

Geodynamics and artificial gravity in spacetime crystals under slow perturbation and deformation

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We present a theory of geodynamics in a spacetime crystal based on an event wave packet constructed from the Floquet-Bloch waves, which involves not only a scalar dispersion function but also a Berry curvature tensor in the phase space manifold of spacetime and the reciprocal quasi-energy-momentum. In the presence of structural deformation, this theory is naturally extended into a covariant form with the introduction of a lattice connection constructed out of the gradients of the local lattice vectors. This geometric framework reveals three distinct gravitational forces: lattice deformation gravity, dispersion gravity, and Berry curvature gravity. These forces manifest as distinct, experimentally tunable connection and metric terms in the geodynamics, establishing a framework for engineering artificial spacetime by controlling lattice strain, band structure, and wave function topology. Our work provides a direct conceptual link between general relativity and condensed matter physics, opening a path toward the laboratory realization of novel gravitational phenomena.

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I. INTRODUCTION

Topological materials have attracted significant attention in recent years due to their rich and experimentally controllable phases. Among these, spacetime crystals—materials exhibiting microscopic translational symmetry in both spatial and temporal dimensions [1,2]—are particularly intriguing due to their highly nontrivial topological characteristics [3–5]. Considerable effort has been dedicated to the creation and investigation of such systems, motivated by their novel topological phenomena [6,7]. From a wave packet dynamics viewpoint, these topological properties profoundly influence the motion of quasiparticles through geometric effects [8,9]. Despite this, the dynamical behavior of quasiparticles in spacetime crystals remains relatively unexplored. In our research, we focus specifically on the propagation of matter waves or excitations governed by linear partial differential equations with periodic dependencies in both position and time.

In the semiclassical limit, a wave packet constructed from Bloch states behaves as a classical point particle over a scale larger than atomic distance, providing a powerful link between quantum and classical mechanics [8]. Its dynamics, however, are fundamentally richer than simple Newtonian motion. The equations of motion are profoundly modified by two key

quantum effects: the energy dispersion, which is periodic in momentum space [10], and the Berry curvature, which acts as an effective gauge field analogous to electromagnetism. This established principle motivates the central problem of our work. By analogy, we propose that a wave packet built from Floquet-Bloch states should behave like an event point tracing out a worldline in an eight-dimensional phase space of spacetime and energy-momentum. This perspective immediately suggests that a deformed spacetime crystal could serve as a background for simulating phenomena from general relativity. Crucially, however, the dynamics within this system can go beyond a simple gravitational analogy; the dispersion and Berry curvatures in the phase space provide the ingredients to engineer effective gravitational theories with complexities and features not found in conventional general relativity.

In this paper, we seek a particle description by constructing a wave packet out of a narrow range of Floquet-Bloch waves of a single band, which is broad in real space compared with the spacetime unit cell but still localized over macroscopic scales. This is the analog of a classical event point in the meta-spacetime, and a set of geodynamic equations of motion for the corresponding worldline in phase space (spacetime and the reciprocal quasi-energy-momentum) can be obtained in the presence of weak external fields and various inhomogeneities. These equations are direct generalizations of equations of motion for electrons in three-dimensional crystals. However, we treat space and time at equal footing and extend all the equations to phase space, and expect to construct artificial gravity through all emergent terms in geodesic lines.

We are also interested in the case of slowly varying deformation of the spacetime crystal and wish to investigate how such deformation affects particle dynamics. With the introduction of a lattice connection from the spacetime lattice vectors and their reciprocals [10], a technique first introduced

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by two of us for the study of electron semiclassical dynamics in 3D crystals under deformation, we are able to obtain covariant geodynamics akin to Plebanski's formulation of general relativity [11]. The geodesic line is of the standard form involving the lattice connection, but both dispersion structure and Berry curvatures enter in higher-order gradients, indicating two additional ways of modification in the effective spacetime structure. There are also Lorentz force terms akin to gauge field effects and correcting the lattice connection.

Synthesizing these elements, our comprehensive geodynamic theory reveals that the effective gravity experienced by a quasiparticle originates from three distinct physical sources. We classify these as (1) lattice deformation gravity, where the nonuniform lattice structure itself defines a background geometry through the lattice connection; (2) dispersion gravity, a quantum effect where the quasiparticle's energy-momentum dispersion introduces an intrinsic effective metric; and (3) Berry curvature gravity, wherein the Berry curvatures of the wave function act as effective gauge fields and connections, further modifying the particle's trajectory. This framework systematically connects the microscopic properties of the spacetime crystal to the macroscopic emergence of gravitational phenomena.

This paper is organized as follows: In Sec. II, we present a general formalism to obtain Floquet-Bloch wave function in spacetime crystals. In Sec. III, there are several direct and explicit examples presenting how the dispersions in different systems depend on the four-momentum. In Sec. IV, an event wave packet is built with the Floquet-Bloch waves. The Lagrangian of the wave packet in phase space is presented, and the semiclassical equations of motion are derived. Section V describes the geodesic line equation based on the semiclassical equations. In Sec. VI, we apply deformation in spacetime crystals and introduce lattice connections to indicate the inhomogeneity. Sections VII and VIII show how the equations of motion and geodesic equations are modified by deformation. We conclude in Sec. VIII with a brief summary and outlook on our results.

II. GEOMETRO-WAVE EQUATION AND DISPERSION

The most general form of the linear equation governing quasiparticle evolution in spacetime crystals can be expressed using a four-coordinate $x = (t, \vec{x})$,

$$\hat{\mathcal{L}}\left(\frac{1}{i}\partial_x, x\right)|\phi\rangle = 0, \quad (1)$$

where $|\phi\rangle$ is the wave function and $\hat{\mathcal{L}}$ is the dispersion operator. Several examples of dispersion operators of various quasiparticles are presented here: For Bloch electrons, the operator is $\hat{\mathcal{L}} = i\partial_t - \hat{H}(\frac{1}{i}\partial_{\vec{r}}; \vec{x}, t)$; for bosons like phonons in a crystal, the operator is $\hat{\mathcal{L}} = \partial_t^2 - \partial_{\vec{r}}^2 + m(\vec{x}, t)$; and for a Dirac fermion, it is $\hat{\mathcal{L}} = iv_F^\mu \partial_\mu - m(\vec{x}, t)$.

Before we get into details of the dispersion operators, symmetry analysis offers important insights into the operators. All the operators in spacetime crystals follow translational symmetry in space and time. The translational symmetry groups are spacetime groups that are discrete [12]. Because of these symmetries, the operators have lattice structure,

$\hat{\mathcal{L}}(\vec{x}, t) = \hat{\mathcal{L}}(\vec{x} + \vec{R}_\alpha, t + T_\alpha)$, from which we can define four-dimensional lattice vectors as $c_\alpha = (T_\alpha, \vec{R}_\alpha)$. According to the Floquet-Bloch theorem, when the wave functions follow the same periodicity in space and time with the corresponding operators, they can be written as product of a periodic part and a phase factor,

$$|\phi(k, x)\rangle = e^{ik_\mu x^\mu} |u(k, x)\rangle, \quad (2)$$

where $|u(k, x)\rangle$ is translational invariant. The inner product of two Floquet-Bloch functions is integrated over a single lattice unit cell and a single period:

$$\langle u_1 | u_2 \rangle = \frac{(2\pi)^4}{TV} \int_{V,T} d^4x u_1^*(x, k) u_2(x, k). \quad (3)$$

Equation (1) can be directly solved by substituting Eq. (6),

$$\hat{\mathcal{L}}\left(\frac{1}{i}\partial_x + k, x\right) |u(k, x)\rangle = 0. \quad (4)$$

The solutions are obtained when four-momentum satisfies $\omega = \omega(\vec{k})$, which are called dispersion relations. The dispersion relation is a hypersurface in the phase space, but since we want to obtain wave packets with separation in the phase space, we may need to introduce a new method to lift the constraint at first without losing any necessary information.

To obtain an independent four-momentum, we introduce some new parameters to lift the dispersion constraint. We first interpret Eq. (1) as the zero “energy” solution of the following eigenvalue problem:

$$\hat{\mathcal{L}}\left(\frac{1}{i}\partial_x, x\right) |\phi(k, x)\rangle = \lambda |\phi(k, x)\rangle, \quad (5)$$

and another parameter conjugated to λ , the “proper time,” τ , is also introduced. These two parameters could be viewed as energy and time generalized to four dimensions. By substituting a state varying with proper time τ , $|\Phi(k, x)\rangle = e^{-i\lambda\tau} |\phi(k, x)\rangle$, the steady equation of states is changed to proper time-dependent equation

$$\hat{\mathcal{L}}\left(\frac{1}{i}\partial_x, x\right) |\Phi(k, x)\rangle = i \frac{d}{d\tau} |\Phi(k, x)\rangle, \quad (6)$$

and λ is conserved to τ . Although the real dispersion $\omega = \omega(\vec{k})$ is dismissed and they are independent, the physical world lies in a shell $\lambda = 0$. We need a four-dimensional spread momentum, so we have to go slightly off-shell with respect to the dispersion and allow components of the event wave packet to lie on a shell with small enough separation around $\lambda = 0$, which causes no problem in the subsequent evolution of the wave packet because of the conservation law of λ .

The dispersion function $\lambda(k)$ determines the allowed propagation modes of a physical system, which are found on the shell defined by $\lambda(k) = 0$. To analyze the system's behavior near this shell, we expand $\lambda(k)$ in a Taylor series around a root point $k_{0\mu}$:

$$\lambda(k) \approx \lambda^\mu (k_\mu - k_{0\mu}) + \frac{1}{2} \lambda^{\mu\nu} (k_\mu - k_{0\mu})(k_\nu - k_{0\nu}). \quad (7)$$

The linear coefficient λ^μ acts as a four-velocity, while the symmetric tensor $\lambda^{\mu\nu}$ captures the quadratic curvature of the dispersion surface. If this tensor is nondegenerate, it can be

interpreted as an effective metric, as we will show in later discussion.

