

Chimera states with multiple coexisting solutions

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We report on the emergence of chimera states in extended systems characterized by the coexistence of more than two spatiotemporal solutions. Specifically, we show that three-domain chimeras may generically appear in reaction-diffusion systems under three conditions, with a crucial role played by the existence of a Maxwell point. We verify the universality of our findings by analyzing five different models, and demonstrate that such new chimeras are independent of the boundary conditions imposed.

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I. INTRODUCTION

The emergence of two-dimensional patterns made of different macroscopic phases coexisting in different spatial domains was originally demonstrated long ago in nonlinear passive optics [1]. During the next ten years, an equivalent phenomenon was observed in distributed systems of coupled oscillators, where, under the right conditions, a mixed state emerged with the coexistence of a synchronized and an incoherent domain [2,3]. Borrowing the name from Greek mythology, such mixed states were termed “chimeras,” and they immediately garnered significant attention from scholars working in a wide range of fields, including physics, mathematics, chemistry, and the life sciences [4].

Research on chimera dynamics predominantly focused on networks of coupled phase oscillators under many different underlying topologies [5–13]. This led to the identification of multiple types of chimera patterns [14–28]. While chimeras described in earlier works all displayed the coexistence of an ordered domain and a disordered one, the meaning of the term soon broadened to include a variety of qualitatively diverse phenomena involving different pairs of states [29], ranging from the coexistence of (partially) incoherent ones [30] to that of fully coherent ones without the simultaneous presence of any incoherent phase [31]. Notice furthermore that similar

states, in which two coherent domains are separated by a so-called kink structure and no incoherent domain is present, have been observed in coupled map lattices as well [32]. Additionally, in the last decade, chimeras consisting of more than two domains have been identified [33–35], where, however, the different domains are in fact just the result of a repetition of a coherent and an incoherent one, rather than being states with fully distinct natures. Thus, despite the vast body of existing research, chimeras consisting of three or more distinct macroscopic phases have been reported very little so far.

In this article, we finally do justice to the mythological Chimera which, in fact, was a hybrid monster with parts of three, and not two, animals: a lion’s head, a goat’s body, and a snake’s tail. Specifically, we reveal the conditions under which a generic reaction-diffusion system can support the existence of a three-domain chimera, showing the fundamental role played by the presence of a Maxwell point. The generality of our claims is validated on five specific systems, highlighting that the emergence of three-domain chimeras is independent of the boundary conditions imposed.

II. RESULTS

We start by considering a generic reaction-diffusion system of n components in one space dimension:

$$\partial_t \mathbf{U}(x, t) = \mathbf{F}(\mathbf{U}, \mathbf{p}) + \mathbf{D} \Delta \mathbf{U}, \quad (1)$$

where $\mathbf{U}$ and $\mathbf{F}$ are n -dimensional vectors describing the system state and the nonlinear response, respectively, $\mathbf{p}$ is a vector of control parameters, $\mathbf{D}$ is the diagonal matrix of diffusion coefficients, and Δ is the Laplacian operator. We consider the system to be bounded in space, so that $x \in [0, L]$, where L is the length of the region. States for which $\mathbf{U}(x, t) = \mathbf{U}_s$ is constant in space and time, and such that $\mathbf{F}(\mathbf{U}_s, \mathbf{p}) = \mathbf{0}$ are fixed points of the dynamics.

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In the absence of diffusion, i.e., when $\mathbf{D} = \mathbf{0}$, and for some range of the parameters, the system may exhibit bistability between a zero-uniform state (ZU), for which $\mathbf{U}_s = \mathbf{0}$, and a non-zero-uniform state (NZU), for which, instead, $\mathbf{U}_s \neq \mathbf{0}$. Then, a solution exists representing a front that propagates at a constant velocity c and connects the two states [36–41]. If c vanishes for some choice of the parameters, the front becomes stationary. The point in parameter space at which this occurs is called the Maxwell point, and the coexistence of the two states corresponds to a two-domain chimera [36,38,42].

