. EXPERIMENTAL FEASIBILITY

This section analyzes the experimental feasibility of the protocol. The proposed model can be realized using the circuit illustrated in Fig. 8, which consists of a transmon qubit coupled to an array of Josephson junctions. This setup facilitates the realization of a two-photon drive and the CD driving through the temporal modulation of the flux traversing the array, enabling parametric amplification and squeezing processes.

The Hamiltonian for the circuit in Fig. 8, under the limit of large resonator capacitance, is given by [81]

$$H = 4E_c(n - n_g)^2 - E_j \cos \varphi + \lambda n N + 4E_c N^2 - N_0 E_j [\Phi(t)] \cos \frac{\phi}{N_0}. \quad (24)$$

Here, $E_j = E_{j,\text{max}} |\cos(\pi \Phi / \Phi_0)|$ is the tunable Josephson energy, E_c is the charging energy, and λ is the coupling constant

between the transmon qubit and the Josephson junction array. This coupling is determined by the capacitive interaction between the transmon and the superconducting quantum interference device (SQUID)-array resonator, mediated by the capacitor C_g . In the typical weak coupling regime, defined by the condition $C_g \ll C_{\Sigma, \text{tr}}, C_{\Sigma, \text{arr}}$, the constant is given by the approximate expression $\lambda \approx 4e^2 C_g / (C_{\Sigma, \text{tr}} C_{\Sigma, \text{arr}})$, where e is the elementary charge, while $C_{\Sigma, \text{tr}} = C_q + C_j + C_g$ and $C_{\Sigma, \text{arr}} = C_B + C_{J, \text{arr}} + C_g$ are the total self-capacitances of the transmon and the array islands, respectively. The effective offset charge of the transmon qubit is denoted by n_g , while n and φ represent the number of Cooper pairs and the phase difference between the superconductors, respectively. For the Josephson junction array, E_C and E_J are the charging energy and Josephson energy of each junction, and ϕ and N denote the number of Cooper pairs and the overall phase across the array, respectively. Additionally, N_0 is the number of SQUIDs in the array, and $\Phi(t)$ is the external magnetic flux used to modulate the parametric oscillators. The transmon circuit can be viewed as a Duffing nonlinear oscillator by assuming the limit $E_c \ll E_j$. When only the lowest two energy levels are considered, the Hamiltonian of the transmon qubit part can be approximated as

$$H_q = \frac{1}{2} \hbar \omega_q \sigma_z, \quad \omega_q = (\sqrt{8E_c E_j} - E_c) / \hbar. \quad (25)$$

For the part involving the array of Josephson junctions, we first assume that the Josephson energy can be expressed as

$$\begin{aligned} E[\Phi(t)] &= E_J + \delta E_J(t) \\ &= E_J + \delta E_J \cos(\omega_p t). \end{aligned} \quad (26)$$

To derive a realistic model, we apply a Taylor expansion to $\cos(\phi/N_0)$ and truncate it to the fourth order, which captures the intrinsic nonlinearity of the resonator. The Hamiltonian becomes

$$\begin{aligned} H_n &\approx 4E_c(n - n_g)^2 - E_j \cos \varphi + \lambda n N \\ &\quad + 4E_c N^2 - N_0 E_J \left(1 - \frac{X}{2} + \frac{X^2}{24}\right) \\ &\quad - N_0 \delta E_J \left(1 - \frac{X}{2}\right) \cos(\omega_p t), \end{aligned} \quad (27)$$

where $X = (\phi/N_0)^2$. Under the premise that the highest energy level is much smaller than the dimension of the Hilbert space, and following the standard quantization procedure for circuits [82,83], we obtain

$$N = -in_0(a - a^\dagger), \quad \phi = \phi_0(a + a^\dagger), \quad (28)$$

where $n_0 = \sqrt[4]{E_J/32N_0E_C}$ and $\phi_0 = \sqrt[4]{2N_0E_C/E_J}$ are the zero-point fluctuations [84]. Similarly, the transmon qubit operators can be expressed in terms of its own harmonic basis operators ($b, b^\dagger$) and zero-point fluctuations:

$$n = in_{q, \text{zpf}}(b^\dagger - b), \quad \varphi = \varphi_{q, \text{zpf}}(b + b^\dagger), \quad (29)$$

$n_{q, \text{zpf}} = \sqrt[4]{E_J/32E_c}$ and $\varphi_{q, \text{zpf}} = \sqrt[4]{2E_c/E_j}$ are the corresponding zero-point fluctuations of the transmon.

We consider a transmon qubit with a large anharmonicity, where the energy separation to higher excited states is much larger than all coupling and drive strengths. This large energy

mismatch suppresses unwanted transitions out of the computational subspace, effectively confining the dynamics to the two levels spanned by the ground ($|g\rangle$) and first excited ($|e\rangle$) states. This justifies projecting the harmonic oscillator operators onto Pauli operators, such that $b \rightarrow \sigma_-$ and $b^\dagger \rightarrow \sigma_+$. The Hamiltonian is rewritten as the laboratory-frame Hamiltonian

$$\begin{aligned} H_{\text{lab}}/\hbar &\approx \omega_r a^\dagger a + \frac{1}{2} \Omega \sigma_z + g(\sigma_- - \sigma_+)(a^\dagger - a) \\ &\quad + \frac{\delta E_J \omega_r}{4\hbar E_J} (a + a^\dagger)^2 \cos(\omega_p t) - K_L (a + a^\dagger)^4, \end{aligned} \quad (30)$$

where the laboratory-frame parameters are defined as $\omega_r = \sqrt{8E_J E_C}/N_0/\hbar$, the coupling strength $g = (\lambda/\hbar)n_{q, \text{zpf}}n_0$, and the laboratory-frame Kerr coefficient $K_L = E_C/(12\hbar N_0^2)$.

To obtain the final effective Hamiltonian, we move to an interaction picture rotating with a frequency ω_d and apply the rotating-wave approximation (RWA). By setting the pump frequency to $\omega_p \approx 2\omega_d$ and neglecting fast-oscillating terms, the laboratory-frame Hamiltonian is simplified.

The term involving $(a + a^\dagger)^2 \cos(\omega_p t)$ generates the two-photon drive, $G(a^{\dagger 2} + a^2)/2$, with a drive strength G given by

$$G = \frac{\delta E_J \omega_r}{4E_J}. \quad (31)$$

The qubit-resonator interaction takes the Jaynes-Cummings form, $g(a^\dagger \sigma_- + a \sigma_+)$. The fourth-order term, $-K_L(a + a^\dagger)^4$, under RWA gives rise to the dominant self-Kerr nonlinearity, $-K(a^\dagger a)^2$. The effective Kerr coefficient K is

$$K = \frac{E_C}{2\hbar N_0^2}. \quad (32)$$

This yields the effective Hamiltonian:

$$\begin{aligned} H_{\text{eff}}/\hbar &= \delta a^\dagger a + \frac{1}{2} \Omega' \sigma_z + g(a^\dagger \sigma_- + a \sigma_+) \\ &\quad + \frac{G}{2} (a^2 + a^{\dagger 2}) - K(a^\dagger a)^2, \end{aligned} \quad (33)$$

where $\delta = \omega_r - \omega_d$ and $\Omega' = \omega_q - \omega_d$ are the detunings. This Hamiltonian, which corresponds to the SDCM studied in Sec. II with the crucial addition of the intrinsic Kerr nonlinearity, forms the basis of our analysis. Its validity relies on the rotating-wave approximation, requiring all coupling and drive strengths to be much smaller than the system's characteristic frequencies ($\{g, G, K\} \ll \omega_r, \Omega$). While this condition holds for the primary drives, the strength of the CD term requires careful consideration. As we will demonstrate in Sec. IV and Appendix B, the CD drive is engineered to operate within the perturbative regime in all physically relevant scenarios, thus ensuring the self-consistency of our model.

