

Phase analysis of Ising machines and their implications on optimization

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Ising machines, which are dynamical systems designed to operate in a parallel and iterative manner, have emerged as a paradigm for solving combinatorial optimization problems. Despite computational advantages, the quality of solutions depends heavily on the form of dynamics and tuning of parameters, which are, in general, set heuristically due to the lack of systematic insights. Here, we focus on optimal Ising machine design by analyzing phase diagrams of spin distributions in the Sherrington-Kirkpatrick model. We find that the ground state can be achieved in the phase where the spin distribution becomes binary, and optimal solutions are produced where the binary phase and gapless phase coexist. Our analysis shows that such coexistence phase region can be expanded by carefully placing a digitization operation, giving rise to a family of superior Ising machines, as illustrated by the proposed algorithm digCIM.

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I. INTRODUCTION

Discrete optimization problems are ubiquitous in science and engineering, encompassing protein folding [1], route planning [2], information processing [3], and machine learning [4], among many others [5]. The Ising model has become a powerful bridge for modeling and analyzing these problems from the viewpoint of statistical physics [6]. However, finding the system states corresponding to the optimal solution of the target problems remains computationally hard [7,8].

An Ising problem, which is equivalent to the class of quadratic unconstrained binary optimization problems (QUBO), generally takes the form of

$$\min_{\sigma \in \{-1, +1\}^N} \left(-\frac{1}{2} \sum_{ij} \sigma_i J_{ij} \sigma_j \right), \quad (1)$$

where $\sigma \in \{\pm 1\}^N$ is the set of N binary spins coupled through the matrix $[J_{ij}]_{N \times N}$. Analog Ising machines (AIMs) are non-von-Neumann computing paradigms designed to find ground-state configurations of Ising models more efficiently than conventional search-based algorithms. In abstract terms, AIMs compute solution states through time evolution of a coupled dynamical system

$$\frac{dx_i}{dt} = \mathcal{F}_i(\mathbf{x}; \boldsymbol{\mu}) + \zeta_i, \quad i \in \{1, \dots, N\}. \quad (2)$$

Here, $\mathbf{x} = (x_1, x_2, \dots, x_N)$, $x_i \in \mathbb{R}$ denotes analog spin values, $\boldsymbol{\mu}$ denotes control parameters, and ζ_i represents noise. The system's dynamics is governed by a driving term $\mathcal{F}_i(\mathbf{x}, \boldsymbol{\mu})$ that is distinct for each AIM variant. Once the system settles into its stationary state, binary spin values σ_i are assigned via the decoding rule $\sigma_i = \text{sgn}(x_i)$ [9], producing a candidate solution to the Ising problem. A number of AIMs have been proposed [10], such as quantum annealers [11–14], optical Ising machines based on pulse lasers [15–17], coupled oscillator networks [18,19], and special-purpose chip implementations on conventional hardware [20–26]. The performance of AIMs, however, can vary significantly, preventing them from large-scale adoption in practical applications.

Considerable efforts have been made to enhance the accuracy by modifying the dynamics or constraining the distribution of analog spins. A prominent example is the coherent Ising machine (CIM), which is characterized by the driving term

$$\mathcal{F}_i^{\text{CIM}}(\mathbf{x}, a, \xi) = -x_i^3 + ax_i + \xi \sum_{\substack{j=1, \\ j \neq i}}^N J_{ij} x_j. \quad (3)$$

The terms correspond (from left to right) to cubic saturation, driving with net gain a , and coupling with interaction strength ξ [17]. Variants of CIM include clipCIM [27] as an optoelectronic version that bounds the amplitude of the analog spins, simCIM [28], which further removes the saturation term from $\mathcal{F}^{\text{CIM}}$ [28], and discrete simulated bifurcation (dsb) that is a momentum-based machine discretizing the analog spins in the interaction term [29] to mitigate errors. However, while promising, accurate solutions require manual fine-tuning of control parameters due to the lack of theoretical guidance.

