

Evaluating criteria for bubbling in coupled chaotic oscillators

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The stability of the synchronized state of coupled time-varying systems can be predicted with good accuracy for noiseless identical oscillators. Such an assumption, however, is unrealistic for real systems where small heterogeneity and noise are always present. In the case of chaotic oscillators, small heterogeneities and/or noise can generate, due to the existence of unstable sets embedded in the attractor, severe desynchronization bursts, known as *bubbling*, even in the coupling regions that standard linear stability analysis identifies as stable. Here, we analyze the bubbling phenomenon in coupled Rössler and Lorenz chaotic oscillators and predict when bubble-free behavior is expected. We present several different criteria, identifying the one that provides the closest prediction of the minimum coupling strength required for high-quality synchronization during a finite-time simulation.

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I. INTRODUCTION

It is well known that two chaotic oscillators can synchronize when they are coupled and the coupling strength is large enough [1,2]. One popular approach for studying their synchronization is to assume that the oscillators are identical and have no external inputs [3]. Under these assumptions, a key finding from research in the early 1980s is that the oscillators can fully synchronize and that the dynamics of each oscillator are restricted to a limited region of the phase space, known as the synchronization manifold [4].

The linear stability of the synchronized state is determined by studying the oscillators' response to perturbations transverse to the synchronization manifold. A popular tool used to perform this analysis and determine the asymptotic stability of the synchronized state is the so-called *master stability function* (MSF) [5]. We provide a brief review of the MSF formalism in Appendix B.

In the MSF approach, the largest transverse Lyapunov exponent $\lambda_{\max}^{\perp}$ determines the linear asymptotic stability of the synchronized state against transverse perturbations [6]. Stability is predicted when $\lambda_{\max}^{\perp} < 0$. Often, the coupling parameter σ —which is a quantity proportional to the coupling strength and depends on the number of oscillators and the way they are connected—is used to control the synchronization stability, with a threshold value σ_{th} predicted by the MSF (see Appendix B).

The MSF has also been widely used to study network synchronization [7–21] because it provides a simple stability criterion. However, it is only one of several possible criteria [5] and, as discussed below, it is a necessary but not sufficient condition for high-quality synchronization if the assumption of noiseless identical oscillators is violated. Moreover, these criteria mainly focus on asymptotic stability and do not exclude the possibility of brief but large-scale desynchronization events. During such an event, which can occur even when $\lambda_{\max}^{\perp} < 0$, there is a departure of the trajectory from the synchronization manifold followed by a return to the manifold.

Attractor bubbling refers to these desynchronization events [22–31] and usually occurs for coupling strengths near the MSF stability boundary, i.e., around the threshold coupling strength σ_{th} for which $\lambda_{\max}^{\perp} = 0$, and for stronger coupling, for which $\lambda_{\max}^{\perp} < 0$. The other ingredient required to observe bubbling is the presence of small oscillator heterogeneity or small noise [22,25,29,32].

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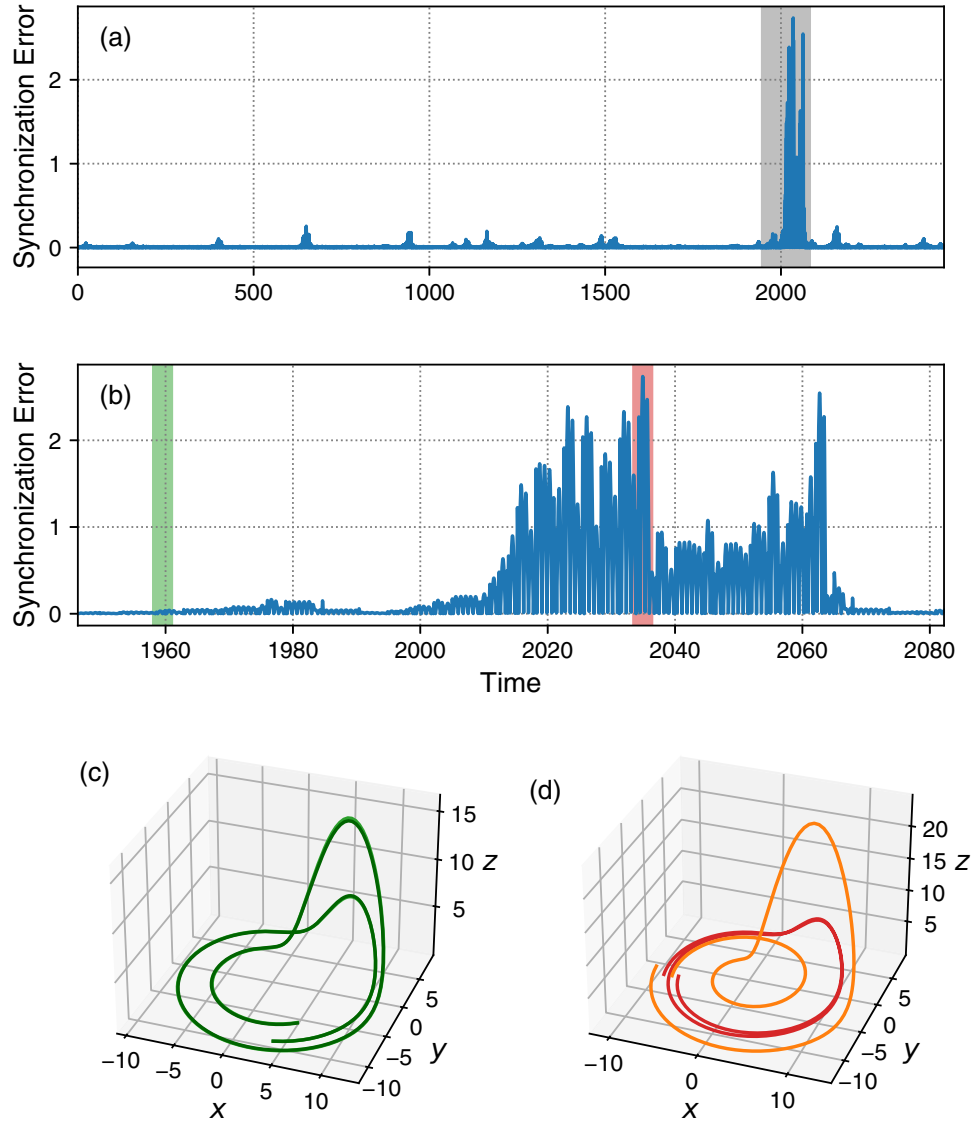


FIG. 1. Attractor bubbling in coupled Rössler oscillators. (a) Bubbling events appear as rare and large fluctuations in the temporal evolution of the synchronization error. (b) Temporal evolution of the synchronization error during the bubble occurring at $t \approx 2000$, indicated by the gray rectangle in panel (a). Trajectories of the individual oscillators during (c) the synchronized precursor [green rectangle in panel (b)] and (d) full development [red rectangle in panel (b)] of the event. The coupling parameter is $\sigma_R = 0.095$; other parameters are given in Appendix A.

