

From coupled $\mathbb{Z}_3$ Rabi models to the $\mathbb{Z}_3$ Potts model

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(Received 17 April 2026; accepted 31 August 2026; published 14 October 2026)

We study a $\mathbb{Z}_3$ -symmetric Rabi model that describes a three-level system coupled to two bosonic modes. We derive a mapping of the two-mode $\mathbb{Z}_3$ Rabi model onto a qubit-boson ring. This mapping allows us to formulate a realistic implementation of the $\mathbb{Z}_3$ Rabi model based on superconducting qubits. It also provides context for the previously proposed optomechanical implementation of the $\mathbb{Z}_3$ Rabi model. In addition, we propose a physical implementation of the $\mathbb{Z}_3$ Potts model via a coupled chain of $\mathbb{Z}_3$ Rabi models.

DOI: [10.1103/ybt7-yh9v](https://doi.org/10.1103/ybt7-yh9v)

I. INTRODUCTION

The field of quantum simulations has entered a mature phase in which a variety of hardware platforms can reproduce many-body Hamiltonians that are analytically intractable. Trapped-ion chains have been used to implement long-range Ising models and lattice gauge theories with tens of qubits [1–6], neutral-atom Rydberg arrays routinely emulate spin models on two-dimensional lattices [7–11], and semiconductor spin-qubit platforms have realized Fermi-Hubbard and other models at the few-plaquette level [12–17]. These successes show that controlled programmable devices can already access a wide variety of interesting physics, such as symmetry breaking, topological order, and nonequilibrium critical phenomena that lie well beyond classical reach. One of the interesting research directions is to simulate models with larger discrete symmetries, e.g., $\mathbb{Z}_3$ symmetry, as they often exhibit richer physics than the models with $\mathbb{Z}_2$ symmetry.

Recent advances in circuit quantum electrodynamics (cQED) provide a complementary route to this goal, focusing on light-matter systems where photons mediate interactions between superconducting (SC) qubits [18–20]. In cQED, the quantum Rabi model [21–24]—which describes a two-level atom coupled to a bosonic mode—serves as the canonical building block and possesses a discrete $\mathbb{Z}_2$ (parity) symmetry. In the ultrastrong-coupling regime, a chain of coupled Rabi models effectively realizes the Ising model [25]. There, photon-induced interactions generate nearly degenerate “cat” ground states with macroscopic qubit-cavity entanglement. Extending this connection between discrete symmetries and entangled phases from $\mathbb{Z}_2$ to $\mathbb{Z}_3$ [26–29] naturally leads to

the paradigmatic three-state Potts model and promises richer symmetry-breaking behavior. Recently, there has been significant interest in the $\mathbb{Z}_3$ quantum Rabi model from both theoretical and experimental perspectives. The generalization of Albert *et al.* of the original ($\mathbb{Z}_2$) Rabi Hamiltonian to an N -state atom [26] established the formal framework for studying generalized Rabi models with higher discrete symmetries. Subsequently, the first proposal for the experimental realization of the $\mathbb{Z}_3$ Rabi model based on the optomechanical (OM) system was discussed in Ref. [28]. Interestingly, it was shown that in the $\mathbb{Z}_3$ Rabi model, the superradiant transition is first order within the mean-field approach, unlike in the $\mathbb{Z}_2$ Rabi model, where it is second order [28]. A recent proposal has shown how a network of Josephson junctions (JJs) can implement the $\mathbb{Z}_3$ Potts model using tunable inductive elements [30]. Here, we introduce a tunable model that connects the $\mathbb{Z}_3$ Rabi model with the $\mathbb{Z}_3$ Potts model. By varying the coupling parameter, the system evolves from a few-level regime into one that spontaneously breaks the threefold rotational symmetry.

The remainder of the paper is organized as follows. In Sec. II, we give a brief overview of the $\mathbb{Z}_3$ Rabi model and derive the mapping of the $\mathbb{Z}_3$ Rabi model onto a certain qubit-bosonic model. We then illustrate how this qubit-bosonic model can be realized by superconducting-qubit and optomechanical platforms. In Sec. III, we show how the $\mathbb{Z}_3$ Potts model can be simulated by a chain of the $\mathbb{Z}_3$ Rabi models. We also illustrate it with superconducting and optomechanical implementations. At the end of Sec. III, we briefly discuss the chiral $\mathbb{Z}_3$ Potts model and the possibility of its implementation based on a chain of the $\mathbb{Z}_3$ Rabi models. Finally, in Sec. IV, we conclude. Some additional details are covered in Appendixes A–C.

II. $\mathbb{Z}_3$ RABI MODEL

A. Theoretical description

The central topic of this paper is the $\mathbb{Z}_3$ -symmetric Rabi and Potts models, as well as the connection between them.

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In this section, we discuss a $\mathbb{Z}_3$ -symmetric model of two bosonic modes coupled to a three-level system (qutrit). The model generalizes the usual Rabi model to the case of multiple bosonic modes and higher discrete symmetry. In the following, we refer to this model as the $\mathbb{Z}_3$ Rabi model. For a more detailed discussion of this model and related ones, we refer the reader to our companion paper [31]. The Hamiltonian reads

$$H_R = \hbar\Omega_R(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2) + B(e^{i\phi}Z + e^{-i\phi}Z^\dagger) - \lambda(\hat{a}_1 + \hat{a}_2^\dagger)X - \lambda(\hat{a}_1^\dagger + \hat{a}_2)X^\dagger, \quad (1)$$

where $\hat{a}_i^\dagger$ ($\hat{a}_i$) with $i = 1, 2$ are boson creation (annihilation) operators [32], and X and Z are 3×3 matrices generalizing the Pauli matrices (usually referred to as the shift and clock matrices, respectively). The parameters have the following physical meaning: Ω_R is the frequency of the two bosonic modes, B and ϕ define the energy levels of the qutrit, λ is the coupling strength between the bosonic modes and the qutrit, and $\hbar$ is the Planck constant. Let us note in passing that in principle one can have two inequivalent ways to write a two-mode $\mathbb{Z}_3$ Rabi model; see the Appendix of the companion paper [31]. We choose the specific form of the Hamiltonian (1) because it can be naturally mapped onto a realistic physical system, as we discuss in Sec. II B.

The $\mathbb{Z}_3$ shift X and clock Z matrices are commonly used to describe qutrit systems. They are defined as

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad (2)$$

where $\omega = \exp[2\pi i/3]$ is the principal cubic root of unity. The clock and shift matrices are unitary but not Hermitian [33].

The boson part of the $\mathbb{Z}_3$ symmetry is simply a residue of the bosonic $U(1)$ symmetry, while its qutrit part is generated by the Z -matrix. As a result, the $\mathbb{Z}_3$ symmetry generator has the following form:

$$\mathcal{P}_R = \exp\left[\frac{2\pi i}{3}(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2)\right]Z, \quad (3)$$

where the minus sign ensures that the operator $\mathcal{P}_R$ commutes with the Hamiltonian H_R . In the following, we provide some additional details about the $\mathbb{Z}_3$ Rabi model that will be useful for our purposes. For a more thorough discussion, we refer to the companion paper [31].

One of the main results in the present paper is an explicit construction of the $\mathbb{Z}_3$ Potts model based on a chain of coupled $\mathbb{Z}_3$ Rabi models, described by Eq. (1) (see Sec. III). One of the key ingredients in this construction is the strong boson-qutrit interaction. In particular, we are interested in what is generally called the *deep-strong-coupling* regime in the Rabi model literature: $\lambda > \hbar\Omega_R$ [34,35]. In addition, we require the limit of low magnetic field: $\hbar\Omega_R \gg B$. This coupling regime leads to the full entanglement between the boson and qutrit degrees of freedom. Specifically, the three lowest eigenstates turn out to be the $\mathbb{Z}_3$ cat states [31]:

$$|\psi_k\rangle = \frac{1}{\sqrt{3}} \sum_{l=0}^2 \omega^{lk} |\omega^{-l}\lambda/\hbar\Omega_R\rangle_{\hat{a}_1} |\omega^l\lambda/\hbar\Omega_R\rangle_{\hat{a}_2} |\omega^{-l}\rangle, \quad (4)$$

where the subscripts $\hat{a}_1$ and $\hat{a}_2$ denote that it is a coherent state of the first and second bosonic modes, respectively. The states $|\psi_k\rangle$ are the $\mathbb{Z}_3$ generalization of the $\mathbb{Z}_2$ cat states $|+\alpha\rangle|+\rangle \pm |-\alpha\rangle|-\rangle$. The latter roughly describe the ground and the first-excited states of the quantum Rabi model [23]. As can be seen, each of the ordinary ($\mathbb{Z}_2$) cat states is a linear combination of two coherent states $|\alpha\rangle$ and $|\alpha\rangle$ entangled with the qubit. In the $\mathbb{Z}_3$ case, each $|\psi_k\rangle$ is a linear combination of *three* coherent states $|\omega^i\alpha\rangle$, with $i = 0, 1, 2$. Similarly, in the two-mode $\mathbb{Z}_3$ Rabi model, the corresponding eigenstates $|\psi_k\rangle$ are joint cat states of the two bosonic modes and the qutrit. In the case of the $\mathbb{Z}_3$ Rabi model, the coherent parameter is expressed through the model's coupling constants, $\alpha = \lambda/\hbar\Omega_R$. In the regime in which we are interested, $\lambda > \hbar\Omega_R$, $B \ll \hbar\Omega_R$, the eigenenergies of the $\mathbb{Z}_3$ cat states are given by

$$\epsilon_k = \langle \psi_k | H_R | \psi_k \rangle = 2Be^{-3(\lambda/\hbar\Omega_R)^2} \cos\left[\frac{2\pi k}{3} + \phi\right] - \frac{2\lambda^2}{\hbar\Omega_R}. \quad (5)$$

In the limit $\lambda/\hbar\Omega_R \rightarrow \infty$, the energy splitting between the states goes to zero. However, we are mostly interested in the intermediate regime with the splitting being small but nonzero. Now that we have reviewed all the necessary details about the $\mathbb{Z}_3$ Rabi model (1), we proceed to map this model onto a physically realistic system.