In this paper, we elucidate how the choice of system dynamics and parameters shapes the spin distribution of Ising machines and determines the solution accuracy. Concretely, we study the static spin distribution of AIMs by applying the replica method to AIMs with the Sherrington-Kirkpatrick

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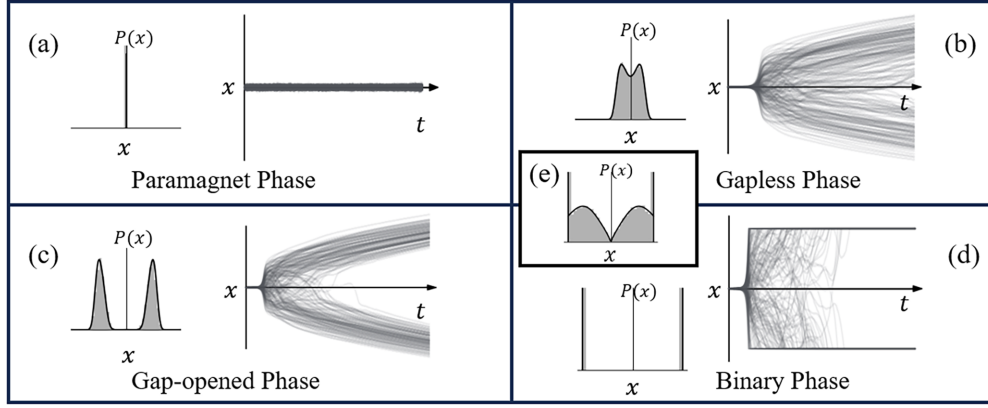


FIG. 1. AIM phases as characterized by distinct modes of spin distribution $P(x)$ together with typical AIM dynamics of x vs t ($T = 0$). (a) Paramagnet phase. (b) Gapless phase. (c) Gap-opened phase. (d) Binary phase. (e) Gapless-binary coexistence region.

(SK) model of spin glasses [30,31], revealing a rich palette of phases that emerge in AIMs, as illustrated in Fig. 1 and to be explained in the following sections. In particular, we discover that a version of AIM, referred to as digCIM, can realize the microscopic equation describing the steady-state solutions of the Ising problem, namely, the TAP equation [32], and find that such solutions can be accessed where the binary phase coexists with the gapless phase. Based on these findings, we propose a superior family of AIMs characterized by digitized driving terms, as illustrated by digCIM that produces optimal solutions in a wide range of the gain parameter.

The rest of the paper is organized as follows. Section II presents the replica analysis of the CIM approach to solve the SK model. Then, in Sec. III, we introduce two analytical tools for the analysis. In Sec. IV, we start with an overview of the possible phases of AIMs, followed by the descriptions of the paramagnetic, gapless, gap-opened, and binary phases. The narrative continues in Sec. V on the gapless-binary transition, which has interesting implications to the dynamical behaviors. In Sec. VI, we explain how digitization enables us to obtain the ground state of the SK model (the TAP equation) using digCIM. In Sec. VII, we discuss the implications of our analysis to the dynamics, leading to the optimality of the coexistence region. In Sec. VIII, we generalize the models to practical problems. The paper concludes in Sec. IX.

II. THE SHERRINGTON-KIRKPATRICK MODEL

To reveal distinct phases that commonly appear in AIMs, we first consider the example of CIM with driving term $\mathcal{F}_{\text{CIM}}$ and white noise ζ_i with temperature T , where $\langle \zeta_i(t) \rangle = 0$ and $\langle \zeta_i(t) \zeta_j(t') \rangle = 2T \delta_{ij} \delta(t - t')$. Equation (3) can then be written as a Langevin equation describing the gradient descent of the Hamiltonian $H = \frac{1}{4} \sum_i x_i^4 - \frac{a}{2} \sum_i x_i^2 - \frac{\xi}{2} \sum_{i \neq j} J_{ij} x_i x_j$.

We consider Ising problems defined by the SK model in which the couplings J_{ij} follow a Gaussian distribution with zero mean and variance J^2/N [30,31]. The SK model is suitable for accuracy analyses, as in the thermodynamic limit the

free energy of the system can be derived analytically using the replica method, and its complex energy landscape is typical of combinatorial optimization problems [33]. The method has been applied to study one of the AIMs [34]. Solutions of the replica method depend on the level of replica symmetry breaking (RSB) [33]. The replica symmetric (RS) solution is stable in the paramagnetic phase. In the spin glass phases, we will quote results of full RSB (FRSB) where possible, and one-step RSB (1RSB) approximation for some numerical results, which produce rather accurate predictions except phase transition points.

