

Strong asymmetric multicluster synchrony in globally coupled simplicial complexes

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In recent years, the investigation of synchronization in higher-order networks has emerged as a prominent area of research in nonlinear dynamics. In this work, we present a substantially generalized model of phase oscillators on globally coupled simplex structures that elucidate the formation mechanisms of strongly asymmetric synchronization clusters in complex systems characterized by many-body interactions. The strong asymmetric cluster synchrony involves three characteristics: (1) each cluster shows a different frequency band for the natural frequencies of the synchronized oscillators, (2) the phases of each cluster are locked to different interval sizes, and (3) the oscillators are nonuniformly distributed between the clusters. Focusing on a paradigmatic example of two coupled 2-simplex (three-body) interactions, we use the mean-field method to identify the existence of strongly asymmetric multicluster synchrony. Their solutions are analytically obtained through self-consistent equations. Our results also demonstrate, both theoretically and numerically, that the presence of asymmetric multicluster synchrony gives rise to multistability. When increasing the coupling strength, the system shows a transition to synchronization, which is then followed by synchronization fading, a complete loss of synchronization, and synchronization re-establishment. Our findings are not only limited to this case, but can also be extended to n -simplex interactions. Furthermore, we highlight that the inclusion of two-body interaction terms can disrupt the stability of the incoherent state, inducing spontaneous synchronization.

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I. INTRODUCTION

Beyond pairwise coupling, higher-order interactions emerge as important tools for characterizing complex network structures [1]. This research expands various domains, such as neuroscience, social networks, and engineering [2,3]. Higher-order interactions have been shown to introduce anomalous percolation [4], biodiversity in ecosystems [5], species coexistence [6], and cooperation in networked groups [7]. In addition, synchronization on higher-order networks can provide a more accurate understanding of synchronization phenomena in complex systems, which requires mathematical tools [3]. In most cases, higher-order interactions (i.e., triadic) cannot be reduced to pairwise ones [8].

In the case of synchronization of Kuramoto phase oscillators, there are several different ways to implement simplicial (many-body) interactions [3]. For example, some papers focus on pure simplex structures [9–11], while others focus on a combination of 1-, 2-, and 3-simplex interactions [12,13]. In the following, we expand some discussion on these different models, emphasizing different choices of parameters since

they require different analytic tools. We notice that a unified model is still lacking.

When replacing the traditional pairwise interactions with triadic ones, the most general form of the Kuramoto model is proposed in Ref. [11], which reads

$$\dot{\theta}_i = \omega_i + \frac{k}{N^2} \sum_{j=1}^N \sum_{k=1}^N \sin(\alpha\theta_j + \beta\theta_k - \gamma\theta_i), \quad (1)$$

where θ_i is the phase of oscillator i , $i = 1, \dots, N$, N is the number of oscillators, and ω_i is the natural frequency of each oscillator, which is drawn randomly from a uni-modal symmetric distribution $g(\omega)$. The coupling strength is represented by k . The coefficients α , β , and γ are integers leading to the 2-simplex interaction, which are often restricted to $\alpha + \beta = \gamma$ for rotational invariance.

The coefficients α , β , and γ in the higher-order coupling function are chosen differently in the literature to facilitate mathematical treatment. For example, the case of $(\alpha, \beta, \gamma) = (1, 1, 2)$ has been widely used in Refs. [9,10,14]. In this case, the most interesting phenomenon is the emergence of extensive multistability and a continuum of abrupt desynchronization [9], which is later rigorously shown in Refs. [10,14]. In addition, there are two clusters located in a unit circle with phase difference π and synchronized oscillators are distributed between these two clusters.

On the other hand, $\alpha + \beta = \gamma = 1$ is used in Ref. [11]. This is also the case for second-order Kuramoto oscillators with inertia [15], adaptive Kuramoto models [16,17], or with an external forcing term [18]. The combination effects of both

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pairwise and higher-order coupling on Chimera states have recently been studied, with different choices for γ for the focal node in Refs. [19,20].

Only one simplex coupling function is considered in all the above-mentioned work. In contrast, the combination of the two-body (1-simplex) and three-body (2-simplex) or multiple higher-order interactions gives rise to both abrupt synchronization and desynchronization transitions, leading to hysteresis loops [12,13,21]. More specifically, synchronization transitions are possible in the following model [12]:

$$\begin{aligned}\dot{\theta}_i = & \omega_i + \frac{k_1}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i) \\ & + \frac{k_2}{N^2} \sum_{j=1}^N \sum_{k=1}^N \sin(2\theta_j - \theta_k - \theta_i) \\ & + \frac{k_3}{N^3} \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N \sin(\theta_j + \theta_k - \theta_l - \theta_i),\end{aligned}\quad (2)$$

where k_1 , k_2 , and k_3 are the coupling strengths of 1- (dyadic), 2- (triadic), and 3-simplex (tetradic) interactions, respectively. For the full system [see Eq. (2)], bifurcation sets for synchronization states have recently been obtained by a self-consistency analysis in Ref. [22]. In addition, the model [Eq. (2)] can be generalized to any n -simplex interactions, provided that the condition $\gamma = 1$ is fulfilled [23].

A unified model that integrates various approaches to implementing simplicial interactions is urgently needed, as current state-of-the-art models are addressed in isolation. It remains largely unclear when other coupling modes of different parameters γ are used in many-body interactions; for example, different γ are used for pairwise and triadic couplings. Currently, it is a big challenge to obtain a deep understanding of the mechanisms underlying clustered synchronization as reported in Refs. [13,21,24]. Therefore, we introduce here a general model of two globally coupled simplex structures that encompasses the various models discussed above [Eqs. (1) and (2)] within a unified framework. This model can also be extended to n -simplex interactions in a straightforward manner.

In this work, we show that strong asymmetric multicluster synchrony is ubiquitous for globally coupled simplex structures. Taking the case of two clusters as an example, the asymmetric cluster synchrony is characterized by the following three properties: (1) two clusters involve different frequency widths for synchronized oscillators, (2) two clusters show different interval sizes of locked phases, and (3) the oscillators are distributed nonuniformly in two clusters. Note that the latter point of property (3) is controlled by an indicator function representing nonuniform distributions, which underlies the core of the original idea of “asymmetric cluster synchrony” [25,26]. To the best of our knowledge, the concept of asymmetric cluster synchrony dates back to a model of identical oscillators [26], where high-mode coupling was introduced. In the case of identical oscillators, two clusters are located at 0 and π . Later, this concept was extended further to the case of nonidentical oscillators [25], where oscillators in two clusters have the same frequency width and are locked

to the same interval sizes of phases. From the traditional perspective, the asymmetric phenomenon is referred to as a nonuniform distribution of oscillators in two clusters. In the following discussion, we denote the behavior fulfilling properties (1)–(3) as “strong asymmetric cluster synchrony,” while the behavior of only property (3) as “weak asymmetric cluster synchrony.” In addition, we show that strong asymmetric cluster synchrony is a general phenomenon in nonidentical oscillator models with many-body interacting structures.

In order to understand the asymmetric multicluster synchrony, we perform a self-consistent analysis by using the example of two globally coupled 2-simplex interactions. Our method is based on the analysis that was originally used for the case of a biharmonic coupling model for pairwise two-body interactions [27,28]. From the viewpoint of a general coupling function, there are different harmonics in the Fourier series. In this respect, the different choices of parameters (α, β, γ) in Eq. (1) share rather similar effects as in the case of high-mode coupling on synchronization [27,28]. High-mode interactions have been shown to naturally benefit the formation of synchronization clusters [27,29].

The structure of the paper is as follows. In Sec. II, we show the general model of two globally coupled 2-simplex structures and self-consistent analysis. Then, we use two examples showing strong asymmetric multicluster synchrony that are observed in the model, namely, two-cluster synchrony in Sec. IV and three-cluster synchrony in Sec. V, respectively. Some conclusions are presented in Sec. VII.

II. GENERAL MODEL OF TWO GLOBALLY COUPLED 2-SIMPLEX STRUCTURES

A. Model

We consider the following model of N number of phase oscillators θ_i ($i = 1, \dots, N$) with a general two globally coupled 2-simplex structures

$$\begin{aligned}\dot{\theta}_i = & \omega_i + \frac{1}{N^2} \sum_{j=1}^N \sum_{k=1}^N [k_1 \sin(\alpha_1 \theta_j + \beta_1 \theta_k - \gamma_1 \theta_i) \\ & + k_2 \sin(\alpha_2 \theta_j + \beta_2 \theta_k - \gamma_2 \theta_i)],\end{aligned}\quad (3)$$

where ω_i is the natural frequency of each oscillator that is drawn randomly from a unimodal symmetric distribution $g(\omega)$, taking a Gaussian distribution with unit width and zero mean without loss of generality.

The model of Eq. (3) involves a higher-order three-body (2-simplex) interaction $(\theta_j, \theta_k, \theta_i)$ with arbitrary coefficients $(\alpha_{1,2}, \beta_{1,2}, \gamma_{1,2})$. Different combinations of $\alpha_{1,2}$, $\beta_{1,2}$, and $\gamma_{1,2}$ represent different modes of many-body interactions in a general coupling function, which often appear naturally in higher-order phase reduced equations [30–35]. In recent work [24], the authors considered a model of identical oscillators coupled on two simplicial structures by choosing a particular set of coefficients $(\alpha_{1,2}, \beta_{1,2}, \gamma_{1,2})$. In addition, we will show later that the general model [Eq. (3)] unifies all different ways of implementing three-body interactions, as we discussed in the Introduction section.