The implementation of the protocol hinges on controlling the parametric drive G and the CD drive. In the proposed circuit QED architecture (Fig. 8), both are realized by modulating the magnetic flux through the SQUID array. The parametric drive G is generated by a fixed-amplitude flux modulation, $\Phi_G(t)$. The CD term, $H_{\text{CD}}(t)$, takes the form of a time-dependent two-photon drive. It is generated by a separate, variable-amplitude flux pulse, $\Phi_{\text{CD}}(t)$, applied through a dedicated drive line. The desired term is engineered by setting the pump frequency to $\omega_p \approx 2\omega_r$ and shaping its profile to

TABLE I. Experimentally demonstrated parameters for key components relevant to our protocol.

References	Description	$\omega_q/2\pi$	$\omega_r/2\pi$	$g/2\pi$	$G/2\pi$ (param. drive)	$K/2\pi$ (Kerr)
[85]	cQED (eng. dissipation)	4.90 GHz	7.58 GHz	65 MHz	111 kHz	4 kHz
[86]	cQED (param. amp.)	–	5.0 GHz	–	–	≈ 0 (tunable)
[84]	cQED (param. amp.)	–	~ 7 GHz	–	up to 20 MHz	17.3 MHz ($N = 10$)
[87]	Trapped ion (param. amp.)	1.69 GHz	5.9 MHz	1.46 kHz	up to 50 kHz	≈ 0

match the precalculated function $\hbar\dot{\epsilon}(t)/2$. The use of such two-photon drives is a standard technique in circuit QED [88–91], and the broader realization of STA protocols is well established across various platforms [40,92,93].

The dominant nonideality in this platform is the residual Kerr nonlinearity, K . A key feature of our proposed SQUID-array design is an intrinsic mechanism for its suppression. As derived, the relative Kerr strength scales as $K/\omega_r \propto N_0^{-3/2}$, allowing it to be engineered to a negligible level by choosing a sufficiently large N_0 . This intrinsic suppression can be further enhanced by advanced circuit elements like the superconducting nonlinear asymmetric inductive element [86,94] or circumvented entirely by implementing the protocol in naturally Kerr-free platforms such as trapped-ion systems [87]. A detailed numerical analysis of the protocol robustness against the residual Kerr effect is presented in Appendix D. The results confirm that even with a realistic Kerr strength (e.g., $K/\omega_r \sim 10^{-5}$), the CD protocol maintains high fidelity and a significant QFI enhancement in the experimentally relevant short-time regime.

Tables I and II summarize a set of experimental parameters from different physical platforms and present a plausible parameter regime for implementing the protocol. These benchmarks indicate that the parameters used in our simulations are experimentally reasonable.

In summary, the detailed derivation, parameter analysis, and numerical simulations provide evidence for the experimental feasibility of the protocol. The proposed circuit QED platform offers an intrinsic mechanism to suppress nonlinearities, and the CD-assisted protocol demonstrates robust, high-performance operation even in the presence of such residual effects.

VI. DECAY EFFECTS AND ROBUSTNESS

After taking into account the decay of the bosonic field, the dynamics of the system is determined by the master equation

$$\dot{\rho}(t) \approx -i[H(t), \rho(t)] + \kappa \mathcal{D}[a]\rho(t), \quad (34)$$

where $\mathcal{D}[o]\rho(t) = o\rho(t)o^\dagger - (o^\dagger o\rho(t) + \rho(t)o^\dagger o)/2$ is the standard Lindblad superoperator and κ is the bosonic field decay rate. We focus on photon loss as the primary error mechanism for two key reasons. First, due to the large qubit frequency and initialization in the spin-down subspace, qubit excitations are strongly suppressed. Second, the relevant protocol times are significantly shorter than the typical coherence times of modern transmon qubits [95,96]. Consequently, qubit decay can be safely neglected, leaving photon loss from the high- Q resonator as the dominant decoherence channel, particularly for the multiphoton states generated by our protocol.

The use of the standard Lindblad form is also a well-justified approximation. While the total Hamiltonian is time-dependent, the validity of the master equation hinges on the driving strengths being much smaller than the system's characteristic frequencies. As rigorously demonstrated in our feasibility analyses (Sec. V and Appendix B), the peak drive strengths remain in the perturbative regime, ensuring that Eq. (34) provides an accurate model for the dominant dissipation channels.

The fidelity of the system, defined as $\mathcal{F}_\kappa(t) = \langle \phi(t) | \rho(t) | \phi(t) \rangle$, is used to evaluate the impact of decay, where $|\phi(t)\rangle$ represents the ideal, noiseless state vector of the evolving state.

The numerical results, summarized in Fig. 9, reveal a stark contrast between the nonassisted and CD-assisted protocols and highlight a key advantage of our approach.

For the nonassisted protocol [Fig. 9(a)], the achievable fidelity is severely constrained by its inherent slowness. Even within a short total evolution time of $\Delta T_{\max} = 100/\omega_r$, the fidelity degrades rapidly as the protocol unfolds. For evolution times $\Delta T > 80/\omega_r$, high fidelity (> 0.9) cannot be maintained, even for the smallest decay rates considered. This illustrates a fundamental dilemma for such protocols: The time required to approach the target state already exposes the system to significant decoherence.

In stark contrast, the CD-assisted protocol [Fig. 9(b)] demonstrates remarkable robustness due to its speed. A large region of high fidelity is maintained across a wide range

TABLE II. A plausible set of parameters for implementing our protocol, derived from the benchmarks in Table I.

Notation	Remarks	Used parameters	Circuit QED	Trapped ion
ω_r	Resonator frequency	Frequency scale	$2\pi \times 7.5$ GHz	$2\pi \times 6$ MHz
ω_q	Qubit frequency	$\omega_q/\omega_r = 0.65$	$2\pi \times 4.9$ GHz	$2\pi \times 1.7$ GHz
g	Coupling strength	$g/\omega_r = 0$ to 6×10^{-4}	$2\pi \times (0-4.5)$ MHz	$2\pi \times (0-3.6)$ kHz
G	Parametric drive strength	$G/\omega_r = 0$ to 9.9×10^{-5}	$2\pi \times (0-0.75)$ MHz	$2\pi \times (0-0.6)$ kHz
K	Kerr nonlinearity	$K/\omega_r = 0.1$ to 1×10^{-5}	$2\pi \times (7.5-75)$ kHz	≈ 0
$ \dot{\epsilon}(t) _{\text{m}}$	Max CD strength	$ \dot{\epsilon}(t) _{\text{m}}/\omega_r = 0$ to 3×10^{-3}	$2\pi \times (0-22.5)$ MHz	$2\pi \times (0-18)$ kHz

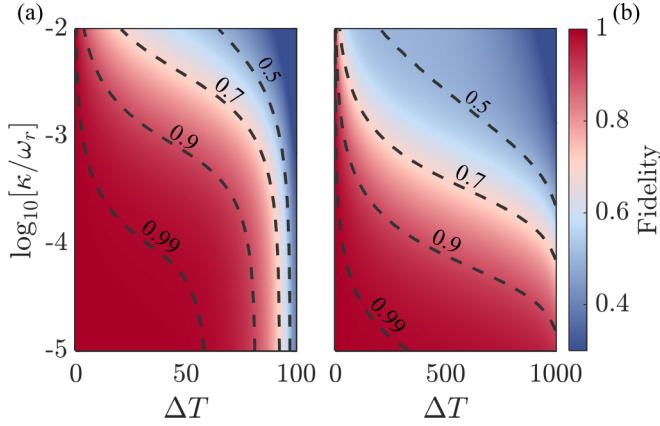


FIG. 9. Fidelity of the final state in the presence of photon loss, as a function of the total evolution time ΔT and the decay rate κ/ω_r . The contour plots compare the performance (a) without CD driving and (b) with CD driving. Each subfigure contains two panels corresponding to different maximum evolution times: $\omega_r \Delta T_{\max} = 10^2$ (left) and $\omega_r \Delta T_{\max} = 10^3$ (right).

of decay rates and evolution times. The right panel shows that even for a much longer total evolution time of $\Delta T_{\max} = 10^3/\omega_r$, high fidelity (>0.9) is preserved for decay rates up to $\log_{10}(\kappa/\omega_r) \approx -4.8$. This result powerfully illustrates the central benefit of our protocol. By enabling high-fidelity state preparation on timescales much shorter than the system's decoherence time, our method significantly mitigates the detrimental effects of environmental noise. The speed afforded by CD driving thus directly translates into a powerful robustness against decoherence.