Assuming that $\lambda^{\mu\nu}$ is invertible with inverse $\lambda_{\mu\nu}$, the linear term can be absorbed by completing the square. This procedure shifts the origin of the momentum expansion, simplifying the dispersion to a pure quadratic form:

$$\lambda(k) = \lambda_0 + \frac{1}{2} \lambda^{\mu\nu} q_\mu q_\nu, \quad (8)$$

$$q_\mu = (k_\mu - k_{0\mu}) + \lambda_{\mu\sigma} \lambda^\sigma, \quad (9)$$

$$\lambda_0 = -\frac{1}{2} \lambda^\mu \lambda_{\mu\nu} \lambda^\nu. \quad (10)$$

This formalism implies that the inverse tensor $\lambda_{\mu\nu}$ serves as an effective metric induced by the dispersion. Conversely, if $\lambda^{\mu\nu}$ is singular (noninvertible), the dispersion remains linear along certain null directions, a characteristic feature of relativistic systems such as those hosting Dirac fermions.

III. DISPERSION EXAMPLES

The Floquet-Bloch function is periodic in space and time $u(k, x + c_\alpha) = u(k, x)$ referring to Floquet-Bloch theorem. We can take the Fourier transformation of the Floquet-Bloch function and use the exact diagonalization method to obtain

$$\begin{bmatrix} \ddots & \dots & \dots & \dots & \dots \\ \dots & \omega - \Omega - 2h \cos((k - \kappa) \cdot a) + V & A/2 & 0 & \dots \\ \dots & A/2 & \omega - 2h \cos(k \cdot a) + V & A/2 & \dots \\ \dots & 0 & A/2 & \omega + \Omega - 2h \cos((k + \kappa) \cdot a) + V & \dots \\ \dots & \dots & \dots & \dots & \ddots \end{bmatrix}. \quad (13)$$

The figure of the dispersion is given in Fig. 1, and the energy bands $\omega = \epsilon(\vec{k})$ can be obtained by adding the constraint $\lambda = 0$, which is also plotted in Fig. 1.

Three points are highlighted in Fig. 1: two are at the extrema (maximum and minimum) of a single dispersion relation band (points B and C) and one is in the middle (point A). At the three points in Fig. 1, we take the expansion mentioned before (see Table I) and find the spatial component λ^{11} of effective metric $\lambda^{\mu\nu}$ is positive for point C and negative for point B. As for point A, since the leading order terms in both k and ω are nonzero, linear terms are dominant. The nonzero expansion coefficients make the quasiparticle act like a massless Fermion. The different expansion parameters make

the spectrum of dispersion. The Fourier transformation is

$$u(k, x) = \sum_n u_n \exp(i b_{n\mu}^\alpha x^\mu), \quad (11)$$

where u_n are Fourier coefficients and b_n^α are the n th reciprocal lattice vectors. Substituting the Fourier series to Eq. (6), we can obtain the dispersion.

To indicate the dispersion spectrum, we select and analyze two representative examples. We focus our discussion on a single dispersion band and present more replica bands of energy under on-shell constraint for simplicity. The first example is a one-dimensional oblique Floquet crystal. In the Wannier basis, the operator is $\hat{\mathcal{L}} = i\partial_t - \hat{H}$, where the Hamiltonian is

$$\hat{H} = \sum_R [(V + A \cos(\kappa R - \Omega t)) |R\rangle \langle R| + h(|R + a\rangle \langle R| + \text{H.c.})]. \quad (12)$$

Here, R labels the lattice point with lattice constant a . The parameter κ and Ω are the outer field's wave vector and frequency that changes the on-site energy, which has a average value of V and a spacetime-dependent varying part $A \cos(\kappa R - \Omega t)$. h is the hopping energy between two neighbor sites. The time-translation symmetry is $H(t + 2\pi/\Omega) = H(t)$. Using Floquet-Bloch theorem [13], we can derive the spectrum of dispersion, which is the eigenvalue of the following matrix:

quasiparticle centering in the region near A, B, and C transport in significantly different trajectories, as will be discussed in subsequent sections.

Another example is one-dimensional Floquet photonic crystal, which is common in Floquet system [14]. The operator for 1D photons with spacetime-translation symmetry is

$$\hat{\mathcal{L}} = \partial_t^2 - \partial_r^2 + A \cos(\Omega t - \kappa r) + M. \quad (14)$$

The external field $A \cos(\Omega t - \kappa r)$ is determined by the wave vector κ and frequency Ω . There is an additional constraint for M and A is given as $(M - A) > 0$ to ensure that the quasiparticles have positive mass. The dispersions are the eigenvalue of the following matrix:

$$\begin{bmatrix} \ddots & \dots & \dots & \dots & \dots \\ \dots & (\omega - \Omega)^2 - (k - \kappa)^2 + M & A/2 & 0 & \dots \\ \dots & A/2 & \omega^2 - k^2 + M & A/2 & \dots \\ \dots & 0 & A/2 & (\omega + \Omega)^2 - (k + \kappa)^2 + M & \dots \\ \dots & \dots & \dots & \dots & \ddots \end{bmatrix}.$$

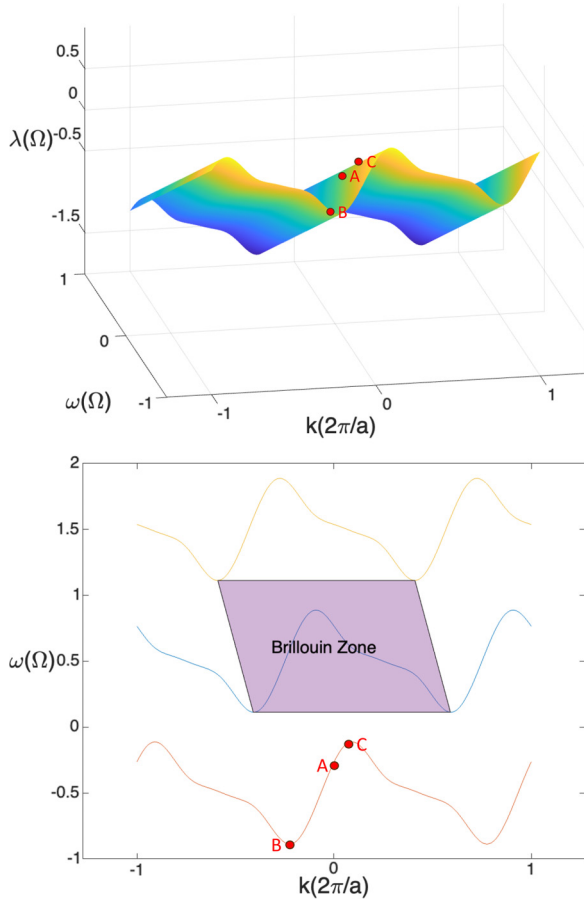


FIG. 1. The figure above is a single-band dispersion in phase space and the figure below is dispersion relation $\omega = \omega(\vec{k})$. This single-band spectrum of the oblique Floquet-Bloch crystal is computed with parameters $h = 0.4, t = 0.5, A = 1.5, \kappa = 1.60$, where the parameters h, t , and A are in units of Ω and κ is in units of $2\pi/a$. a is the lattice constant and Ω is the frequency of outer fields, which are set to be 1. The dispersion relation $\omega = \omega(\vec{k})$ can be solved by taking $\lambda = 0$. The oblique Brillouin zone is indicated by shaded region in the plot below, which is consistent with our previous analysis.

Two points are marked as D and E. These dispersion relations of these two points are hyperbolic curves in the (ω, k) plane, which are induced by the eigenvalues of the effective metric tensor $\lambda^{\mu\nu}$. The eigenvalues are one positive and one negative, and we can find such dispersions for particles moving on the spacelike manifold. Like what we find in the first example, the

parameters in expansion govern the motion of the quasiparticles.

IV. EVENT WAVE PACKET AND GEODYNAMICS UNDER SLOWLY VARYING PERTURBATION

With the independent four-momentum, the four-dimensional wave packet can be constructed as

$$|W\rangle = \int d^4k a(k, \tau) e^{ik_\mu x^\mu} |u(k, x)\rangle, \quad (15)$$

where $|W\rangle$ is the wave packet state function and $a(k, \tau)$ is the distribution function localized in the center of the wave packet with normalization to make $\langle W|W\rangle = 1$. The position of the center of the wave packet, where the wave packet is gathered in real spacetime, can be calculated by the expectation value of the position operator $\hat{x}$ as $\langle W|\hat{x}|W\rangle = x_c$. The momentum of the wave packet is also the expectation value from the function $a(k, \tau)$,

$$k_c = \int d^4k |a(k, \tau)|^2 k. \quad (16)$$

When a wave packet propagates through a slowly varying potential, where the adiabatic condition holds, the total dispersion λ for the wave packet is given by the expectation value of the full dispersion operator, $\lambda = \langle \phi|\hat{\mathcal{L}}|\phi\rangle$. Under the adiabatic condition, we can expand this expectation value around the wave packet's center, x_c , to first order:

$$\lambda \approx \langle \phi|\hat{\mathcal{L}}_c|\phi\rangle + \langle \phi|\Delta\hat{\mathcal{L}}|\phi\rangle. \quad (17)$$

This expansion separates the total dispersion into two physically distinct contributions.