Examples of bubbling events for two coupled chaotic Rössler oscillators [33] are shown in Fig. 1 using model parameters given in Appendix A. Here, we take the noise-free case with small heterogeneity $|\delta\mu_i|/|\bar{\mu}_i| = 0.1\%$, where $\bar{\mu}_i$ is a vector of model parameters and $\delta\mu_i$ is the parameter difference, and $\sigma_R > \sigma_{th}$ (nmemonic R for Rössler). Figure 1(a) shows that long epochs of small synchronization error (SE) are interrupted by large desynchronization events, where

$$SE(t) = |x_1(t) - x_2(t)|. \quad (1)$$

Here, x_i denotes the x variable of the i th oscillator and is equal to zero if the two oscillators are synchronized.

We limit our study to small parameter mismatch because large heterogeneity can induce new behaviors. For example, they can enhance stability and promote synchronization in time-delay, forced, and spatially extended systems [34–39].

Because the two oscillators are chaotic, aperiodic bubbles occur, such as the one shown in Fig. 1(b). Figures 1(c) and 1(d) show the details of the dynamics of the two oscillators right before and during the bubble. While in panel (c) the SE is still very small and the trajectories of the two oscillators are indistinguishable (high-quality synchronization), the distance between the two becomes comparable to the size of the attractor when the bubble is fully developed, as shown in panel (d).

The origin of the instability is generally understood as a sudden amplification of small transverse perturbations—generated by noise or heterogeneities—by transversely unstable invariant sets embedded in the synchronization manifold. Nevertheless, a robust criterion to predict the minimum coupling strength for high-quality synchronization is still missing.

In this work, we show how standard criteria for predicting the domain of bubbling do not agree with numerical simulations when the simulations are (necessarily) performed over a finite interval. To address this issue, we introduce new stability criteria that incorporate the time interval that the system resides in the transversely unstable regions of the synchronization manifold. One criterion—based on the finite-time transverse Lyapunov exponents and the duration of the averaging window—is the one that most precisely captures the bubbling domain evaluated for a finite simulation time.

This work is organized as follows. Section II describes the methodology used to detect the domain of bubbling. In Sec. III, we introduce the relevant concepts to analyze the transverse stability of the synchronization manifold. Section IV presents the results obtained when we systematically evaluate different criteria for predicting regions of high-quality synchronization, considering two coupled Rössler oscillators [33] (coupled Lorenz chaotic oscillators [40] are discussed in Appendix D). Finally, Sec. V presents the discussion of the results, the limitations of our study, and open problems for future research.

II. DETECTING BUBBLING

A possible method for detecting the presence of bubbling is to compare the maximum value of the synchronization error, denoted by Max SE, to the Median SE, which is less sensitive to rare large outliers than other possible indicators of synchronization, such as the root mean squared error [23,26]. Figure 2 shows that Max SE stays higher than the Median SE for considerably larger values of coupling parameter σ_R (mnemonic *R* for Rössler) for a pair of heterogeneous, coupled Rössler oscillators, and only approaches 0 for $\sigma_R \approx 0.125$ above which there is apparent bubble-free, high-quality synchronization over the duration of the simulation. The bubbling region can be delineated as the region where the Median SE ≈ 0 , while Max SE remains significantly above 0 (see the red hatched region in Fig. 2).

The issue with measuring Max SE is that it depends on the observation time. In Fig. 1, we see that large bubbles can be rare. Nevertheless, a larger bubble may appear if the observation window is extended.

Figure 3 shows the SE for two coupled Rössler oscillators as a function of σ_R for different observation times. We see that the bubbling domain, where the Max SE is large compared to the Median SE, enlarges as the integration time increases. However, the increase is small: Going from $T = 10^5$ —which was used in Fig. 2—to $T = 10^6$, the transition to high-quality synchronization only shifts to higher coupling strength by $\Delta\sigma_R \simeq 0.05$. Similarly, decreasing the integration time shifts the transition to high-quality synchronization toward a lower coupling strength. We also observe that the standard deviation of Max SE, found from simulations with different initial conditions, is large in the bubbling region for all simulation times.

III. TRANSVERSE STABILITY OF THE SYNCHRONIZATION MANIFOLD

In this section, we define the relevant concepts and notation.

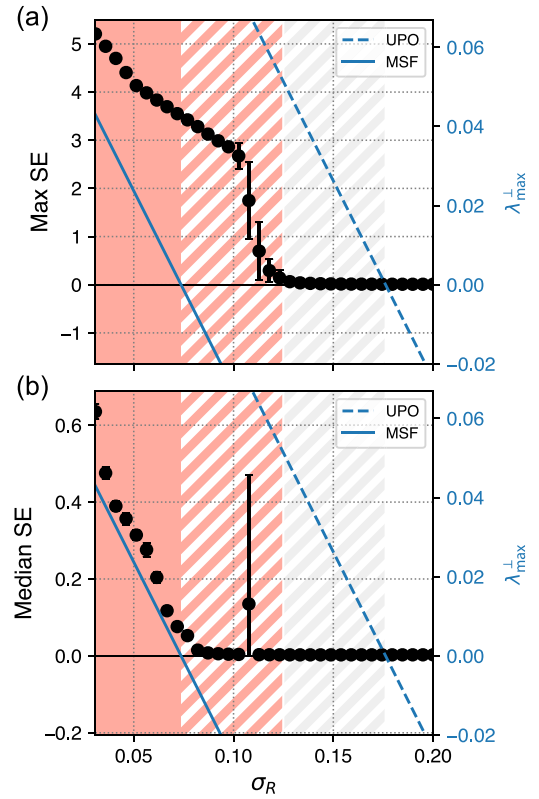


FIG. 2. The domain of bubbling. The maximum [panel (a)] and median [panel (b)] values of the synchronization error (left axis) in the x direction as a function of the coupling parameter σ_R for two heterogeneous coupled Rössler oscillators, calculated from 100 simulations with different initial conditions. Heterogeneity in the parameters is 0.1%. Details regarding the simulations are provided in Appendix A. Note the different scales of the left vertical axes. The right vertical axis shows the largest transverse Lyapunov exponent (labeled MSF, solid blue line) and the largest transverse Lyapunov exponent of the most unstable invariant sets embedded in the synchronization manifold (dashed blue line). The solid red region is where synchronization is unstable according to the MSF analysis ($\lambda_{\perp}^{\max} > 0$), while the hatched area indicates the region where there is at least one transversely unstable set. This area is divided into a red-hatched region, where we observe bubbling, and a gray region, where we do not. The boundary at $\sigma_R \sim 0.125$ is a guide to the eye to indicate when Max SE drops to a low value; it is not derived from any mathematical condition. The anomalous value of the Median SE for $\sigma_R \simeq 0.11$ is due to a periodic orbit that the two oscillators follow, but with a lag between them.