VII. CONCLUSION

In summary, we have proposed and analyzed a robust and physically realizable protocol based on the CD driving for the high-fidelity preparation of critical ground states. The central result is that this approach enables rapid state preparation on experimentally feasible timescales, overcoming the limitations of conventional adiabatic methods near a quantum critical point. A key aspect of this work is the analysis under a physical constraint on the maximum CD driving strength. We have shown that even when the drive is limited to a realistic perturbative regime, for instance, a maximum strength below $2\pi \times 7.5$ MHz for a circuit QED implementation ($\zeta < 10^{-3}$), the protocol yields a significant enhancement in metrological precision. On timescales of tens of nanoseconds ($\Delta T < 10^3/\omega_r$), this results in a QFI enhancement factor exceeding 10^3 and an SNR enhancement of over 10^6 compared to the nonassisted protocol.

These findings establish a practical pathway for harnessing quantum criticality. The protocol's speed provides an intrinsic robustness against environmental decoherence, a notable advantage over slow adiabatic approaches. Furthermore, our analysis of a circuit QED implementation confirms the protocol's feasibility and its robustness against primary non-idealities such as the Kerr effect. This work opens several promising directions for future investigation. Future work could explore the application of this CD protocol to other

critical many-body systems. Another promising direction is to combine the analytical strength of our CD approach with the numerical power of optimal control and machine learning techniques to further enhance performance and robustness in complex experimental environments.

ACKNOWLEDGMENTS

Y.-H.C. was supported by the National Natural Science Foundation of China under Grants No. 12304390 and No. 12574386, the Fujian 100 Talents Program, and the Fujian Minjiang Scholar Program. Y.X. was supported by the National Natural Science Foundation of China under Grant No. 62471143, the Key Program of National Natural Science Foundation of Fujian Province under Grant No. 2024J02008.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: DERIVATION OF THE TIME-DEPENDENT EFFECTIVE HAMILTONIAN

In this Appendix, we provide a detailed derivation of the effective time-dependent Hamiltonian, including the leading-order nonadiabatic corrections, using a time-dependent Schrieffer-Wolff transformation. This procedure rigorously justifies the form of the Hamiltonian used in our main analysis.

Our starting point is the system Hamiltonian in a corotating frame where the two-photon drive is time-independent. The Hamiltonian is given by

$$H(t) = H_0 + V(t), \quad (\text{A1})$$

$$H_0 = \delta a^\dagger a + \frac{\Omega'}{2} \sigma_z + \frac{G}{2} (a^{\dagger 2} + a^2), \quad (\text{A2})$$

$$V(t) = g(t)(a^\dagger \sigma_- + a \sigma_+), \quad (\text{A3})$$

where δ and Ω' are the cavity and qubit detunings, respectively. The interaction term $V(t)$ contains the time-dependent coupling strength $g(t)$, which is assumed to vary slowly compared to the physical detuning $\Delta = \Omega' - \delta$.

The goal of the SW transformation is to find a unitary operator $U(t) = e^{iS(t)}$ that systematically eliminates the off-diagonal interaction $V(t)$. The transformed Hamiltonian is given by $H_{\text{eff}}(t) = U^\dagger(t)H(t)U(t) - i\hbar U^\dagger(t)\dot{U}(t)$. Following the standard formalism for time-dependent SW transformations [68,69], we expand the generator $S(t)$ in orders of g/Δ . The first- and second-order generators, chosen to cancel the original interaction and the leading-order nonadiabatic off-diagonal term, are

$$S_1(t) = -i \frac{g(t)}{\hbar \Delta} (a^\dagger \sigma_- - a \sigma_+), \quad (\text{A4})$$

$$S_2(t) = \frac{\dot{g}(t)}{\hbar \Delta^2} (a^\dagger \sigma_- + a \sigma_+). \quad (\text{A5})$$

Carrying out the transformation up to the third order in g/Δ yields the key diagonal corrections. The second-order

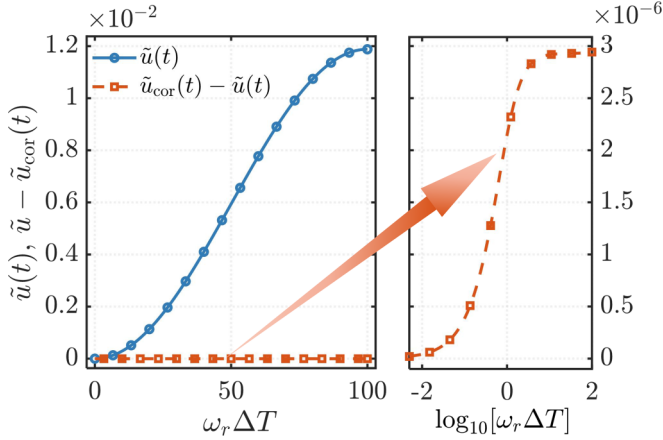


FIG. 10. Comparison of the primary dispersive coupling term (from ac Stark shift) and the nonadiabatic correction term for an evolution time of $\Delta T = 100/\omega_r$, where the couplings are rescaled as $\tilde{u}_{\text{cor}} = \sqrt{u_{\text{cor}}}e^i$. The left panel shows $\tilde{u}(t)$ and the difference $\tilde{u}_{\text{cor}}(t) - \tilde{u}(t)$. The right panel magnifies this difference, confirming it is $\sim 10^{-4}$ times the primary term.

correction is the well-known ac Stark shift:

$$H_{\text{eff,D}}^{(2)}(t) = -\frac{g(t)^2}{2\Delta} [I + (2a^\dagger a + 1)\sigma_z]. \quad (\text{A6})$$

The third-order correction arises entirely from the nonadiabatic dynamics:

$$H_{\text{eff,D}}^{(3)}(t) = \frac{g(t)\ddot{g}(t) - \dot{g}(t)^2}{2\Delta^3} [I + (2a^\dagger a + 1)\sigma_z]. \quad (\text{A7})$$

The total effective Hamiltonian is the sum of the noninteracting part and these diagonal corrections, $H_{\text{eff}}(t) \approx H_0 + H_{\text{eff,D}}^{(2)}(t) + H_{\text{eff,D}}^{(3)}(t)$. After collecting terms proportional to $a^\dagger a \sigma_z$ and disregarding global energy shifts, we arrive at the final working Hamiltonian:

$$H_{r,\text{cor}}(t) \approx \delta a^\dagger a + \frac{\Omega'(t)}{2} \sigma_z + u_{\text{cor}}(t) a^\dagger a \sigma_z + \frac{G}{2} (a^{\dagger 2} + a^2). \quad (\text{A8})$$

Here, the corrected dispersive coupling strength incorporates both corrections:

$$u_{\text{cor}}(t) = \frac{g(t)^2}{\Delta} - \frac{g(t)\ddot{g}(t) - \dot{g}(t)^2}{\Delta^3}. \quad (\text{A9})$$

The qubit frequency is also renormalized to $\Omega'(t)$. This expression for $u_{\text{cor}}(t)$ is used in the main text to validate the robustness of our protocol. To justify the use of an idealized Hamiltonian (which neglects the nonadiabatic term) for our protocol design, we perform a quantitative analysis of these corrections. As shown in Fig. 10, the nonadiabatic correction term is several orders of magnitude smaller than the primary ac Stark term throughout the evolution. This confirms that the nonadiabatic effects constitute a minor perturbation, and the idealized model, which includes only the ac Stark term, serves as a highly accurate foundation for both the system analysis and the control protocol design in the main text.