Furthermore, upon reintroducing the diffusion term in the system, one of the uniform states may become unstable. When this occurs, the system undergoes a Turing bifurcation, leading to the appearance of a spatially periodic patterned state (PP state). If such a bifurcation is subcritical, the PP state is drawn toward the stable uniform state, resulting in the formation of a homoclinic snaking structure and numerous localized patterns. Within this region of the parameter space, the front connecting the two states remains stationary [36,39,43–46], which causes, once again, the emergence of two-domain chimeras.

But then, by judiciously tuning the parameters, one may place the system in a state where the snaking structure forms at the Maxwell point, resulting in both fronts becoming stationary at the same time. In turn, such a concurrence would lead to the coexistence of three domains, namely the ZU, the NZU, and the PP states, causing the formation of a three-domain chimera.

Thus, we argue that three conditions must be met for the emergence of such chimeras:

- (i) Bistability must occur between the ZU and NZU states, which is a prerequisite for the existence of a Maxwell point.
- (ii) The ZU or the NZU state must undergo a subcritical Turing bifurcation, resulting in a further bistability with a PP state.
- (iii) A snaking structure must exist at the Maxwell point, triggering the coexistence of the ZU, NZU, and PP states.

To illustrate our claims, we refer to one of the possibly most celebrated models in physics, the generalized Swift-Hohenberg (SH) model:

$$\partial_t u = \lambda u + a_2 u^2 - a_3 u^3 - d_2 \nabla^2 u - d_4 \nabla^4 u. \quad (2)$$

Originally derived to describe emerging patterns in thermal convection [47], the SH model is, indeed, of general interest in pattern formation and statistical physics, as it has applications in fields as diverse as lasers dynamics [48], curved elastic bilayer materials [49], and theoretical ecology [38].

In the following, we demonstrate that the SH model satisfies all three conditions described above, and therefore it supports the emergence of three-domains chimeras.

When diffusion terms are absent ($d_2 = d_4 = 0$), the system exhibits bistability between the ZU and an NZU state, as long as the control parameter λ lies in the interval $[-\frac{a_2^2}{4a_3}, 0]$ (see details in the Supplemental Material [50]), thus fulfilling condition (i). Moreover, $\lambda = \lambda_{sn} = -\frac{a_2^2}{4a_3}$ is a saddle-node bifurcation point, whereas $\lambda = \lambda_c = 0$ is a transcritical bifurcation point.

Furthermore, the system has two Turing bifurcation points λ_{T1} and λ_{T2} , and depending on the value of the parameter

d_2 , the bifurcation can be supercritical or subcritical, thereby satisfying condition (ii).

Figure 1 reports the bifurcation diagrams [panels (a) and (b)] obtained by using AUTO-07p [51] and illustrates the different dynamical behaviors of the system. The steady states are the ZU state u_0 [panel (a3)] and a NZU state u_2 [panel (a1)]. Turing instabilities emerge for $\lambda \in [\lambda_{sn}, \lambda_{T1}]$ and $\lambda \in [\lambda_{T2}, \lambda_c]$. The main difference between the supercritical and the subcritical regimes concerns the periodic solutions that bifurcate from λ_{T2} . In the supercritical case, a periodic solution emerges toward an unstable ZU state, and the system self-organizes to form PP states, as shown in Fig. 1(a2). In the subcritical regime, instead, periodic solutions bifurcate towards a stable ZU state, giving rise to the homoclinic snaking state that, in turn, generates numerous local structures, illustrated in Figs. 1(b2) and 1(b3).

More specifically, in the hybrid state of Fig. 1(b2), the PP state coexists with the NZU one, whereas in the one of Fig. 1(b3) it coexists with the ZU state. Additionally, perturbing the ZU state within the Turing instability region leads to the emergence of a spatially periodic steady state, eventually forming Turing patterns, as shown in Figs. 1(a4) and 1(a2), respectively. Finally, perturbing the ZU state within the bistability region of the ZU and PP state ultimately results in the formation of two-domain chimeras, shown in Fig. 1(b4), in which the ZU state and the PP one coexist, as also illustrated in Fig. 1(b3).