The transverse Lyapunov exponents quantify the exponential separation of trajectories over time when starting from two different initial conditions close to the synchronization manifold. A classic way to estimate the largest exponent from discretely sampled data (sample period dt) is to observe the trajectories briefly, estimate the growth or decay rate, reset the separation, and repeat many times around the synchronization manifold [41,42]. At each step, the distance $|\delta\mathbf{x}|$ between the trajectories scales as

$$|\delta\mathbf{x}| = |\delta\mathbf{x}_0| \exp(\lambda_i^{\perp} dt), \quad (2)$$

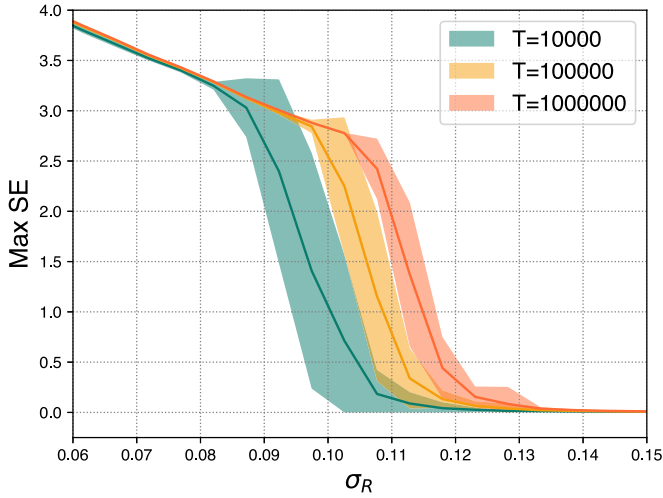


FIG. 3. Maximum synchronization error in the x direction between two coupled Rössler oscillators for three different integration times as a function of the coupling parameter. Heterogeneity in the parameters is 0.1%. Shaded areas represent an interval of ± 1 standard deviation obtained by integrating the system for 100 different initial conditions.

where $|\delta \mathbf{x}_0|$ is the initial perturbation, and the *transverse local Lyapunov exponent* (LLE) [43] is denoted by $\lambda_i^\perp$. It is well known that local Lyapunov exponents have large variation on a chaotic attractor [44] and we show below that this is also the case for the synchronization manifold.

The largest *finite-time transverse Lyapunov exponent* [41,44] is the average of the LLEs over a time interval $\tau_m = m \, dt$ and is given by

$$\bar{\lambda}_m^\perp = \frac{1}{m} \sum_{i=1}^m \lambda_i^\perp. \quad (3)$$

The largest transverse Lyapunov exponent, mentioned in the Introduction, is given by

$$\lambda_{\max}^\perp = \lim_{m \rightarrow \infty} \bar{\lambda}_m^\perp. \quad (4)$$

Transverse stability can also be assessed by considering invariant sets embedded in the attractor on the synchronization manifold. For the Rössler oscillator considered here, the invariant sets related to bubbling are unstable periodic orbits (UPOs) [23]. The trajectories of the oscillators visit the j th UPO for a typical time given by [25]

$$\tau_j^{\text{UPO}} = 1/\lambda_j^{\parallel, \text{UPO}}, \quad (5)$$

where $\lambda_j^{\parallel, \text{UPO}}$ is the largest Lyapunov exponent associated with this UPO on the synchronization manifold (indicated by $\parallel$). Here, $\lambda_j^{\parallel, \text{UPO}}$ is found by forcing the trajectories to stay in the neighborhood of the j th UPO, and the UPOs can be found by searching for near-repeated points on the attractor (see Appendix B). After a time τ_j^{UPO} , the trajectory leaves the UPO and visits another UPO, a process that repeats indefinitely [23]. These orbits are also unstable with respect to synchronization, and their transverse stability is determined by their associated transverse Lyapunov exponents $\lambda_j^{\perp, \text{UPO}}$.

We provide a summary of the notation used in this paper regarding Lyapunov exponents in Table I.

IV. RESULTS

Using the concepts introduced in the previous section, we now evaluate the validity of several criteria for high-quality, bubble-free synchronization.

Criterion 1: the master stability function. The MSF predicts that a pair of identical coupled oscillators would synchronize when

$$\lambda_{\max}^\perp < 0. \quad (6)$$

The typical method to assess whether the MSF correctly predicts synchronized behavior is to observe the dynamics over a long time for each σ_R . The MSF is validated if $\text{SE}(t)$ remains small when criterion (6) is satisfied.

From Fig. 2 we can see that this is not the case if the oscillators are slightly heterogeneous. Figure 2(a) shows the Max SE (left axis, black) and $\lambda_{\max}^\perp$ (right axis, solid blue line) as a function of σ_R . The threshold coupling strength predicted by the MSF where $\lambda_{\max}^\perp = 0$, $\sigma_{\text{th}} \simeq 0.075$, can be read off from this plot. However, we see that the Max SE approaches zero at a higher value, indicating that the MSF does not correctly predict the domain of synchronization.

Criterion 2: transversely unstable periodic orbits. Previous studies of bubbling demonstrate that it can arise from transversely unstable invariant sets embedded in the synchronization manifold [22,23]. Thus, bubble-free behavior is predicted when

$$\lambda_{\max}^{\perp, \text{UPO}} < 0, \quad (7)$$

where max indicates the largest transverse Lyapunov exponent for all UPOs [29], which is typically related to a UPO having a low period [23].

The most transversely unstable UPO for the two Rössler oscillators has an associated exponent indicated by the blue dashed line in Fig. 2. We see that it becomes negative above $\sigma \sim 0.175$. Thus, we expect bubbling to occur for $\sigma_{\text{th}} < \sigma < 0.175$. However, we do not observe any bubbling in the gray-hatched region in Fig. 2 for our finite simulation time, where there are transversely unstable UPOs, but Max SE and Median SE are very small.

Therefore, we conclude that this criterion is too restrictive for coupled Rössler oscillators; it is a sufficient but not necessary condition.

Criterion 3: cascaded UPOs. Bubble formation has two conditions: The trajectories on the synchronization manifold must be near a transversely unstable UPO ($\lambda_j^{\perp, \text{UPO}} > 0$) and they need to remain near this UPO long enough for the bubble to grow, fed by the transverse instability of the UPO. The residence time is approximately given by τ_j^{UPO} defined in Eq. (5). We find that the trajectories approach transversely unstable UPOs, but do not remain long enough to generate a bubble when σ is in the gray-hatched region in Fig. 2.

Consistent with a previous study [23], we find that the trajectory can approach a transversely unstable UPO followed by an approach to another transversely unstable UPO shortly after. We call this *cascaded amplification* as it

TABLE I. Summary of the Lyapunov exponents used in this work. Further details are provided in Appendix C.

Symbol	Invariant object/trajectory	Type	Timescale	Criterion
$\lambda_{\max}^{\perp}$	Full attractor	Transverse	Asymptotic	1
$\lambda_{\max}^{\perp, \text{UPO}}$	Most unstable UPO	Transverse	Finite time	2
$\lambda_i^{\perp, \text{UPO}}$	i th UPO	Transverse	Finite time	3
$\lambda_i^{\parallel, \text{UPO}}$	i th UPO	Longitudinal	Finite time	3
$\lambda_i^{\perp}$	Full attractor (local, at time t_i)	Transverse	Local	4
$\bar{\lambda}_m^{\perp}$	Full attractor (window of m steps)	Transverse	Finite time	5

increases the probability of a bubble by extending the time that the trajectories are in a transversely unstable region of the synchronization manifold. When σ_R is not too large, there are many transversely unstable UPOs and the probability of cascaded amplification is high. As σ_R increases, more UPOs become transversely stable and, therefore, fewer transversely unstable orbits are available to produce cascaded amplification.

To quantify this relation, we define the cascaded amplification factor that describes the worst-case bubble that can be generated, which occurs when the trajectory only visits transversely unstable UPOs, and visits all of them (i.e., the “perfect storm” scenario). In this case, the cascade amplification factor is given by

$$C = \exp \left\{ \sum_i \tau_i^{\text{UPO}} \lambda_i^{\perp, \text{UPO}} \right\}. \quad (8)$$

Here, the sum is over all transversely unstable UPOs, and hence C is in the range $1 \leq C < \infty$, with $C = 1$ when there are no transversely unstable UPOs.