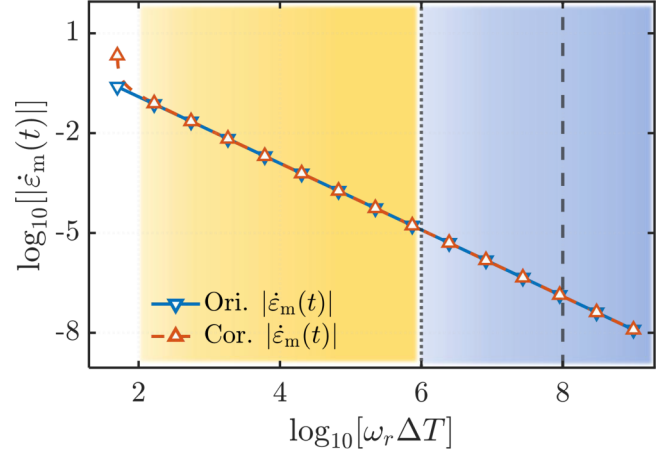


FIG. 11. Peak CD driving strength $|\dot{\epsilon}(t)|_{\text{max}}$ as a function of total evolution time ΔT for a target state at $\tilde{u}_f = 0.999\tilde{u}_c$. The yellow shaded region indicates the diabatic regime, where the nonassisted protocol fails to prepare the target state with high fidelity. The blue shaded region represents the regime where adiabatic preparation becomes progressively feasible. The vertical dotted and dashed lines mark the minimum time required for the nonassisted protocol to achieve high fidelity for targets at $\tilde{u}_f = 0.99\tilde{u}_c$ and $\tilde{u}_f = 0.999\tilde{u}_c$, respectively. The results for the original (Ori.) and corrected (Cor.) models are nearly identical.

APPENDIX B: PHYSICAL FEASIBILITY OF THE COUNTERDIABATIC DRIVE

In this Appendix, we provide a detailed numerical analysis of the required CD driving strength, $|\dot{\epsilon}(t)|$, to validate the physical feasibility of our protocol. A crucial aspect of any shortcut-to-adiabaticity protocol is ensuring that the required control fields remain within experimentally accessible limits. We investigate the trade-offs between protocol duration, proximity to the critical point, and the peak driving strength.

First, we analyze the peak driving strength required to reach a state very close to the critical point ($\tilde{u}_f = 0.999\tilde{u}_c$) as a function of the total protocol time ΔT . As shown in Fig. 11, a clear trade-off exists: Shorter preparation times necessitate stronger drives. However, the required strength decreases rapidly with time. For an evolution time of $\Delta T \approx 10^3/\omega_r$, the peak driving strength is already 2 orders of magnitude smaller than the resonator frequency ω_r ($|\dot{\epsilon}(t)|_{\text{m}} < 10^{-2}\omega_r$). This is well within the perturbative regime, ensuring the validity of our theoretical model and the experimental feasibility. Notably, even at this timescale, the nonassisted adiabatic protocol completely fails to prepare the target state, as indicated by the yellow shaded “diabatic regime.” This demonstrates that our CD protocol can achieve what is impossible for the adiabatic counterpart within the same timeframe. We also observe that the nonadiabatic corrections to the Hamiltonian have a negligible impact on the required driving strength, as the curves for the original and corrected models are perfectly overlaid.

Next, we explore these trade-offs from different perspectives in Fig. 12. Figure 12(a) shows the temporal profile of the CD drive for a fixed, short duration of $\Delta T = 100/\omega_r$. As the target state approaches the critical point (from $0.9\tilde{u}_c$ to $0.999\tilde{u}_c$), the required peak strength increases. For a target

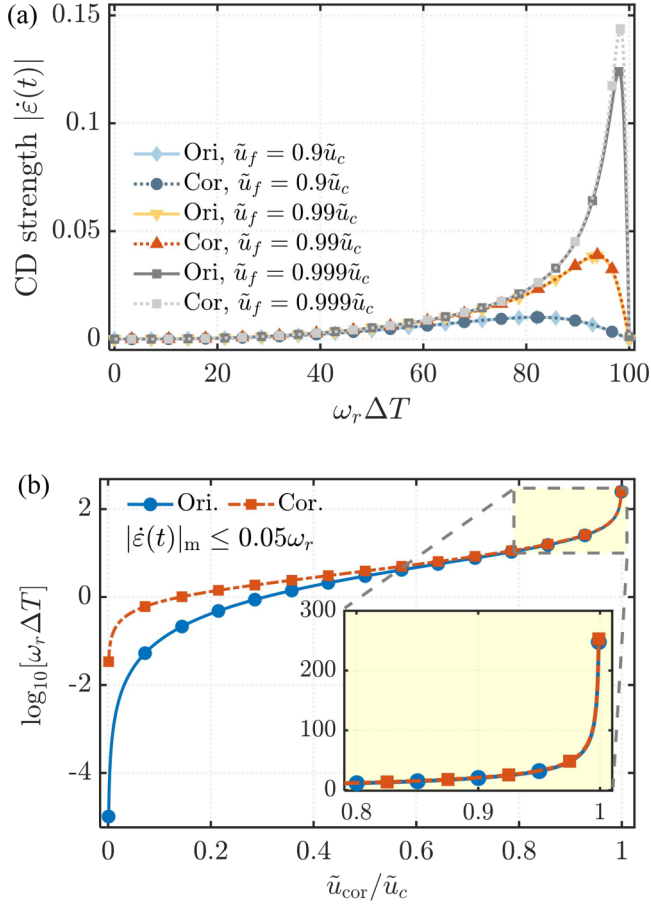


FIG. 12. Trade-offs in the CD protocol. (a) Instantaneous CD driving strength $|\dot{\epsilon}(t)|$ as a function of time for a fixed total duration $\Delta T = 100/\omega_r$. Different curves correspond to different target proximities to the critical point ($\tilde{u}_f/\tilde{u}_c$). (b) Minimum evolution time required to reach a target state, subject to a constraint on the maximum driving strength, $|\dot{\epsilon}(t)|_m \leq 0.05\omega_r$.

at $0.99\tilde{u}_c$, which already provides substantial metrological resources as shown in the main text, the peak strength is only about $0.05\omega_r$. This confirms that significant quantum enhancement can be achieved with modest control fields.

Figure 12(b) investigates the protocol from a complementary viewpoint by showing the minimum evolution time required to prepare a target state under a fixed maximum driving strength. For a limit of $|\dot{\epsilon}|_{\text{max}} \leq 0.05\omega_r$, a high-fidelity preparation of a state very close to the critical point ($\tilde{u}_f = 0.999\tilde{u}_c$) can be achieved in a time of $\Delta T \approx 300/\omega_r$. At this timescale, the nonassisted protocol is operating far from the adiabatic limit and is ineffective. This result highlights the substantial performance advantage of the CD-assisted method.