B. Qubit-boson ring

Although our ultimate goal is to present a physically realistic model for the realization of the Rabi model $\mathbb{Z}_3$, we first discuss a construction that serves as a link between the Rabi model and specific microscopic physical implementations. Namely, we show how the two-mode $\mathbb{Z}_3$ Rabi model (1) appears in a particular model describing a three-site qubit-boson (QB) ring. This model is a central construction of this paper. Although still purely theoretical, it is more suitable for mapping to concrete physical platforms than the original $\mathbb{Z}_3$ Rabi model.

Consider the QB ring with the Hamiltonian [28]

$$\begin{aligned} H_{QB} &= \epsilon \sum_{j=0}^2 \sigma_j^z + \hbar\Omega_{QB} \sum_{j=0}^2 \hat{b}_j^\dagger \hat{b}_j + V_{QB}, \\ V_{QB} &= g \sum_{j=0}^2 (\sigma_j^+ \sigma_{j+1}^- e^{i(\hat{x}_j - \hat{x}_{j+1})} + \text{H.c.}) \\ &= g \sum_{j=0}^2 (\sigma_j^+ \sigma_{j+1}^- e^{i\frac{\eta}{\sqrt{2}}(\hat{b}_j + \hat{b}_j^\dagger - \hat{b}_{j+1} - \hat{b}_{j+1}^\dagger)} + \text{H.c.}), \end{aligned} \quad (6)$$

where the Pauli matrix σ_j^z corresponds to the qubit at site j , $\hat{b}_j^\dagger$ ($\hat{b}_j$) is the creation (annihilation) operator of the j th boson, described by a harmonic oscillator on the real line $\mathbb{R}$, ϵ defines the Zeeman splitting of the qubit energy levels, g is the coupling strength, and Ω_{QB} is the boson frequency. The periodic boundary conditions are chosen; therefore, the position index j is valued modulo 3, e.g., $j+3 \equiv j$. We can express the

dimensionless coordinate and momentum operators for each bosonic mode through the creation and annihilation operators

$$\hat{x}_j = \frac{\eta}{\sqrt{2}}(\hat{b}_j + \hat{b}_j^\dagger), \quad (7)$$

$$\hat{p}_j = \frac{i}{\sqrt{2}\eta}(\hat{b}_j^\dagger - \hat{b}_j). \quad (8)$$

The creation and annihilation operators here diagonalize the Hamiltonian of the harmonic oscillator in the following form:

$$H_j = \hbar\Omega_{\text{QB}} \left(\frac{\hat{x}_j^2}{2\eta^2} + \frac{\eta^2 \hat{p}_j^2}{2} \right), \quad (9)$$

where η is the dimensionless squeezing parameter. We prefer this over a more conventional notation since it maps naturally onto the case of the superconducting circuits; i.e., $\hat{x}_j$ corresponds to the normalized flux $\hat{\phi}_j$ and $\hat{p}_j$ corresponds to the normalized charge $\hat{n}_j$. The dimensionless parameter η is important here since it gives us an additional parameter to tune the system.

The system has a $U(1)$ symmetry that conserves the z component of the total spin, $\sum_j \sigma_j^z$, as it clearly commutes with the Hamiltonian (6). Thus, rather than working in the full Hilbert space, we restrict ourselves to the single-excitation sector, i.e., the subspace with exactly one qubit in the $|\uparrow\rangle$ state. In other words, the single-excitation sector is given by

$$\mathcal{H}_{\text{QB},1} = \text{Span}\{|\uparrow\downarrow\downarrow\rangle, |\downarrow\uparrow\downarrow\rangle, |\downarrow\downarrow\uparrow\rangle\}. \quad (10)$$

In order to prove that the QB ring [Eq. (6)] restricted to the single-excitation sector $\mathcal{H}_{\text{QB},1}$ [Eq. (10)] is equivalent to the $\mathbb{Z}_3$ Rabi model, we first apply several algebraic transformations to the Hamiltonian H_{QB} [Eq. (6)]. To start with, we use the spin-dependent momentum translation to simplify the interaction term V_{QB} [Eq. (6)]. Then, we apply the Fourier transform, which decouples one of the bosonic modes from the rest of the system. Afterward, the resulting Hamiltonian is restricted to the single-excitation sector $\mathcal{H}_{\text{QB},1}$ [Eq. (10)]. We obtain the two-mode $\mathbb{Z}_3$, which is brought to canonical form with some final adjustments.

a. Spin-dependent momentum translation. Applying the unitary operator $S = \prod_{j=0}^2 e^{i\sigma_j^z \hat{x}_j/2}$ removes the exponentials in $\hat{V}_{\text{QB}}$ and yields

$$\begin{aligned} S^{-1}H_{\text{QB}}S = & \epsilon \sum_{j=0}^2 \sigma_j^z + \hbar\Omega_{\text{QB}} \sum_{j=0}^2 \hat{b}_j^\dagger \hat{b}_j + \frac{\eta^2 \hbar\Omega_{\text{QB}}}{2} \sum_{j=0}^2 \hat{p}_j \sigma_j^z \\ & + \frac{3}{4} \eta^2 \hbar\Omega_{\text{QB}} + g \sum_{j=0}^2 (\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+). \end{aligned} \quad (11)$$

After transformation, the interaction term V_{QB} involves only qubit degrees of freedom, at the price of an additional spin-boson term proportional to $\hat{p}_j \sigma_j^z$.

b. Fourier transform. To proceed, we define momentum-space operators via the Fourier transform,

$$\hat{b}(k) = \frac{1}{\sqrt{3}} \sum_{j=0}^2 e^{-2\pi i j k/3} \hat{b}_j, \quad S^z(k) = \sum_{j=0}^2 e^{-2\pi i j k/3} \sigma_j^z, \quad (12)$$

where $k = 0, 1, 2$. Here, we use a different normalization for the Fourier transform of the spin operators for the following to

be true: $S^z(0) = \sum_j \sigma_j^z$. The Fourier transform components of the spin ladder operators $S^\pm(k)$ are defined analogously. We use parentheses to distinguish momentum-space operators, which depend on the wave vector k , from real-space operators, which carry a site index as a subscript. In terms of these operators, Eq. (11) takes the form

$$\begin{aligned} H_{\text{FT}} = & \epsilon S^z(0) + \hbar\Omega_{\text{QB}} \sum_{k=0}^2 \hat{b}^\dagger(k) \hat{b}(k) \\ & + \frac{\eta^2 \hbar\Omega_{\text{QB}}}{2\sqrt{3}} \sum_{k=0}^2 \hat{p}(k) S^z(-k) + \frac{3}{4} \eta^2 \hbar\Omega_{\text{QB}} \\ & + \frac{g}{3} \sum_{k=0}^2 (e^{2\pi i k/3} S^+(k) S^-(-k) + \text{H.c.}), \end{aligned} \quad (13)$$

where the zero-momentum bosonic mode ($k = 0$) couples to $S^z(k = 0)$, which is conserved. Hence, the zero-momentum boson actually decouples from the rest of the system. We are left with only two bosonic modes, exactly as required for the $\mathbb{Z}_3$ Rabi model.

c. Restriction to $\mathcal{H}_{\text{QB},1}$. To restrict the Hamiltonian (13) to the single-excitation sector $\mathcal{H}_{\text{QB},1}$ [Eq. (10)], we identify the single-excitation spin states with qutrit states, $|0\rangle \equiv |\uparrow\downarrow\downarrow\rangle$, $|1\rangle \equiv |\downarrow\uparrow\downarrow\rangle$, and $|2\rangle \equiv |\downarrow\downarrow\uparrow\rangle$. After discarding constant terms and the decoupled zero-momentum mode, we obtain

$$\begin{aligned} H_{\text{QB},1} = & \hbar\Omega_{\text{QB}} \sum_{k=1}^2 \hat{b}^\dagger(k) \hat{b}(k) + g(e^{2\pi i/3} X + e^{-2\pi i/3} X^\dagger) \\ & + i \frac{\eta \hbar\Omega_{\text{QB}}}{\sqrt{6}} [-(\hat{b}(1) - \hat{b}^\dagger(2))Z - (\hat{b}(2) - \hat{b}^\dagger(1))Z^\dagger], \end{aligned} \quad (14)$$

where X and Z matrices act on the qutrit formed by the single-excitation sector $\mathcal{H}_{\text{QB},1}$ [Eq. (10)].

d. Final rearrangement. The Hadamard transform on the spin degrees of freedom, combined with a global bosonic $U(1)$ phase rotation, brings the Hamiltonian to the canonical $\mathbb{Z}_3$ two-mode Rabi form:

$$\begin{aligned} H_{\text{R}} = & \hbar\Omega_{\text{QB}}(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) + g(e^{2\pi i/3} Z + e^{-2\pi i/3} Z^\dagger) \\ & - \frac{\eta \hbar\Omega_{\text{QB}}}{\sqrt{6}} [(\hat{a}_1 + \hat{a}_2^\dagger)X + (\hat{a}_2 + \hat{a}_1^\dagger)X^\dagger], \end{aligned} \quad (15)$$

where we identified $\hat{a}_k = \hat{b}(k)$ and $\hat{a}_k^\dagger = \hat{b}^\dagger(k)$ for $k = 1, 2$.

Comparing Eq. (15) with Eq. (1), one explicitly identifies the parameters of the QB ring model with the corresponding parameters of the Rabi model:

$$\Omega_{\text{R}} = \Omega_{\text{QB}}, \quad B = g, \quad \phi = \frac{2\pi}{3}, \quad \lambda = \frac{\eta \hbar\Omega_{\text{QB}}}{\sqrt{6}}. \quad (16)$$

Consequently, the deep-strong-coupling regime, $\lambda > \hbar\Omega_{\text{R}}$, corresponds to the large squeezing parameter, $\eta > \sqrt{6}$. For future convenience, we provide an explicit matrix representation of the “magnetic” term,

$$g(e^{2\pi i/3} Z + e^{-2\pi i/3} Z^\dagger) = \begin{pmatrix} -g & 0 & 0 \\ 0 & -g & 0 \\ 0 & 0 & 2g \end{pmatrix}. \quad (17)$$

As shown, the single-excitation sector $\mathcal{H}_{\text{QB},1}$ of the QB ring [Eq. (6)] indeed realizes the $\mathbb{Z}_3$ Rabi model (1). Moreover, the mapping is exact. We note that superconducting qubit platforms are natural for engineering the interaction [Eq. (6)], although analogous constructions are possible for other platforms as well. For example, the optomechanical system implements slightly more general interaction V_{QB} than that of Eq. (6). However, it can still be mapped to the $\mathbb{Z}_3$ Rabi model [28]. The general V_{QB} is discussed in Appendix A.