Figure 4 shows that C gradually decreases with σ_R as expected. The curve is smooth and has no particular feature to indicate the start of the instability. Once criterion (7) is satisfied, $C = 1$. Thus, a criterion for bubble-free synchronization

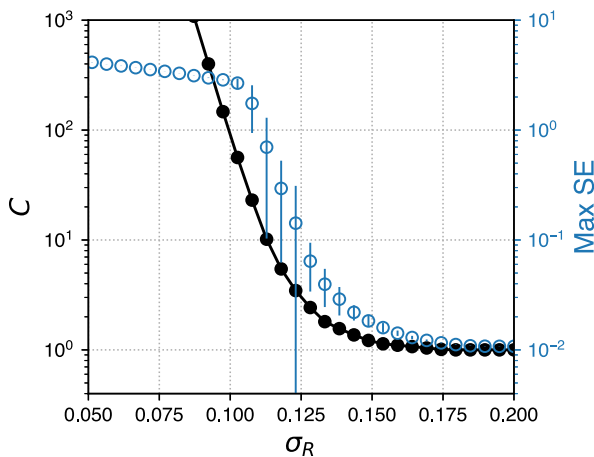


FIG. 4. Cascaded amplification induces bubbles. The cascade amplification factor C (log scale) quantifies the error amplification produced by the sequence of transitions between transversely unstable UPOs, as a function of the coupling parameter, σ_R . Here, $C = 1$ for $\sigma \geq 0.175$ because there are no transversely unstable UPOs above this coupling strength (see the blue dashed line in Fig. 2). As a comparison, we report the Max SE in blue (right axis) in log scale.

likely takes the form

$$C < C_{\text{th}}, \quad (9)$$

with C_{th} certainly greater than 1.

Some amplification of transverse perturbations can be tolerated as long as the trajectories remain in a linear neighborhood of the synchronization manifold.

Here, we take an empirical approach by finding C at the bubble–bubble-free transition occurring at $\sigma_R \sim 0.125$. At this boundary, $C_{\text{th}} \sim 3$, which is plausible.

Criterion 4: local Lyapunov exponents. Another proposed criterion for high-quality synchronization [5,24,28] is

$$\lambda_i^{\perp} < 0 \quad \text{for all } i. \quad (10)$$

That is, every point on the synchronization manifold must be transversely stable. As mentioned above, we expect that there will be substantial variation of the LLEs on the synchronization manifold [41,44].

Figure 5 shows the probability density for $\lambda_i^{\perp}$ for different values of the coupling parameters. For $\sigma_R = 0$ [panels (a) and (b)], the distribution of transverse LLEs is identical to the distribution of longitudinal LLEs and is sharply peaked at a positive value (note the log-log scale in the first panel). As expected, the average of this distribution is positive because each uncoupled oscillator is chaotic.

For $\sigma_R = \sigma_{\text{th}}$ [Fig. 5(c)], which is the MSF synchronization threshold given in Eq. (6), the distribution is similar to the uncoupled case, but is shifted toward negative values. The average of the distribution is zero, as expected, because $\sigma_R = \sigma_{\text{th}}$ is the MSF synchronization boundary. However, we see a long tail of positive LLEs, which might generate bubbling.

For much higher coupling shown in panel (d), the average of the distribution moves to negative values, but a long tail of large positive transverse LLEs remains. While the positive exponents are a potential source of transverse instability, we do not observe bubbling for this coupling value. Therefore, we conclude that criterion (10) is overly conservative: It is a sufficient but not a necessary condition.

Criterion 5: finite-time transverse Lyapunov exponents. To generate a bubble, the oscillators’ trajectories must reside in a transversely unstable part of the synchronization manifold for a long enough time to allow the instability to grow above the baseline SE. However, we do not know the unstable regions or how long the trajectory spends in these regions.

To identify the regions of the attractor that have positive LLEs, Fig. 6 shows a typical trajectory on the attractor (i.e., on the synchronization manifold), and in this figure, each point is colored according to the value of $\lambda_i^{\perp}$, where red (blue)

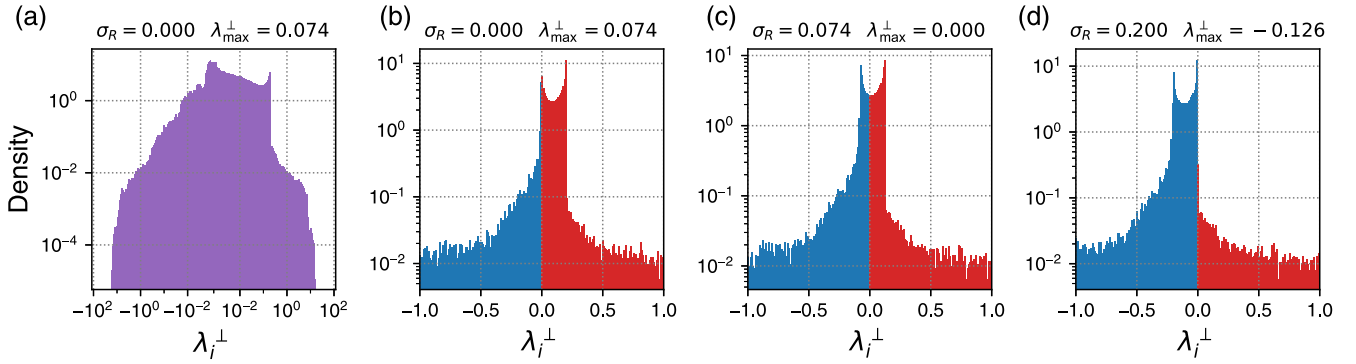


FIG. 5. Indicators of transverse instability. Distribution of the transverse local Lyapunov exponents for no coupling [panels (a) and (b)] and for values of the coupling parameter (c) at (d) and above the MSF synchronization threshold. The negative (positive) exponents are shown in blue (red). The average of the distribution, which is the largest transverse Lyapunov exponent, is also indicated in each panel.

indicates transversely unstable (stable) regions. A similar heterogeneous distribution of local Lyapunov exponents is found for Lorenz oscillators, as shown in Fig. 7.

In both attractors, regions with positive and negative LLEs alternate. Considering that a bubble develops over several cycles of the attractor (see Fig. 2) and that the trajectory passes multiple times through stable and unstable regions, the time spent in the unstable regions must be, on average, longer than the time spent in the stable regions to create a bubble.