In summary, this detailed analysis confirms that our CD-assisted protocol is not only theoretically powerful but also physically feasible. For experimentally relevant parameters, the required driving fields are well behaved and modest in strength, allowing for the rapid preparation of critical states in a timescale where traditional adiabatic methods fail.

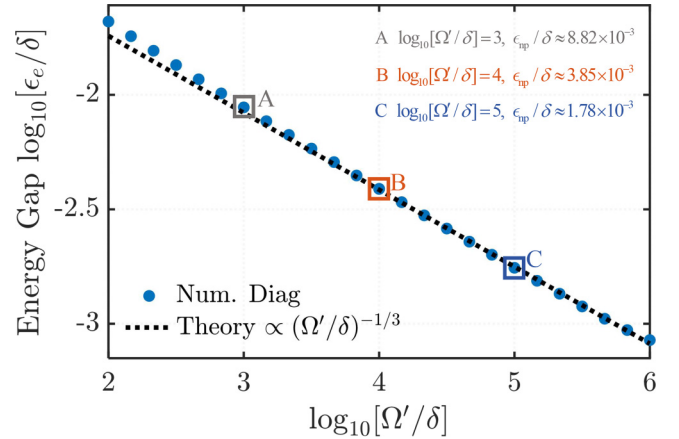


FIG. 13. Energy gap at the critical point as a function of the system's effective size, quantified by the frequency ratio Ω'/δ . The blue dots represent results from numerical diagonalization, which are in excellent agreement with the theoretical scaling prediction $\epsilon_e \propto (\Omega'/\delta)^{-1/3}$ (black dotted line). The labels A, B, and C highlight specific data points discussed in the text.

APPENDIX C: FINITE-SIZE EFFECT ON THE CRITICAL ENERGY GAP

The divergence of the CD driving strength at the critical point, as discussed in the main text, is a feature of the idealized model that assumes a thermodynamic limit where the ratio of energy scales $\Omega'/\delta \rightarrow \infty$. In any realistic physical system, however, this ratio is finite. This finite size of the system introduces a fundamental cutoff that prevents the energy gap from strictly closing, thereby regularizing the divergence of the CD drive. In this Appendix, we analyze this finite-size effect and connect it to the parameter choices in our main analysis.

Our theoretical model predicts an energy gap between the ground and first excited states of the form

$$\epsilon_e = \sqrt{\Delta_r(\Delta_r - \tilde{u}^2)}, \quad (\text{C1})$$

which vanishes at the critical point $\tilde{u}_c = \sqrt{\Delta_r}$. However, the derivation of this effective model relies on a series of approximations that are valid in the limit $\Omega'/\delta \rightarrow \infty$.

To investigate the behavior for finite Ω'/δ , we have performed numerical diagonalization of the full Hamiltonian. The results are presented in Fig. 13, where we plot the minimum energy gap at the critical point ($\tilde{u} = \tilde{u}_c$) as a function of the frequency ratio Ω'/δ . The numerical data (blue dots) show excellent agreement with a theoretical scaling of $\epsilon_e \propto (\Omega'/\delta)^{-1/3}$. This confirms that for any finite ratio, the energy gap remains nonzero. This finite gap can be mapped to an effective proximity to the critical point in our idealized model. We define an effective proximity parameter η such that the finite-size gap ϵ_{num} is equivalent to the gap of the ideal model at a coupling strength of $\tilde{u} = \eta\tilde{u}_c$. This yields the relation

$$\epsilon_{\text{num}} = \Delta_r \sqrt{1 - \eta^2}. \quad (\text{C2})$$

For a typical experimental ratio of $\Omega'/\delta = 10^4$ (Point B in Fig. 13), our numerical results give a gap of $\epsilon_{\text{num}}/\delta \approx 3.85 \times 10^{-3}$. Using the parameters from our main text ($G = 0.99\delta$,

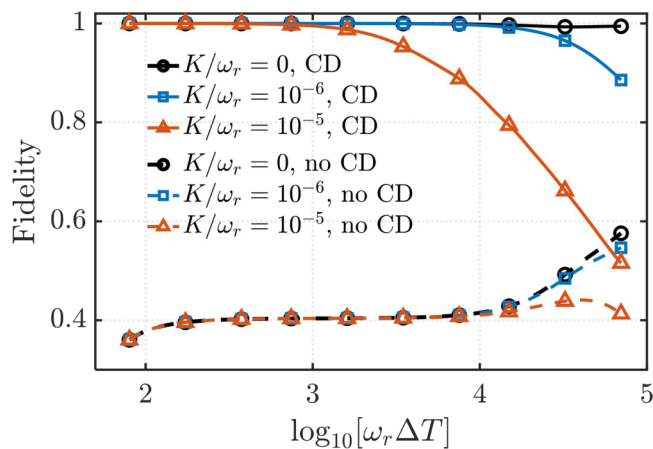


FIG. 14. State fidelity vs evolution time ΔT in the presence of Kerr nonlinearity. The performances of the CD-assisted (solid lines) and nonassisted (dashed lines) protocols are compared for Kerr strengths of $K = 0$ (black), $K/\omega_r = 10^{-6}$ (blue), and $K/\omega_r = 10^{-5}$ (orange).

which gives $\Delta_r \approx 0.1411\delta$), we can solve for the effective proximity, yielding

$$\eta = \sqrt{1 - (\epsilon_{\text{num}}/\Delta_r)^2} \approx 0.9996. \quad (\text{C3})$$

This analysis provides two crucial insights. First, it confirms that the requirement for an infinite CD driving strength is an artifact of the thermodynamic limit. Any physically realistic system possesses a nonvanishing energy gap that inherently bounds the required control field. Second, it provides a strong justification for our choice of parameters in the main text. A realistic system with an energy scale ratio of 10^4 is physically equivalent to our ideal model being driven to a proximity of $\eta \approx 0.9996$ to the critical point. Therefore, our analysis using target proximities of 0.99 and 0.999 is not only a conservative choice but also directly relevant to physically achievable experimental conditions.

APPENDIX D: ROBUSTNESS AGAINST KERR NONLINEARITY

This Appendix provides a detailed quantitative analysis of the protocol robustness against the residual Kerr nonlinearity, which is the dominant nonideality in the circuit QED implementation discussed in the main text.

The numerical simulations are governed by the effective Hamiltonian derived in Sec. V [Eq. (33)], which includes the intrinsic Kerr nonlinearity term, $-K(a^\dagger a)^2$. The time dependence is introduced through the coupling strength $g(t)$,

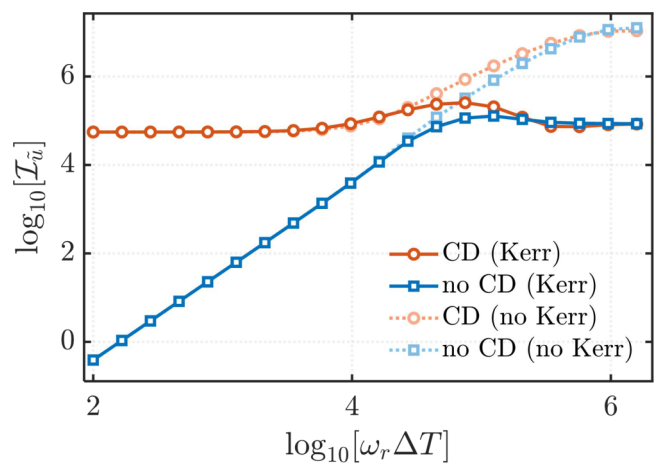


FIG. 15. QFI vs evolution time ΔT . The plot compares the performance of the CD-assisted and nonassisted protocols, both with (solid lines, $K/\omega_r = 10^{-5}$) and without (dotted lines, $K = 0$) Kerr nonlinearity.

which follows the pulse shape defined in Eq. (10). Based on the experimental parameters summarized in Tables I and II, relative Kerr strengths in the range of $K/\omega_r \sim 10^{-5}$ to 10^{-6} are selected for the simulations.