We restrict ourselves to the case of integer coefficients and $\alpha_n + \beta_n = \gamma_n$, $n = 1, 2$, which guarantee rotational

invariance. In addition, for simplicity purposes, we consider the case of positive couplings between the two interacting simplex structures (i.e., $k_{1,2} \geq 0$). Our results in the following can be generalized to the cases of mixture combinations of positive and negative coupling as well.

In order to characterize the synchronization, we first define the order parameter as follows:

$$Z_m = R_m e^{i\Theta_m} = \frac{1}{N} \sum_{j=1}^N e^{im\theta_j}, \quad (4)$$

where m is the integer and chosen as $m = 1, 2$, and R_m and Θ_m are the amplitudes and, respectively, the arguments of the order parameters. Note that the first-order parameter Z_1 is generally sufficient to describe the macroscopic synchronization dynamics in the case of a single synchronized cluster. In the case of traditional two-cluster synchrony, the second-order parameter Z_2 is often required to measure the level of cluster synchronization, while Z_1 measures the asymmetry between two clusters [25]. However, in the case of strongly asymmetric multicluster synchrony for properties (1) and (2), $Z_{1,2}$ are influenced by the asymmetry of multiclusters.

Then, the model [Eq. (3)] can be rewritten as

$$\dot{\theta}_i = \omega_i - k_1 R_{\alpha_1} R_{\beta_1} \sin \gamma_1 \theta_i - k_2 R_{\alpha_2} R_{\beta_2} \sin \gamma_2 \theta_i. \quad (5)$$

In the mean-field equation (5), we assume $\Theta_1 = \Theta_2 = 0$, for the symmetry of the coupling function and of the distribution.

In the case of a unique $\gamma_1 = \gamma_2 = \gamma$, Eq. (5) is further simplified as a mean-field model as

$$\dot{\theta}_i = \omega_i - F(k_{1,2}, R_{\alpha_{1,2}}, R_{\beta_{1,2}}) \sin \gamma \theta_i. \quad (6)$$

where $F(k_{1,2}, R_{\alpha_{1,2}}, R_{\beta_{1,2}}) = k_1 R_{\alpha_1} R_{\beta_1} + k_2 R_{\alpha_2} R_{\beta_2}$.

Based on Eqs. (5) and (6), we see that $\gamma_{1,2}$ plays a decisive role in finding the number of fixed points, while $\alpha_{1,2}$ and $\beta_{1,2}$ change the amplitudes of $\sin \gamma_1 \theta_i$ and $\sin \gamma_2 \theta_i$ through the order parameters $R_{\alpha_{1,2}}$ and $R_{\beta_{1,2}}$. In consequence, we show in the following that the qualitative results of the system, i.e., the number of multiclusters, are determined by $\gamma_{1,2}$. However, the quantitative results, for example, the critical coupling threshold values $k_{1,2}$ for synchronization transitions, are influenced by $\alpha_{1,2}$ and $\beta_{1,2}$. In addition, we intentionally avoid discussing the case of the unique choice for $\gamma_1 = \gamma_2 = \gamma$ as this has been largely discussed in the literature:

(1) The choice of $\gamma_1 = \gamma_2 = 1$ has been reported in Refs. [11,12,22]. Traditional self-consistent analysis has recently been reported in Ref. [22] for the model (2). In addition, the unique choice of $\gamma = 1$ allows us to study analytically higher-order couplings of any order [23], which is crucial to the use of the Ott-Antonsen (OA) ansatz [36].

(2) The other case of choosing $\gamma_1 = \gamma_2 = 2$ has been studied in Refs. [9,10,14], where one finds two weak asymmetric clusters of synchronization. In this case, the most interesting phenomenon in coupled phase oscillators is the emergence of extensive multistability and a continuum of abrupt desynchronization [9].

Finally, we emphasize that the case of unique γ is essentially a representation of one mean field for the system [Eq. (6)], which will be mentioned in the following analysis when necessary. In addition, for the general case of $\gamma_1 \neq \gamma_2$, the OA ansatz is not applicable.

B. Self-consistent analysis

In this section, we generalize a self-consistent analysis for two globally coupled simplex structures, which was originally used to analyze lower-order pairwise coupled phase oscillators [27,28].

From the mean-field description [Eq. (5)], the steady solutions are obtained as

$$\omega_i = k_1 R_{\alpha_1} R_{\beta_1} \sin \gamma_1 \theta_i + k_2 R_{\alpha_2} R_{\beta_2} \sin \gamma_2 \theta_i. \quad (7)$$

It is difficult to get explicit solutions directly from Eq. (7) for the general case of $\gamma_1 \neq \gamma_2$. Therefore, introducing a transformation $R = \sqrt{(k_1 R_{\alpha_1} R_{\beta_1})^2 + (k_2 R_{\alpha_2} R_{\beta_2})^2}$, Eq. (7) is then written as

$$x = \sin \phi \sin \gamma_1 \theta + \cos \phi \sin \gamma_2 \theta = y(\phi, \theta), \quad (8)$$

where

$$x = \frac{\omega}{R}, \quad \sin \phi = \frac{k_1 R_{\alpha_1} R_{\beta_1}}{R}, \quad \cos \phi = \frac{k_2 R_{\alpha_2} R_{\beta_2}}{R}. \quad (9)$$

The above definitions result in only two parameters for the next analysis, namely, R and ϕ . Here, x is considered as the rescaled individual frequencies of the oscillators, while $\sin \phi$ and $\cos \phi$ play the role of relative coupling strengths determined by the actual coupling strength and order parameters.

Phase-locked oscillators correspond to the solutions of Eq. (8). Note that x on the left-hand side of Eq. (8) is independent of ϕ . Therefore, the solutions of the mean-field equation are viewed as intersections between the horizontal line x and the curve $y(\phi, \theta)$. In addition, it is sufficient to obtain all possible solutions when we scan $\phi \in [0, \pi/2]$, and $\phi = \pi/2$ corresponds to the individual case of $\sin \gamma_1 \theta$ and $\phi = 0$ for $\sin \gamma_2 \theta$. Multiple clusters of synchronized oscillators are obtained when more intersections are realized between x and $y(\phi, \theta)$. In addition, we have to verify the stability condition $\partial y(\phi, \theta)/\partial \theta > 0$ that follows from Eq. (5). More details will be explained by the case studies in the following when finding a stable solution of θ as a function of x , which is assisted by the curves of $y(\phi, \theta)$. Therefore, the interval $[y_{\min}(\phi, \theta), y_{\max}(\phi, \theta)]$ determines the phase-locking condition; that is, the oscillators with $|\omega| \leq y_{\max} R$ are locked. In contrast, the frequencies of drifting oscillators are outside this interval.

All oscillators can be separated into locked and drifting ones. The locked phases can be found via Eq. (8) and the distribution function is denoted as $\rho_l(\theta|x)$. On the other hand, the distribution function $\rho_d(\theta|x)$ of the drifting oscillators is inversely proportional to the phase velocity; that is, $\rho_d(\theta|x) = C|x - y|^{-1}$, where $C(x, \phi) = 1/\int_{-\pi}^{\pi} d\theta |x - y|^{-1}$ is the normalization constant. Therefore, in the thermodynamic limit, the self-consistent equation is expressed as

$$F_m(R, \phi) = \int_{-\pi}^{\pi} d\theta \int_{|x| \leq y_{\max}} dx \cos m\theta g(Rx) \rho_l(\theta|x) + \int_{-\pi}^{\pi} d\theta \int_{|x| > y_{\max}} dx \cos m\theta g(Rx) \rho_d(\theta|x), \quad (10)$$

where $F_m = R_m/R$. In Eq. (10), the first term corresponds to phase-locked oscillators, while the second term refers to drifting oscillators.

For each pair of parameters (R, ϕ) , we calculate $F_m(R, \phi)$ and, therefore, the order parameters and coupling constants are further presented as

$$R_m = RF_m(R, \theta), \quad k_1 = \frac{R \sin \phi}{R_{\alpha_1} R_{\beta_1}}, \quad k_2 = \frac{R \cos \phi}{R_{\alpha_2} R_{\beta_2}}. \quad (11)$$

Equations (10) and (11) solve the problem of finding the stationary states in analytic form. Appendix A presents the key steps of the self-consistent approach by using the numerical example of $\gamma_1 = 1, \gamma_2 = 2$, which is written in the MATLAB programming language.

III. NUMERICAL MODELS OF TWO GLOBALLY COUPLED 2-SIMPLEX STRUCTURES

In the next sections, we demonstrate how the self-consistent approach is applied to disclose the mechanisms of strong asymmetric multicluster synchrony when different combinations of coefficients are chosen. We choose specific values $\alpha_{1,2}, \beta_{1,2}, \gamma_{1,2}$ for the numerical analysis taking into account the following aspects: (1) The values of $\alpha_{1,2}$ and $\beta_{1,2}$ are chosen to preserve rotational symmetry. (2) We emphasize that the qualitative results, that is, the number of asymmetric synchronized clusters, are determined by $\gamma_{1,2}$, which does not change for different values of $\alpha_{1,2}$ and $\beta_{1,2}$. On the other hand, the quantitative results for the critical thresholds for the synchronization transitions are affected by different $\alpha_{1,2}$ and $\beta_{1,2}$. (3) We choose $\alpha_{1,2}, \beta_{1,2}$ that are widely used in the literature and hence for better comparison.