As for condition (iii), Fig. 2 reports the bifurcation diagram for $d_2 = 0.6$, showing a Maxwell point emerging at $\lambda = \lambda_m = -0.125$. As seen in Fig. 2(a), the homoclinic snaking structures still persist at λ_m . As the front between the ZU and NZU states becomes stationary at λ_m , it induces a pattern consisting of domains characterized by the two states separated by the two fronts, as shown in Fig. 2(a3). These are a pair of symmetric solutions of the model, one representing the ZU state propagating into the NZU one, as in Fig. 2(a1), and the other representing the opposite situation, as in Fig. 2(a2). Notice that, if one imposes periodic continuity only on the derivative of the solution (Neumann boundary conditions), rather than on the function itself, the NZU state does not extend to both ends of the spatial domain.

The spatiotemporal patterns created by u at λ_m and at $\lambda_m \pm 0.005$ are reported in panel (b) of Fig. 2, under periodic boundary conditions (upper row) and under Neumann boundary conditions (lower row).

Confirming our prediction, when $\lambda = \lambda_m$, a type of spatial pattern emerges, consisting of the simultaneous presence of the ZU, NZU, and PP states, as seen in panels (b2) and (b5). The observed pattern, which has not been previously documented in the literature, has all the characteristics of a chimera state, and we refer to it as a three-domain chimera. Notice that when λ deviates from the value of the Maxwell point, the three-domain chimera dissipates and is supplanted by a two-domain one. In more detail, when $\lambda = \lambda_m - 0.005$ one obtains a classic chimera consisting of a combination of the ZU and PP states, illustrated in panels (b1) and (b4), whereas when $\lambda = \lambda_m + 0.005$ the resulting classic chimera, shown in panels (b3) and (b6), is characterized by the coexistence of the NZU and PP states.