Figure 14 shows the state fidelity as a function of evolution time for different Kerr strengths. Even with a significant Kerr strength ($K/\omega_r = 10^{-5}$), the CD-assisted protocol maintains high fidelity in the short-time regime, where the nonassisted protocol fails. While the Kerr term introduces a mismatch that degrades fidelity at long times, the advantage of the CD protocol in the practical finite-time regime remains.

The robustness of the protocol is essential for metrological applications. Figure 15 analyzes the impact of the Kerr effect on the quantum Fisher information. The results reveal a nuanced picture: In the short-time regime, the high QFI achieved by the CD protocol is largely unaffected by the Kerr nonlinearity. At longer evolution times, a performance degradation becomes apparent due to the accumulating effect of the mismatch caused by the Kerr term. Nevertheless, the key advantage of the protocol, which is to achieve high QFI on rapid timescales, is robustly preserved.

In conclusion, the numerical simulations confirm that the CD-assisted protocol is robust against realistic residual Kerr nonlinearities. While the Kerr effect introduces performance degradation at long evolution times, the essential advantage of the protocol in the experimentally relevant short-time regime is maintained. This provides strong support for the feasibility of the protocol in practical circuit QED implementations.

- [1] D. S. Simon, G. Jaeger, and A. V. Sergienko, *Quantum Metrology, Imaging, and Communication* (Springer International Publishing, Cham, Switzerland, 2016).
- [2] J. P. Dowling, A. G. J. MacFarlane, and G. J. Milburn, Quantum technology: The second quantum revolution, *Philos. Trans. R. Soc. London Ser. A* **361**, 1655 (2003).

- [3] A. Acín, I. Bloch, H. Buhrman, T. Calarco, C. Eichler, J. Eisert, D. Esteve, N. Gisin, S. J. Glaser, F. Jelezko, S. Kuhr, M. Lewenstein, M. F. Riedel, P. O. Schmidt, R. Thew, A. Wallraff, I. Walmsley, and F. K. Wilhelm, The quantum technologies roadmap: A European community view, *New J. Phys.* **20**, 080201 (2018).

- [4] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum metrology, *Phys. Rev. Lett.* **96**, 010401 (2006).
- [5] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, *Nat. Photonics* **5**, 222 (2011).
- [6] R. Schnabel, N. Mavalvala, D. E. McClelland, and P. K. Lam, Quantum metrology for gravitational wave astronomy, *Nat. Commun.* **1**, 121 (2010).
- [7] H. Katori, Optical lattice clocks and quantum metrology, *Nat. Photonics* **5**, 203 (2011).
- [8] M. Skotiniotis, P. Sekatski, and W. Dür, Quantum metrology for the Ising Hamiltonian with transverse magnetic field, *New J. Phys.* **17**, 073032 (2015).
- [9] B. M. Escher, R. L. de Matos Filho, and L. Davidovich, General framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology, *Nat. Phys.* **7**, 406 (2011).
- [10] R. Chaves, J. B. Brask, M. Markiewicz, J. Kołodyński, and A. Acín, Noisy metrology beyond the standard quantum limit, *Phys. Rev. Lett.* **111**, 120401 (2013).
- [11] P. Zanardi, M. G. A. Paris, and L. Campos Venuti, Quantum criticality as a resource for quantum estimation, *Phys. Rev. A* **78**, 042105 (2008).
- [12] D. Braun, G. Adesso, F. Benatti, R. Floreanini, U. Marzolino, M. W. Mitchell, and S. Pirandola, Quantum-enhanced measurements without entanglement, *Rev. Mod. Phys.* **90**, 035006 (2018).
- [13] C. Invernizzi, M. Korbman, L. Campos Venuti, and M. G. A. Paris, Optimal quantum estimation in spin systems at criticality, *Phys. Rev. A* **78**, 042106 (2008).
- [14] P. A. Ivanov and D. Porras, Adiabatic quantum metrology with strongly correlated quantum optical systems, *Phys. Rev. A* **88**, 023803 (2013).
- [15] M. Tsang, Quantum transition-edge detectors, *Phys. Rev. A* **88**, 021801 (2013).
- [16] G. Salvatori, A. Mandarino, and M. G. A. Paris, Quantum metrology in Lipkin-Meshkov-Glick critical systems, *Phys. Rev. A* **90**, 022111 (2014).
- [17] K. Macieszczak, M. Guță, I. Lesanovsky, and J. P. Garrahan, Dynamical phase transitions as a resource for quantum enhanced metrology, *Phys. Rev. A* **93**, 022103 (2016).
- [18] S. Fernández-Lorenzo and D. Porras, Quantum sensing close to a dissipative phase transition: Symmetry breaking and criticality as metrological resources, *Phys. Rev. A* **96**, 013817 (2017).
- [19] T. L. Heugel, M. Biondi, O. Zilberberg, and R. Chitra, Quantum transducer using a parametric driven-dissipative phase transition, *Phys. Rev. Lett.* **123**, 173601 (2019).
- [20] L. Garbe, M. Bina, A. Keller, M. G. A. Paris, and S. Felicetti, Critical quantum metrology with a finite-component quantum phase transition, *Phys. Rev. Lett.* **124**, 120504 (2020).
- [21] Y. Chu, S. Zhang, B. Yu, and J. Cai, Dynamic framework for criticality-enhanced quantum sensing, *Phys. Rev. Lett.* **126**, 010502 (2021).
- [22] Y.-H. Chen, Y. Qiu, A. Miranowicz, N. Lambert, W. Qin, R. Stassi, Y. Xia, S.-B. Zheng, and F. Nori, Sudden change of the photon output field marks phase transitions in the quantum Rabi model, *Commun. Phys.* **7**, 12 (2024).
- [23] X. Zhu, J.-H. Lü, W. Ning, F. Wu, L.-T. Shen, Z.-B. Yang, and S.-B. Zheng, Criticality-enhanced quantum sensing in the anisotropic quantum Rabi model, *Sci. China Phys. Mech. Astron.* **66**, 250313 (2023).
- [24] X. Zhu, J.-H. Lü, W. Ning, L.-T. Shen, F. Wu, and Z.-B. Yang, Quantum geometric tensor and critical metrology in the anisotropic Dicke model, *Phys. Rev. A* **109**, 052621 (2024).
- [25] P. Zanardi and M. Rasetti, Holonomic quantum computation, *Phys. Lett. A* **264**, 94 (1999).
- [26] N. V. Vitanov, T. Halfmann, B. W. Shore, and K. Bergmann, Laser-induced population transfer by adiabatic passage techniques, *Annu. Rev. Phys. Chem.* **52**, 763 (2001).
- [27] K. Kim, M.-S. Chang, S. Korenblit, R. Islam, E. E. Edwards, J. K. Freericks, G.-D. Lin, L.-M. Duan, and C. Monroe, Quantum simulation of frustrated Ising spins with trapped ions, *Nature (London)* **465**, 590 (2010).
- [28] H. Landa, M. Schiró, and G. Misguich, Multistability of driven-dissipative quantum spins, *Phys. Rev. Lett.* **124**, 043601 (2020).
- [29] P. Brookes, G. Tancredi, A. D. Patterson, J. Rahamim, M. Esposito, T. K. Mavrogordatos, P. J. Leek, E. Ginossar, and M. H. Szymanska, Critical slowing down in circuit quantum electrodynamics, *Sci. Adv.* **7**, eabe9492 (2021).
- [30] G. Beaulieu, F. Minganti, S. Frasca, V. Savona, S. Felicetti, R. Di Candia, and P. Scarlino, Observation of first- and second-order dissipative phase transitions in a two-photon driven Kerr resonator, *Nat. Commun.* **16**, 1954 (2025).
- [31] S. L. Sondhi, S. M. Girvin, J. P. Carini, and D. Shahar, Continuous quantum phase transitions, *Rev. Mod. Phys.* **69**, 315 (1997).
- [32] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Colloquium: Nonequilibrium dynamics of closed interacting quantum systems, *Rev. Mod. Phys.* **83**, 863 (2011).
- [33] K. Gietka, Squeezing by critical speeding up: Applications in quantum metrology, *Phys. Rev. A* **105**, 042620 (2022).
- [34] D. Guéry-Odelin, A. Ruschhaupt, A. Kiely, E. Torrontegui, S. Martínez-Garaot, and J. G. Muga, Shortcuts to adiabaticity: Concepts, methods, and applications, *Rev. Mod. Phys.* **91**, 045001 (2019).
- [35] M. V. Berry, Transitionless quantum driving, *J. Phys. A: Math. Theor.* **42**, 365303 (2009).
- [36] M. Demirplak and S. A. Rice, Adiabatic population transfer with control fields, *J. Phys. Chem. A* **107**, 9937 (2003).
- [37] A. del Campo, Shortcuts to adiabaticity by counterdiabatic driving, *Phys. Rev. Lett.* **111**, 100502 (2013).
- [38] J. Kölbl, A. Barfuss, M. S. Kasperczyk, L. Thiel, A. A. Clerk, H. Ribeiro, and P. Maletinsky, Initialization of single spin dressed states using shortcuts to adiabaticity, *Phys. Rev. Lett.* **122**, 090502 (2019).
- [39] A. C. Santos, A. Nicotina, A. M. Souza, R. S. Sarthour, I. S. Oliveira, and M. S. Sarandy, Optimizing NMR quantum information processing via generalized transitionless quantum driving, *Europhys. Lett.* **129**, 30008 (2020).
- [40] T. Yan, B.-J. Liu, K. Xu, C. Song, S. Liu, Z. Zhang, H. Deng, Z. Yan, H. Rong, K. Huang, M.-H. Yung, Y. Chen, and D. Yu, Experimental realization of nonadiabatic shortcut to non-Abelian geometric gates, *Phys. Rev. Lett.* **122**, 080501 (2019).
- [41] N. N. Hegade, X. Chen, and E. Solano, Digitized counterdiabatic quantum optimization, *Phys. Rev. Res.* **4**, L042030 (2022).
- [42] S. Puri, S. Boutin, and A. Blais, Engineering the quantum states of light in a Kerr-nonlinear resonator by two-photon driving, *npj Quantum Inf.* **3**, 18 (2017).
- [43] J.-W. Yu, K.-X. Yan, Y. Qiu, J. Song, Y.-H. Chen, and Y. Xia, Efficient generation of arbitrary photon-number-squeezed light via shortcuts to adiabaticity, *Opt. Express* **33**, 26356 (2025).