The first term, $\lambda_c(x_c, k_c) = \langle \phi|\hat{\mathcal{L}}_c|\phi\rangle$, is the local dispersion. It arises from the local dispersion operator, $\hat{\mathcal{L}}_c = \hat{\mathcal{L}}(\hat{x}, \hat{p}; x_c)$, which describes a uniform crystal whose parameters match those of the perturbed crystal at the specific point x_c . This local operator and its eigenvalue λ_c are defined by the local eigenvalue equation [15]:

$$\hat{\mathcal{L}}\left(\frac{1}{i}\partial_x, x; x_c\right) |\phi(k, x; x_c)\rangle = \lambda_c(k, x; x_c) |\phi(k, x; x_c)\rangle. \quad (18)$$

The second term in the expansion is the gradient correction, $\Delta\lambda = \langle \phi|\Delta\hat{\mathcal{L}}|\phi\rangle$. This correction originates from the gradient correction operator $\Delta\hat{\mathcal{L}}$:

$$\Delta\hat{\mathcal{L}} = \frac{1}{2} \left[(\hat{x} - x_c) \cdot \frac{\partial \hat{\mathcal{L}}}{\partial x} + \frac{\partial \hat{\mathcal{L}}}{\partial x} \cdot (\hat{x} - x_c) \right]_{x=x_c}. \quad (19)$$

TABLE I. Expansion parameters of different points listed in Figs. 1 and 2. The first column refers to different points in figures, and the first row refers to different parameters. $k_{0\mu}$ are zero points of $\lambda(k, \omega)$, λ^μ are linear expansion parameters, and $\lambda^{\mu\nu}$ are quadratic parameters. A_μ are momentum shift vectors. \ means that in this direction the dispersion cannot be simplified to quartic forms.

	k_{01}	k_{00}	λ^1	λ^0	λ^{11}	λ^{10}	λ^{01}	λ^{00}	A_1	A_0
A	-0.03	-0.47	7.47	-1	-8.68	0	0	0	-0.89	\
B	-0.14	-0.69	0	-1	-152.39	0	0	0	0	\
C	0.044	-0.11	0	-1	153.97	0	0	0	0	\
D	3.07	-0.71	0.62	0.03	0.20	-6.88	-6.88	28.72	2.64	-0.81
E	1.04	-1.21	1.06	-0.89	1.64	-9.10	-9.10	63.16	3.87	-0.82

The explicit form of the gradient correction to the dispersion was derived in Ref. [15] and is generalized here to four dimensions:

$$\Delta\lambda = -\text{Im}\left[\left\langle\partial_{x^\mu}u\right|\lambda_c - \hat{\mathcal{L}}_c\left|\frac{\partial u}{\partial k_\mu}\right\rangle\right]\Bigg|_{k=k_c}. \quad (20)$$

Due to the presence of spacetime inhomogeneity, Berry connections associated with spacetime coordinates do not vanish. They are defined as follows:

$$\begin{aligned} A_{x^\mu} &= i\langle u(k, x)|\partial_{x^\mu}|u(k, x)\rangle, \\ A_{k_\mu} &= i\langle u(k, x)|\partial_{k_\mu}|u(k, x)\rangle. \end{aligned} \quad (21)$$

The total Berry curvatures are introduced in the eight-dimensional phase space by differentiating these Berry connections with respect to the four-momentum and four-coordinates:

$$\begin{aligned} \Omega_{k_\mu k_\nu} &= i(\langle\partial_{k_\mu}u|\partial_{k_\nu}u\rangle - \langle\partial_{k_\nu}u|\partial_{k_\mu}u\rangle), \\ \Omega_{k_\mu x^\nu} &= i(\langle\partial_{k_\mu}u|\partial_{x^\nu}u\rangle - \langle\partial_{x^\nu}u|\partial_{k_\mu}u\rangle), \\ \Omega_{x^\mu x^\nu} &= i(\langle\partial_{x^\mu}u|\partial_{x^\nu}u\rangle - \langle\partial_{x^\nu}u|\partial_{x^\mu}u\rangle). \end{aligned} \quad (22)$$

The semiclassical action for the particlelike wave packet is given by $S = \int L d\tau$, where the Lagrangian L is defined as $L = \langle W|i\frac{d}{d\tau} - \hat{\mathcal{L}}|W\rangle$. Explicitly, the Lagrangian can be expressed as [15]

$$L = -\lambda + k_{c\mu}\dot{x}_c^\mu + \dot{k}_{c\mu}A_{k_{c\mu}} + \dot{x}_c^\mu A_{x_c^\mu}. \quad (23)$$

This formulation generalizes the three-dimensional lattice scenario, incorporating the dependence on wave packet center positions, to an extended description applicable to a phase space with variational corrections in the dispersion operator.

Using the variation principle, the corresponding equations of motion can be derived:

$$\dot{x}^\mu = \frac{\partial\lambda}{\partial k_\mu}\Big|_{\lambda=0} - \Omega_{k_\mu x^\nu}\dot{x}^\nu - \Omega_{k_\mu k_\nu}\dot{k}_\nu, \quad (24)$$

$$\dot{k}_\mu = \frac{\partial\lambda}{\partial x^\mu}\Big|_{\lambda=0} + \Omega_{x^\mu x^\nu}\dot{x}^\nu + \Omega_{x^\mu k_\nu}\dot{k}_\nu, \quad (25)$$

where $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$ and $\dot{k}_\mu = \frac{dk_\mu}{d\tau}$. In subsequent discussions, since the physical results assume the condition $\lambda = 0$, this constraint will not be explicitly stated. An alternative method of presenting this constraint using an auxiliary field is explored in the Appendix.

V. GEODESIC EQUATIONS AND ARTIFICIAL GRAVITY

The phase-space equation of motion [Eq. (25)] can be viewed as describing particles propagating on a four-dimensional manifold, with extra “force” terms arising from the underlying dispersion band structure and geometry. A key question is whether these additional couplings genuinely modify the effective spacetime manifold through which the particles move. In what follows, we will demonstrate that the dispersion derivative terms and Berry-curvature contributions are effectively in the form of spacetime connections and gauge fields.

To illustrate the dynamics clearly, we begin by neglecting all Berry curvatures, focusing solely on the pure dynamical

effect of wave packet motion. Under this simplification, the equations of motion reduce to

$$\dot{x}^\mu = \frac{\partial\lambda}{\partial k_\mu}, \quad \dot{k}_\mu = -\partial_{x^\mu}\lambda. \quad (26)$$

To derive the geodesic equation, which reveals how the underlying spacetime manifold is curved, we need an equation dependent solely on coordinates and their derivatives. By differentiating the coordinate component once more and substituting the momentum expression, we obtain the second-order equation:

$$\ddot{x}^\mu = \dot{k}_\nu\partial_{k_\mu}\dot{x}^\nu - \dot{x}^\nu\partial_{k_\mu}\dot{k}_\nu. \quad (27)$$

The left-hand side represents the acceleration, while the right-hand side can be interpreted as a rotationlike term in phase space.

For the sake of later discussion, we introduce the concept of a free particle here. For a free particle, no external force arises from spatial or temporal variations in dispersion. This physical condition is mathematically expressed by requiring that the dispersion function λ has no explicit dependence on the spacetime coordinates, implying

$$\partial_{x^\mu}\lambda(k, x) = 0. \quad (28)$$

Moreover, this condition remains valid around the momentum center of the wave packet. As a free particle propagates, it experiences no force induced by derivatives of dispersion in momentum space, leading to

$$\partial_{k_\nu}\partial_{x^\mu}\lambda(k, x) = 0. \quad (29)$$

Under these conditions, the equations of motion simplify significantly:

$$\ddot{x}^\mu = 0. \quad (30)$$

This result confirms that the acceleration of a free particle must vanish, aligning with a generalized form of Newton’s first law.

For a more general discussion, we may expand the dispersion relation around a zero point to various orders. Focusing specifically on neighborhoods of zero points with nonzero first-order derivatives, such as point A in Fig. 1, the dispersion can be expressed in a linear form as $\lambda = \lambda^\mu k_\mu + \lambda_0$. Here, the Fermi velocity λ^μ and the deformation potential λ_0 depend solely on the spatial coordinates x . A similar dispersion relationship is found near Dirac points in Floquet graphene [16,17]. Under these conditions, the equation of motion in real space simplifies to

$$\dot{x}^\mu = \lambda^\mu, \quad (31)$$

illustrating explicitly that the wave packet is moving with Fermi velocity.

When quadratic terms are taken into account, the metric terms $\lambda^{\mu\nu}$ lead to modifications in the effective spacetime structure. To simplify the quadratic expansion, we introduce a shifted momentum $q_\mu = k_\mu - A_\mu$. Substituting this shift, the dispersion relation becomes

$$\begin{aligned} \ddot{x}^\mu + \tilde{\Gamma}_{\rho\sigma}^\mu \dot{x}^\rho \dot{x}^\sigma &= \lambda^{\mu\nu}\partial_\nu\lambda_0 - \lambda^{\mu\nu}\mathbf{F}_{\nu\rho}\dot{x}^\rho, \\ \tilde{\Gamma}_{\rho\sigma}^\mu &= \frac{1}{2}\lambda^{\mu\beta}(\partial_\beta\lambda_{\rho\sigma} - \partial_\rho\lambda_{\beta\sigma} - \partial_\sigma\lambda_{\beta\rho}). \end{aligned} \quad (32)$$

Here, the new connections $\tilde{\Gamma}_{\rho\sigma}^\mu$, which serve as Levi-Civita symbols induced from the effective metric $\tilde{g}_{\mu\nu} = \lambda_{\mu\nu}$, explicitly link the spacetime geometry to the dispersion relation. Additionally, the emergent Abelian gauge fields $\mathbf{F}_{\nu\rho} = \partial_\nu \mathbf{A}_\rho - \partial_\rho \mathbf{A}_\nu$ produce a term analogous to the Lorentz force; the variation of scalar field λ_0 provides an acceleration.