More specifically, in Sec. IV we show asymmetric two-cluster synchrony when choosing $\gamma_1 = 1, \gamma_2 = 2$ in the following model:

$$\begin{aligned} \dot{\theta}_i = \omega_i + \frac{1}{N^2} \sum_{j=1}^N \sum_{k=1}^N [k_1 \sin(2\theta_j - \theta_k - \theta_i) \\ + k_2 \sin(\theta_j + \theta_k - 2\theta_i)]. \end{aligned} \quad (12)$$

Note that the 2-simplex (three-body) interacting structures in the above model are discussed separately in the literature. In particular, $\sin(2\theta_j - \theta_k - \theta_i)$ is studied in Ref. [12] and $\sin(\theta_j + \theta_k - 2\theta_i)$ is reported in Ref. [9]. In Sec. IV D, we compare the results of Eq. (12) with the following model that has been used in the literature [13,21,37]:

$$\begin{aligned} \dot{\theta}_i = \omega_i + \frac{k_1}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i) \\ + \frac{k_2}{N^2} \sum_{j=1}^N \sum_{k=1}^N \sin(\theta_j + \theta_k - 2\theta_i). \end{aligned} \quad (13)$$

This model has been initially discussed in Refs. [13,21] and is recently discussed in a different context [24]. In the review process of this work for resubmission, we notice that this particular model [Eq. (13)] is analyzed in Ref. [37]. In this comparative model consisting of 1- and 2-simplex structures, we focus on the influence of lower-order pairwise interaction on asymmetric two-cluster synchrony. In Sec. IV E, we discuss the special case of $\gamma_1 = \gamma_2 = \gamma$ in Eq. (12). We further

compare our results to the model of Ref. [9], which is retrieved by setting $k_1 = 0$ in our model [Eq. (12)].

In Sec. V, strong asymmetric three-cluster synchrony is shown when choosing $\gamma_1 = 1, \gamma_2 = 3$ in the general model. More specifically, in particular, we consider the following model:

$$\begin{aligned} \dot{\theta}_i = \omega_i + \frac{1}{N^2} \sum_{j=1}^N \sum_{k=1}^N [k_1 \sin(2\theta_j - \theta_k - \theta_i) \\ + k_2 \sin(2\theta_j + \theta_k - 3\theta_i)]. \end{aligned} \quad (14)$$

In Sec. VI, we generalize our results from the case of two coupled 2-simplex structures to the case of three- and four-body interactions. More specifically, the model reads:

$$\begin{aligned} \dot{\theta}_i = \omega_i + \frac{k_1}{N^2} \sum_{j=1}^N \sum_{k=1}^N \sin(2\theta_j - \theta_k - \theta_i) \\ + \frac{k_2}{N^3} \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N \sin(\theta_j + \theta_k + \theta_l - 3\theta_i). \end{aligned} \quad (15)$$

We note that phase diagrams of Eq. (15) are qualitatively similar to Eq. (14), while $\alpha_{1,2}$ and $\beta_{1,2}$ only affect the relative coupling strength through the amplitudes of $\sin \gamma_1 \theta_i$ and $\sin \gamma_2 \theta_i$.

IV. STRONG ASYMMETRIC TWO-CLUSTER SYNCHRONY

A. Phase diagrams for Eq. (12)

The mean field of Eq. (12) is as follows:

$$y(\phi, \theta) = \sin \phi \sin \theta + \cos \phi \sin 2\theta. \quad (16)$$

When searching $\phi \in [0, \pi/2]$, there are two cases for stable branches of solutions, which are shown in Fig. 1. Meanwhile, the stability condition is determined by $\partial y(\phi, \theta)/\partial \theta$. More specifically, we have the following two cases:

(1) A single peak for $y(\phi, \theta)$ in Fig. 1(a): When $\tan \phi > 2$, there is only one stable branch $\theta \in [-\theta_1, \theta_1]$, where $\theta_1 = \arccos[(-\tan \phi + \sqrt{\tan^2 \phi + 32})/8]$. This forms a single cluster of synchronized oscillators whose natural frequencies satisfy $|\omega_i| < \omega_{c_1} = \gamma_1(\phi, \theta_1)R$. We use the notation C_1 to represent only one cluster of synchronized oscillators in the following.

In this case of C_1 [Fig. 1(a)], Eq. (10) is calculated as

$$\begin{aligned} F_m(R, \phi) = \int_{-\theta_1}^{\theta_1} d\theta \cos m\theta g(Ry) \frac{\partial y}{\partial \theta} \\ + \int_{-\pi}^{\pi} d\theta \int_{|x|>y_1} dx \cos m\theta C(x, \phi) g(Rx) |x - y|^{-1}. \end{aligned} \quad (17)$$

(2) Two peaks for $y(\phi, \theta)$ in Fig. 1(b): When $\tan \phi < 2$, there is an additional stable branch $\theta \in [\theta_2, \pi] \cup [-\pi, -\theta_2]$, where $\theta_2 = \arccos[(-\tan \phi - \sqrt{\tan^2 \phi + 32})/8]$. This forms another cluster of synchronized oscillators whose natural frequencies satisfy $|\omega_i| < \omega_{c_2} = \gamma_2(\phi, \theta_2)R$. These two clusters are asymmetric, showing different widths for oscillator natural frequencies ($\Delta\omega_{c_1} \neq \Delta\omega_{c_2}$) and different interval

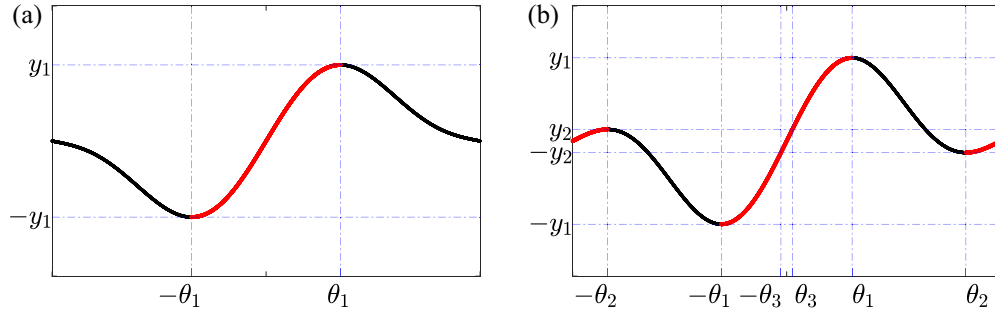


FIG. 1. Stable phase branches (red bold line) vs. frequency $x = \omega/R$. (a) a single peak for the C_1 cluster when $\phi \in [\arctan 2, \pi/2]$ and (b) two peaks for the two-cluster state C_2^s when $\phi \in [0, \arctan 2]$.

sizes for locked phases ($\Delta\theta_{c_1} = 2\theta_1$, $\Delta\theta_{c_2} = 2(\pi - \theta_2)$, and $\Delta\theta_{c_1} \neq \Delta\theta_{c_2}$). Therefore, two synchronized asymmetric clusters coexist within the synchronized region, which is denoted C_2^s below.

Note that the case of $\phi = 0$ in Eq. (16) yields the same frequency width $\Delta\omega_{c_1} = \Delta\omega_{c_2}$ and equal intervals of locked phases $\Delta\theta_{c_1} = \Delta\theta_{c_2} = \pi/2$, resulting from the case of $k_1 = 0$ and $k_2 \neq 0$ in Eq. (12). Indeed, this corresponds to $\gamma = 2$ as reported in Ref. [9], which will be termed as weak asymmetric two-cluster synchrony C_2^w since the frequency width and locked phases in one cluster are the same as in the other cluster. The asymmetry is referred to as the nonuniform distribution of oscillators in two clusters.

In the case of C_2^s [Fig. 1(b)], the self-consistent equation (10) is written as

$$\begin{aligned}
 F_m(R, \phi) = & 2 \int_{\theta_3}^{\theta_1} d\theta \cos m\theta g(Ry) \frac{\partial y}{\partial \theta} \\
 & + 2\sigma \int_0^{\theta_3} d\theta \cos m\theta g(Ry) \frac{\partial y}{\partial \theta} \\
 & + 2(1 - \sigma) \int_{\theta_2}^{\pi} d\theta \cos m\theta g(Ry) \frac{\partial y}{\partial \theta} \\
 & + \int_{-\pi}^{\pi} d\theta \int_{|x| > y_1} dx \cos m\theta C(x, \phi) g(Rx) |x - y|^{-1},
 \end{aligned} \quad (18)$$

where θ_3 fulfills $y(\phi, -\theta_3) = y(\phi, \theta_2)$. In Eq. (18), σ is an arbitrary indicator function ($\sigma \in [0, 1]$), which describes the distribution of oscillators in the first group (respectively, $1 - \sigma$ in the second group) in the overlapping regions of C_2^s . In addition, the drifting oscillators are $|x| > y_{\max} = y_1(\phi, \theta_1)$. In consequence, $F_m(R, \phi)$ explicitly depends on σ , which yields the result that the macroscopic order parameters $R_{1,2}$ are influenced by the appearance of the strong asymmetric multiclusters.