To quantify this effect, we define an amplification factor $\mathcal{A}_m$, which varies around the attractor. It is defined in terms of the finite-time Lyapunov exponents $\bar{\lambda}_m^\perp$ for a time interval τ_m [see Eq. (3)] and is given by

$$\mathcal{A}_m = \exp(\bar{\lambda}_m^\perp \tau_m). \quad (11)$$

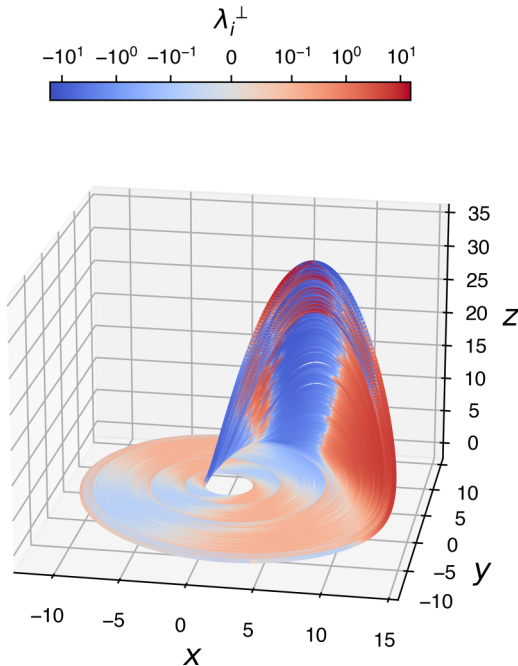


FIG. 6. Visualizing the indicators of instability. The color bar indicates the value of the transverse local Lyapunov exponents $\lambda_i^\perp$ for $\sigma_R = 0.075$.

To find $\mathcal{A}_m$, we search for the interval $\tau_{\mathcal{A}^\text{max}}$ that achieves the largest amplification $\mathcal{A}^\text{max}$, which occurs at one region of the synchronization manifold.

Figure 8 shows $\mathcal{A}^\text{max}$ and $\tau_{\mathcal{A}^\text{max}}$ as a function of the coupling parameter using an integration time of 10^5 , as in Fig. 2. Below the MSF threshold ($\sigma_R < \sigma_\text{th}$), $\mathcal{A}^\text{max}$ is very large ($\gg 10^4$) and $\tau_{\mathcal{A}^\text{max}}$ tends to 10^5 . This means there is positive amplification during most time intervals, given that the integration time is 10^5 , consistent with the MSF prediction of transverse instability.

Close to the MSF threshold ($\sigma_R \simeq \sigma_\text{th}$), there is an abrupt change in the slope of the curve, and $\tau_{\mathcal{A}^\text{max}}$ starts decreasing. For σ_R slightly larger than σ_th , $\tau_{\mathcal{A}^\text{max}}$ exhibits a second slope change. At this point, $\tau_{\mathcal{A}^\text{max}} \approx 120$, which corresponds to ≈ 20 cycles around the attractor. Here, the average period $T_\text{ave} = 5.9$, which is estimated by averaging 100 time intervals between consecutive crossings of the $x = 0$ plane. In this

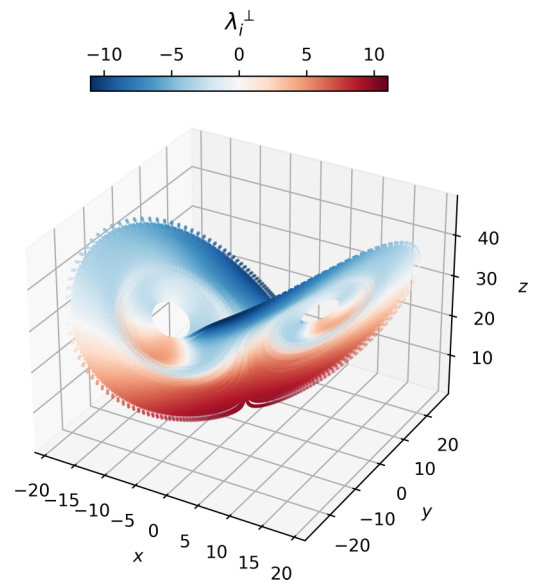


FIG. 7. Local Lyapunov exponent of the Lorenz attractor. The color bar indicates the value of the transverse local Lyapunov exponents along a trajectory of two synchronized Lorenz oscillators at the MSF boundary.

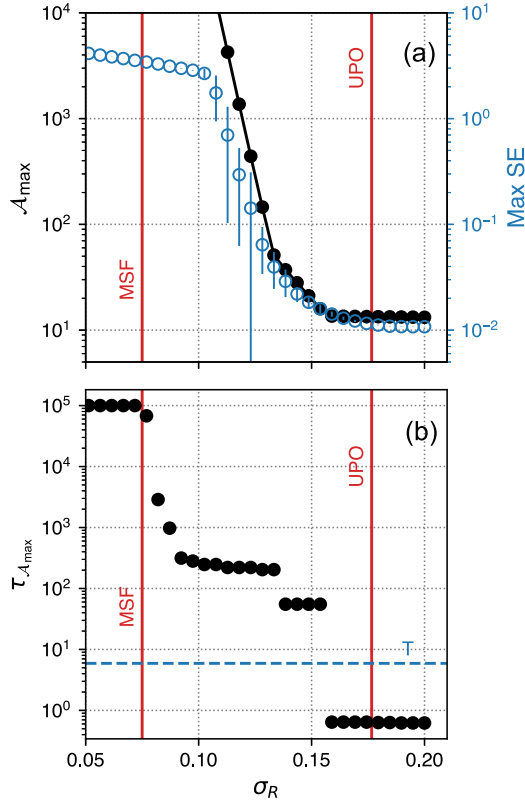


FIG. 8. Finite-time transverse Lyapunov exponents identify bubbling region. Finite-time amplification factor $\mathcal{A}_{\max}$ as a function of σ_R [panel (a)] and corresponding window size of maximum amplification $\tau_{\mathcal{A}_{\max}}$ as a function of σ_R [panel (b)] for Rössler attractors. The average oscillators' pseudoperiod, T , is reported as a horizontal dashed blue line. Vertical red lines mark the coupling values where the MSF and the most unstable sets embedded in the attractors become transversely stable. The calculation of the Lyapunov exponents and Max SE used a 10^5 time units long trajectory.

coupling region, $\mathcal{A}_{\max}$ is in the range of 10^1 – 10^3 , which is consistent with the transverse instability needed for bubbling.

At $\sigma_R \approx 0.125$, which is where we observe the bubble–bubble-free transition (i.e., the coupling parameter above which no bubbling is observed), $\mathcal{A}_{\max}$ changes slope again to a value ≈ 0 , and $\tau_{\mathcal{A}_{\max}}$ jumps to a value of ≈ 50 . This corresponds to ≈ 7 cycles around the attractor. After this transition, $\tau_{\mathcal{A}_{\max}}$ stays fixed at this value until it exhibits another sudden drop around $\sigma_R \approx 0.155$.

At this point, $\mathcal{A}_{\max}$ changes slope again to a value ≈ 0 , and $\tau_{\mathcal{A}_{\max}}$ reaches a value of ≈ 0.6 . This timescale is shorter than the average pseudoperiod around the attractor and is similar to the time that the trajectory spends in the most unstable part of the attractor.

Therefore, Fig. 8 shows that the finite-time transverse Lyapunov exponents, which determine $\mathcal{A}_{\max}$ and $\tau_{\mathcal{A}_{\max}}$, complement our understanding of bubbling. The changes in the slope of $\mathcal{A}_{\max}$ and jumps in $\tau_{\mathcal{A}_{\max}}$ are distinct and better delineate the regions of instability, bubbling, and high-quality synchronization in comparison to the MSF and UPO.