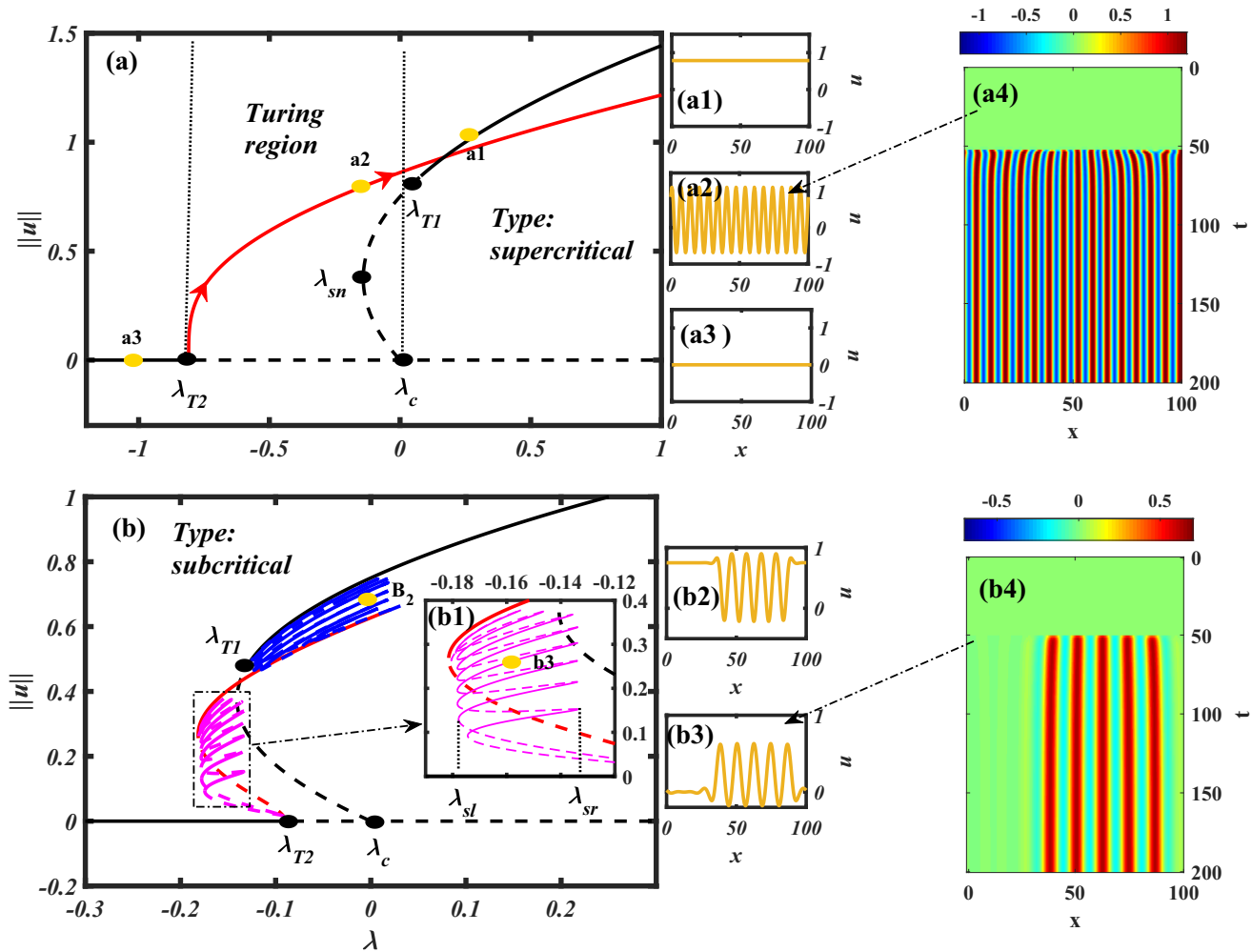


FIG. 1. Steady states in the generalized Swift-Hohenberg model. Panel (a) shows the supercritical Turing bifurcation regime, obtained for $d_2 = 1.8$, while panel (b) illustrates the subcritical regime, corresponding to $d_2 = 0.6$. The other parameters are $a_2 = 0.75$, $a_3 = 1$, and $d_4 = 1$. The point λ_{sn} is a saddle-node bifurcation point, whereas λ_c is a transcritical bifurcation point, and λ_{T1} and λ_{T2} are the Turing bifurcation points. Solid (dashed) lines denote stable (unstable) solutions. Black lines represent the zero-uniform and non-zero-uniform states; red lines correspond to periodic patterned states; blue and magenta curves represent the homoclinic snaking structures, and the corresponding region is magnified in the inset (b1). The spatial distributions of the steady states are shown in panels (a1)–(a3) and in panels (b2) and (b3). Finally, panels (a4) and (b4) report, in space (horizontal) and time (vertical), the periodic patterned state and the snaking structure emerging, respectively, in the supercritical and subcritical scenario.

As shown in Fig. 3(a), within the Turing instability region, one can have PP states originating at different bifurcation points with different wavelengths, illustrated in panels (a1)–(a3), where one can also notice that the wavelength decreases with the value of λ at the corresponding bifurcation point. Moreover, one finds that the amplitude of the wave, defined as the difference between the maximum and minimum values of the PP state and shown in panel (a4), initially increases sharply with wavelength, before oscillating and eventually converging to a finite value, suggesting that the system reaches a unimodal standing wave for sufficiently large wavelengths. Also in this regime, one can obtain classic chimeras characterized by the coexistence of two distinct wavelengths, shown in panel (a5).

It is worth noting that the three-domain chimera state exists as a strictly stationary solution only at a specific parameter value, namely the Maxwell point $\lambda = \lambda_m$. However, to evaluate its robustness with respect to parameter changes, we have

carried out extensive numerical simulations, whose results indicate that this structure remains clearly observable within a narrow interval around λ_m . Specifically, for deviations on the order of $\pm 5 \times 10^{-5}$, the three-domain chimera can persist in a stable form over a long period. Significant deformation and eventual transition into a two-domain state occur only when the parameter lies outside this interval and for sufficiently long evolution times, as illustrated in panels (b1) and (b3) of Fig. 2. These observations suggest that the three-domain chimera features a certain degree of robustness to parameter perturbations, and its prolonged persistence may be sufficient to ensure its experimental observability.