Then, for each pair of parameters (R, ϕ) , we obtain $F_m(R, \phi)$ from the self-consistent equation. In addition, we use Eq. (11) to find the phase diagram in the coupling (k_1, k_2) plane, which is shown in Fig. 2. In this figure, the black line L_1 separates the synchronous states from the asynchronous ones, corresponding to a saddle-node bifurcation. In the upper right coherent region, the black line L_2 separates the C_1 from the C_2^s region, which is determined by the critical value $\tan \phi = 2$. In the C_1 region, we get only a single cluster of synchrony,

while in the C_2^s regime, we obtain asymmetric two-cluster synchrony.

In addition, C_2^s is extensive multistable because the order parameters and the critical coupling strength are determined by the arbitrary indication function $\sigma \in [0, 1]$. Note that σ is necessary only for the overlapping domain denoting the redistribution of oscillators over the stable branches, and a continuum abrupt desynchronization transition occurs from the C_2^s state for each different value of σ . Therefore, in the C_2^s region of Fig. 2, we show several values for σ .

B. Strong asymmetric two-cluster synchrony

Based on the phase diagram of Fig. 2, we show in this section that asymmetric multicluster synchrony is ubiquitous for globally coupled simplex interactions. To this end, we choose three representative coupling pairs (k_1, k_2) corresponding to C_1 , C_2^s , and C_2^w to show in Fig. 3 the main phenomena of

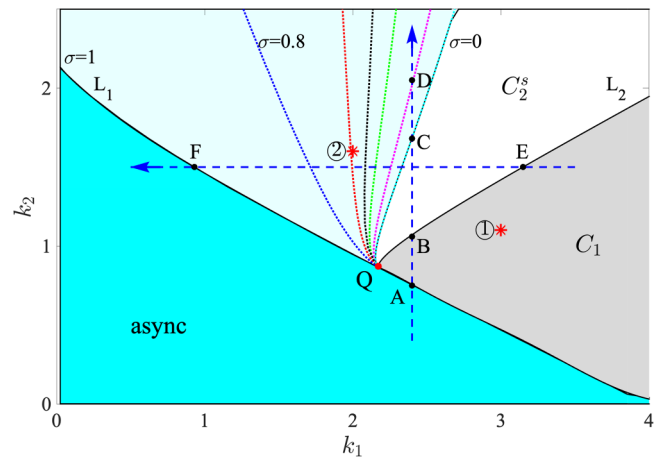


FIG. 2. Phase diagram of different synchronous states in dependence on coupling parameters (k_1, k_2) resulting from the analytical solutions of Eq. (11). The bold black line L_1 is the border of synchronous state, while line L_2 separates a single cluster C_1 from the coexistence domain of asymmetric clusters C_2^s . Above L_1 , dotted lines represent the onset of C_2^s synchrony for $\sigma = 1, 0.8, 0.6, 0.5, 0.4, 0.2, 0$ (from left to right). The asterisk points ① and ② represent two typical pairs (k_1, k_2) that are used in Fig. 3. The vertical arrow lines at position $k_1 = 2.4$ and horizontal one at $k_2 = 1.5$ correspond to the cases in Fig. 5.

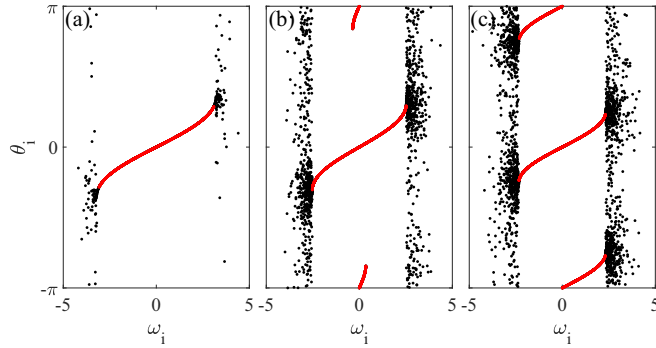


FIG. 3. The single cluster C_1 and asymmetric clusters C_2^s, C_2^w states of the system [Eq. (12)] when choosing different coupling strength. (a) $(k_1, k_2) = (3, 1.1)$ in C_1 region, (b) $(k_1, k_2) = (2, 1.6)$ in C_2^s region, showing the coexistence of two strong asymmetric clusters, and (c) $(k_1, k_2) = (0, 4)$ showing the traditional two weak asymmetric clusters of synchrony C_2^w . The circular representation is shown in Fig. 4.

strongly asymmetric two-cluster synchrony (the trivial case of the asynchronous state is not shown). The asymmetric nature of the clusters of synchrony is alternatively shown by the circular representation of the oscillator configurations in Fig. 4, where each oscillator is placed on the unit circle according to its instantaneous phases and color coded by the natural frequencies of locked oscillators.

When the coupling values (k_1, k_2) are in the C_1 region, the model shows a single cluster of synchronized oscillators as demonstrated in Figs. 3(a) and 4(a), resulting in a single mode in the probability density function (PDF) of locked phases [Fig. 4(d)].

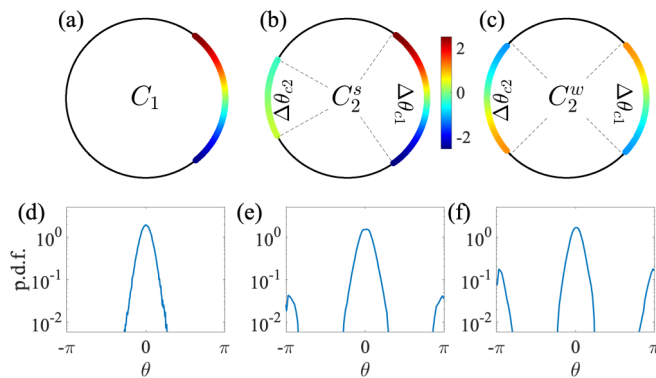


FIG. 4. Circular representation of oscillator configurations of coherent clusters corresponding to different states of the system [Eq. (14)]. (a) $(k_1, k_2) = (3, 1.1)$ in C_1 region, showing a single cluster of coherence. (b) Two strong asymmetric clusters of synchrony when $(k_1, k_2) = (2, 1.6)$, which is denoted as C_2^s . Two clusters have different interval size of locked phases ($\Delta\theta_{c1} \neq \Delta\theta_{c2}$) and different frequency width $\Delta\omega_{c1} \neq \Delta\omega_{c2}$ as captured by color bars. (c) Two weak asymmetric clusters C_2^w when $(k_1, k_2) = (0, 4)$, where $\Delta\theta_{c1} = \Delta\theta_{c2}$ and natural frequencies locked oscillators are in the same intervals $\Delta\omega_{c1} = \Delta\omega_{c2}$. (d)–(f) The PDF of coherent phases. Drifting oscillators are not shown. Numerical details: $N = 1 \times 10^5$ and $\sigma = 0.9$.

When the coupling values (k_1, k_2) are in the C_2^s region, we observe strong asymmetric two-cluster synchrony in Eq. (12) [see Figs. 3(b) and 4(b)]. There are three properties for the asymmetric two-cluster synchrony:

(1) The oscillators in two clusters have different frequency widths, i.e., $\Delta\omega_{c1} \neq \Delta\omega_{c2}$ [as captured by different colorbars in Fig. 4(b)].

(2) Two clusters show different interval sizes of locked phases [annotated by $\Delta\theta_{c1} \neq \Delta\theta_{c2}$ in Fig. 4(b)].

(3) The oscillators are nonuniformly distributed in the overlapping regions of two clusters, yielding different probability densities [Fig. 4(e)].

Note that the last point of property (3) is controlled by an arbitrary indicator function σ representing nonuniform distribution of oscillators in the overlapping regions of the clusters. Due to $\Delta\omega_{c1} \neq \Delta\omega_{c2}$ and $\Delta\theta_{c1} \neq \Delta\theta_{c2}$ for two clusters, σ is required only for the overlapping regions, i.e., for those oscillators that have a natural frequency $|\omega_i| \leq \omega_{c2}$. For the purpose of comparison, we denote the behavior fulfilling properties (1)–(3) as “strong asymmetric cluster synchrony,” while the behavior of property (3) as “weak asymmetric cluster synchrony.”

When the coupling values $(k_1 = 0, k_2)$ are in the C_2^w region, we observe weak asymmetric two-cluster synchrony [Fig. 3(c), and Figs. 4(c) and 4(f)]. In this case, two clusters have the same frequency width $\Delta\omega_{c1} = \Delta\omega_{c2}$ [the same colorbar for each cluster in Fig. 4(c)] and the same sizes of locked phases ($\Delta\theta_{c1} = \Delta\theta_{c2}$). The asymmetric phenomenon is referred to by a nonuniform distribution of oscillators in two clusters leading to different probability densities [Fig. 4(f)]. We note that this corresponds to the original notation of “asymmetric cluster synchrony” in Ref. [25].

Bimodal distributions are expected for both C_2^s and C_2^w [Figs. 4(e) and 4(f)] and the difference between them is as follows. In the case of C_2^s , the peak heights of the two modes are mainly due to the different interval sizes $\Delta\theta_{c1}$ and $\Delta\theta_{c2}$ [Fig. 4(e)]. The indication function σ is necessary only for the overlapping regions of C_2^s . However, in the case of C_2^w , the peak heights of two modes are fully controlled by σ [Fig. 4(f)].

C. Typical desynchronization transition processes

Next, we show the dependence of the order parameters R_1 and R_2 on the coupling strength by illustrating two typical synchronization transition processes as annotated by dashed arrow lines in Fig. 2. Some phase transitions are formed there because of the presence of an asymmetric multicuster.