Why so complicated?

The suggested implementation of the $\mathbb{Z}_3$ Rabi model from the previous section is considerably more complicated than the setups for the usual $\mathbb{Z}_2$ Rabi model. Indeed, in order to obtain the $\mathbb{Z}_2$ Rabi model, it is enough to simply couple a qubit with a boson [36]. Therefore, one may wonder whether these complications are truly necessary. We argue that they are, because straightforward generalizations of the $\mathbb{Z}_2$ Rabi model to higher $\mathbb{Z}_n$ symmetries encounter fundamental obstacles. In this section, we provide some arguments to explain why this is the case.

There are two most common ways to get a two-level quantum system. It is possible to use a spin as a two-level system [37–40] or two levels in an anharmonic oscillator. For instance, in the latter case, the Rabi model describes a superconducting qubit coupled to a cavity [29,35,41–45]. Therefore, in order to get a $\mathbb{Z}_2$ Rabi model, one has to take either a spin or an anharmonic oscillator and couple it to a bosonic mode using the interaction term $\hat{V} = \sigma^x \hat{x}_{\text{boson}}$, where σ^x is the Pauli matrix.

Both ways are easily generalized to the $\mathbb{Z}_3$ case. Using a spin-1 system might seem natural, but in this case the interaction is realized by the spin-1 s^x matrix:

$$s^x = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}. \quad (18)$$

It is easy to see that s^x is not $\mathbb{Z}_3$ symmetric. In other words, the nice properties of the spin-1/2 (two-level) Pauli σ^x do not generalize to the spin-1 case. This does not prove that building a $\mathbb{Z}_3$ -symmetric system (or higher $\mathbb{Z}_n$ -symmetric systems) out of spins is impossible (we have shown one way to do it in the previous section). However, it illustrates why more intricate constructions are required to realize higher discrete symmetries.

Similarly, if we try to use the three lowest levels of an anharmonic oscillator, the truncated coordinate operator is not $\mathbb{Z}_3$ symmetric,

$$\hat{x}_{\text{an}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}. \quad (19)$$

As a result, the $\mathbb{Z}_2$ Rabi model based on the anharmonic oscillator does not directly generalize to the higher $\mathbb{Z}_n$ Rabi models as well.

These obstacles make an experimental realization of $\mathbb{Z}_n$ Rabi models nontrivial, which motivates our proposal. The rationale provided in this section is in no way a formal proof. However, the argument should serve as an intuition as to why

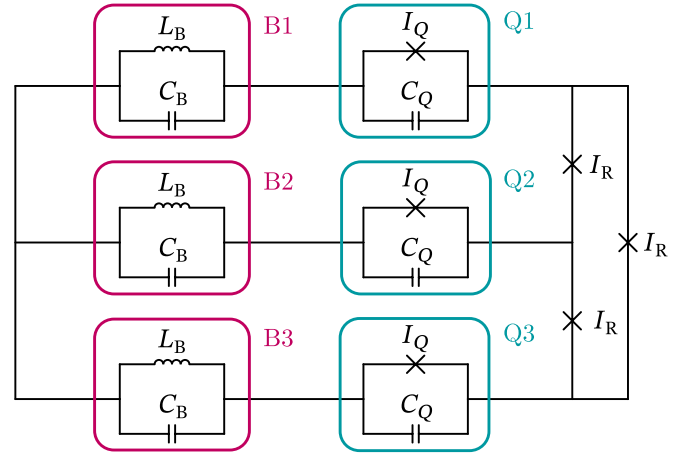


FIG. 1. Superconducting-circuit implementation of the two-mode $\mathbb{Z}_3$ Rabi model H_R defined in Eq. (1). The full circuit is described by the Hamiltonian of the qubit-boson ring H_{QB} , defined by Eq. (6), while the $\mathbb{Z}_3$ Rabi model H_R corresponds to its single-excitation subsector. The LC circuits (L_B , C_B) host the bosonic modes B1, B2, and B3 (marked by the magenta color); the charge qubits (I_Q , C_Q) correspond to the qubit degrees of freedom Q1, Q2, and Q3 (marked by the cyan color); and Josephson junctions I_R give rise to the interaction term in the qubit-boson-ring Hamiltonian V_{QB} [see Eq. (6)]. Note that the qubit elements here are represented by Cooper pair boxes (CPBs) for simplicity, while the actual implementation requires a more sophisticated variant of a charge qubit; see Appendix B for details.

the $\mathbb{Z}_3$ Rabi model is less straightforward to engineer than its more well-known $\mathbb{Z}_2$ analog. The model we proposed in the previous section solves this problem by having an inherent $\mathbb{Z}_3$ symmetry originating from the translational symmetry of the qubit ring.

C. Superconducting-circuit implementation

In this section, we discuss how to implement the QB-ring Hamiltonian (6) in a superconducting qubit system. The obvious choice for bosonic modes is a cavity (modeled by an LC circuit for our purposes). For the degrees of freedom of the qubits σ_i^α , we consider superconducting charge qubits [46–49], because we require the qubit eigenstates to be charge eigenstates, as we explain in the following. An extensive discussion on the charge qubits that we use is provided in Appendix B. Here, we only want to comment that the qubit eigenstates are $|0\rangle$ and $|1\rangle$ defined by $\hat{n}|0\rangle = 0|0\rangle$, $\hat{n}|1\rangle = 1|1\rangle$, where $\hat{n}$ is the capacitor charge operator (in units of $2e$). Below, to denote a linear span of these states, we use $\mathcal{V}_k = \text{Span}\{|0\rangle_k, |1\rangle_k\}$, where k is the qubit index on the ring.

Figure 1 shows a superconducting realization of the three-site QB ring (Sec. II B). Each horizontal branch of the circuit corresponds to one site $i = 0, 1, 2$ of the ring. The i th boson B_i and qubit Q_i are implemented by the LC circuit and the SC qubit on the i th branch, respectively. The Josephson junctions in the vertical segments (right side of the figure) couple the branches and are responsible for the interaction term in H_{QB} from Eq. (6) [50–54]. In Fig. 1, the characteristic parameters of each circuit element are indicated. Each LC circuit has

capacitance C_B and inductance L_B . The island of each qubit has a JJ with a critical current I_Q and a capacitance C_Q . The JJs in the coupling on the right have a critical current I_R . For clarity, we omit any elements that are necessary for an actual experimental implementation, e.g., readout resonators, flux-bias lines, and filtering components.

Next, we argue why this circuit models the Hamiltonian (6). Without the coupling JJs, the LC circuits and qubits do not interact with each other and are described by a free Hamiltonian:

$$H = \sum_{i=0}^2 H_{LC,i}(\hat{\phi}_i, \hat{q}_i) + \sum_{i=0}^2 H_{Q,i}(\hat{\phi}_i, \hat{n}_i), \quad (20)$$

where $\hat{\phi}_i = \hat{\Phi}_i/\Phi_0$ and $\hat{q}_i$ are the normalized magnetic flux and the capacitor charge of the i th LC circuit, respectively. Here, $\Phi_0 = h/(2e)$ is the magnetic flux quantum, with h being the (nonreduced) Planck constant and e the electron charge. On the other hand, $\hat{\phi}_i$ and $\hat{n}_i$ are the JJ superconducting phase and the capacitor charge of the i th qubit. The Hamiltonian of the LC circuit is $H_{LC,i} = \hat{q}_i^2/(2C_B) + \hat{\phi}_i^2/(2L_B)$. We leave the microscopic details of the qubit Hamiltonian $H_{Q,i}$ unspecified (beyond being a charge qubit) so that this framework can accommodate different qubit designs. However, we do require that in the two-dimensional subspace spanned by $|0_i\rangle, |1_i\rangle$, the qubit Hamiltonian reduces simply to

$$H_Q \Big|_{\mathcal{V}} = \begin{pmatrix} \langle 0_i | H_Q | 0_i \rangle & \langle 0_i | H_Q | 1_i \rangle \\ \langle 1_i | H_Q | 0_i \rangle & \langle 1_i | H_Q | 1_i \rangle \end{pmatrix} = \epsilon \sigma_z, \quad (21)$$

where the qubit states $|0_i\rangle, |1_i\rangle$ are the charge operator eigenstates $\hat{n}_i|0_i\rangle = N|0_i\rangle$ and $\hat{n}_i|1_i\rangle = (N+1)|1_i\rangle$, with $N \in \mathbb{Z}$. The qubit Hilbert space $\mathcal{V}_i = \text{Span}\{|0_i\rangle, |1_i\rangle\}$ is a computational subspace of the full Hilbert space of the system implementing the qubit, $H_{Q,i}$. Due to this property, the operator $e^{i\hat{\phi}_i}$ acts on the qubit subspace $\mathcal{V}_i$ as a raising operator:

$$e^{i\hat{\phi}_i} \Big|_{\mathcal{V}_i} = \begin{pmatrix} \langle 0_i | e^{i\hat{\phi}_i} | 0_i \rangle & \langle 0_i | e^{i\hat{\phi}_i} | 1_i \rangle \\ \langle 1_i | e^{i\hat{\phi}_i} | 0_i \rangle & \langle 1_i | e^{i\hat{\phi}_i} | 1_i \rangle \end{pmatrix} = \sigma^+, \quad (22)$$

since we have $[e^{i\hat{\phi}}, \hat{n}] = e^{i\hat{\phi}}$.