Given the abrupt drop in $\tau_{\mathcal{A}_{\max}}$ to a value less than the average period around the attractor, we propose a new criterion

for bubble-free synchronization given as

$$\tau_{\mathcal{A}_{\max}} < T_{\text{th}}. \quad (12)$$

Here, T_{th} depends on the oscillator dynamics. For our study of chaotic Rössler oscillators, a good criterion is obtained for $T_{\text{th}} = T_{\text{ave}}$. As discussed above, this result makes physical sense because the initiation of a burst takes place over several cycles around the attractor.

To assess the generality of this criterion for other types of oscillators, in Fig. 10, we show $\mathcal{A}_{\max}$ and $\tau_{\mathcal{A}_{\max}}$ for a pair of chaotic Lorenz oscillators [40] and find a jump to small values in $\tau_{\mathcal{A}_{\max}}$ like that seen in Fig. 8(b). However, $\tau_{\mathcal{A}_{\max}} < T_{\text{ave}}$ before the jump and reduces to dt after. Thus, $T_{\text{th}} < (1 + \varepsilon)dt$, where ε is a small positive number. We conjecture that this difference is due to a bubble being initiated when the Lorenz trajectories visit a neighborhood of the saddle (with zero period) embedded in the synchronization manifold.

The results of the analysis are summarized in Table II. Here, we present the mathematical condition, physical interpretation, predicted threshold, and computational requirements for each criteria.

V. DISCUSSION

We demonstrate that large-scale desynchronization events are prevalent when two heterogeneous Rössler oscillators are coupled, for coupling strengths where synchronization is expected according to MSF analysis. We remind that MSF analysis is based on a linear stability analysis of identical and noise-free oscillators.

MSF analysis was extended to nonidentical oscillators and it has been shown that small parameter heterogeneities only produce a residual desynchronisation error that scales linearly with the size of parameter mismatches [45]. Since in our work we consider very small parameter mismatches, we expected a small effect on the synchronization threshold predicted by MSF analysis. However, we find that, despite the fact that the level of heterogeneity of the parameter is very small (0.1%), the MSF synchronization threshold is considerably lower than the threshold needed for high-quality, bubble-free synchronization.

We show that it is possible to predict the region of coupling strengths where bubbling occurs. There are many criteria, and we make a systematic comparison. Criterion 1 (MSF) does not predict high-quality synchronization, but its converse, $\lambda_{\max}^{\perp} > 0$, predicts the domain of transverse instability.

Criterion 2 [Eq. (7)] predicts the bubbling domain by considering the transverse Lyapunov exponents associated with UPOs embedded in the attractor on the synchronization manifold. However, we conclude that this criterion is overly conservative for a finite observation time, because we find bubble-free synchronization where Criterion 2 predicts bubbling. In Appendix D, we show that Criterion 2 also fails for two coupled heterogeneous Lorenz oscillators, where the most unstable set embedded in the attractor is represented by a saddle.

We introduced Criterion 3 to account for the time that the system resides in a neighborhood of consecutive transversely unstable UPOs, introducing the cascaded amplification factor C , which accounts for the system visiting one unstable UPO

TABLE II. Comparison of the five synchronization criteria.

Criterion	Condition	Physical interpretation	Predicted threshold	Computational requirements
1	$\lambda_{\max}^{\perp} < 0$	The synchronized manifold must be transversely stable.	$\sigma_{\text{th}} = 0.075$	Integration of the system's tangent space [Eq. (B3)] and periodic Gram-Schmidt orthonormalizations [43].
2	$\lambda_{\max}^{\perp, \text{UPO}} < 0$	The most unstable invariant set embedded in the attractor must be transversely stable.	$\sigma_{\text{th}} = 0.175$	Finding all the unstable sets embedded in the synchronization manifold. In the case of Rössler oscillators, finding as many UPOs as possible and then perform a Floquet analysis.
3	$C < C_{\text{th}}$	Transversely unstable orbits available for cascaded amplification.	$\sigma_{\text{th}} \approx 0.15\text{--}0.175$	Same as Criterion 2.
4	$\lambda_i^{\perp} < 0$ for all i	Every point on the synchronization manifold must be transversely stable.	$\sigma_{\text{th}} = 15.4$	Same as Criterion 1.
5	$\tau_{\mathcal{A}^{\max}} < T_{\text{th}}$	Amplification timescale is smaller than the average period of the oscillators.	$\sigma_{\text{th}} = 0.155$	Same as Criterion 1, followed by a search over overlapping windows of different size.

after another. While predicting an appropriate threshold value C_{th} likely requires a nonlinear stability analysis, we conjecture that this criterion is also inaccurate because it assumes that the trajectories visit every unstable UPO consecutively and exactly once. In fact, the real sequence of UPOs might include the same orbit multiple times, and therefore the resulting amplification factor might be underestimated. In fact, we observe in Fig. 4 that the amplification factor predicted by the cascaded amplification shows slightly slower growth than Max SE.

Moreover, such a calculation requires finding all the UPOs embedded in the attractor, which is a time-consuming task and there is no way to know in advance how many there are.

Criterion 4 considers the local transverse Lyapunov exponents and predicts bubble-free behavior when $\lambda_i^{\perp} < 0$ for all i . Very large values of the coupling parameter σ_R are required to satisfy this criterion due to the long tail in the distribution of $\lambda_i^{\perp}$ [see Fig. 2(a)], yet we find bubble-free synchronization within this domain. Thus, this criterion is even more conservative than Criterion 3 and is not a necessary condition. However, we speculate that it is the safest criterion when stability is required.

We hypothesize that Criteria 2 and 4 overestimate the bubbling domain because they do not consider the time that the system resides in an unstable region of the synchronization manifold. We therefore introduce Criterion 5, which is based on the finite-time transverse Lyapunov exponents. By averaging over a finite-time window, we capture both the strength of the instability and the residence time. Such a criterion incorporates also Criterion 3, as it follows the system trajectory in its possible close passages through multiple UPOs. However, using the trajectory, it follows the natural order selected by the dynamics rather than every single UPO only once. Nevertheless, the bubbling threshold identified by this criterion is likely conservative because even a large amplification factor might not result in observable bubbles if the initial perturbation induced by the heterogeneity is small.

We introduce an amplification factor $\mathcal{A}^{\max}$ and the averaging time $\tau_{\mathcal{A}^{\max}}$ that achieves this maximum. We find that $\mathcal{A}^{\max}$ shows distinct scaling changes with σ_R at the bubbling boundaries, and that there is a sharp transition in $\tau_{\mathcal{A}^{\max}}$ at the bubbling–bubble-free boundary. This leads us to the conjecture that bubble-free synchronization is achieved when $\tau_{\mathcal{A}^{\max}} < T_{\text{th}}$, corresponding to the first significant jump in $\tau_{\mathcal{A}^{\max}}$, at which point the most efficient instability transitions abruptly from the microscopic scale of the local transverse Lyapunov exponents to the macroscopic scale of the finite-time Lyapunov exponents.

Aside from providing a good estimation of the bubbling region, Criterion 5 is also easy to implement, as the calculation of the transverse local Lyapunov exponents is the base of the calculation of the MSF. Therefore, a simple modification of an algorithm that computes the MSF can be easily implemented to obtain both $\mathcal{A}^{\max}$ and $\tau_{\mathcal{A}^{\max}}$.