To validate the generality of our results, we studied four further reaction-diffusion systems, as summarized in Table I. The verification processes for the extended Klausmeier model [52,53], the Guttal-Jayaprakash model [54], the Hardenberg model [55], and the FitzHugh-Nagumo model [56] are detailed in the Supplemental Material

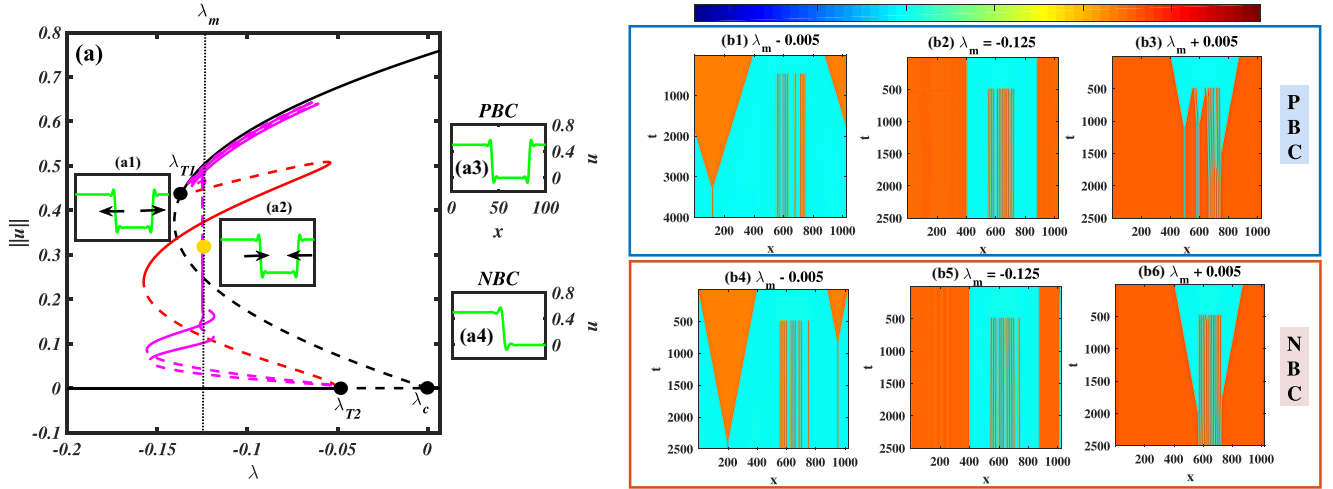


FIG. 2. Emergence of three-domain chimeras. Panel (a): the bifurcation diagram for $a_2 = 0.75$, $a_3 = 1$, $d_2 = 0.45$, and $d_4 = 1$ shows the emergence of a Maxwell point λ_m . The style and color codes for the lines are the same as used in Fig. 1. The spatial distribution of u at steady state is shown in panels (a1)–(a4). Panel (b): the spatiotemporal patterns formed by u when $\lambda = \lambda_m$ and when $\lambda = \lambda_m \pm 0.005$, for periodic boundary conditions (PBC, first row) and for Neumann boundary conditions (NBC, second row). Specifically, panels (b1), (b3), (b4), and (b6) show classic two-domain chimeras, while panels (b2) and (b5) demonstrate the occurrence of the three-domain chimeras.