Connecting the i th and $(i+1)$ th circuit branches with a JJ closes a superconducting loop. The flux quantization condition for the loop is

$$\hat{\phi}_i + \hat{\phi}_i + \hat{\phi}_R - \hat{\phi}_{i+1} - \hat{\phi}_{i+1} = 2\pi N, \quad \text{where } N \in \mathbb{Z}. \quad (23)$$

In other words, the superconducting phase $\hat{\phi}_R$ of the coupling JJ is not an independent degree of freedom; it is fixed by the phases of the adjacent nodes. Therefore, for the corresponding term in the Hamiltonian, we obtain

$$\begin{aligned} V_{\text{SC QB},j} &= \frac{\Phi_0 I_R}{2\pi} \cos(\hat{\phi}_R) \\ &= \frac{\Phi_0 I_R}{2\pi} \cos(\hat{\phi}_j + \hat{\phi}_j - \hat{\phi}_{j+1} - \hat{\phi}_{j+1}). \end{aligned} \quad (24)$$

Projecting this coupling term onto the two-level subspaces of qubits j and $j+1$ yields the following:

$$\begin{aligned} V_{\text{SC QB},j} \Big|_{\mathcal{V}_j \otimes \mathcal{V}_{j+1}} &= \frac{\Phi_0 I_R}{4\pi} (e^{i(\hat{\phi}_j - \hat{\phi}_{j+1})} e^{i(\hat{\phi}_j - \hat{\phi}_{j+1})} + \text{H.c.}) \Big|_{\mathcal{V}_j \otimes \mathcal{V}_{j+1}} \\ &= \frac{\Phi_0 I_R}{4\pi} (e^{i(\hat{\phi}_j - \hat{\phi}_{j+1})} \sigma_j^+ \sigma_{j+1}^- + \text{H.c.}). \end{aligned} \quad (25)$$

Here, we used the fact that $e^{\pm i\hat{\phi}}$ act as ladder operators for the charge qubit. This justifies the need specifically for charge qubits rather than flux or phase qubits.

The Josephson junction coupling between two pairs of an LC circuit and a charge qubit gives us precisely the interaction term we wanted. As a result, we conclude that the system depicted in Fig. 1 is described by the Hamiltonian (6). The couplings of the QB-ring Hamiltonian can now be expressed via the circuit parameters in the following way:

$$\begin{aligned} \epsilon &= \frac{4e^2 n_{\text{off}}}{C_Q}, \quad \Omega_{\text{QB}} = \frac{1}{\sqrt{L_B C_B}}, \\ \eta^2 &= \frac{4e^2}{\hbar} \sqrt{\frac{L_B}{C_B}} = 2\pi \frac{Z_B}{R_Q}, \quad g = \frac{\Phi_0 I_R}{4\pi}, \end{aligned} \quad (26)$$

where we introduced the superconducting resistance quantum $R_Q = h/4e^2 \approx 6.5 \text{ k}\Omega$ and the LC circuit impedance $Z_B = \sqrt{L_B/C_B}$. For details on the computation of ϵ , see Appendix B. Applying the argument from Sec. II B, we deduce that the circuit is described by the $\mathbb{Z}_3$ Rabi model.

It is important to highlight that in the proposed circuit the qubit is coupled to the LC circuit inductively via a JJ. However, the $\mathbb{Z}_3$ Rabi model does not fully describe the circuit, but is only *simulated* on this circuit. Consequently, the effective coupling between the qubit and the boson modes in the $\mathbb{Z}_3$ Rabi model behaves as if it is a capacitive coupling, i.e., scales with $Z_B^{1/2}$ instead of $Z_B^{-1/2}$. This is explained by the fact that we first transform the Hamiltonian to the form shown in Eq. (11), where the interaction term $\sum_j \hat{p}_j \sigma_j^z$ looks like a longitudinal capacitive coupling (in cQED, the momentum is the charge). As a result, to reach the deep-strong-coupling regime, we require a high-impedance resonator. Specifically, since we want to have the squeezing parameter to be large, $\eta > \sqrt{6}$ [cf. discussion after Eq. (16)], we need the impedance to be larger than the superconducting resistance quantum, $Z_B > (6/2\pi)R_Q \approx R_Q$. For the standard resonator with $Z_B = 50 \Omega$, the deep-strong-coupling regime is not achievable since the squeezing parameter is small, $\eta^2 \approx 0.05$. However, the large impedance Z_B can be achieved via superinductors. They evade the vacuum-impedance bound because their inductance is kinetic [55,56] or ultracompact geometric [57]. Superinductor-based resonators with impedances of several R_Q have been demonstrated [57,58], and kinetic superinductances with microwave impedance above $160 \text{ k}\Omega \approx 25R_Q$ have been realized [59]. Therefore, while the requirements to achieve the deep-strong-coupling regime are challenging, they are within the experimentally feasible range.

Finally, two practical notes are in order. (1) Replacing each coupling JJ by a dc superconducting quantum interference

device (SQUID) makes $g \propto I_R$ tunable and allows us to imprint controlled phase offsets around the ring, thereby realizing the “magnetic” term with a phase different from that in Eq. (17). (2) The device can be initialized at $g \simeq 0$ with the three qubits and three LC modes effectively decoupled; the adiabatic ramp of g then reaches the target operating point, while the standard dispersive readout on the auxiliary resonators provides state discrimination. Taken together, these considerations make the circuit in Fig. 1 a realistic platform for probing $\mathbb{Z}_3$ Rabi model dynamics across coupling regimes without invoking a rotating-wave approximation (RWA). The effects of the disorder on the present construction are discussed in Appendix C.

D. Optomechanical implementation

Another possible physical platform for the $\mathbb{Z}_3$ Rabi model is an optomechanical system consisting of three ions whose vibration modes are bosonic degrees of freedom. They are connected by a chiral circular waveguide that acts as an interaction medium between the spins and the vibrational phonons (first proposed in Ref. [28]). Although challenging to realize, this platform illustrates the generality of our approach. In this section, we outline how the $\mathbb{Z}_3$ Rabi model arises in this model; for the extended discussion, we refer the reader to the original paper [28].

The optomechanical model in question is described by the following Hamiltonian:

$$H_{\text{OM}} = \sum_k \zeta_k \hat{c}_k^\dagger \hat{c}_k + \epsilon \sum_{j=0}^2 \sigma_j^z + \hbar \Omega_{\text{OM}} \sum_{j=0}^2 \hat{b}_j^\dagger \hat{b}_j + \gamma \sum_{j=0}^2 \sum_{k=0}^\infty [\sigma_j^+ \hat{c}_k e^{ik[R\phi_j + \hat{x}_j]} + \text{H.c.}], \quad (27)$$

where $\hat{c}_j^\dagger$ ($\hat{c}_j$) is the chiral waveguide photon creation (annihilation) operator; σ_j^z , σ_j^+ , and σ_j^- act on the ion qubit degree of freedom; $\hat{b}_j^\dagger$ ($\hat{b}_j$) are the ion vibrational phonon creation (annihilation) operators, with $\hat{x}_j = \sqrt{\hbar/(2M\Omega_{\text{OM}})}(\hat{b}_j + \hat{b}_j^\dagger)$ being the canonical coordinate operator; and $R\phi_j$ is the equilibrium (average) position of the j th ion, with R being the radius of the waveguide. The waveguide photon dispersion is $\zeta_k = vk$, ϵ is the Zeeman splitting, Ω_{OM} is the phonon frequency, and γ is the coupling strength. The photon momentum is quantized, and due to the chiral nature of the waveguide, the photons travel only in one direction.

Using the Schrieffer-Wolff transformation [60], we can integrate out the photon degrees of freedom to get the Hamiltonian in the desired form:

$$H_{\text{OM,eff}} = \epsilon \sum_{j=0}^2 \sigma_j^z + \hbar \Omega_{\text{OM}} \sum_{j=0}^2 \hat{b}_j^\dagger \hat{b}_j - \frac{\gamma^2 R}{2\hbar v} \sum_{i<j} \times [i\sigma_i^+ \sigma_j^- e^{i(\epsilon/v)R\phi_{ij}} e^{i\eta(\hat{b}_i + \hat{b}_i^\dagger - \hat{b}_j - \hat{b}_j^\dagger)} + \text{H.c.}], \quad (28)$$

where $\eta = (\epsilon/v)\sqrt{\hbar/(2M\Omega_{\text{OM}})}$, M is the mass of the trapped atoms, and $\phi_{ij} = \phi_i - \phi_j$. The Hamiltonian is a bit different from Eq. (6). It is treated in Appendix A and gives rise to the

following $\mathbb{Z}_3$ Rabi model:

$$H_R = -\frac{\gamma^2 R}{2\hbar v} (e^{-\pi i/6} Z + e^{\pi i/6} Z^\dagger) + \hbar \Omega_{\text{OM}} (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) - \frac{\hbar \Omega_{\text{OM}} \eta}{6^{1/2}} [(\hat{a}_1 + \hat{a}_2^\dagger) X + (\hat{a}_2 + \hat{a}_1^\dagger) X^\dagger], \quad (29)$$

which differs from the $\mathbb{Z}_3$ Rabi model we obtained in the superconducting-circuit realization only by the finite phase $\varphi = -5\pi/6$ of the $\mathbb{Z}_3$ “magnetic” term. For convenience, we provide the explicit matrix realization of the “magnetic” term:

$$\frac{-\gamma^2 R}{2\hbar v} (e^{-\pi i/6} Z + e^{\pi i/6} Z^\dagger) = \frac{\sqrt{3}\gamma^2 R}{2\hbar v} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (30)$$

As one can see, the nonzero phase φ leads to all the eigenvalues being nondegenerate [cf. Eq. (17)].

Let us briefly comment on the magnitude of η attainable in this platform. For optical transitions of trapped ions in standard traps, the squeezing parameter reaches $\eta \sim 0.1$. While larger η is not strictly impossible, it remains an experimental challenge. We therefore regard the optomechanical construction (27) primarily as an illustration of the generality of the qubit-boson-ring mechanism (cf. Sec. II B), while the superconducting circuit of Sec. II C constitutes the quantitatively feasible platform.

III. $\mathbb{Z}_3$ POTTS MODEL

A. Theoretical description

The $\mathbb{Z}_3$ Potts model [61] is a one-dimensional chain of three-level systems (qutrits) with a global $\mathbb{Z}_3$ symmetry. Originally, it appeared in the context of statistical physics as a generalization of the Ising model [61,62]. Today, the Potts model is relevant for qudit-based quantum computation [63,64]. However, direct experimental realization of the Potts model has not yet been achieved. One of the goals of this paper is to propose an experimental realization using $\mathbb{Z}_3$ Rabi models as building blocks. To achieve this, we generalize the idea of building an Ising model by coupling a chain of $\mathbb{Z}_2$ Rabi models, proposed by Hwang and Choi [25], to the $\mathbb{Z}_3$ -symmetric case.