We now discuss the limitations of our study. We necessarily study bubbling over a finite observation time. This results in Max SE, C , $\mathcal{A}^{\max}$, and $\tau_{\mathcal{A}^{\max}}$ being lower bounds for their true values. Hence, the coupling strength needed to observe bubble-free synchronization may be higher than the values presented here as suggested by the results shown in Fig. 3. However, we remark that performing long simulations with constant parameters is unrealistic because slow parameter drifts in real-world systems are unavoidable and, therefore, the system's long-term asymptotic behavior can be expected to be different from the real dynamics (in the same way that identical oscillators are an idealization that can fail to represent real-world systems).

Second, as in many synchronization studies, we focus on two coupled Rössler oscillators because they are popular “toy models” to represent deterministic but irregular oscillatory behavior. In Appendix D, we analyze two coupled Lorenz oscillators and find similarities with the Rössler oscillators, but also differences that we attribute to an unstable fixed point that the synchronized trajectories often approach.

Finally, our most promising criterion is based on the finite-time Lyapunov exponents, which are not invariant under a change in metric. As a result, the criterion is not universal. However, these exponents are nearly invariant, with the deviation from universality decreasing as the averaging time increases [41]. In the bubbling region, the averaging time spans tens of typical oscillations, which suggests that deviations are minimal.

Our work has important implications for the experimental study or design of coupled oscillators because any physical device has small heterogeneity or noise, and the existing criteria either do not work or are overly conservative.

Work is currently underway to explore the applicability of these criteria in networks of nonidentical chaotic oscillators [46], as well as their performance in other chaotic systems.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: MODELS

In this work, we considered pairs of nearly identical oscillators coupled through all the dynamical variables. Their evolution equations read

$$\dot{\mathbf{x}}_1 = \mathbf{F}(\mathbf{x}_1, \mu_1) + K(\mathbf{x}_2 - \mathbf{x}_1), \quad (\text{A1})$$

$$\dot{\mathbf{x}}_2 = \mathbf{F}(\mathbf{x}_2, \mu_2) + K(\mathbf{x}_1 - \mathbf{x}_2), \quad (\text{A2})$$

where $\mathbf{x}_i$ represents the state of the i th oscillator, K is the coupling strength, and $\mathbf{F}$ represents the evolution of an isolated oscillator depending on a set of parameters μ . Since the oscillators are nearly identical, we can write $\mu_1 = \bar{\mu} + \delta\mu$ and $\mu_2 = \bar{\mu} - \delta\mu$ with $|\delta\mu_i| \ll |\bar{\mu}_i|$ for each parameter. The ratio $|\delta\mu_i|/|\bar{\mu}_i|$ sets the heterogeneity level.

1. Rössler oscillators

For Rössler oscillators [33], the evolution equation is

$$F_x = -y - z, \quad (\text{A3})$$

$$F_y = x + ay, \quad (\text{A4})$$

$$F_z = b + x(z - c), \quad (\text{A5})$$

where $a = b = 0.2$ and $c = 7$, for which the dynamics of the two oscillators are chaotic. Simulations have been carried out with a heterogeneity level $|\delta\mu_i|/|\bar{\mu}_i| = 0.1\%$, using an Livermore solver for ordinary differential equations, with automatic method switching (LSODA) integration implemented in the Python package Scipy using a relative and absolute tolerance of 10^{-10} , for 10^5 time units and discarding 10^3 time units of transient. The simulations were repeated 100 times for every coupling strength.

2. Lorenz oscillators

For Lorenz oscillators [40], whose evolution is given by

$$F_x = s(y - x), \quad (\text{A6})$$

$$F_y = x(\rho - z) - y, \quad (\text{A7})$$

$$F_z = xy - \beta z, \quad (\text{A8})$$

we focused on the case $\beta = 8/3$, $\rho = 28$, $s = 10$, to ensure chaotic dynamics. As for the Rössler oscillators, simulations have been carried out with a heterogeneity level $|\delta\mu_i|/|\bar{\mu}_i| = 0.1\%$. Since with these parameters the characteristic timescale of the Lorenz oscillators is faster compared to the Rössler ones, we ran the simulations for 10^4 time units. Also in this case, the simulations were repeated 100 times for every coupling strength.

APPENDIX B: MASTER STABILITY FUNCTION

The MSF formalism allows to study the stability of a synchronized state by linearizing the dynamics in the direction transverse to the synchronization manifold $\mathbf{x}_\parallel$.

In particular, using the approximation that the oscillators are identical ($\delta\mu = 0$), one can write

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + K(\mathbf{L} \otimes \mathbf{I})\mathbf{x}, \quad (\text{B1})$$

where $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2]$ and $\mathbf{F}(\mathbf{x}) = [\mathbf{F}(\mathbf{x}_1, \bar{\mu}), \mathbf{F}(\mathbf{x}_2, \bar{\mu})]$, and $\mathbf{L}$ is the Laplacian matrix.

Linearizing this equation along the synchronization manifold, we arrive at an equation for the first-order perturbation of the trajectory $\delta\mathbf{x}$ given by

$$D\dot{\mathbf{x}} = [D\mathbf{F}(\mathbf{x}_\parallel) + K\mathbf{L} \otimes \mathbf{I}] \delta\mathbf{x}, \quad (\text{B2})$$

where D is the Jacobian operator. This equation can be further simplified with a change of coordinates in the basis that diagonalizes the coupling matrix. With such a change of coordinates, $\delta\mathbf{x}$ separates into a component that is tangent to the synchronization manifold, neutrally stable, and one that is perpendicular to the manifold, and whose stability is determined by

$$D\dot{\mathbf{x}}_\perp = [D\mathbf{F}(\mathbf{x}_\parallel) - \sigma\mathbf{I}] \delta\mathbf{x}_\perp, \quad (\text{B3})$$

where $\sigma = Ku_\perp$ is the coupling parameter, and $u_\perp$ is the eigenvalue of $\mathbf{L}$ associated with the perpendicular direction. Hence, for two coupled oscillators we have

$$\sigma = 2K. \quad (\text{B4})$$

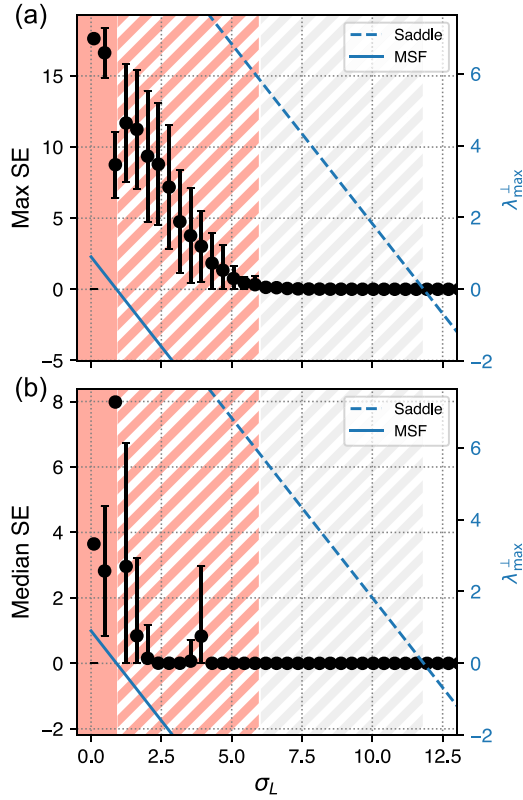


FIG. 9. The domain of bubbling for Lorenz oscillators. As Fig. 2 but for two heterogeneous coupled Lorenz oscillators.