- [44] Z. Wei and M. Ying, Quantum adiabatic computation and adiabatic conditions, *Phys. Rev. A* **76**, 024304 (2007).
- [45] H. Pu, P. Maenner, W. Zhang, and H. Y. Ling, Adiabatic condition for nonlinear systems, *Phys. Rev. Lett.* **98**, 050406 (2007).
- [46] S. Deffner, C. Jarzynski, and A. del Campo, Classical and quantum shortcuts to adiabaticity for scale-invariant driving, *Phys. Rev. X* **4**, 021013 (2014).
- [47] S. Y. Chen, Y. N. Zhang, J. Yang, H. D. Liu, and H. Y. Sun, Quantum-classical correspondence for adiabatic shortcut in two- and three-level atoms, *Int. J. Theor. Phys.* **58**, 836 (2019).
- [48] X. Yang, Y. Xie, J. Zhang, M. Zhang, C. Wu, W. Wu, T. Chen, and P.-X. Chen, Manipulation of phonon states in ion traps by shortcuts to adiabaticity, *New J. Phys.* **25**, 083050 (2023).
- [49] S. Ashhab and F. Nori, Qubit-oscillator systems in the ultrastrong-coupling regime and their potential for preparing nonclassical states, *Phys. Rev. A* **81**, 042311 (2010).
- [50] D. Ballester, G. Romero, J. J. García-Ripoll, F. Deppe, and E. Solano, Quantum simulation of the ultrastrong-coupling dynamics in circuit quantum electrodynamics, *Phys. Rev. X* **2**, 021007 (2012).
- [51] L. Garziano, R. Stassi, A. Ridolfo, O. Di Stefano, and S. Savasta, Vacuum-induced symmetry breaking in a superconducting quantum circuit, *Phys. Rev. A* **90**, 043817 (2014).
- [52] S. L. Braunstein and C. M. Caves, Statistical distance and the geometry of quantum states, *Phys. Rev. Lett.* **72**, 3439 (1994).
- [53] R. Di Candia, F. Minganti, K. V. Petrovnin, G. S. Paraoanu, and S. Felicetti, Critical parametric quantum sensing, *npj Quantum Inf.* **9**, 23 (2023).
- [54] L.-T. Shen, C.-Q. Tang, Z. Shi, H. Wu, Z.-B. Yang, and S.-B. Zheng, Squeezed-light-induced quantum phase transition in the Jaynes-Cummings model, *Phys. Rev. A* **106**, 023705 (2022).
- [55] C.-S. Hu, Z.-B. Yang, H. Wu, Y. Li, and S.-B. Zheng, Twofold mechanical squeezing in a cavity optomechanical system, *Phys. Rev. A* **98**, 023807 (2018).
- [56] C.-S. Hu, Z.-Q. Liu, Y. Liu, L.-T. Shen, H. Wu, and S.-B. Zheng, Entanglement beating in a cavity optomechanical system under two-field driving, *Phys. Rev. A* **101**, 033810 (2020).
- [57] C.-S. Hu, X.-Y. Lin, L.-T. Shen, W.-J. Su, Y.-K. Jiang, H. Wu, and S.-B. Zheng, Improving macroscopic entanglement with nonlocal mechanical squeezing, *Opt. Express* **28**, 1492 (2020).
- [58] J. R. Schrieffer and P. A. Wolff, Relation between the Anderson and Kondo Hamiltonians, *Phys. Rev.* **149**, 491 (1966).
- [59] A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf, Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation, *Phys. Rev. A* **69**, 062320 (2004).
- [60] L.-T. Shen, J.-W. Yang, Z.-R. Zhong, Z.-B. Yang, and S.-B. Zheng, Quantum phase transition and quench dynamics in the two-mode Rabi model, *Phys. Rev. A* **104**, 063703 (2021).
- [61] U. Alushi, A. Coppo, V. Brosco, R. Di Candia, and S. Felicetti, Collective quantum enhancement in critical quantum sensing, *Commun. Phys.* **8**, 74 (2025).
- [62] X. Xie, C. Liu, L.-L. Jiang, and J.-F. Huang, Controlling the superradiant phase transition in the quantum Rabi model, *Phys. Rev. A* **111**, 023708 (2025).
- [63] M.-J. Hwang and M. B. Plenio, Quantum phase transition in the finite Jaynes-Cummings lattice systems, *Phys. Rev. Lett.* **117**, 123602 (2016).
- [64] M.-J. Hwang, R. Puebla, and M. B. Plenio, Quantum phase transition and universal dynamics in the Rabi model, *Phys. Rev. Lett.* **115**, 180404 (2015).
- [65] T. Kato, On the adiabatic theorem of quantum mechanics, *J. Phys. Soc. Jpn.* **5**, 435 (1950).
- [66] K.-P. Marzlin and B. C. Sanders, Inconsistency in the application of the adiabatic theorem, *Phys. Rev. Lett.* **93**, 160408 (2004).
- [67] S. Jansen, M.-B. Ruskai, and R. Seiler, Bounds for the adiabatic approximation with applications to quantum computation, *J. Math. Phys.* **48**, 102111 (2007).
- [68] M. Malekakhlagh, E. Magesan, and D. C. McKay, First-principles analysis of cross-resonance gate operation, *Phys. Rev. A* **102**, 042605 (2020).
- [69] Z. Xiao, E. Doucet, T. Noh, L. Ranzani, R. W. Simmonds, L. C. G. Govia, and A. Kamal, Perturbative diagonalization for time-dependent strong interactions, *Phys. Rev. Appl.* **18**, 024009 (2022).
- [70] Y.-H. Chen, Y. Xia, Q.-C. Wu, B.-H. Huang, and J. Song, Method for constructing shortcuts to adiabaticity by a substitute of counteradiabatic driving terms, *Phys. Rev. A* **93**, 052109 (2016).
- [71] K.-X. Yan, Y. Qiu, Y. Xiao, Y.-H. Chen, and Y. Xia, Generating three-photon Rabi oscillations without a large-detuning condition, *Phys. Rev. A* **110**, 043711 (2024).
- [72] K.-X. Yan, Y. Qiu, Y. Xiao, J. Song, Y.-H. Chen, and Y. Xia, Spontaneous emission in Casimir-Rabi oscillations through a weak optomechanical coupling, *Opt. Express* **33**, 39283 (2025).
- [73] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum-enhanced measurements: Beating the standard quantum limit, *Science* **306**, 1330 (2004).
- [74] L. Garbe, O. Abah, S. Felicetti, and R. Puebla, Critical quantum metrology with fully-connected models: From Heisenberg to Kibble-Zurek scaling, *Quantum Sci. Technol.* **7**, 035010 (2022).
- [75] U. Alushi, W. Górecki, S. Felicetti, and R. Di Candia, Optimality and noise resilience of critical quantum sensing, *Phys. Rev. Lett.* **133**, 040801 (2024).
- [76] W. Górecki, F. Albarelli, S. Felicetti, R. Di Candia, and L. Maccone, Interplay between time and energy in bosonic noisy quantum metrology, *PRX Quantum* **6**, 020351 (2025).
- [77] K. Gietka, F. Metz, T. Keller, and J. Li, Adiabatic critical quantum metrology cannot reach the Heisenberg limit even when shortcuts to adiabaticity are applied, *Quantum* **5**, 489 (2021).
- [78] M. P. da Silva, D. Bozyigit, A. Wallraff, and A. Blais, Schemes for the observation of photon correlation functions in circuit QED with linear detectors, *Phys. Rev. A* **82**, 043804 (2010).
- [79] R. Di Candia, E. P. Menzel, L. Zhong, F. Deppe, A. Marx, R. Gross, and E. Solano, Dual-path methods for propagating quantum microwaves, *New J. Phys.* **16**, 015001 (2014).
- [80] G. Tóth and I. Apellaniz, Quantum metrology from a quantum information science perspective, *J. Phys. A: Math. Theor.* **47**, 424006 (2014).
- [81] A. Blais, A. L. Grimsmo, S. M. Girvin, and A. Wallraff, Circuit quantum electrodynamics, *Rev. Mod. Phys.* **93**, 025005 (2021).
- [82] J. Koch, T. M. Yu, J. Gambetta, A. A. Houck, D. I. Schuster, J. Majer, A. Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, Charge-insensitive qubit design derived from the Cooper pair box, *Phys. Rev. A* **76**, 042319 (2007).