The spacetime interval induced by the effective metric $\lambda_{\mu\nu}$ is thus given by

$$ds^2 = \lambda_{\mu\nu} dx^\mu dx^\nu. \quad (33)$$

In a principal coordinate frame, this metric can be diagonalized:

$$ds^2 = \eta_0(dx'^0)^2 + \eta_1(dx'^1)^2 + \eta_2(dx'^2)^2 + \eta_3(dx'^3)^2. \quad (34)$$

Here, $\eta_\mu(x')$ are the local eigenvalues of the metric tensor, which could be adjusted to different values of models. This tunability provides a unique tabletop platform to explore physics typically confined to the realms of cosmology or high-energy string theory. For instance, a metric with two positive and two negative eigenvalues, corresponding to a $(+, +, -, -)$ signature, realizes a geometry explored in “2-time physics” models [18]. Furthermore, a metric that retains the standard Lorentzian signature $(+, -, -, -)$ but with unequal spatial eigenvalue magnitudes provides a direct analog to the anisotropic Bianchi I universe [19,20]. Crucially, the symmetries of the effective metric are controlled by engineering the underlying Floquet spacetime crystal. This establishes a remarkable tabletop platform to simulate and study quasiparticle dynamics within a diverse range of model universes, including those with exotic signatures and anisotropies.

Now we add Berry curvatures back and keep the free particle constraint. The equations of motion (25) may be simplified to the form

$$(\delta_\mu^\nu - \Omega_{x^\mu k_\nu}) \dot{k}_\nu = \Omega_{x^\mu x^\nu} \dot{x}^\nu. \quad (35)$$

The equations above show that the particle is moving under the artificial gauge field, which could induce topological Hall effect [21]. As for the spatial part, they are

$$\dot{x}^\mu = \frac{\partial \lambda}{\partial k_\mu} - \Omega_{k_\mu x^\nu} \dot{x}^\nu - \Omega_{k_\mu k_\nu} \dot{k}_\nu. \quad (36)$$

Taking an additional derivative of the coordinate equation, the geodesic equation can be further refined to

$$\begin{aligned} \ddot{x}^\rho + \partial_{x^\sigma} \Omega_{k_\rho x^\nu} \dot{x}^\sigma \dot{x}^\nu \\ = (\Omega_{x^\lambda x^\alpha} \partial_{k_\alpha} \partial_{k_\rho} \lambda + \Omega_{k_\rho k_\nu} \partial_{x^\lambda} \partial_{x^\nu} \lambda) \dot{x}^\lambda. \end{aligned} \quad (37)$$

Equation (37) is a second-order correction of Eq. (30), incorporating the effects induced by gauge fields. The terms $\partial_{k_\alpha} \partial_{k_\rho} \lambda$ and $\partial_{x^\lambda} \partial_{x^\nu} \lambda$ act as effective metrics in phase space that couple to the dual Berry curvatures. Berry curvatures with indices in real space ($\Omega_{x^\lambda x^\alpha}$) or momentum space ($\Omega_{k_\rho k_\nu}$) are similar to electromagnetic fields in general relativity [22]. Additionally, the Berry curvature $\Omega_{k_\rho x^\nu}$ modifies the density of states in phase space in the absence of external fields [23]. Its spatial covariant derivative contributes directly to alterations in wave packet trajectories, demonstrating the impact of phase-space dynamics on real-space motion.

VI. GEOMETRY OF DEFORMATION IN SPACETIME CRYSTAL

In addition to perturbations induced by the wave packet center, deformations of the crystal structure also play a crucial role. Spacetime deformations, easily controllable in certain cold-atom systems, significantly affect the motion of quasiparticles. Extending our description from a flat spacetime crystal to a deformed spacetime crystal necessitates specifying local crystal variations explicitly. Consequently, introducing a lattice connection becomes essential to characterize these local structural changes precisely.

The spacetime lattice connection captures the inhomogeneity of spacetime crystals. When a spacetime crystal is deformed, the lattice vectors become vector fields varying with the position and time, which means $c_\alpha = c_\alpha(x)$. The form is a direct generalization of local lattice connection in three-dimensional deformed lattices [10,24]. Because of the orthogonal condition between lattice vectors and reciprocal lattice vectors, the reciprocal lattice vectors are also vector fields $b^\alpha = b^\alpha(x)$. To derive the form of lattice connection, consider the lattice vectors change $dc_\alpha^\mu = \partial_\nu c_\alpha^\mu dx^\nu$, we define the lattice connection to make the change of lattice vector fields proportional to the product of lattice connection and lattice vector fields, where $\Gamma_{\sigma m}^\mu$ is the connection. By substituting reciprocal lattice vectors,

$$\Gamma_{\sigma m}^\mu = b_\sigma^\alpha \partial_m c_\alpha^\mu. \quad (38)$$

The adiabatic condition of deformation requires a relatively small deformation, so all the derivatives of lattice vector fields are small terms $|\frac{\partial_\mu c_\alpha(x)}{c_\alpha(x)}| \ll 1$. In momentum space, all the reciprocal lattice vectors are also slowly deformed and satisfy $|\frac{\partial_\mu b^\alpha(x)}{b^\alpha(x)}| \ll 1$.

Under slow, long-wavelength deformation, the lattice potential no longer maintains strict periodicity. However, it retains local periodicity within small spacetime regions. Consider the one-dimensional boson model as an example, where the lattice potential is expressed as $A \cos(\Omega t - \kappa r)$. Under deformation, we set $\Omega(x, t) = 5 + 0.12 \cos(t)$, $\kappa(x, t) = 5 + 0.12 \sin(x)$, and $a(x, t) = 1 + 0.1 \sin(0.01t)$. All parameters here are dimensionless. In Fig. 3, we illustrate the potential over an extended range to highlight its deformation clearly, and we select a smaller region to demonstrate that the potential experienced by a wave packet locally still obtains translational symmetry approximately.

The lattice connection defined earlier may exhibit an antisymmetric component due to the discrete nature of the spacetime lattice points. Nonetheless, wave packet dynamics remain valid for describing localized spacetime wave packets, provided that there are no topological defects in the spacetime crystal. The presence of topological defects introduces singularities characterized by the equation

$$\oint_C b_\mu^\alpha dx^\mu = Z^\alpha, \quad (39)$$

where Z^α is a nonzero integer and the integral is evaluated around an arbitrary loop containing the defect. These singularities break local translational symmetry, invalidating the wave packet description. Our current discussion assumes explicitly that no such singularities exist.

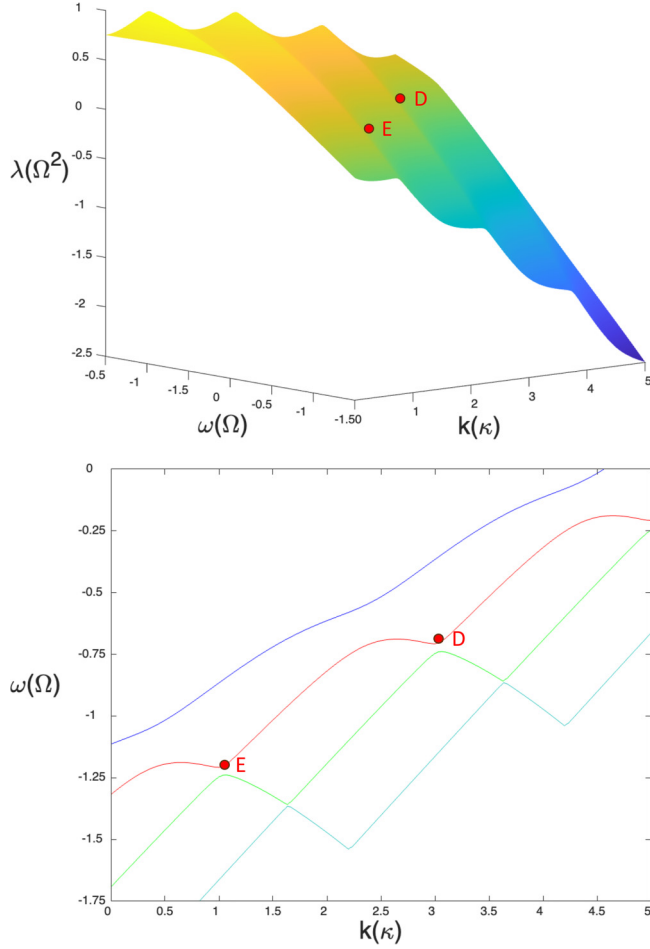


FIG. 2. Dispersion of boson with time-translation symmetry. We select only one band to present. The parameters are $A = 0.5$, $M = 0.75$, where k is in units of κ and Ω , A , and M are in units of Ω . λ is in units of Ω^2 . The dispersion relation can be solved by taking $\lambda = 0$. The figure below presents the dispersion of the band containing points D and E.