(1) For a fixed $k_1 = 2.4$ while increasing k_2 (left column in Fig. 5): There are three different regions: nonsynchronous, a single cluster C_1 , and asymmetric two-cluster synchrony C_2^s . Point “A” separates nonsynchronous from the C_1 region. In $A < k_2 < B$, we have a single-cluster state C_1 , which is σ independent. However, after passing point “B,” multiple synchronized solutions emerge that are dependent on σ , indicating extensive multistability in the regime of C_2^s , which further results in significant fading of the synchronization of the decreased order parameters $R_{1,2}$. In addition, a complete loss of synchronization occurs for the branches of $\sigma < 0.5$

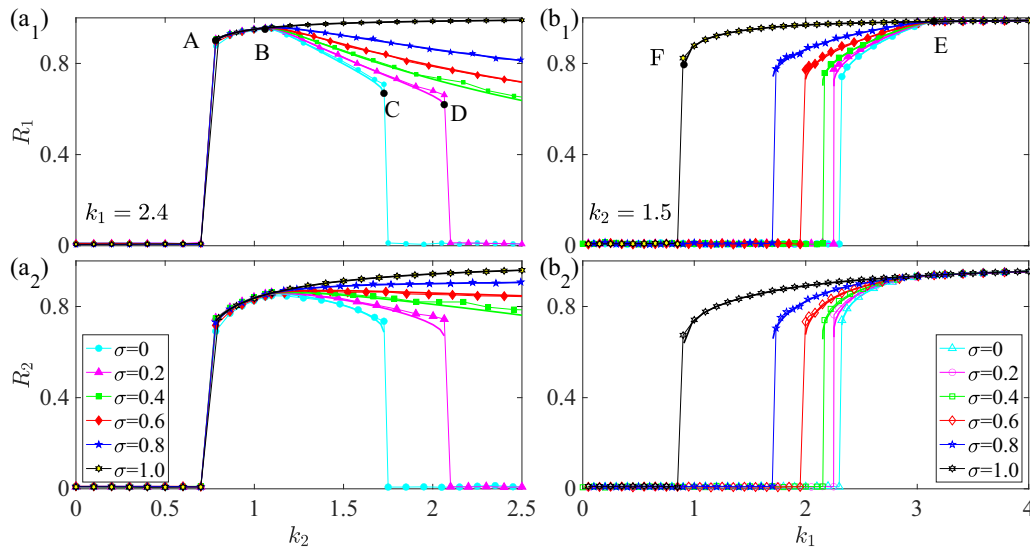


FIG. 5. Two typical synchronization transition processes. Left column: $k_1 = 2.4$ varying k_2 and right column: $k_2 = 1.5$ varying k_1 . Upper row is for order parameter R_1 and the bottom row is for R_2 . In each case, the macroscopic order parameters $R_{1,2}$ depend on σ , which are shown by six values for the ratio $\sigma = 1.0$ (black), 0.8 (blue), 0.6 (red), 0.4 (green), 0.2 (purple), and 0.0 (light blue). Number of oscillators $N = 1 \times 10^5$.

at different thresholds. In Fig. 5(a), we denote two critical couplings “C” for $\sigma = 0$ and “D” for $\sigma = 0.2$.

(2) For a fixed $k_2 = 1.5$ while decreasing k_1 (right column in Fig. 5): after point “E” (L_2), strong asymmetric two cluster C_2^s emerges from C_1 . However, a continuum desynchronization transition occurs for C_2^s when k_2 is further decreased, which is dependent on σ .

D. Comparative model [Eq. (13)]

Let us discuss the effects of lower-order pairwise interactions on the simplex structures by studying the model of Eq. (13). In this case, spontaneous transitions from incoherence to synchronization are observed. More details of the stability analysis for the incoherent state is provided in Appendix B.

In this case, Eq. (13) has the same mean field $y(\phi, \theta) = \sin \phi \sin \theta + \cos \phi \sin 2\theta$, noting here a different definition for ϕ with Eq. (16) of Eq. (12). Therefore, we obtain qualitatively similar phase diagrams for synchronous states in the (k_1, k_2) plane (Fig. 6).

The specific result of Eq. (13) is the following: Since we use a Gaussian distribution of frequencies $g(\omega) = \frac{1}{2\pi} e^{-\omega^2/2}$, the linear stability theory shows that the incoherent state becomes unstable at coupling $k_{1c} = 2/\pi g(0) = 2\sqrt{2/\pi}$ [38], forming a spontaneous synchronization transition. The co-existence of incoherent and synchronized states disappears when $k_1 > k_{1c}$, which is different from Fig. 3. Therefore, a hysteresis is expected between the synchronization and desynchronization transitions.

E. Special cases when $\gamma_{1,2} = \gamma$

As we discussed above, asymmetric two clusters of coherence do not appear when $\gamma_{1,2} = 1$, which is reported in Ref. [12]. Note that the critical transition lines related to

L_2 and the indication function σ disappear from the (k_1, k_2) plane, resulting only in a single cluster C_1 .

When $k_1 = 0$ in Eq. (12) yielding to $\phi = 0$ in Eq. (16), the results of weak asymmetric two-cluster synchrony with phase difference π are recovered ($\sigma \in [0.5, 1]$), which are reported in Refs. [9,10]. The global collective behavior is significantly influenced by the nonuniform distribution of oscillators in the two clusters. In this case, the most interesting phenomenon in coupled phase oscillators is the emergence of extensive multistability and a continuum of abrupt desynchronization, which has potential applications in memory and information storage [9]. That is, starting from a synchronized state and the choice of proper initial conditions, the system

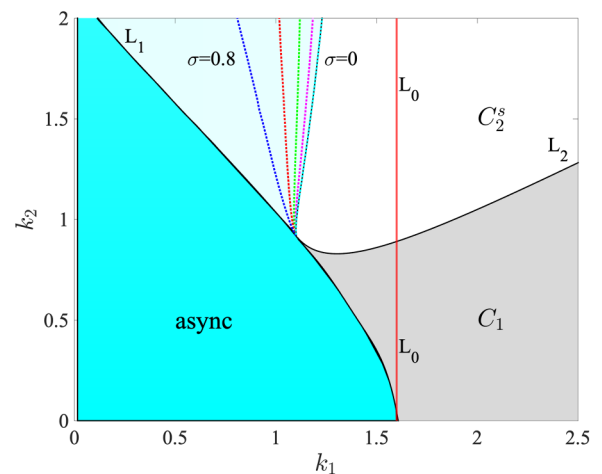


FIG. 6. Phase diagram of different synchronized states in dependence on coupling parameters (k_1, k_2) for Eq. (13) (1- and 2-simplex). The dotted lines represent the onset of two-cluster synchrony for $\sigma = 0.8, 0.6, 0.4, 0.2, 0$ (from left to right). Note that the vertical line L_0 corresponds to $k_{1c} = 2/\pi g(0)$.

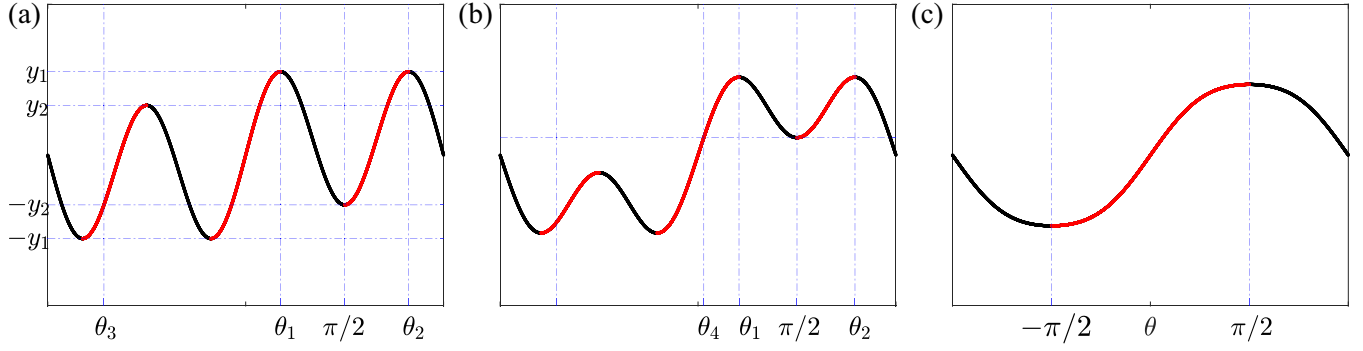


FIG. 7. There are two subclasses of C_3^s state: (a) $\phi \in [0, \pi/4]$ resulting in C_3^{23} state and (b) $\phi \in (\pi/4, \arctan 9]$ leading to C_3^{12} state. In panel (c), only a single C_1 state exists for $\phi \in (\arctan 9, \pi/2]$.

shows discontinuous jumps at different coupling thresholds from a synchronized state to an incoherent state. In contrast, there is no complementary spontaneous synchronization transition from an incoherent state to a coherent state on the increase of coupling, since the incoherent one is stable for any coupling k .