The Hamiltonian of the $\mathbb{Z}_3$ Potts model with open boundary conditions is given by

$$H_P = f_P \sum_{m=1}^L (e^{i\phi} Z_m + e^{-i\phi} Z_m^\dagger) + J_P \sum_{m=1}^{L-1} (X_m X_{m+1}^\dagger + X_{m+1} X_m^\dagger), \quad (31)$$

where X_m and Z_m are the shift and clock matrices (2) that act on the m th qutrit. Here, f_P describes the strength of a single-site potential and J_P is the coupling strength of $\mathbb{Z}_3$ -symmetric nearest-neighbor interaction.

As we have already discussed, it is difficult to obtain the symmetry $\mathbb{Z}_3$ in a chain of an arbitrary three-level system. However, we have already learned how to construct a $\mathbb{Z}_3$ Rabi model, which we can now use as a building block for the

$\mathbb{Z}_3$ Potts model. In the rest of this section, we proceed in the following way. We first assemble a chain of the $\mathbb{Z}_3$ Rabi models. Then, the three cat states in the m th $\mathbb{Z}_3$ Rabi model (4) are identified with the three states on the m th site of the Potts model $|j\rangle_m = |\psi_j\rangle_m$ for $j = 0, 1, 2$. In the deep-strong-coupling limit of the Rabi model (1), $\lambda > \hbar\Omega_R$, the higher energy states are effectively decoupled from the $|\psi_j\rangle$ states.

It is convenient to build the Potts model from the Rabi models, since the boson degrees of freedom in the latter allow us to obtain a $\mathbb{Z}_3$ -symmetric interaction. The reasoning is simple: The Rabi model's $\mathbb{Z}_3$ symmetry acting on bosons is a residue of a usual boson $U(1)$ symmetry. Consequently, a $U(1)$ -symmetric coupling between the Rabi models also satisfies the $\mathbb{Z}_3$ symmetry. The simplest $U(1)$ -symmetric boson interaction is just a hopping term $\hat{a}_m^\dagger \hat{a}_{m+1} + \text{H.c.}$ As a result, in order to obtain the $\mathbb{Z}_3$ Potts model, we need to take a chain of $\mathbb{Z}_3$ Rabi models and couple them by the boson hopping term:

$$H_{\text{R chain}} = \sum_{m=1}^L H_{\text{R},m} + J \sum_{m=1}^{L-1} \sum_{k=1}^2 (\hat{a}_{m,k}^\dagger \hat{a}_{m+1,k} + \hat{a}_{m+1,k}^\dagger \hat{a}_{m,k}), \quad (32)$$

where $m = 1, \dots, L$ labels the chain sites and $k = 1, 2$ labels the two bosonic modes. We claim that this Hamiltonian gives us precisely the $\mathbb{Z}_3$ Potts model when we restrict it to the Hilbert space $\bigotimes_m \mathcal{R}_m$ generated by the three cat states on the m th site.

Within the Rabi qutrit subspace $\mathcal{R}_m$, the operator $\hat{a}_{m,k}$ acts as a permutation between the cat states, as can be seen from Eq. (4):

$$\begin{aligned} \hat{a}_{m,1} |\psi_i\rangle_m &= (\lambda/\hbar\Omega_R) |\psi_{i+1}\rangle_m, \\ \hat{a}_{m,2} |\psi_i\rangle_m &= (\lambda/\hbar\Omega_R) |\psi_{i-1}\rangle_m. \end{aligned} \quad (33)$$

The first and second mode annihilation operators act as opposite cyclic permutations. The action of the creation operator is slightly more complicated:

$$\begin{aligned} \hat{a}_{m,1}^\dagger |\psi_i\rangle_m &= (\lambda/\hbar\Omega_R) |\psi_{i-1}\rangle_m + |\delta_{\perp,1}\rangle, \\ \hat{a}_{m,2}^\dagger |\psi_i\rangle_m &= (\lambda/\hbar\Omega_R) |\psi_{i+1}\rangle_m + |\delta_{\perp,2}\rangle, \end{aligned} \quad (34)$$

where $|\delta_{\perp,1}\rangle$ and $|\delta_{\perp,2}\rangle$ are components orthogonal to the cat-state subspace $\mathcal{R}_m$, which satisfy $\langle \delta_{\perp,k} | \delta_{\perp,k} \rangle = 1$. In the deep-strong-coupling regime $\lambda/\hbar\Omega_R > 1$ (see Refs. [65,66]), we can neglect $|\delta_{\perp}\rangle$. As a result, within the Rabi qutrit subspace $\mathcal{R}_m$, both the creation and annihilation operators act as cyclic permutations,

$$\begin{aligned} \hat{a}_{m,1} |\mathcal{R}_m\rangle &= (\lambda/\hbar\Omega_R) X_m, \\ \hat{a}_{m,2} |\mathcal{R}_m\rangle &= (\lambda/\hbar\Omega_R) X_m^\dagger, \\ \hat{a}_{m,1}^\dagger |\mathcal{R}_m\rangle &\approx (\lambda/\hbar\Omega_R) X_m^\dagger, \\ \hat{a}_{m,2}^\dagger |\mathcal{R}_m\rangle &\approx (\lambda/\hbar\Omega_R) X_m. \end{aligned} \quad (35)$$

Consequently, in this sector the chain of coupled Rabi models is equivalent to the Potts model:

$$\begin{aligned} J \sum_{k=1}^2 (\hat{a}_{m,k}^\dagger \hat{a}_{m+1,k} + \hat{a}_{m+1,k}^\dagger \hat{a}_{m,k}) \\ = 2(\lambda/\hbar\Omega_R)^2 J (X_m X_{m+1}^\dagger + X_{m+1} X_m^\dagger). \end{aligned} \quad (36)$$

In summary, a chain of $\mathbb{Z}_3$ Rabi models coupled by boson hopping implements the nearest-neighbor term of the $\mathbb{Z}_3$ Potts Hamiltonian, in which the degrees of freedom of the qutrit are formed by the $\mathbb{Z}_3$ Rabi cat states (4). Comparing Eqs. (36) and (31), we see that the coupling parameters are related as $J_P = 2(\lambda/\hbar\Omega_R)^2 J$. On the other hand, the single-site terms $f_P(e^{i\phi} Z_m + \text{H.c.})$ originate from the small energy splitting of the $\mathbb{Z}_3$ cat states [Eq. (5)], which is controlled by the parameter B . The mapping requires the deep-strong-coupling regime $\lambda/\hbar\Omega_R > 1$ and the scale separation $J_P, f_P \ll \hbar\Omega_R$.

Qubit-boson-ring point of view

Previously, we showed how to construct the $\mathbb{Z}_3$ Potts model out of a chain of $\mathbb{Z}_3$ Rabi models. However, each Rabi model is implemented by a QB ring (see Sec. II B). Here, we show how these QB rings should be coupled to obtain the boson hopping [Eq. (32)]. It turns out to be quite straightforward. The hopping terms between the QB ring bosons $\hat{b}_{m,j}$ and $\hat{b}_{m+1,j}^\dagger$ give us precisely the hopping terms between the $\mathbb{Z}_3$ Rabi model bosons $\hat{a}_{m,k}$ and $\hat{a}_{m+1,k}^\dagger$ since the Fourier transform does not change the form of the boson hopping:

$$\begin{aligned} \sum_{j=0}^2 (\hat{b}_{m,j}^\dagger \hat{b}_{m+1,j} + \hat{b}_{m+1,j}^\dagger \hat{b}_{m,j}) \\ = \sum_{k=0}^2 (\hat{b}_m^\dagger(k) \hat{b}_{m+1}(k) + \hat{b}_{m+1}^\dagger(k) \hat{b}_m(k)), \end{aligned} \quad (37)$$

where $\hat{b}_m(k)$ and $\hat{b}_m^\dagger(k)$ are the Fourier-transformed QB ring bosonic modes. Taking into account that *by definition* we have $\hat{a}_{m,k} = \hat{b}_m(k)$ and $\hat{a}_{m,k}^\dagger = \hat{b}_m^\dagger(k)$ for $k = 1$ and 2 , as written after Eq. (15), we immediately obtain the desired form of the interaction [Eq. (32)]. Also, we have a hopping term for the zero-momentum bosonic modes $\hat{b}_m(0)$ and $\hat{b}_m^\dagger(0)$. Since it decouples from the rest of the system (cf. Sec. II B), we ignore it.

Using Eqs. (5), (16), and (36), one can check that the couplings in the Potts Hamiltonian can be related to the underlying QB ring parameters in the following way:

$$f_P = g \exp\left(-\frac{\eta^2}{2}\right), \quad J_P = \frac{\eta^2 J}{3}. \quad (38)$$

Thus, we see that the $\mathbb{Z}_3$ Potts model can be realized using a chain of the QB ring units, each implementing the $\mathbb{Z}_3$ Rabi model. In the following, we illustrate this proposal with a specific superconducting circuit (Fig. 3).

B. Coupled superconducting $\mathbb{Z}_3$ Rabi models as the Potts model

Having explored the theoretical construction of the Potts model based on a chain of coupled Rabi models, here we show how to implement the Potts model using superconducting circuits. We consider a chain of superconducting circuits that simulate the $\mathbb{Z}_3$ Rabi models (see Sec. II C). Next, we need to couple the neighboring Rabi models with a boson hopping term to get the Hamiltonian (32). As we have shown in Sec. III A, we must couple the boson modes of the neighboring QB rings pairwise to achieve this. Therefore, we connect

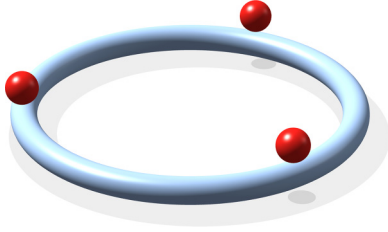


FIG. 2. Schematic illustration of the optomechanical implementation [28] of the two-mode $\mathbb{Z}_3$ Rabi model H_R , given by Eq. (29). The red spheres depict trapped ions carrying two-level systems; their vibrational modes are the bosons from the qubit-boson-ring Hamiltonian H_{QB} in Eq. (6). The blue ring is the chiral waveguide that facilitates the interaction in the qubit-boson ring. The full system is described by the Hamiltonian H_{OM} as defined in Eq. (27).

the corresponding LC circuits of the neighboring QB rings by capacitive coupling (Fig. 3).