In general, for the 1D boson with deformation $\Omega = \Omega(x, t)$, $\kappa = \kappa(x, t)$, $a = a(x, t)$, the lattice vectors are $(\frac{2\pi}{\Omega}, 0)$ and $(\frac{\kappa a}{\Omega}, a)$, and the reciprocal lattice vectors are $(\frac{\Omega}{2\pi}, \frac{\kappa}{2\pi})$ and $(0, -\frac{1}{a})$. By substituting the expression of the lattice connection, we can find that all the connection components are

$$\begin{aligned} \Gamma_{00}^0 &= -\frac{\partial_t \Omega}{\Omega}, & \Gamma_{01}^0 &= -\frac{\partial_x \Omega}{\Omega}, & \Gamma_{10}^0 &= -\frac{a \partial_t \kappa + \kappa \partial_t a}{a \Omega}, \\ \Gamma_{11}^0 &= -\frac{a \partial_x \kappa + \kappa \partial_x a}{a \Omega}, & \Gamma_{00}^1 &= 0, & \Gamma_{01}^1 &= 0, \\ \Gamma_{10}^1 &= -\frac{\partial_t a}{a}, & \Gamma_{11}^1 &= -\frac{\partial_x a}{a}. \end{aligned} \quad (40)$$

The lattice connections above show that the connections are not torsion-free because $\Gamma_{\rho\sigma}^\mu \neq \Gamma_{\sigma\rho}^\mu$. The torsion defined as the antisymmetric part of connection is $S_{\rho\sigma}^\mu = \frac{1}{2}(\Gamma_{\rho\sigma}^\mu - \Gamma_{\sigma\rho}^\mu)$. From the definition of torsion, we can obtain $S_{\rho\sigma}^\mu = -S_{\sigma\rho}^\mu$, so there are only two independent components of tor-

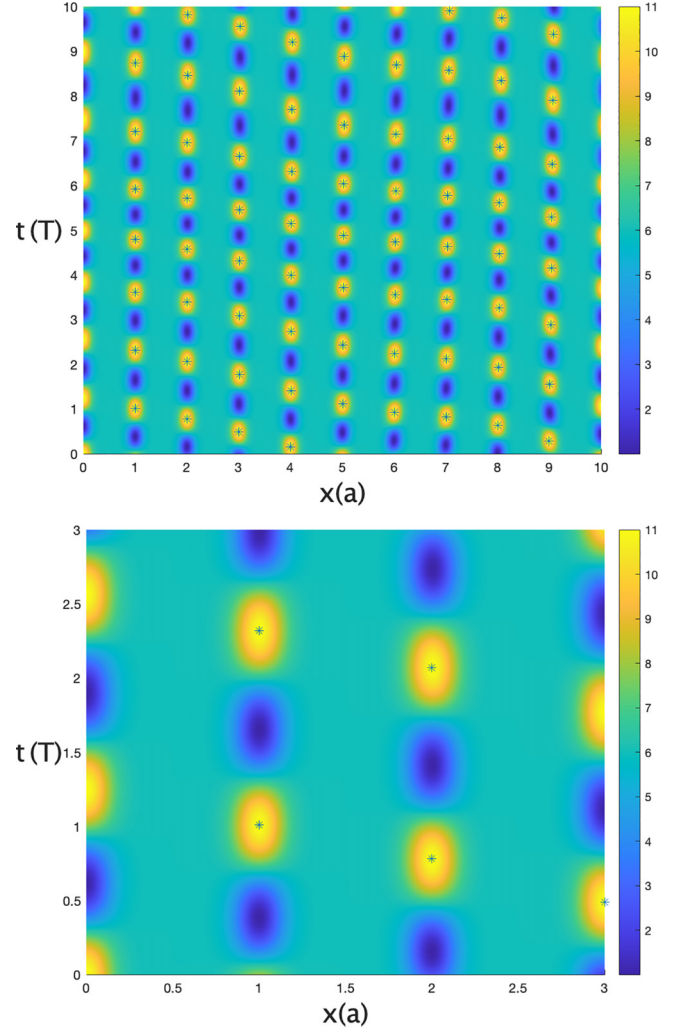


FIG. 3. Graphs of the lattice potential of 1D boson space-time crystal. The parameter is $\Omega = 5 + 0.12 \cos(x)$, $\kappa = 5 + 0.12 \sin(x)$, $a = 1 + 0.1 \sin(0.01t)$. The maximum point of lattice potential is marked by a star label. The figure above is the potential over a large spacetime zone, where significant variation of periodicity can be found. The figure below shows the potential in a small region where the potential forms an approximately constant lattice structure.

sion, S_{01}^0 and S_{01}^1 , which are

$$S_{01}^0 = \frac{a \partial_t \kappa + \kappa \partial_t a - a \partial_x \Omega}{2a\Omega}, \quad S_{01}^1 = \frac{\partial_t a}{2a}. \quad (41)$$

Torsion may couple to spin and angular momentum in general relativity [25], but since we just take one dispersion band into consideration, the coupling with spin or angular momentum may be neglected. However, we can find the torsion of connection may couple to effective gauge fields and lead to non-Abelian situations.

VII. GEODYNAMICS OF EVENT WAVE PACKET UNDER DEFORMATION

A free particle in curved spacetime moves along a geodesic line, following the straightest possible path determined solely by spacetime curvature [22]. To isolate the purely

dynamical motion along the path, we introduce the lattice covariant derivative, which is required to satisfy the local periodicity condition for any derivative,

$$\nabla_{x^\mu} f|(k+2\pi b(x),x) = \nabla_{x^\mu} f|(k,x), \quad (42)$$

where f is an arbitrary periodic function satisfying $f(k+2\pi b(x),x) = f(k,x)$ and ∇_x is the covariant derivative. This condition, which ensures that our description is independent of the local choice of unit cell, leads uniquely to the following expression for the covariant derivative:

$$\nabla_{x^\mu} f = \partial_{x^\mu} f(k,x) - k_l \Gamma_{\mu\nu}^l \partial_{k_\nu} f(k,x). \quad (43)$$

The expressions derived here have a similar form to those presented in Ref. [26]. By selecting the linear and quadratic momentum functions $f = \lambda^\mu k_\mu$ and $f = \lambda^{\mu\nu} k_\mu k_\nu$, respectively, we obtain covariant derivatives for the first- and second-order expansion coefficients of the dispersion relation λ :

$$\begin{aligned} \nabla_{x^\sigma} \lambda^\mu &= \partial_{x^\sigma} \lambda^\mu - k_l \Gamma_{m\sigma}^l \partial_{k_m} \lambda^\mu - \Gamma_{\alpha\sigma}^\mu \lambda^\alpha \\ \nabla_{x^\sigma} \lambda^{\mu\nu} &= \partial_{x^\sigma} \lambda^{\mu\nu} - k_l \Gamma_{m\sigma}^l \partial_{k_m} \lambda^{\mu\nu} - \Gamma_{\sigma m}^\mu \lambda^{\nu m} - \Gamma_{\sigma m}^\nu \lambda^{\mu m}. \end{aligned} \quad (44)$$

These covariant derivatives, compared with those in general relativity, contain additional terms required to keep periodicity in phase space.

To achieve a complete phase-space representation of the equations of motion, we introduce the covariant total derivative of momentum. We require that the total derivative of a function can be expressed as

$$df(k,x) = \nabla_{x^\mu} f dx^\mu + \partial_{k_\mu} f Dk_\mu. \quad (45)$$

Therefore, the total derivative of quasimomentum, under spacetime deformation, is defined by

$$Dk_\mu = dk_\mu + k_l \Gamma_{\nu\mu}^l dx^\nu. \quad (46)$$

The total derivative thus incorporates influences from both spacetime deformation and the intrinsic periodicity in momentum space.

The Lagrangian of quasiparticle of a deformed crystal can be generalized from Eq. (23) by changing all terms to covariant forms. Following the rule above, the total derivative of k is changed to Dk to indicate the pure dynamic effect. As for the dispersion, besides the terms proportional to gradient correction, there is an extra term because of the deformation. To obtain the dispersion, we turn to the dispersion operator correction first, the correction is (see Appendix for details)

$$\Delta \hat{\mathcal{L}}_c = \frac{1}{2} [(\hat{x} - x_c) \cdot \nabla_{x_c} \hat{\mathcal{L}}_c + \nabla_{x_c} \hat{\mathcal{L}}_c \cdot (\hat{x} - x_c)] + \frac{1}{2} \Gamma_{\mu\sigma}^m \hat{\mathcal{O}}_m^{\mu\sigma}, \quad (47)$$

where we replace all derivatives by covariant forms, and $\hat{\mathcal{O}}_m^{\mu\sigma} = \sum_l \frac{\partial \hat{\mathcal{L}}(\{\tilde{R}_l - x\})}{\partial \tilde{R}_l^m} (\tilde{R}_l - x)^\sigma (\tilde{R}_l - x)^\mu$ is the additional term from the deformation. The dispersion correction $\langle \phi | \Delta \hat{\mathcal{L}} | \phi \rangle$ is

$$\Delta \lambda = -Im \left[\langle \nabla_{x_c} u | \lambda_c - \hat{\mathcal{L}}_c \left| \frac{\partial u}{\partial k} \right\rangle \right] \Big|_{k=k_c} + \frac{1}{2} \Gamma_{\mu\sigma}^m \langle \hat{\mathcal{O}}_m^{\mu\sigma} \rangle \Big|_{k=k_c}. \quad (48)$$

With the modified dispersion, we have the covariant Lagrangian,

$$L = -\lambda + k_{c\mu} \dot{x}_c^\mu + \dot{k}_{c\mu} A_{k_{c\mu}} + \dot{x}_c^\mu A_{x_c^\mu}. \quad (49)$$

Using the covariant variational method $\delta f = \nabla_{x^\mu} f \delta x^\mu + \partial_{k_\mu} f \delta Dk_\mu$, where $\delta Dk_\mu = \delta k_\mu + k_l \Gamma_{\nu\mu}^l \delta x^\nu$. The covariant equations of motion is

$$\dot{x}^\mu = \frac{\partial \lambda}{\partial k_\mu} \Big|_{\lambda=0} - \Omega_{k_\mu x^\nu} \dot{x}^\nu - \Omega_{k_\mu k_\nu} \dot{k}_\nu, \quad (50)$$

$$\dot{k}_\mu = -\nabla_{x^\mu} \lambda \Big|_{\lambda=0} + \Omega_{x^\mu x^\nu} \dot{x}^\nu + \Omega_{x^\mu k_\nu} \dot{k}_\nu, \quad (51)$$

where $\dot{k}_\mu = \frac{Dk_\mu}{d\tau}$ and the equation is in the covariant form. x and k are the wave packet center position and momentum. Here we omit the subscript c for simplicity.