V. STRONG ASYMMETRIC THREE-CLUSTER SYNCHRONY

A. Phase diagrams for Eq. (14)

The mean field of Eq. (14) is written as

$$y(\phi, \theta) = \sin \phi \sin \theta + \cos \phi \sin 3\theta, \quad (19)$$

where $\sin \phi = k_1 R_1 R_2 / R$, $\cos \phi = k_2 R_1 R_2 / R$, and $R = \sqrt{(k_1 R_1 R_2)^2 + (k_2 R_1 R_2)^2}$. There are two subclasses for strong asymmetric three-cluster synchrony C_3^s , which will be explained and annotated separately in the following. From Eq. (19), the interval $\phi \in [0, \pi/2]$ is divided into three critical cases (Fig. 7): (1) when $\phi \in [0, \pi/4]$ [Fig. 7(a)], we have asymmetric three clusters synchrony C_3^{23} , (2) when $\phi \in (\pi/4, \arctan 9]$ [Fig. 7(b)], we have asymmetric three clusters synchrony C_3^{12} , and (3) when $\phi \in (\arctan 9, \pi/2]$ [Fig. 7(c)], we have only one stable branch for C_1 .

Note that the separatrix of $\pi/4$ and $\arctan 9$ is calculated, respectively, by $y(\phi, \pi/2) = 0$ and $\partial^2 y(\phi, \theta) / \partial \theta^2|_{\theta=\pi/2} = 0$. In addition, the phases of the synchronized oscillators for the asymmetric clusters C_3^{23} and C_3^{12} are restricted in $[-\theta_1, \theta_1] \cup [\pi/2, \theta_2] \cup [-\theta_2, -\pi/2]$, where $\theta_1 = \arctan(\sqrt{(3 \cos \phi + \sin \phi) / (9 \cos \phi - \sin \phi)})$ and $\theta_2 = \pi - \theta_1$, which are obtained by $\partial y(\phi, \theta) / \partial \theta = 0$. In contrast, for the C_1 state, the phases are constrained in $[-\pi/2, \pi/2]$.

The self-consistent equation can be further calculated as follows:

(1) C_3^{23} state when $\phi \in [0, \pi/4]$ [Fig. 7(a)].

There are no solutions in the intervals of $(-\infty, -y_1] \cup (y_1, +\infty)$, two stable solutions in $(y_2, y_1] \cup [-y_1, -y_2]$, and three stable solutions in $[-y_2, y_2]$, which are annotated by C_3^{23} . More specifically, oscillators of natural frequency $|\omega_i| \leq \omega_{c2} = y(\phi, -\pi/2)R$ form three clusters, while those with natural frequency $\omega_{c2} < |\omega_i| \leq \omega_{c1} = y(\phi, \theta_1)R$ form two clusters, and the others $|\omega_i| > \omega_{c1}$ drift. Therefore, Eq. (10)

is calculated as follows:

$$F_m(R, \phi) = 2\sigma \int_0^{\theta_1} d\theta \cos m\theta g(Ry) \frac{\partial y}{\partial \theta} + (1 - \sigma) \int_{\pi/2}^{\theta_2} d\theta \cos m\theta g(Ry) \frac{\partial y}{\partial \theta} + (1 - \sigma) \int_{-\theta_2}^{\theta_3} d\theta \cos m\theta g(Ry) \frac{\partial y}{\partial \theta}, \quad (20)$$

where $\theta_3 \in [-\pi, -\pi/2]$ satisfying the condition $y_3(\phi, \theta_3) = y(\phi, \pi/2)$. Note that the integral of the drifting oscillators vanishes due to symmetry.

(2) C_3^{12} state when $\phi \in (\pi/4, \arctan 9]$ [Fig. 7(b)].

When varying ϕ , the coexistence region of three branches transforms into the region of a single branch, resulting in C_3^{12} . In addition, Eq. (10) is calculated as follows:

$$F_m(R, \phi) = 2 \int_0^{\theta_4} d\theta \cos m\theta g(Ry) \frac{\partial y}{\partial \theta} + 2\sigma \int_{\theta_4}^{\theta_1} d\theta \cos m\theta g(Ry) \frac{\partial y}{\partial \theta} + 2(1 - \sigma) \int_{\pi/2}^{\theta_2} d\theta \cos m\theta g(Ry) \frac{\partial y}{\partial \theta}, \quad (21)$$

where $\theta_4 \in [0, \theta_1]$, which is obtained by the condition $y_4(\phi, \theta_4) = y(\phi, \pi/2)$.

(3) C_1 state when $\phi \in (\arctan 9, \pi/2]$ [Fig. 7(c)].

In this case, only a single cluster of synchronized oscillators is observed, and Eq. (10) is easily calculated as

$$F_m(R, \phi) = \int_{-\pi/2}^{\pi/2} d\theta \cos m\theta g(Ry) \frac{\partial y}{\partial \theta}. \quad (22)$$

Note that the phase-locking range is $\theta \in [-\pi/2, \pi/2]$ and is invariant when the coupling strength is changed.

Based on self-consistent equations, the phase diagram in the coupling (k_1, k_2) plane is shown in Fig. 8(a). The transition line L_1 is the border of the synchronized state, line L_2 separates a single cluster C_1 from C_3^{12} , and line L_3 separates C_3^{23} . In the region of C_3^s , the onset of a strong asymmetric three-cluster synchrony is influenced by σ . We show several representative values $\sigma = [1, 0]$ with step size $\Delta\sigma = 0.1$. However, the critical lines L_2 and L_3 are independent of σ and are

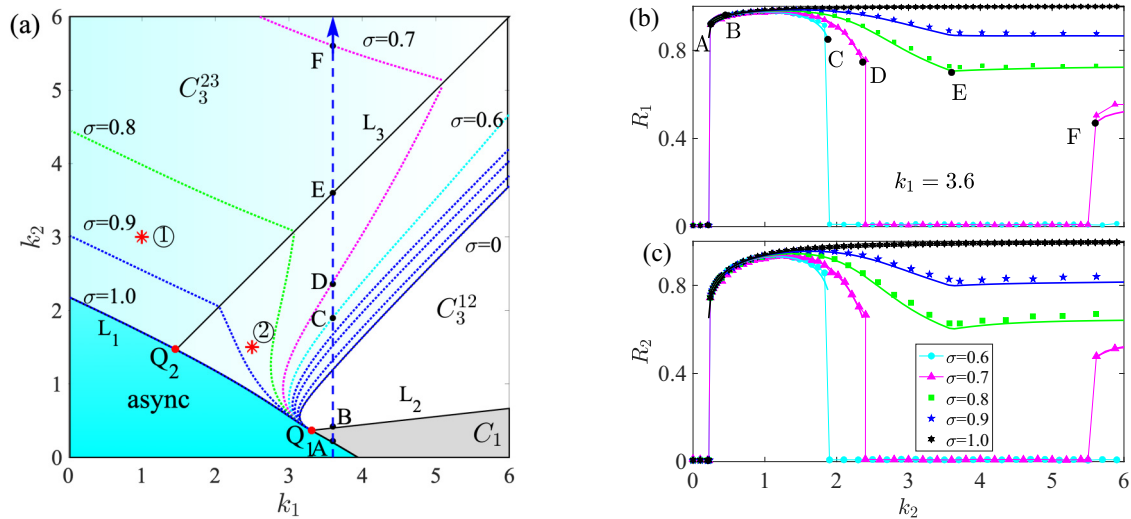


FIG. 8. (a) Phase diagram of different synchronous states in dependence on coupling parameters (k_1, k_2) for Eq. (14), which are resulted from the analytical solutions of Eq. (11). The asterisk points ① and ② represent the typical pairs (k_1, k_2) that are used in Fig. 9. The vertical arrow lines at position $k_1 = 3.6$ correspond to the synchronization transition route in panels (b) and (c), while k_2 is varied. The dotted lines represent the onset of three-cluster synchrony for $\sigma = 1, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.2, 0$ (from left to right). (b) Order parameter R_1 and panel (c) is for R_2 . Five representative values of σ are shown as denoted by the legend. Note that the asynchronous state becomes synchronized again at $k_2 = 5.7$. Number of oscillators $N = 1 \times 10^5$.

determined, respectively, by $\tan \phi = k_1/k_2$, where $\phi = \arctan 9$ and $\pi/4$.

Note that we generally need two indicator functions when there are three clusters. In our case, however, only one indicator function σ is required to describe the distribution of oscillators over the stable branches because of the symmetry $y(\phi, -\theta) = -y(\phi, \theta)$, which further leads to the symmetric phase-locked intervals about the origin and they are independent of γ and ϕ . Therefore, we use σ to represent the proportion of oscillators in the central cluster, while $1 - \sigma$ is for the rest of oscillators in the other clusters that make the same contribution to the order parameters.

B. Strong asymmetric three-cluster synchrony

Based on the phase diagram of Fig. 8(a), we choose three representative coupling strengths corresponding to three different states for the system, which are shown in Fig. 9. The asymmetric nature of three synchronized clusters is alternatively shown in Fig. 10. Again, the asymmetric cluster synchrony is characterized by (1) the frequency widths of oscillators ($\Delta\omega_{c1,c2,c3}$) in each cluster are different [color bar in Figs. 10(a) and 10(b)], (2) each cluster shows different intervals of locked phases [$\Delta\theta_{c1,c2,c3}$ in Figs. 10(a) and 10(b)], and (3) a nonuniform distribution of oscillators in the clusters [Figs. 10(d) and 10(e)]. In the case of weak asymmetric cluster synchrony C_3^w , we only observe the nonuniform distribution of oscillators in three clusters; that is, both $\Delta\theta_{c1,c2,c3}$ and $\Delta\omega_{c1,c2,c3}$ are the same for each cluster in Figs. 10(c) and 10(f).