In the text, for a clearer distinction, when analyzing two Rössler oscillators, we refer to σ_R , and when analyzing two Lorenz oscillators, we refer to σ_L .

The Lyapunov exponents of Eq. (B3) determine the stability of the transverse direction and will be a function of σ . The largest exponent, $\lambda_{\max}^{\perp}(\sigma)$, is called the master stability function, and for those coupling strengths that show $\lambda_{\max}^{\perp}(\sigma) < 0$, the synchronized solution is asymptotically stable. For the case considered here, where the coupling is linear and acting in all the variables, there is only one threshold value of the coupling parameter for which $\lambda_{\max}^{\perp}(\sigma_{\text{th}}) = 0$, and the synchronized solution is stable for all $\sigma > \sigma_{\text{th}}$.

We computed the Lyapunov exponents of Eq. (B3) using the Benettin algorithm [6,43] with an integration step of 0.001 and renormalizing every ten steps. The length of the trajectories matched those of the simulations of the paired dynamics. At each renormalization step, we obtain the transverse local Lyapunov exponents of the dynamics $\lambda_i^{\perp}$, which can therefore be obtained “for free” during the calculation of the MSF.

Unstable periodic orbits and their stability

The UPOs for the Rössler oscillators have been numerically calculated following two steps. First, we computed a Poincaré map using the section $x = 0$ for a very long trajectory, where we identified regions of high density that could be related to UPOs. Second, we used these regions as starting points for a Newton method to find closed orbits in the Poincaré map.

Duplicated UPOs were removed, and we were left with 27 orbits, with periods ranging from period 1 to period 8. We did not search for orbits above period 8, as the most unstable ones are usually those with lower periods, as stated in the main text.

We determined the level of instability of each orbit with Floquet analysis. The formalism of the MSF can be easily extended to the UPOs to determine the transverse Floquet multipliers and the transverse Lyapunov exponents from them.

APPENDIX C: LYAPUNOV EXPONENTS' NOTATION SUMMARY

(1) $\lambda_{\max}^{\perp}$: Largest transverse Lyapunov exponent that, in the MSF approach, determines the linear asymptotic stability of the synchronized state against transverse perturbations.

(2) $\lambda_i^{\perp}$: Transverse local Lyapunov exponent in the region of the attractor that is visited at time t_i ; $\lambda_{\max}^{\perp}$ is the average of the distribution of $\lambda_i^{\perp}$ values.

(3) $\lambda_i^{\parallel}$: LLE that describes the evolution of a small perturbation along the synchronization manifold, in the region of the attractor that is visited at time t_i . Without coupling, they are the same as the transverse LLEs.

(4) $\lambda_i^{\perp, \text{UPO}}$: Lyapunov exponent of the i th unstable periodic orbit on the synchronization manifold (also known as parallel or longitudinal Lyapunov exponent).

(5) $\lambda_i^{\perp, \text{UPO}}$: Transverse Lyapunov exponent of the i th unstable periodic orbit.

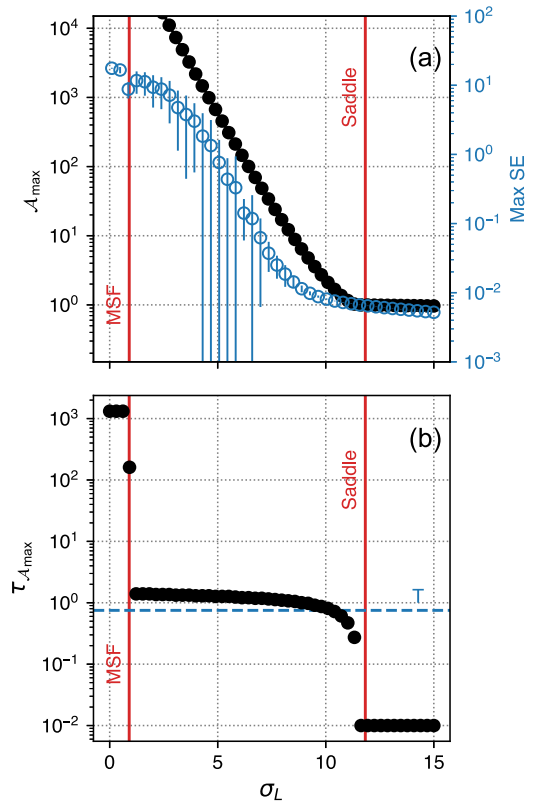


FIG. 10. Finite-time transverse growth for coupled Lorenz oscillators. As Fig. 8 but for coupled Lorenz oscillators.

TABLE III. Summary of synchronization thresholds and qualitative behavior for the Rössler and Lorenz systems. For the Lorenz system, Criterion 3 was not evaluated because analyzing all unstable invariant sets is computationally expensive.

	Rössler	Lorenz
Max SE bubbling boundary	$\sigma_R \approx 0.125$	$\sigma_L \approx 6.0$
Most unstable invariant set	Period-1 UPO	Saddle
Criterion 1	$\sigma_R = 0.075$	$\sigma_L = 1.0$
Criterion 2	$\sigma_R = 0.175$	$\sigma_L = 11.5$
Criterion 3	$\sigma_R \approx 0.15-0.175$	Not evaluated
Criterion 4	$\sigma_R = 15.4$	$\sigma_L = 11.3$
Criterion 5	$\tau_{A_{\max}}$ decreases with σ_R , exhibiting multiple sharp drops and saturating for σ_R smaller than the value predicted by the UPO	$\tau_{A_{\max}}$ decreases with σ_L , exhibiting sharp drops near the MSF and saddle thresholds; it saturates at a small value beyond the saddle threshold.

(6) $\lambda_m^\perp$: Finite-time transverse Lyapunov exponent, which is the average of $\lambda_i^\perp$ over a window of m discrete time steps, $\lambda_m^\perp = (1/m)\sum_{i=1}^m \lambda_i^\perp$.

APPENDIX D: COUPLED LORENZ OSCILLATORS

In Fig. 9, we report the maximum and the Median SE for two coupled Lorenz oscillators as a function of the coupling parameter σ_L (mnemonic L for Lorenz). As for the Rössler oscillators, the heterogeneity of their parameters is set to 0.1%. Also in this case, the maximum SE detects bubbling well beyond the synchronization threshold predicted by the MSF. In this case, bubbling is so severe that the Median SE is also significantly affected at least up to coupling strengths that are twice the MSF synchronization threshold. For the Lorenz

attractor, the most unstable embedded invariant set is represented by the saddle in the origin, whose separatrix $x = y = 0$ pierces the attractor in the conjunction of the two lobes. The saddle becomes transversely stable around $\sigma_L \simeq 11.5$. Like the Rössler oscillators, the coupling threshold above which bubbling vanishes sits more or less halfway between the MSF threshold and the saddle transition to transverse stability.

Figure 10 shows the linear finite-time growth of transverse perturbations for two coupled Lorenz oscillators, analogously to Fig. 8. The finite-time growth predicts a sharper transition than the one observed in the simulation (blue dots), but roughly captures the scaling of the maximum SE with the coupling parameter σ_L . Table III summarizes these results and presents a comparison of the criteria applied to the Rössler and Lorenz systems.

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