Our framework applies generally to various quantum systems without requiring the explicit form of their action. However, for an important subset of quantum systems governed by the Schrödinger equation, the Lagrangian has a particularly simple structure. In these cases, the dispersion introduced by the operator $\hat{\mathcal{L}} = i\partial_t - \hat{H}$ always takes the form $\lambda = -\omega + \epsilon(\vec{k}, x, t)$, where $\epsilon(\vec{k}, x, t)$ is the eigenvalue of $\hat{H}$ including gradient corrections. With this dispersion relation, the equation of motion for the temporal component simplifies to

$$i = \frac{\partial \lambda}{\partial(-\omega)} = 1. \quad (52)$$

The linear dependence of ω leads directly to $t = \tau$. Additionally, since the state function is independent of ω , the Berry connection in ω is zero. Substituting $t = \tau$ into the Lagrangian, we obtain

$$L = -\epsilon(\vec{x}, \vec{k}) + \vec{k} \cdot \dot{\vec{x}} + (A_t + \dot{\vec{x}} \cdot \vec{A}_x) + \dot{\vec{k}} \cdot \vec{A}_k \quad (53)$$

which matches the known expression for wave packet dynamics in a three-dimensional deformed lattice as discussed in Refs. [10,24]. Our advancement extends this scenario to explicitly include discrete time periodicity within a deformed lattice. The conventional time-independent scenario can thus be viewed as the continuum limit of this periodic structure as the time period $T \rightarrow 0$.

VIII. GEODESIC LINE AND ARTIFICIAL GRAVITY WITH DEFORMATION

In previous sections, we identified effective gravity induced by intrinsic dispersion of quasiparticles. However, when external deformation is introduced, emergent gravity may result from the interplay between this deformation and intrinsic dispersion of the quasiparticles. Deformation primarily determines the modification of the spacetime structure, while the geometric properties of quasiparticles and dispersion relations provide higher-order corrections under the assumption of adiabatic approximation.

Following the approach established in previous sections, to account for deformation, we replace all terms with their covariant forms, analogous to the process described in Eq. (51). Upon this substitution, we note that the commutator between the momentum derivative and the covariant derivative in coordinate space is nonzero, specifically

$$[\nabla_{x^\nu}, \partial_{k_\mu}] f(k,x) = \Gamma_{\nu\mu}^\mu \partial_{k_\mu} f(k,x). \quad (54)$$

Applying these covariant replacements allows us to derive the equations of motion consistent with those deformations.

First, we may find the equation of motion for a free particle. When we ignore all Berry curvatures, Eq. (51) is simplified to

$$\begin{aligned}\dot{x}^\mu &= \frac{\partial \lambda}{\partial k_\mu}, \\ \dot{k}_\mu &= -\nabla_{x^\mu} \lambda,\end{aligned}\quad (55)$$

and taking an extra derivative of Eq. (55) yields

$$\ddot{x}^\mu - \Gamma_{\rho\sigma}^\mu \dot{x}^\rho \dot{x}^\sigma = \dot{k}_\nu \partial_{k_\mu} \dot{x}^\nu - \dot{x}^\nu \partial_{k_\mu} \dot{k}_\nu, \quad (56)$$

The left-hand side of Eq. (56) describes covariant transport, where the lattice connection plays a similar role as the Levi-Civita symbol in general relativity, and the right-hand side of Eq. (56) is a rotation in phase space like we find in previous sections.

The constraint defining free particles is that the covariant derivative of the dispersion should vanish,

$$\nabla_{x_c^\mu} \lambda(k_c, x_c) = 0. \quad (57)$$

This constraint is still valid within a certain neighborhood around x_c and k_c , leading to the additional condition $\partial_{k_{cv}} \nabla_{x_c^\mu} \lambda(k_c, x_c) = 0$. Under this constraint, Eq. (56) reduces to

$$\ddot{x}^\rho - \Gamma_{\nu\sigma}^\rho \dot{x}^\sigma \dot{x}^\nu = 0, \quad (58)$$

which is the standard geodesic equation. Although the entire system is embedded in a flat laboratory spacetime manifold, the curved lattice structure makes quasiparticles behave as if they are moving in a curved universe. Unlike the Levi-Civita connection typically encountered in general relativity, the lattice connection described here is experimentally accessible and can be manipulated, especially in cold atom systems. The deformed lattice connection thus provides a source of first-order artificial gravity, causing observable deviations in geodesic trajectories from those predicted by the free particle geodesic equation.

Assuming a dispersion relation that is linear in momentum, $\lambda = \lambda^\mu k_\mu + \lambda_0$, the velocity of the wave packet is given by

$$\dot{x}^\mu = \lambda^\mu. \quad (59)$$

While this first-order equation describes the motion, the full dynamics are revealed in the second-order equation for acceleration. Here, the effects of deformation, which are intrinsically linked to the underlying lattice, manifest as an effective force:

$$\ddot{x}^\mu - \Gamma_{\rho\sigma}^\mu \dot{x}^\rho \dot{x}^\sigma = -\dot{x}^\nu \nabla_{x^\nu} \lambda^\mu. \quad (60)$$

This modified geodesic equation shows how covariant derivatives of the dispersion coefficients cause the particle's trajectory to deviate from a standard geodesic.

Consider the dispersion up to the quadratic term; we still perform a momentum shift $q_\mu = k_\mu - A_\mu$ for simplicity, so that the dispersion is $\lambda = \lambda_0 + \frac{1}{2} \lambda^{\mu\nu} q_\mu q_\nu$. With such form of dispersion, the geodesic equation is

$$\begin{aligned}\ddot{x}^\mu - (\Gamma_{\rho\sigma}^\mu - \tilde{\Gamma}_{\rho\sigma}^\mu) \dot{x}^\rho \dot{x}^\sigma &= \lambda^{\mu\nu} \partial_\nu \lambda_0 - \lambda^{\mu\nu} F_{\nu\rho} \dot{x}^\rho, \\ \tilde{\Gamma}_{\rho\sigma}^\mu &= \frac{1}{2} \lambda^{\mu\beta} (\nabla_\beta \lambda_{\rho\sigma} - \nabla_\rho \lambda_{\beta\sigma} - \nabla_\sigma \lambda_{\beta\rho}).\end{aligned}\quad (61)$$

Compared with Eq. (32), there are two differences, the first one is we have lattice connection together with connection induced by covariant derivatives of the dispersion as an effective connection and the other is the gauge field may be non-Abelian when there is torsion. The covariant gauge field is $F_{\nu\rho} = \nabla_\nu A_\rho - \nabla_\rho A_\nu$, which is

$$F_{\nu\rho} = \partial_\nu A_\rho - \partial_\rho A_\nu + 2S_{\nu\rho}^\mu A_\mu. \quad (62)$$

The covariant equation for a free particle with Berry curvatures is

$$\begin{aligned}\ddot{x}^\rho + (\nabla_{x^\sigma} \Omega_{k_\rho x^\nu} - \Gamma_{\nu\sigma}^\rho) \dot{x}^\sigma \dot{x}^\nu \\ = (\Omega_{x^l x^\alpha} \partial_{k_\alpha} \partial_{k_\rho} \lambda + \Omega_{k_\rho k_\nu} \nabla_{x^l} \nabla_{x^\nu} \lambda) \dot{x}^l.\end{aligned}\quad (63)$$

Equation (63) is the covariant form of Eq. (37). All the Berry curvatures are obtained using covariant derivative to be consistent with deformation, which is also true for the new effective connection $\nabla_{x^\sigma} \Omega_{k_\rho x^\nu}$ and metrics $\nabla_{x^l} \nabla_{x^\nu} \lambda$.

After obtaining the geodesic equations under various approximations and constraints, it is instructive to clarify the similarities and differences between geodesic lines in spacetime crystals and those encountered in general relativity. A key similarity is that both types of geodesic equations involve connections, highlighting the deformation of the underlying spacetime manifold. In the case of spacetime crystals, an effective connection emerges, influenced by the dispersion relations and geometric characteristics of the system, whereas in general relativity, the connection is uniquely determined by the spacetime metric itself. The first crucial difference arises from the effective force term. In spacetime crystals, this effective force originates from momentum shifts and Berry curvatures, both of which are gauge fields coupled to the new dispersion metric. Another difference is that because of the rich dispersion in meta-spacetime crystals, all the effective metric terms induced by dispersion or Berry curvatures in phase space could simulate spacetime structures that are hard to realize in general relativity.

In summary, the effective spacetime structure experienced by quasiparticles in spacetime crystals is influenced by three factors: the deformation of the base manifold, the dispersion characteristics of quasiparticles, and the geometrical properties associated with these particles. The first aspect introduces lattice connections and covariant derivatives. The latter two aspects represent extensions beyond the standard formulation of general relativity by providing modification in spacetime connection ($\tilde{\Gamma}_{\rho\sigma}^\mu, \nabla_{x^\sigma} \Omega_{k_\rho x^\nu}$) and Lorentz force terms arising from the interplay between new gauge fields and their dual metrics in phase space ($\Omega_{x^l x^\alpha} \partial_{k_\alpha} \partial_{k_\rho} \lambda + \Omega_{k_\rho k_\nu} \nabla_{x^l} \nabla_{x^\nu} \lambda$).