C. Typical synchronization transition processes

Next, we show the dependence of the order parameters R_1 and R_2 following the vertical dashed arrow lines as annotated

in Fig. 8(a). That is, we vary the coupling strength k_2 for a fixed $k_1 = 3.6$, which is shown in Figs. 8(b) and 8(c).

In addition to the asynchronous regime, there are three distinct synchronous regimes: C_1 , C_3^{12} , and C_3^{23} , leading to more complex transitions when the coupling strength k_2 changes [Figs. 8(b) and 8(c)]. More specifically, for different σ , the synchronization transitions can be visually divided into two types: (1) For large $\sigma \in [0.8, 1]$, the system undergoes state transitions from incoherence to C_1 , C_3^{12} , and C_3^{23} accordingly. Along this route, the order parameter, respectively, undergoes a synchronization transition at point “A,” then exhibits multistability at point “B,” and finally, a significant synchronization fading occurs at point “E,” resulting in decreased values of $R_{1,2}$. (2) For small $\sigma \in [0.6, 0.7]$, the system undergoes state transitions from incoherence to C_1 , C_3^{12} , incoherence, and C_3^{23} . In this case, multistability of C_3^{12} yields a complete loss of

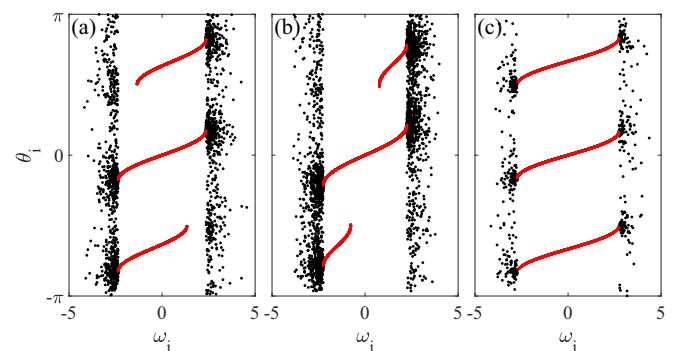


FIG. 9. Asymmetric three clusters of synchrony states of the system [Eq. (14)] when choosing different coupling strength. (a) $(k_1, k_2) = (1, 3)$ in the C_3^{23} region, (b) $(k_1, k_2) = (2.5, 1.5)$ in C_3^{12} region, and (c) $(k_1, k_2) = (0, 4)$ in the weak asymmetric cluster C_3^w region. The circular representation is shown in Fig. 10.

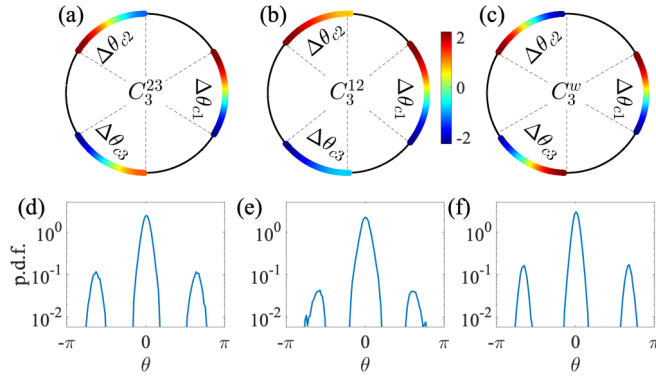


FIG. 10. Circular representation of oscillator configurations of three coherent clusters of Fig. 9. (a) $(k_1, k_2) = (1, 3)$ in the C_3^{23} region, (b) $(k_1, k_2) = (2.5, 1.5)$ in C_3^{12} region, and (c) $(k_1, k_2) = (0, 4)$ in the weak asymmetric C_3^w region. (d)–(f) the PDF of coherent phases. Numerical details: $N = 1 \times 10^5$ and $\sigma = 0.9$.

synchronization state at points “C” and “D,” resulting in $R_{1,2} = 0$. The synchronized state C_3^{23} is re-established at point “F” when coupling k_2 is further increased to $k_2 = 5.7$. Generally speaking, Figs. 8(b) and 8(c) show anomalous synchronization fading and complete loss of synchronization, which are mainly due to the existence of asymmetric multi-cluster, that is, C_3^{12} and C_3^{23} . Meanwhile, it is σ that makes it possible to observe the fading phenomenon.

VI. GENERALIZATION TO THE CASE OF THREE- AND FOUR-BODY INTERACTIONS

Based on the mean field of Eq. (5), we show that $\gamma_{1,2}$ plays an important role in forming different numbers of stable branches of synchronized solutions. Once $\gamma_{1,2}$ are fixed, the model [Eq. (3)] can be generalized to the case of any-body

coupling, which will be further illustrated by considering the example of both three- and four-body interactions, that is, the model [Eq. (15)].

The mean-field equation for Eq. (15) is given by $y(\phi, \theta) = \sin \phi \sin \theta + \cos \phi \sin 3\theta$, where $\sin \phi = k_1 R_1 R_2 / R$, $\cos \phi = k_1 R_1^3 / R$, and $R = \sqrt{(k_1 R_1 R_2)^2 + (k_2 R_1^3)^2}$. Note that this equation has different definitions for $\cos \phi$ and $\sin \phi$, which is, however, qualitatively the same as Eq. (19). In consequence, qualitatively, we obtain the same three critical cases as in Fig. 7 when $\phi \in [0, \pi/2]$. Therefore, we have the same scenario for asymmetric three-cluster synchrony as discussed for the case of $\gamma_1 : \gamma_2 = 1:3$.

The phase diagram of synchronous states in Eq. (15) is shown in Fig. 11(a), which is qualitatively the same as Fig. 8(a). Meanwhile, we show three representative L_3 lines that correspond to $\sigma = 0.8, 0.9$, and 1.

Compared with Figs. 11 and 8, the difference occurs due to different definitions of ϕ . More specifically, the separatrix L_3 between the domains of C_3^{12} and C_3^{23} depends on the indicator function σ , which is obtained by $k_1 R_2 / k_2 R_1^2 = 1$.

In addition, we follow the vertical dashed arrow line in Fig. 11(a), showing the synchronization fading phenomena of Eq. (15) while the coupling strength k_2 is increased. At point “A,” the C_1 state occurs via a saddle-node bifurcation, which is followed by the appearance of the C_3^{12} state at point “B” while crossing L_2 . The C_3^{12} state is significantly influenced by σ and loses stability to incoherent states at critical values at points “C” and “D.” In addition, the C_3^{23} states occur at points “E” and “F,” respectively, if we choose $\sigma = 0.9, 0.8$.

VII. CONCLUSIONS

In summary, we propose a substantially general Kuramoto model that includes two globally coupled simplex structures.

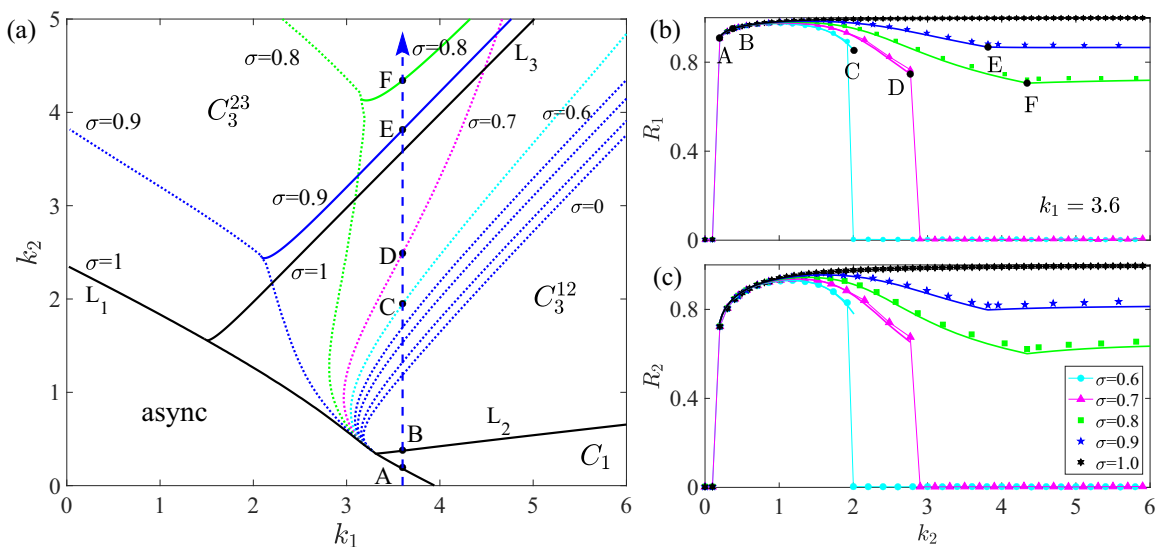


FIG. 11. (a) Phase diagram of synchronous states in dependence on coupling parameters (k_1, k_2) for Eq. (15). The vertical arrow line at position $k_1 = 3.6$ corresponds to the simulations in Eq. (15) while varying k_2 . (b) Order parameter R_1 and panel (c) is for R_2 . Five representative values of σ are shown as denoted by the legend (note that the sync transition revives at a larger coupling $k_2 \gg 6$). Dots are for the numerical simulations, while lines are from the theory. Number of oscillators $N = 1 \times 10^5$.