After going from the Lagrangian to the Hamiltonian description, the nearest-neighbor capacitive coupling induces an all-to-all coupling among the canonical momenta. However, this complication can be ignored assuming that the capacitance responsible for the coupling of the nearest-neighbor Rabi models is much smaller than the cavity capacitance, $C_P \ll C_B$ [36]. In this limit, the nearest-neighbor coupling turns out to be proportional to C_P/C_B^2 ; capacitors are

$$\begin{aligned} V_{P,m,j} &= -\frac{C_P}{2C_B^2} (\hat{q}_{m+1,j} - \hat{q}_{m,j})^2 \\ &= -\frac{2e^2 C_P}{C_B^2} (\hat{n}_{m,j}^2 + \hat{n}_{m+1,j}^2) \\ &\quad - \frac{e^2 C_P}{\eta^2 C_B^2} (\hat{b}_{m,j}^\dagger \hat{b}_{m+1,j}^\dagger - \hat{b}_{m,j}^\dagger \hat{b}_{m+1,j} + \text{H.c.}), \end{aligned} \quad (39)$$

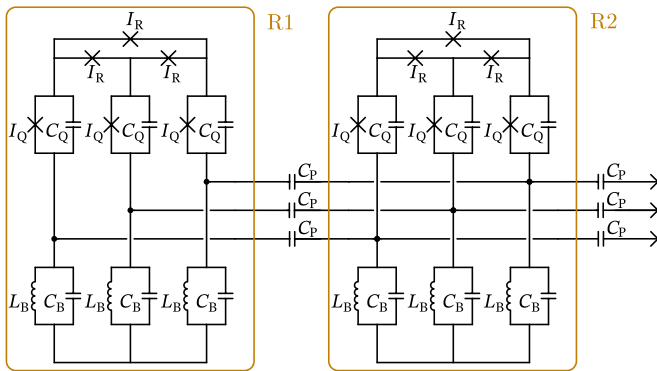


FIG. 3. Superconducting-circuit implementation of the $\mathbb{Z}_3$ Potts model (31). It is composed of a chain of coupled superconducting circuits R_i , with $i = 1, 2, \dots, L$ (only the first two are shown explicitly in the figure), each representing the $\mathbb{Z}_3$ Rabi model (see Fig. 1). The full circuit is described by the Hamiltonian (32). In the notation for circuit elements, the subscript “B” denotes elements that correspond to boson degrees of freedom; the subscript “Q” denotes elements hosting qubits; the subscript “R” denotes elements responsible for the interaction in the qubit-boson ring; and finally, the subscript “P” denotes elements that give rise to the interaction in the Potts model.

where we expressed the charge $\hat{q}_{m,j}$ through the creation and annihilation operators, $\hat{q}_{m,j} = 2e\hat{n}_{m,j} = i2e\eta(\hat{b}_{m,j}^\dagger - \hat{b}_{m,j})/\sqrt{2}$.

We now neglect the terms $\hat{n}_{m,j}^2$ and $\hat{n}_{m+1,j}^2$ in Eq. (39) because they simply renormalize the parameters of the Rabi model. Additionally, the capacitor coupling produces unbalanced terms $\hat{b}_{m,j}^\dagger \hat{b}_{m+1,j}^\dagger$ and $\hat{b}_{m,j} \hat{b}_{m+1,j}$. However, we can eliminate them with the rotating-wave approximation. For RWA to be applicable, we require $\hbar\Omega_{QB} \gg e^2 C_P / \eta^2 C_B^2$, which, after simplification, turns out to be equivalent to $C_P \ll C_B$. After applying the RWA, we are left precisely with the hopping term between the bosonic modes of the neighboring Rabi models. Consequently, according to Sec. III A, the superconducting circuit in Fig. 3 implements the $\mathbb{Z}_3$ Potts model. The coupling parameters of the resulting Potts model are related to the circuit parameters as follows:

$$f_P = \frac{\Phi_0 I_R}{4\pi} \exp\left(-\frac{2e^2}{\hbar} \sqrt{\frac{L_B}{C_B}}\right), \quad J_P = \frac{e^2 C_P}{3C_B^2}. \quad (40)$$

In summary, we propose the superconducting circuit (see Fig. 3) that simulates the $\mathbb{Z}_3$ Potts model. The required parameter range includes

- (1) Rabi deep-strong-coupling regime,

$$\frac{\eta}{\sqrt{6}} = \sqrt{\frac{2e^2}{3\hbar}} \sqrt{\frac{L_B}{C_B}} > 1; \quad (40a)$$

- (2) Potts model parameters must keep the dynamics within the cat-state qutrit subspace,

$$\begin{aligned} J_P &= \frac{e^2 C_P}{3C_B^2} \ll \frac{\hbar}{\sqrt{L_B C_B}}, \\ f_P &= \frac{\Phi_0 I_R}{4\pi} \exp\left(-\frac{2e^2}{\hbar} \sqrt{\frac{L_B}{C_B}}\right) \ll \frac{\hbar}{\sqrt{L_B C_B}}; \end{aligned} \quad (40b)$$

- (3) RWA being valid and the capacitive coupling being local,

$$\frac{C_P}{C_B} \ll 1. \quad (40c)$$

All these requirements should be achievable for the present stage of superconducting-circuit platforms. As a result, we believe that our setup is feasible for experimental realization.

Concrete example

As a concrete example, take $L_B \approx 0.3 \mu\text{H}$ and $C_B \approx 3 \text{ fF}$, which lie within the demonstrated device space of Josephson-array, granular-aluminum, and geometric superinductors [56–58]. This yields $\Omega_{QB}/2\pi \approx 5.3 \text{ GHz}$ and $Z_B \approx 10 \text{ k}\Omega \approx 1.6 R_Q$; hence, $\eta \approx 3.1$ and $\lambda/\hbar\Omega_R \approx 1.3$. Coupling junctions with $I_R \approx 1.6 \text{ nA}$ ($E_{J,R}/\hbar \approx 0.8 \text{ GHz}$, tunable *in situ* when implemented as dc SQUIDs) give $B/\hbar = g/\hbar \approx 0.4 \text{ GHz}$, i.e., $B/\hbar\Omega_{QB} \approx 0.08$ and $f_P/\hbar \approx 3 \text{ MHz}$. Inter-ring coupling capacitances $C_P \approx 3\text{--}30 \text{ aF}$ give $J_P/\hbar \approx 4\text{--}40 \text{ MHz}$. All conditions (40a)–(40c) are then satisfied with at least two orders of magnitude of margin ($f_P/\hbar\Omega_{QB}$, $J_P/\hbar\Omega_{QB} \lesssim 10^{-2}$; $C_P/C_B \leq 10^{-2}$, so that the counterrotating scale $e^2 C_P / \eta^2 C_B^2$ eliminated by the RWA remains in the megahertz range),

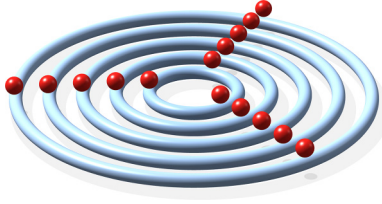


FIG. 4. Schematic illustration of the OM implementation of the Potts model. It consists of several concentric OM implementations of the Rabi model, i.e., several concentric cyclic chiral waveguides and three trapped ions on top of each waveguide. The notations follow those in Fig. 2.

while f_p and J_p exceed by one to three orders of magnitude the linewidths $\kappa/2\pi \sim 10\text{--}100$ kHz typical of high-impedance resonators with internal quality factors $10^4\text{--}10^6$ [56–58]. We therefore conclude that the circuit of Fig. 3 is feasible with present-day superconducting-circuit technology.

C. Coupled optomechanical $\mathbb{Z}_3$ Rabi models as the Potts model

In principle, we can construct an optomechanical implementation of the $\mathbb{Z}_3$ Potts model by similarly coupling multiple $\mathbb{Z}_3$ Rabi units. However, the geometry is a bit more peculiar due to the use of a circular waveguide in the optomechanical Rabi setup. To build the optomechanical $\mathbb{Z}_3$ Potts model, we arrange the optomechanical $\mathbb{Z}_3$ Rabi models concentrically (see Fig. 4). On the one hand, the radius of the waveguide does not affect the parameters of the model. Furthermore, placing the ions of neighboring Rabi models close to each other generates an effective phonon-phonon coupling via the Coulomb interaction [67,68]. The phonon-phonon coupling between the neighboring Rabi models leads to the same effective Hamiltonian (32), realizing the $\mathbb{Z}_3$ Potts model as described in Sec. III.

D. Chiral clock model and parafermions

In this subsection, we briefly review the $\mathbb{Z}_3$ chiral clock model. It is a generalization of the $\mathbb{Z}_3$ Potts model that has a chiral nearest-neighbor interaction:

$$H_{\text{cl}} = f' \sum_{m=1}^L (e^{i\phi'} Z_m + e^{-i\phi'} Z_m^\dagger) + J' \sum_{m=1}^{L-1} (e^{i\theta'} X_m X_{m+1}^\dagger + e^{-i\theta'} X_{m+1} X_m^\dagger). \quad (41)$$

In other words, it is obtained from the $\mathbb{Z}_3$ Potts model (31) by allowing for a complex nearest-neighbor coupling $J' e^{i\theta'}$.

In the construction described in Sec. III, the interaction parameter J' originates from the interaction between bosons of the neighboring QB rings. Hence, if we want to engineer the chiral clock model, we have to find a way to create a chiral boson hopping. Chirality arises when hopping between bosonic modes acquires a complex amplitude, so that excitations accumulate a nontrivial phase when moving around a closed loop. In general, such a complex hopping term

$$H_{nm} = g e^{i\phi_{nm}} \hat{a}_n^\dagger \hat{a}_m + \text{H.c.} \quad (42)$$

corresponds to a synthetic gauge field, with the directed phase factors ϕ_{nm} acting as Peierls phases. The net phase around a plaquette defines an effective magnetic flux that breaks time-reversal symmetry and leads to the chiral transport of photons or phonons.