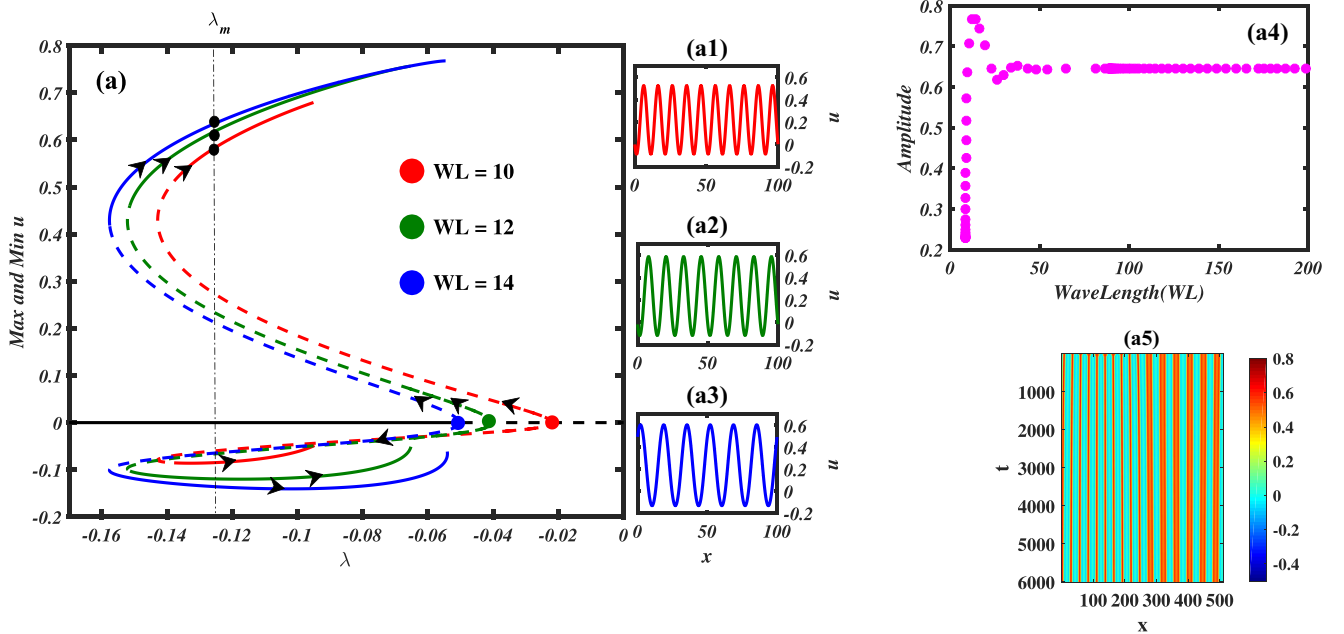


FIG. 3. Periodic patterned states with different wavelength at the Maxwell point. Panel (a): The bifurcation diagram around the Maxwell point, for $a_2 = 0.75$, $a_3 = 1$, $d_2 = 0.45$, and $d_4 = 1$, shows the existence of different Turing bifurcation points, indicated by the solid colored dots. The spatial patterns of the corresponding periodic states for $\lambda = \lambda_m$, sketched in (a1)–(a3), display an increasing wavelength for decreasing values of λ at the bifurcation point. After an initial oscillation, the amplitude of the steady-state solution as a function of the wavelength converges to a finite value, as shown in (a4). Panel (a5) illustrates the spatiotemporal pattern of u for a chimera featuring the coexistence of two periodic patterned states with different wavelength.

TABLE I. Three-domain chimera states are observed only if all conditions are fulfilled.

Model	Condition (i)	Condition (ii)	Condition (iii)	Three-domain chimera
Generalized Swift-Hohenberg	✓	✓	✓	Yes
Extended Klausmeier	✓	✓	✓	Yes
Guttal-Jayaprakash	✓	✓	✓	Yes
Hardenberg	×	✓	×	No
FitzHugh-Nagumo	✓	✓	×	No

[50]. Our analysis confirms that three-domain chimeras occur only when all three conditions are simultaneously satisfied.

III. CONCLUSIONS

In conclusion, starting from a general form for reaction-diffusion systems, we described the possibility of the emergence of a chimera state consisting of three different domains. We emphasize that this is made possible only by the fulfillment of three mathematical conditions that are independent from the specific details of the model or dynamical system considered. Notably, one of the conditions is that the system allows the existence of a Maxwell point, and that the dynamics take place at the corresponding regime.

We would like to stress that, while the states we observe do bear some qualitative resemblance to spatiotemporal patterns such as amplitude chimeras, chimera death, and related structures [6,20,21,23,25–27], these domains essentially result from the repetition of two types of coherent states. In contrast, the three-domain chimeras described in this article consist of different types of states, and are therefore fundamentally distinct from previously reported phenomena.

This offers an explanation of why three-domain chimeras had not yet been observed, notwithstanding the vast body of research that chimera states have generated, since they result from a particularly fine-tuned set of parameters. Our findings are nonetheless universal, as confirmed by the detailed analysis of five different reaction-diffusion systems, thereby offering an additional insight into the significant role that this mechanism may play in morphogenesis.

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