IX. SUMMARY AND OUTLOOK

In this paper, we have developed a method for constructing spacetime wave packets to represent events within spacetime crystals and have derived the corresponding equations of motion for the particle worldlines. Additionally, we have introduced a geometric framework to characterize deformation within spacetime crystals, leading to a covariant formulation of the equations of motion in phase space. Our detailed analysis of the geodesic equations revealed that the effective gravity

experienced by a quasiparticle emerges from a rich interplay of classical and quantum geometric effects.

The principal result of our formalism is the decomposition of this effective gravity into three distinct physical sources. The first, lattice deformation gravity, provides the background geometry, where the lattice connection functions as the primary spacetime connection. The second, dispersion gravity, introduces a quantum geometric correction by defining an intrinsic effective metric, which in turn induces its own Levi-Civita-like connection. Finally, Berry curvature gravity contributes in a dual capacity: It not only generates Lorentz-like forces through its gauge field but also directly modifies the geodesic equation by contributing an effective connection term, which is derived from the spatial gradient of the Berry curvature. This tripartite structure systematically shows how a particle's trajectory is shaped by the interplay between the classical lattice, the quantum dispersion, and the topological geometry of the wave function.

A natural and intriguing extension of this work would be to generalize our method to non-Abelian cases, such as quasiparticles exhibiting spin textures or other internal degrees of freedom, which could couple explicitly to the torsion component of the lattice connection. Furthermore, addressing the Einstein field equations for the underlying spacetime manifold represents a compelling direction for future research. Incorporating external dynamical elements such as domain walls and electromagnetic fields may enable explicit derivation of these equations within our framework.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [27].

APPENDIX

1. Gradient expansion to dispersion

Viewed in the laboratory frame, the total dispersion depends on a deformed spacetime lattice, which is difficult to handle directly. Since the variation is small, we expand the dispersion to first order in the deformation; the zeroth-order dispersion is locally periodic along the spatial and temporal axes. To obtain a periodic frame (the “lattice frame”), we introduce a smooth field parameterized by lattice-frame coordinates. Given the four-positions of lattice points in the laboratory frame, R_l (where l is a four-component index locating a lattice point), we define a smooth lattice field $R(x')$ satisfying

$$R_l = R(l), \quad (\text{A1})$$

and relate laboratory-frame coordinates to lattice-frame coordinates by

$$x = R(x'). \quad (\text{A2})$$

By this definition, any deforming crystal is mapped to a periodic unit-cell lattice in the lattice frame. The lattice vector fields then follow naturally:

$$c_\alpha^\mu \left(\frac{l + (l + 1)_\alpha}{2} \right) = R_{l+1_\alpha} - R_l. \quad (\text{A3})$$

Returning to the laboratory frame, this is consistent with the definition used in the main text.

The local dispersion operator $\hat{\mathcal{L}}(k, r, x_c)$ depends on the deformation of four-coordinates in lattice and four-coordinates of quasiparticle. Generally, in the laboratory frame, we can find that the spatially related part has the form like $\hat{\mathcal{L}}(\tilde{R}_l - x)$, where x is the center position of wave packet. With the lattice field $R_l = R(l)$ and $\tilde{R}_l = l^\alpha c_\alpha(x)$, expanding l with respect to the electron position in lattice frame x' , we have

$$R_l - x = R(l) - R(x') \approx (l - x')^\alpha c_\alpha + \frac{1}{2}(l - x')^\alpha (l - x')^\beta (c_\beta \cdot \partial) c_\alpha, \quad (\text{A4})$$

and the gradient correction of effective due to deformation is

$$\hat{\mathcal{L}}(\tilde{R}_l - x) \approx \hat{\mathcal{L}}((l^\alpha - x'^\alpha) c_\alpha(x)) + \frac{1}{2} \Gamma_{\mu\sigma}^m \hat{\mathcal{O}}_m^{\mu\sigma}, \quad (\text{A5})$$

where $\hat{\mathcal{O}}_m^{\mu\sigma} = \sum_l \frac{\partial \hat{\mathcal{L}}(\{\tilde{R}_l - x'^\alpha c_\alpha\})}{\partial \tilde{R}_l^m} (\tilde{R}_l - x'^\alpha c_\alpha)^\sigma (\tilde{R}_l - x'^\alpha c_\alpha)^\mu$, if we return to laboratory frame we can find that the correction of deformation is $\hat{\mathcal{O}}_m^{\mu\sigma} = \sum_l \frac{\partial \hat{\mathcal{L}}(\{\tilde{R}_l - x\})}{\partial \tilde{R}_l^m} (\tilde{R}_l - x)^\sigma (\tilde{R}_l - x)^\mu$, where x is four-coordinate of wave packet center in the laboratory frame.

In addition to deformation of lattice points, the dispersion also depends on the position of the center spacetime position of the wave packet. Using the covariant derivative, the gradient correction due to the center position is

$$\Delta \hat{\mathcal{L}}' = \frac{1}{2} [(\hat{x} - x_c) \cdot \nabla_{x_c} \hat{\mathcal{L}}_c + \nabla_{x_c} \hat{\mathcal{L}}_c \cdot (\hat{x} - x_c)], \quad (\text{A6})$$

so the total correction is

$$\Delta \hat{\mathcal{L}}_c = \frac{1}{2} [(\hat{x} - x_c) \cdot \nabla_{x_c} \hat{\mathcal{L}}_c + \nabla_{x_c} \hat{\mathcal{L}}_c \cdot (\hat{x} - x_c)] + \frac{1}{2} \Gamma_{\mu\sigma}^m \hat{\mathcal{O}}_m^{\mu\sigma}. \quad (\text{A7})$$

With the effective correction, the first-order correction of eigenvalue λ is the expectation value of $\Delta \hat{\mathcal{L}}_c$ in the wave packet state,

$$\Delta \lambda = \langle W | \Delta \hat{\mathcal{L}}_c | W \rangle. \quad (\text{A8})$$

Within Sundaram-Niu wave packet formalism,

$$\Delta \lambda = -\text{Im} \left[\langle \nabla_{x_c} u | \lambda_c - \hat{\mathcal{L}}_c \left| \frac{\partial u}{\partial k} \right\rangle \right]_{k=k_c} + \frac{1}{2} \Gamma_{\mu\sigma}^m \langle \hat{\mathcal{O}}_m^{\mu\sigma} \rangle_{k=k_c}. \quad (\text{A9})$$

2. Geodesic line equation in general

In the main text, we obtained geodesic line equations with different corrections to examine their effects. Here, we present a general geodesic line equation indicating the coupling of dispersion and Berry curvatures. Using another set of coordinates

for simplicity, the equations of motion are

$$M_{\mu\nu}^{ij} \dot{\xi}_i^v = \partial_{\xi_j^\mu} \lambda, \quad (\text{A10})$$

where $M_{\mu\nu}$ is an 8×8 matrix

$$M_{\mu\nu}^{ij} = \begin{bmatrix} \eta_{\mu\nu} + \Omega_{k^\mu k^\nu} & \Omega_{k^\mu k^\nu} \\ -\Omega_{x^\mu x^\nu} & \eta_{\mu\nu} - \Omega_{x^\mu k^\nu} \end{bmatrix} \quad (\text{A11})$$

ξ collects the phase-space coordinates,

$$\xi_j^v = \begin{bmatrix} x^v \\ k^v \end{bmatrix}. \quad (\text{A12})$$

Both the matrix and the coordinate vector are partitioned into real-space (top) and momentum-space (bottom) components; each is a four-vector. For simplicity, we lower the indices of k and x with the metric. Repeated Latin indices i, j and Greek indices μ, ν are summed. The partial derivatives take the covariant form

$$\partial_{\xi_j^\mu} = \begin{bmatrix} \nabla_{x^\mu} \\ \partial_{k^\mu} \end{bmatrix}. \quad (\text{A13})$$

Under the adiabatic approximation of deformation and perturbation, the general geodesic equation is

$$M_{\mu\nu}^{ij} \ddot{\xi}_j^v + (T_{\rho\mu\nu}^{ijk} + \Gamma_{\rho\mu}^{m;k} A_{m\nu}^{ij} + \Gamma_{\rho\nu}^{m;k} A_{\mu m}^{ij} - \Gamma_{\mu\nu}^{m;k} M_{\mu\rho}^{ij}) \dot{\xi}_k^\rho \dot{\xi}_j^v = \partial_{\xi_j^\mu} \partial_{\xi_i^\nu} \lambda \dot{\xi}_j^v, \quad (\text{A14})$$

where $A_{\mu\nu}^{ij}$ is the Berry curvature matrix in phase space,

$$A_{\mu\nu}^{ij} = \begin{bmatrix} \Omega_{k^\mu k^\nu} & \Omega_{k^\mu k^\nu} \\ 0 & -\Omega_{x^\mu k^\nu} \end{bmatrix}, \quad (\text{A15})$$

and the tensor $T_{\rho\mu\nu}^{ijk} = \partial_{\xi_i^\rho} A_{\mu\nu}^{jk}$. The connection here $\Gamma_{\rho\sigma}^{m;k}$ is a vector,

$$\Gamma_{\rho\sigma}^{m;k} = [\Gamma_{\rho\sigma}^m \quad 0]. \quad (\text{A16})$$

Since there is deformation only in real space, the component of momentum in the connection vector vanishes. This general form shows that the geometric phases couple to lattice deformation and induce higher-order modifications in the effective connection.