In trapped-ion arrays, the bosons are vibrational phonons. Coupling between the phonons is engineered through the Coulomb interaction combined with laser-induced forces. By introducing a static gradient in trap frequencies and applying periodic modulation (photon-assisted tunneling), the effective phonon hopping becomes resonant and inherits the phase of the driving field. Adjusting laser phases thus imprints controlled Peierls phases on the phonon couplings, producing synthetic fluxes that enable chiral phonon propagation [67,69–73].

In superconducting circuits, the bosons are microwave photons in resonators or qubits. Here, Josephson junctions provide nonlinear couplers whose effective interaction strength can be modulated parametrically by flux or multitone microwave pumping. When the pump frequency matches the detuning between two modes, frequency-converting tunneling occurs with an amplitude set by the pump strength and a phase set by the pump phase. Embedding such complex hoppings in a ring of modes yields a net synthetic magnetic flux, leading to unidirectional photon circulation [74–77].

Assuming that we obtained the chiral coupling in one of the ways described above, we now want to discuss what physical effects the complex coupling causes in the chiral clock model. The most interesting consequence is parafermion physics [78–81]. The duality between the chiral clock model and the parafermion chain is a generalization of the duality between the Ising model and the Kitaev chain that displays Majoranas [82]. The mapping between the degrees of freedom of qutrit and parafermion is achieved via a generalization of the Jordan-Wigner transformation, the so-called Fradkin-Kadanoff (FK) transformation [83]:

$$\Gamma_m = \prod_{m' < m} Z_{m'} X_{m'}, \quad \Delta_m = \prod_{m' \leq m} Z_{m'} X_{m'}. \quad (43)$$

Here, Γ_j, Δ_j are parafermion operators that satisfy the following commutation relations: $\Gamma_m \Gamma_{m'} = \omega^{\text{sign}(m-m')} \Gamma_{m'} \Gamma_m$, $\Delta_m \Delta_{m'} = \omega^{\text{sign}(m-m')} \Delta_{m'} \Delta_m$, and $\Gamma_m \Delta_{m'} = \omega^{\text{sign}(m-m')} \Delta_{m'} \Gamma_m$.

Using the FK transformation, it is easy to show that the chiral clock model is equivalent to a parafermion chain. The issue is that the FK transformation is nonlocal. In terms of parafermion operators, the chiral clock model Hamiltonian (44) reads

$$H_{\text{chP}} = f' \sum_{m=1}^L (e^{i\phi'} \Gamma_m \Delta_m^\dagger + e^{-i\phi'} \Delta_m \Gamma_m^\dagger) + J' \sum_{m=1}^{L-1} (e^{i\theta'} \Delta_m \Gamma_{m+1}^\dagger + e^{-i\theta'} \Gamma_{m+1} \Delta_m^\dagger). \quad (44)$$

On the other hand, it is well known that the parafermion chain hosts a topological phase with the parafermion edge

modes [78]:

$$\Gamma_{\text{edge}} = \Gamma_1 + \frac{f'}{J' \sin(3\theta')} \mathcal{O} + \dots, \quad (45)$$

where $\mathcal{O}$ is a known combination of parafermion operators at the first lattice site and the ellipses represent higher-order terms. This expression also clarifies why we require the chiral coupling (44). For the parafermion edge mode to be localized, the expansion parameter must satisfy $f'/(J' \sin(3\theta')) < 1$. However, in the nonchiral case, $\theta' = 0$, the expansion parameter diverges.

IV. CONCLUSION

We established a hierarchy of $\mathbb{Z}_3$ -symmetric models. First, we have shown that the single-excitation sector of the qubit-boson ring [Eq. (6)] can be mapped to the $\mathbb{Z}_3$ Rabi model (1). Then, we considered a chain of coupled $\mathbb{Z}_3$ Rabi models, each hosting an individual qutrit degree of freedom. The resulting chain simulates the $\mathbb{Z}_3$ Potts model. In the QB ring, the $\mathbb{Z}_3$ symmetry appears as a discrete rotation; upon projecting to the single-excitation manifold, it acts as an internal symmetry that is inherited by the coupled-block construction of the $\mathbb{Z}_3$ Potts Hamiltonian.

Our main goal was to suggest a way toward an experimental realization of the $\mathbb{Z}_3$ Rabi and Potts models. The superconducting circuit we proposed is experimentally feasible. However, this approach does not have to be limited only to superconducting-circuit platforms. For instance, we believe that a similar strategy could work for spin-qubit systems, though this remains a topic for future research.

More broadly, our work highlights the intriguing potential of $\mathbb{Z}_n$ -symmetric systems within condensed matter physics. These systems present fertile ground for discovering novel quantum phenomena, many of which remain to be fully understood or described.

ACKNOWLEDGMENTS

The authors thank Daria Kalacheva, Henry Legg, Katharina Laubscher, and Ilia Luchnikov for fruitful discussions and useful comments. This work was supported as part of NCCR SPIN, a National Center of Competence in Research, funded by the Swiss National Science Foundation (Grant No. 225153). This work has received funding from the Swiss State Secretariat for Education, Research and Innovation (SERI) under Contract No. M822.00078. D.L. acknowledges the Deanship of Research and the Quantum Center for the support received under Grants No. CUP25102 and No. INQC2600, respectively.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: QB RING GENERALIZATION TO AN ARBITRARY INTERACTION MATRIX

In this Appendix, we summarize how the derivation in Sec. II B extends to a QB ring with the most general $\mathbb{Z}_3$ -symmetric form of the interaction between different qubit-boson pairs. The system is described by a direct generalization of the Hamiltonian (6), which reads

$$H_{\text{gen}} = \epsilon \sum_{j=0}^2 \sigma_j^z + \hbar \Omega \sum_{j=0}^2 \hat{a}_j^\dagger \hat{a}_j + V_{\text{gen}}, \quad (A1)$$

$$V_{\text{gen}} = g \sum_{j,k=0}^2 A_{jk} \sigma_j^+ \sigma_k^- e^{i(\hat{x}_j - \hat{x}_k)},$$

where the position indices are periodic, $j = k = 0 = 3$, and the interaction matrix is Hermitian $A^\dagger = A$. Applying the same sequence of steps as in Sec. II B, i.e., the momentum translation S , the Fourier transform, the restriction to the single-excitation sector $\mathcal{H}_{\text{QB},1}$, and final basis transformations, we obtain a Hamiltonian that has the same form as that in Eq. (15). The only difference is that the qutrit term $g(e^{2\pi i/3} Z + e^{-2\pi i/3} Z^\dagger)$ in Eq. (15) is replaced by $gX^\dagger H A H^\dagger X$, where H is the Hadamard matrix. Taking into account that A is Hermitian and commutes with $\mathbb{Z}_3$ symmetry, the qutrit term can be written in the canonical form $gH A H^\dagger = \tilde{g}(e^{i\phi} Z + e^{-i\phi} Z^\dagger)$, where the parameters $\tilde{g}$ and ϕ depend on the matrix A .

In the superconducting implementation, one has $A = X + X^\dagger$ (here, A is a 3×3 matrix), thereby recovering the special case treated in Sec. II B. On the other hand, the optomechanical realization of the QB ring in Eq. (28) gives rise to a different interaction matrix, $A = -iX + iX^\dagger$. Nevertheless, the logic stays exactly the same. As a result, in the optomechanical implementation of the $\mathbb{Z}_3$ Rabi model, we obtain a different phase for the qutrit term, $gH A H^\dagger = g(e^{-\pi i/6} Z + e^{\pi i/6} Z^\dagger)$ [see Eqs. (29) and (30)].

APPENDIX B: CHARGE QUBIT FOR THE $\mathbb{Z}_3$ RABI MODEL

Here, we provide details on the superconducting qubit architecture used in Sec. II C. While the CPB qubit is probably the most well-known type of a charge qubit, it is not suitable for our purposes because its eigenstates are not charge eigenstates but rather their symmetric/antisymmetric superpositions. In other words, in the charge basis, the CPB Hamiltonian is proportional to σ^x . This is undesirable because in this case the Hamiltonian of the qubit chain does not commute with the total qubit-excitation-number operator S^z .

Therefore, we need some other type of charge qubit. Recall that in the CPB Hamiltonian, the σ^x term arises from the Josephson junction. Thus, removing JJ, one obtains a Hamiltonian proportional to σ^z in the charge basis. While this approach solves the issue with the $\mathbb{Z}_3$ symmetry, such a system cannot be called a qubit because of the absence of nonlinearity.

Thus, we want two things simultaneously: (1) The Hamiltonian must be proportional to σ^z in the charge basis, and (2) in the qubit circuit, there must be JJ that provides the nonlinearity. The solution satisfying both requirements turns out to be a higher-harmonic Josephson junction. Usually it

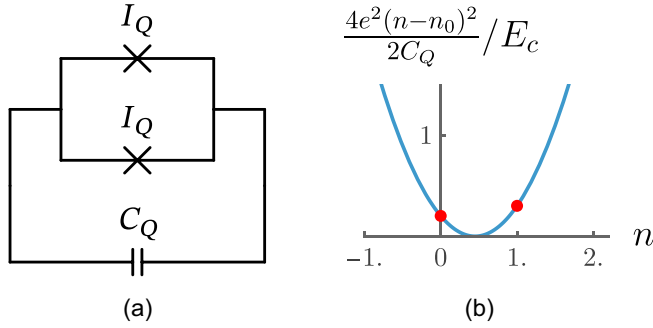


FIG. 5. Second-harmonic CPB qubit used for a superconducting-circuit realization of the $\mathbb{Z}_3$ Rabi model, as shown in Fig. 1. (a) The electric circuit of the second-harmonic CPB qubit. (b) The capacitor potential in Eq. (B4) for $n_0 = 0.45 = 0.5 - 0.05$. The red dots indicate the eigenenergies of the second-harmonic CPB qubit.

is neglected, but strictly speaking the JJ Hamiltonian always includes higher harmonics:

$$H_{JJ} = E_{J,1} \cos \hat{\phi} + E_{J,2} \cos(2\hat{\phi}) + \dots, \quad (\text{B1})$$

where $\hat{\phi}$ is the superconducting phase and $E_{J,i} = I_{Q,i} \Phi_0 / (2\pi)$, with $\Phi_0 = 2\pi \hbar / 2e$ being the superconducting magnetic flux quantum and $I_{Q,i}$ being the critical current of the i th harmonic of the Josephson junction.