- [83] J. Q. You, X. Hu, S. Ashhab, and F. Nori, Low-decoherence flux qubit, *Phys. Rev. B* **75**, 140515 (2007).
- [84] Z. Wang, M. Pechal, E. Alex Wollack, P. Arrangoiz-Arriola, M. Gao, N. R. Lee, and A. H. Safavi-Naeini, Quantum dynamics of a few-photon parametric oscillator, *Phys. Rev. X* **9**, 021049 (2019).
- [85] Z. Leghtas, S. Touzard, I. M. Pop, A. Kou, B. Vlastakis, A. Petrenko, K. M. Sliwa, A. Narla, S. Shankar, M. J. Hatridge, M. Reagor, L. Frunzio, R. J. Schoelkopf, M. Mirrahimi, and M. H. Devoret, Confining the state of light to a quantum manifold by engineered two-photon loss, *Science* **347**, 853 (2015).
- [86] N. E. Frattini, U. Vool, S. Shankar, A. Narla, K. M. Sliwa, and M. H. Devoret, 3-wave mixing Josephson dipole element, *Appl. Phys. Lett.* **110**, 222603 (2017).
- [87] S. C. Burd, R. Srinivas, H. M. Knaack, W. Ge, A. C. Wilson, D. J. Wineland, D. Leibfried, J. J. Bollinger, D. T. C. Allcock, and D. H. Slichter, Quantum amplification of boson-mediated interactions, *Nat. Phys.* **17**, 898 (2021).
- [88] W. Wustmann and V. Shumeiko, Parametric resonance in tunable superconducting cavities, *Phys. Rev. B* **87**, 184501 (2013).
- [89] P. Krantz, A. Bengtsson, M. Simoen, S. Gustavsson, V. Shumeiko, W. D. Oliver, C. M. Wilson, P. Delsing, and J. Bylander, Single-shot read-out of a superconducting qubit using a Josephson parametric oscillator, *Nat. Commun.* **7**, 11417 (2016).
- [90] S.-W. Xu, Z.-Y. Zhang, J.-T. Ye, Z.-Z. Zhang, Y.-Y. Guo, K.-X. Yan, Y.-H. Chen, and Y. Xia, Optimal robust control of cat-state qubits against parameter imperfections, *Opt. Express* **33**, 40755 (2025).
- [91] Y.-H. Chen, Z.-C. Shi, F. Nori, and Y. Xia, Error-tolerant amplification and simulation of the ultrastrong-coupling quantum Rabi model, *Phys. Rev. Lett.* **133**, 033603 (2024).
- [92] A. Vepsäläinen, S. Danilin, and G. S. Paraoanu, Superadiabatic population transfer in a three-level superconducting circuit, *Sci. Adv.* **5**, eaau5999 (2019).
- [93] A. Smith, Y. Lu, S. An, X. Zhang, J.-N. Zhang, Z. Gong, H. T. Quan, C. Jarzynski, and K. Kim, Verification of the quantum nonequilibrium work relation in the presence of decoherence, *New J. Phys.* **20**, 013008 (2018).
- [94] A. Grimm, N. E. Frattini, S. Puri, S. O. Mundhada, S. Touzard, M. Mirrahimi, S. M. Girvin, S. Shankar, and M. H. Devoret, Stabilization and operation of a Kerr-cat qubit, *Nature (London)* **584**, 205 (2020).
- [95] A. P. M. Place, *et al.*, New material platform for superconducting transmon qubits with coherence times exceeding 0.3 milliseconds, *Nat. Commun.* **12**, 1779 (2021).
- [96] M. Kjaergaard, M. E. Schwartz, J. Braumüller, P. Krantz, J. I.-J. Wang, S. Gustavsson, and W. D. Oliver, Superconducting qubits: Current state of play, *Annu. Rev. Condens. Matter Phys.* **11**, 369 (2020).