3. An example in one-dimensional spacetime crystal

To illustrate the geodesic equation and the role of deformation more directly, consider a (1+1)D spacetime crystal with arbitrary dispersion. We write the geodesic equation using t instead of τ as the parameter. The two geodesic equations [see Eq. (58)] for free particles are

$$\ddot{t} + \Gamma_{00}^0 \dot{t}^2 + (\Gamma_{01}^0 + \Gamma_{10}^0) \dot{t} \dot{x} + \Gamma_{11}^0 \dot{x}^2 = 0, \quad (\text{A17})$$

$$\ddot{x} + \Gamma_{00}^1 \dot{t}^2 + (\Gamma_{01}^1 + \Gamma_{10}^1) \dot{t} \dot{x} + \Gamma_{11}^1 \dot{x}^2 = 0. \quad (\text{A18})$$

Using $\dot{x} = \dot{x}'$ with $x' = \frac{dx}{dt}$, we have the equation of x :

$$x'' + \Gamma_{00}^1 + (\Gamma_{01}^1 + \Gamma_{10}^1 - \Gamma_{00}^0) x' + (\Gamma_{11}^1 - \Gamma_{01}^0 - \Gamma_{10}^0) x'^2 - \Gamma_{11}^0 x'^3 = 0, \quad (\text{A19})$$

which gives the acceleration law in the deformed spacetime crystal.

To make the simple model more concrete and highlight the spacetime structure in the geodesic equation, consider the one-dimensional oblique spacetime crystal with Hamiltonian

$$\hat{H} = \sum_R [(V + A \cos(\kappa R - \Omega t)) |R\rangle \langle R| + h(|R + a\rangle \langle R| + \text{H.c.})], \quad (\text{A20})$$

where the lattice spacing a , frequency Ω , and κ all depend on space and time. The lattice vectors are $(\frac{2\pi}{\Omega}, 0)$ and $(\frac{\kappa a}{\Omega}, a)$. The reciprocal lattice vectors are $(\frac{\Omega}{2\pi}, \frac{\kappa}{2\pi})$ and $(0, -\frac{1}{a})$. The lattice connections of this spacetime lattice are

$$\Gamma_{00}^0 = -\frac{\partial_t \Omega}{\Omega}, \quad \Gamma_{01}^0 = -\frac{\partial_x \Omega}{\Omega}, \quad \Gamma_{10}^0 = -\frac{a \partial_t \kappa + \kappa \partial_t a}{a \Omega},$$

$$\Gamma_{11}^0 = -\frac{a \partial_x \kappa + \kappa \partial_x a}{a \Omega}, \quad (\text{A21})$$

$$\Gamma_{00}^1 = 0, \quad \Gamma_{01}^1 = 0, \quad \Gamma_{10}^1 = -\frac{\partial_t a}{a}, \quad \Gamma_{11}^1 = -\frac{\partial_x a}{a}. \quad (\text{A22})$$

The motion of a wave packet in a (1+1)D oblique spacetime crystal, neglecting contributions from the Berry curvature, is governed by an effective geodesic equation. The spatial component of this equation, which describes the particle's acceleration, is given by

$$x'' + \left(\frac{\partial_t \Omega}{\Omega} - \frac{\partial_t a}{a} \right) x' + \left(-\frac{\partial_x a}{a} + \frac{\partial_x \Omega}{\Omega} + \frac{a \partial_t \kappa + \kappa \partial_t a}{a \Omega} \right) x'^2 + \frac{a \partial_x \kappa + \kappa \partial_x a}{a \Omega} x'^3 = 0. \quad (\text{A23})$$

Here, $x(t)$ is the particle's trajectory and primes denote derivatives with respect to time ($x' = dx/dt$). The particle's dynamics are determined by the spacetime variations of the crystal's geometric parameters: a (a lattice parameter), Ω (the unit-cell area), and κ (which characterizes the lattice's obliqueness).

A particularly insightful case arises when the crystal parameters are spatially uniform, varying only in time. This implies that all spatial derivatives vanish ($\partial_x a = \partial_x \Omega = \partial_x \kappa = 0$). The geodesic equation (A23) simplifies considerably to

$$x'' + \left(\frac{\dot{\Omega}}{\Omega} - \frac{\dot{a}}{a} \right) x' = 0, \quad (\text{A24})$$

where the dot notation now represents the time derivative (e.g., $\dot{\Omega} = d\Omega/dt$). This first-order linear differential equation for the velocity $v(t) = x'(t)$ can be integrated directly, yielding the solution

$$v(t) = C \frac{a(t)}{\Omega(t)}, \quad (\text{A25})$$

where C is a constant of integration.

This result presents a compelling analogy to cosmological particle dynamics. The quantity $\Omega(t)/a(t)$ plays the role of an effective cosmological scale factor. The particle's velocity is inversely proportional to this scale factor, akin to how a particle's momentum redshifts in an expanding universe.

4. Gauge transformation of τ

The parameter τ in the main text plays the role of time but can be chosen arbitrarily since the form of λ is not unique. We require the action $S = \int d\tau L$ to be invariant under a reparametrization $\tau' = f(\tau)$, so that the quasiparticle action becomes $S = \int d\tau' L'$. With the explicit Lagrangian

$$L = -\lambda + k_{c\mu} \dot{x}_c^\mu + \dot{k}_{c\mu} A_{k_{c\mu}} + \dot{x}_c^\mu A_{x_c^\mu}, \quad (\text{A26})$$

only λ lacks a first-order τ derivative; the other terms are τ independent and thus invariant under the transformation of τ . To encode the gauge transformation of λ , introduce an auxiliary field $\mu = \mu(\tau)$ and rewrite the Lagrangian as

$$L = -\mu\lambda + k_{c\mu} \dot{x}_c^\mu + \dot{k}_{c\mu} A_{k_{c\mu}} + \dot{x}_c^\mu A_{x_c^\mu}. \quad (\text{A27})$$

Under $\tau' = f(\tau)$, the auxiliary field transforms as $\mu(\tau') = \frac{d\tau}{d\tau'} \mu(\tau)$, which leaves the action invariant because $\mu(\tau') d\tau' = \mu(\tau) d\tau$.

The equations of motion with this auxiliary field are

$$\lambda = 0, \quad (\text{A28})$$

$$\dot{x}^\mu = \mu \frac{\partial \lambda}{\partial k_\mu} - \Omega_{k_\mu x^\nu} \dot{x}^\nu - \Omega_{k_\mu k_\nu} \dot{k}_\nu, \quad (\text{A29})$$

$$\dot{k}_\mu = -\mu \nabla_{x^\mu} \lambda + \Omega_{x^\mu x^\nu} \dot{x}^\nu + \Omega_{x^\mu k_\nu} \dot{k}_\nu. \quad (\text{A30})$$

Equation (A28) is derived from the variation of μ . To simplify the set of equations, we fix the gauge $\mu(\tau) = 1$ in the main text and apply the constraint $\lambda = 0$ in equation of motion.

5. Deformation coupling to the momentum shift

Under the expansion of dispersion near the zero of $\lambda(k)$, we may find that the dispersion is dependent on the moment after a shift of zero point $\lambda(k) \rightarrow \lambda(k - A)$. The new variable $q = k - A$ is also defined in the main text. If there is deformation in real space, the zero points may be spacetime dependent and behave like a gauge field, which leads to an additional change to the original covariant derivative ∇_{x^μ} . We may define a new

covariant derivative $\tilde{\nabla}$ here,

$$\tilde{\nabla}_\mu = \partial_{x^\mu} |_{q_\mu=k_\mu-A_\mu} - \frac{\partial A_m}{\partial x^\mu} \partial_{q_m} - k_l \Gamma_{m\sigma}^l \partial_{k_m}, \quad (\text{A31})$$

$$= \partial_{x^\mu} - \left(q_l \Gamma_{m\sigma}^l \partial_{k_m} + \left(A_l \Gamma_{m\sigma}^l + \frac{\partial A_m}{\partial x^\mu} \right) \partial_{q_m} \right), \quad (\text{A32})$$

$$= \partial_{x^\mu} - \left(q_l \Gamma_{m\sigma}^l \partial_{k_m} + \left(A_l \Gamma_{m\sigma}^l + \frac{\partial A_m}{\partial x^\mu} \right) \partial_{q_m} \right), \quad (\text{A33})$$

$$= \partial_{x^\mu} - (q_l \Gamma_{\mu m}^l + \nabla_{x^\mu} A_m) \partial_{q_m}. \quad (\text{A34})$$

So, we have a new covariant derivative with modification induced by gauge fields. In addition, the total derivative of momentum q can be upgraded as

$$\tilde{D}q_\mu = dq_\mu + (q_l \Gamma_{\mu m}^l + \nabla_{x^\mu} A_m) dx^m. \quad (\text{A35})$$

The equation of motion is still the same form: We just need to replace the variables from (x, k) to (x, p) ; the motion coupling to the effective gauge fields hides in the new covariant derivatives. By the substitution, Eq. (55) is

$$\dot{x}^\mu = \frac{\partial \lambda}{\partial k_\mu}, \quad (\text{A36})$$

$$\dot{q}_\mu = -\tilde{\nabla}_{x^\mu} \lambda, \quad (\text{A37})$$

where $\dot{q}$ is $\frac{\tilde{D}q}{d\tau}$. When we take a step back and write it in the original covariant derivative ∇_{x^μ} to obtain a more clear insight, we will find the equation of motion in momentum component there is a term like the Lorentz force. For example, Eq. (55) may be changed to

$$\dot{x}^\mu = \frac{\partial \lambda}{\partial k_\mu}, \quad (\text{A38})$$

$$\dot{q}_\mu = -\nabla_{x^\mu} \lambda - F_{\mu\rho} \dot{x}^\rho, \quad (\text{A39})$$

we are using the original covariant derivative here. The effective field strength $F_{\nu\rho}$ is defined as the strength of gauge fields $F_{\nu\rho} = \partial_\nu A_\rho - \partial_\rho A_\nu + 2S_{\nu\rho}^\mu A_\mu$.

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