In practice, the higher harmonics are usually considerably weaker: $E_{J,2} \ll E_{J,1}$. However, if we completely eliminate the first harmonic, the second harmonic becomes dominant. The second harmonic, on the one hand, can help to suppress the noise and, at the same time, does not induce any coupling in the qubit subspace:

$$\cos 2\hat{\phi}|_{\nu} = \frac{1}{2}(e^{2i\hat{\phi}} + e^{-2i\hat{\phi}})|_{\nu} = \frac{1}{2}((\sigma^+)^2 + (\sigma^-)^2) = 0. \quad (\text{B2})$$

In order to eliminate the first harmonic, we can use a superconducting quantum interference device [84]. If the magnetic flux through the SQUID is tuned to $\Phi = \pi$, then the SQUID Hamiltonian reads

$$\begin{aligned} H_{\text{SQUID}} &= E_{J,1} \cos(\hat{\phi}) + E_{J,2} \cos(2\hat{\phi}) \\ &\quad + E_{J,1} \cos(\hat{\phi} + \pi) + E_{J,2} \cos(2\hat{\phi} + 2\pi) \\ &= 2E_{J,2} \cos(2\hat{\phi}), \end{aligned} \quad (\text{B3})$$

which has the desired form with only the second harmonic being present.

Consequently, using the second-harmonic JJ we can construct a charge qubit, which we refer to as the second-harmonic CPB qubit, described by the Hamiltonian (see Fig. 5)

$$H = \frac{4e^2}{2C_Q} (\hat{n} - n_0)^2 + 2E_{J,2} \cos(2\hat{\phi}). \quad (\text{B4})$$

Here, we have the capacitor energy $E_C = 2e^2 / C_Q$. By biasing the offset charge slightly away from the symmetric point ($n_0 = 0.5 - n_{\text{off}}$ with $n_{\text{off}} \ll 1$), the low-energy Hamiltonian of the qubit becomes

$$H_Q = \frac{4e^2 n_{\text{off}}}{C_Q} \sigma^z. \quad (\text{B5})$$

Hence, we obtain the qubit Hamiltonian proportional to σ_z , as desired. Here, we assume that $\frac{4e^2}{2C_Q} \gg 2E_{J,2}$; i.e., the charging energy dominates over the Josephson energy, as is usually the case for a charge qubit.

To conclude, the suggested charge qubit consists of a capacitor and a second-harmonic JJ. One can think of it as a modified version of a Cooper pair box qubit. Here, we only briefly describe its blueprint to further support the implementation proposed in Sec. IIC.

APPENDIX C: EFFECTS OF DISORDER IN THE SUPERCONDUCTING IMPLEMENTATION OF THE $\mathbb{Z}_3$ RABI MODEL

1. U(1) symmetry-breaking disorder

In this Appendix, we discuss the effects of disorder on the qubit-boson ring [Eq. (6)]. First, consider a U(1) symmetry-breaking term. It breaks the conservation of $\hat{S}^z$, the total number of qubit excitations. The U(1) symmetry is broken by a small misalignment of the Zeeman terms with the $\hat{z}$ direction:

$$H_{\text{QB,dis}} = H_{\text{QB}} + \sum_{j=0}^2 \Delta_j \sigma_j^x. \quad (\text{C1})$$

As explained in Appendix B, for the superconducting-circuit realization of the qubit-boson ring, the condition of vanishing disorder $\Delta_j = 0$ relies on the fine-tuning of the SQUID magnetic field. Therefore, in a realistic setting, a small detuning is inevitable: $\Delta_j = I_{Q,1} \Phi_0 / (2\pi) \sin(\Delta \Phi_j)$.

As a result, the qubit-boson-ring Hamiltonian no longer commutes with the S^z operator [85], $[H_{\text{QB,dis}}, S^z] \neq 0$. This means that the exact mapping to the $\mathbb{Z}_3$ Rabi model derived in the main text does not hold anymore. However, assuming that the disorder is considerably weaker than other parameters in the Hamiltonian, we can still hope that the dynamics remains approximately that of the Rabi model. An analytic estimate is somewhat cumbersome, and we rely on numerical simulations of the qubit-boson ring. The results are shown in Fig. 6. The figure shows the time evolution of $\langle S^z(t) \rangle$ for various strengths of the disorder. The simulation indicates that for small disorder the evolution of the qubit-boson ring stays sufficiently close to the single-excitation subspace $\mathcal{H}_1$, as $\langle S^z(t) \rangle \approx -1$ at all simulated times. We speculate that even when S^z is not strictly conserved, the spins of the system collectively precess around the $S^z = -1$ direction, effectively keeping $\langle S^z \rangle$ near -1 .

2. $\mathbb{Z}_3$ symmetry-breaking disorder

Now, assuming that conservation of S^z is maintained ($\Delta_j = 0$), we examine a different type of disorder: spatial variations in the system parameters. In an inhomogeneous qubit-boson ring, the $\mathbb{Z}_3$ symmetry is obviously broken. However, we want to investigate how this disorder manifests itself after the projection onto the single-excitation subspace. The

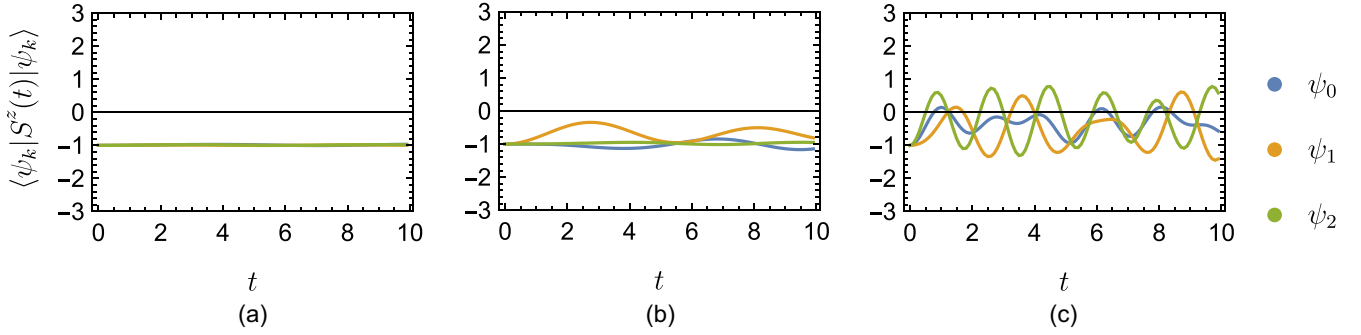


FIG. 6. Time dependence of the z component of the total spin operator in the presence of a symmetry-breaking disorder [Eq. (C1)]. In order to illustrate that the Rabi model implementation is robust, we plot $\langle S^z(t) \rangle$ for three disorder strengths. We use normally distributed disorder Δ_j with zero mean and fixed standard deviation σ . (a) $\sigma/\Omega_{\text{QB}} = 0.1$; (b) $\sigma/\Omega_{\text{QB}} = 0.3$; and (c) $\sigma/\Omega_{\text{QB}} = 1$.

disordered Hamiltonian in the full Hilbert space reads

$$H'_{\text{QB}} = \sum_{j=0}^2 \epsilon_j \sigma_j^z + \sum_{j=0}^2 \hbar \Omega_{\text{QB},j} \hat{a}_j^\dagger \hat{a}_j + \sum_{j=0}^2 g_{j,j+1} [\sigma_j^+ \sigma_{j+1}^- e^{i(\hat{x}_j - \hat{x}_{j+1})} + \text{H.c.}], \quad (\text{C2})$$

where we introduced spatial nonuniformity of the system parameters. The positional index j is valued modulo 3, $j+3 = j$.

In the superconducting implementation, the couplings become $\epsilon_j = 4e^2 n_{\text{off},j} / (2C_{Q,j})$, $\Omega_{\text{QB},j} = 1/\sqrt{L_{B,j} C_{B,j}}$, and $g_{j,j+1} = I_{R,j} \Phi_0 / (4\pi)$, where the subscript j labels the parameters of the elements belonging to the j th branch of the circuit (see Fig. 1).

For convenience, we decompose the coupling parameters into a homogeneous part and a disorder: $\epsilon_j = \epsilon + \Delta\epsilon_j$, $\Omega_{\text{QB},j} = \Omega_{\text{QB}} + \Delta\Omega_{\text{QB},j}$, and $g_{j,j+1} = g + \Delta g_{j,j+1}$, where we assume that the spatial average of the disorder vanishes: $\sum_{j=0}^2 \Delta\epsilon_j = \sum_{j=0}^2 \Delta\Omega_{\text{QB},j} = \sum_{j=0}^2 \Delta g_{j,j+1} = 0$. This can always be achieved by adjusting the homogeneous parts of the couplings.

Then, after projecting to the single-excitation sector, the Hamiltonian $H_{\text{QB}'}$ reduces to

$$H'_R = H_R + \epsilon(2)X + \epsilon(1)X^\dagger + (X + X^\dagger) \sum_{k=1}^2 \Delta g(k) Z^k + \hbar \Omega_{\text{QB}}(1) \sum_{k=0}^2 \hat{a}^\dagger(k+1) \hat{a}(k) + \hbar \Omega_{\text{QB}}(2) \sum_{k=0}^2 \hat{a}^\dagger(k) \hat{a}(k+1) + \sum_{l=1}^2 \frac{\eta \hbar \Omega_{\text{QB}}(l)}{2^{3/2}} \sum_{k=0}^2 [\hat{a}(k+l) + \hat{a}^\dagger(-k-l)] Z^k, \quad (\text{C3})$$

where the bosonic frequencies satisfy the following property: $\Omega_{\text{QB}}(1) = (\Omega_{\text{QB}}(2))^*$. One can clearly see that, in the presence of disorder, the first bosonic mode is no longer decoupled. We expect that the $\mathbb{Z}_3$ cat state structure of the eigenstates should remain intact for sufficiently small disorder. However, a detailed investigation lies beyond the scope of this paper.

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