Depending on the choice of parameters (α, β, γ) , this model unifies different ways of implementing higher-order interactions that have been addressed separately in the literature. In order to characterize the multicluster synchrony as observed in this model for a general setting $\gamma_1 \neq \gamma_2$, we further develop a self-consistent analysis to obtain the synchronized solutions and the corresponding phase diagrams in the coupling space (k_1, k_2) . We find that strong asymmetric multicluster synchrony is observed in two coupled simplicial structures. From the phenomenological point of view, the novelty of strong asymmetric cluster synchrony shows differences in the following three aspects: (1) each cluster shows a different frequency band for the natural frequencies of synchronized oscillators, (2) the phases of each cluster are locked to different interval sizes, and (3) the oscillators are nonuniformly distributed between clusters. We note that the traditional concept of weak asymmetric cluster synchrony is referred to as property (3) only [24–26].

We also emphasize that the multistability induced by asymmetric clusters creates complex synchronization transitions. More specifically, when increasing coupling strength, the system first undergoes a transition from an incoherent state to a coherent one, which is then followed by synchronization fading involving a decrease in the order parameters and a complete loss of synchronization involving zero values for the order parameters. If the coupling continues to increase, the system that has completely desynchronized or experienced synchronization fading returns to a synchronization state, showing synchronization re-establishment. The finding of all these transitions in one system with a simple variation of coupling strength suggests that two globally coupled simplex structures show very rich and complex dynamical behavior.

Although we focus on two mean fields of 2-simplex interactions, we show that our model can be used to study n -simplex interactions. We find that the phenomena of strong asymmetric multicluster of synchronized oscillators hold for a wide class of systems that involve many-body interactions. Further cases of $k_{1,2} < 0$ are certainly interesting in showing the competition between repulsive and attractive couplings, which will be the subject of future work. In addition, it is a challenging task to consider network connectivity structures, while the present work considers all-to-all coupling only.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: EXAMPLE MATLAB CODE COMPUTING PHASE DIAGRAM

We show the main steps of the self-consistent methodology used in this work to compute the phase diagram for the case $\gamma_1 = 1, \gamma = 2$ [Eq. (12)].

```
% Define the mean field equation for the oscillator system
y = @(phi, theta) sin(phi) .* sin(theta) + cos(phi) .* sin(2 .* theta);

sigma = 1.0; % Indication function sigma in [0, 1]

% Main loop for the phi variable
for phi = 0.001:0.001:pi/2
    % Calculate phase locked intervals for the two clusters
    theta1 = acos(tan(phi) ./ 8 + sqrt(tan(phi) .* tan(phi) + 32) ./ 8);
    theta2 = acos(tan(phi) ./ 8 - sqrt(tan(phi) .* tan(phi) + 32) ./ 8);

    % Determine the phase locked interval for the overlapping region
    fs = @(theta_x) y(phi, theta_x) - y(phi, theta2);
    theta3 = fzero(fs, 0);

    % Store the boundary value for the drifting oscillators
    y1 = y(phi, theta1);

    % Loop through the order parameter R
    for R = 0.01:0.01:5
        % Calculate the first and second order integrators
```

```

F1 = F(1, phi, R, theta1, y1, theta2, theta3, sigma);
F2 = F(2, phi, R, theta1, y1, theta2, theta3, sigma);

% Calculate the order parameters
R1 = R .* F1;
R2 = R .* F2;

% Calculate the coupling strengths
k1 = sin(theta) .* R ./ R1;
k2 = cos(theta) .* R ./ (R1 .* R1);

% Plot the results
plot(k1, k2, 'ko');
hold on;
end
end

% Function to compute the self-consistent equation (18)
function F = F(m, phi, R, theta1, y1, theta2, theta3, sigma)
    f = @(theta) cos(m .* theta) .* g(R .* y(phi, theta)) .* (sin(phi) .* cos(theta)
        + 2 .* cos(phi) .* cos(2 .* theta));

    % Integrate over different intervals based on locked and drifting oscillators
    Int_lock = 2 .* integral(f, 0, theta1);
    Int_lock2 = 2 .* integral(f, theta2, pi);
    Int_lock11 = 2 .* integral(f, theta3, 0);

    g1 = @(x, theta) cos(m .* theta) .* C(x, phi) .* g(R .* x) ./ (x - y(phi, theta)
        );
    Int_drift = integral2(g1, y1, inf, 0, 2 .* pi);

    % Combine the integrals to get the final result
    F = Int_lock + (sigma - 1) .* Int_lock11 + (1 - sigma) .* Int_lock2 + 2 .*
        Int_drift;
end

```

APPENDIX B: STABILITY OF INCOHERENT STATE

In the thermodynamic limit $N \rightarrow \infty$, we introduce the density function $\rho(\theta, \omega, t)$ that describes the density of oscillators in the frequency interval $(\omega, \omega + d\omega)$ having phases $(\theta, \theta + d\theta)$ at time t . In addition, $\rho(\theta, \omega, t)$ fulfills the condition

$$\int_{-\infty}^{+\infty} \int_{-\pi}^{\pi} \rho(\theta, \omega, t) d\theta d\omega = 1. \quad (\text{B1})$$

Then, the order parameter Z_m [Eq. (4)] is expressed as

$$Z_m = \int_{-\infty}^{+\infty} \int_{-\pi}^{\pi} e^{im\theta} \rho(\theta, \omega, t) g(\omega) d\theta d\omega. \quad (\text{B2})$$

The function $\rho(\theta, \omega, t)$ fulfills the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial \theta} = 0, \quad (\text{B3})$$

where $v(t)$ is the velocity given by $v = \omega - \sum_{n=1}^2 k_n R_{\alpha_n} R_{\beta_n} \sin \gamma_n \theta = \omega + \sum_{n=1}^2 k_n (Z_{\alpha_n} Z_{\beta_n} e^{-i\gamma_n \theta} + c.c.)/2$, and $c.c.$ represents the complex conjugates.

An incoherent state with a uniform distribution of phases yields $\rho_0(\theta, \omega, t) = \frac{1}{2\pi}$ and vanishing order parameters $R_{1,2}$, which are always a solution of the system [Eq. (5)]. Next, we consider the stability of the incoherent solution $\rho_0(\theta, \omega, t)$ by introducing a perturbation function $\eta(\theta, \omega, t)$ in the following form:

$$\rho(\theta, \omega, t) = \frac{1}{2\pi} + \epsilon \eta(\theta, \omega, t), \quad (\text{B4})$$

where ϵ is small. Then, the order parameter Z_m is written as

$$Z_m = \epsilon \int_{-\infty}^{+\infty} \int_{-\pi}^{\pi} e^{im\theta} \eta(\theta, \omega, t) g(\omega) d\theta d\omega = \epsilon Z'_m. \quad (\text{B5})$$

Inserting Eqs. (B4) and (B5) in the continuity equation [Eq. (B3)] and neglecting the higher-order terms $O(\epsilon^2)$, we get the following equation for the small perturbation:

$$\frac{\partial \eta}{\partial t} + \omega \frac{\partial \eta}{\partial \theta} = 0. \quad (\text{B6})$$

From Eq. (B6), it is easy to see that the higher-order simplex interactions have no effect on the stability of the incoherent

solution $\rho_0(\theta, \omega, t)$ since the coupling $k_{1,2}$ is absent from the perturbation [Eq. (B6)]. To see this, we show that the velocity is expressed as

$$v = \omega + \epsilon^2 \sum_{n=1}^2 \frac{k_n}{2} (Z'_{\alpha_n} Z'_{\beta_n} e^{-i\gamma_n \theta} + \text{c.c.}), \quad (\text{B7})$$

which shows that the simplex interactions for the perturbation are of $O(\epsilon^2)$, not contributing to the stability of the incoherent state. Therefore, it is essential to introduce pairwise interaction to destabilize the incoherence state, for example, by $\alpha_1 = 0$ or $\beta_1 = 0$ and $\gamma_1 = 1$. Then, it follows

$$v = \omega + \epsilon \frac{k_1}{2} (Z'_1 e^{-i\theta} + \text{c.c.}) + \epsilon^2 \frac{k_2}{2} (Z'_{\alpha_2} Z'_{\beta_2} e^{-i\gamma_2 \theta} + \text{c.c.}). \quad (\text{B8})$$

To further show the stability of the incoherent state, we expand the perturbation $\eta(\theta, \omega, t)$ into Fourier series as

$$\eta(\theta, \omega, t) = \sum_{m=0}^{\infty} \eta_m(\omega, t) e^{im\theta}, \quad (\text{B9})$$

where $\eta_m = \int_{-\pi}^{\pi} \eta(\theta, \omega, t) e^{-im\theta} d\theta$. Inserting Eq. (B9) into Eq. (B6), we get

$$\dot{\eta}_1(\omega, t) = -i\omega \eta_1 + \frac{k_1}{2} \int_{-\infty}^{+\infty} \eta_1(\omega', t) g(\omega') d\omega'. \quad (\text{B10})$$

Equation (B10) shows that it is only the pairwise coupling strength k_1 that matters, leading to the critical coupling strength $2/\pi g(0)$, which is applicable to $\gamma_1 > 1$ for high-mode coupling. The incoherent state is neutrally stable in the absence of pairwise interaction with $k_1 = 0$, even though k_2 is nonzero. In the case of pure simplex interaction for a fixed γ_n , it is not possible for the system to show a spontaneous synchronization transition while increasing the coupling k_2 .

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