To begin with, consider the CIM for optimizing the Hamiltonian of the SK model,

$$H = -\frac{1}{2} \sum_{i \neq j} J_{ij} \sigma_i \sigma_j. \quad (4)$$

It is a fully connected Ising spin glass model with the randomly quenched couplings J_{ij} following independent Gaussian distributions with mean $\langle J_{ij} \rangle = 0$ and variance $\langle J_{ij}^2 \rangle = \delta_{ij} J^2/N$. The corresponding analog Ising machine replaces the Ising spins $\{\sigma_i\}$ by continuous variables $\{x_i\}$ and adds two extra terms. The first extra term is quadratic in x_i with the gain a tunable by the pump rate of the CIM. The second one is a fourth-order term representing the nonlinearity that restricts the magnitude of $\{x_i\}$. This yields the Hamiltonian given by

$$H = -\frac{a}{2} \sum_i x_i^2 + \frac{1}{4} \sum_i x_i^4 - \frac{\xi}{2} \sum_{i \neq j} J_{ij} x_i x_j. \quad (5)$$

Let $Z(T) = \int \exp(-\beta H(\vec{x})) d(\vec{x})$ be the partition function where $\beta \equiv 1/T$ is the inverse temperature. After replicating the system n times and performing disorder average (denoted as $[\dots]_J$), we arrive at the partition function that is dominated by the saddle point of the parameters $q_{\alpha\alpha}$ and $q_{\alpha\beta}$ in the thermodynamic limit (see Sec. 1 of the Supplemental

Material (SM) [35] for details),

$$[Z^n]_J = \exp \left\{ -\frac{N\beta^2 J^2 \xi^2}{4} \sum_{\alpha=1}^n q_{\alpha\alpha}^2 - \frac{N\beta^2 J^2 \xi^2}{2} \sum_{\substack{\alpha, \beta=1, \\ \alpha < \beta}}^n q_{\alpha\beta}^2 + N \ln \prod_{\alpha=1}^n \left(\int dx^\alpha \right) \exp \left[-\frac{\beta}{4} \sum_{\alpha=1}^n (x^\alpha)^4 + \frac{\beta a}{2} \sum_{\alpha=1}^n (x^\alpha)^2 \right. \right. \\ \left. \left. + \frac{\beta^2 J^2 \xi^2}{2} \sum_{\alpha=1}^n q_{\alpha\alpha} (x^\alpha)^2 + \beta^2 J^2 \xi^2 \sum_{\substack{\alpha, \beta=1, \\ \alpha < \beta}}^n q_{\alpha\beta} x^\alpha x^\beta \right] \right\}. \quad (6)$$

The solution in the RS ansatz is applicable to the low-gain regime. In this ansatz, $q_{\alpha\alpha} = Q$ and $q_{\alpha\beta} = q$ for $\alpha \neq \beta$. In the limit $n \rightarrow 0$, we can obtain the expression of the average free energy per node f using the replica formula

$$-\beta f = \frac{1}{N} [\ln Z]_J = \lim_{n \rightarrow 0} \frac{[Z^n]_J - 1}{Nn}. \quad (7)$$

However, when the system enters the spin glass phase, the RS solution becomes unstable, as verified in Sec. 2 of the SM [35]. The 1RSB ansatz provides a more appropriate framework [33]. In the 1RSB ansatz (see Sec. 3 of the SM [35]), the n replicas of the matrix of $q_{\alpha\beta}$ is divided into n/m blocks of size m . The diagonals take the value Q , while other elements in the diagonal blocks take the value q_1 , and the elements in the off-diagonal blocks are q_0 . The partition function reduces to

$$[Z^n]_J = \exp \left[N \left(-\frac{1}{4} \beta^2 J^2 \xi^2 [n^2 q_0^2 + nm(q_1^2 - q_0^2) + n(Q^2 - q_1^2)] + \ln \prod_{\alpha=1}^n \left(\int dx_\alpha \right) \exp \left\{ -\frac{\beta}{4} \sum_{\alpha=1}^n x_\alpha^4 \right. \right. \right. \\ \left. \left. + \frac{\beta}{2} a \sum_{\alpha=1}^n x_\alpha^2 + \frac{1}{2} \beta^2 J^2 \xi^2 \left[q_0 \left(\sum_{\alpha=1}^n x_\alpha \right)^2 + (q_1 - q_0) \sum_{k=0}^{n/m-1} \left(\sum_{\alpha=km+1}^{(k+1)m} x_\alpha \right)^2 + (Q - q_1) \sum_{\alpha} x_\alpha^2 \right] \right\} \right) \right]. \quad (8)$$

In the limit $n \rightarrow 0$, we can obtain the expression of the average free energy per node f using the replica formula. Due to the symmetry between the blocks, products of terms involving individual replicas become identical, and all replica indices of x can be omitted,

$$-\beta f = -\frac{1}{4} \beta^2 J^2 \xi^2 [m(q_1^2 - q_0^2) + Q^2 - q_1^2] \\ + \frac{1}{m} \int Dv \ln \int Du \left(\int dx e^{-\beta g} \right)^m, \quad (9)$$

where $Dv \equiv dv \exp(-v^2/2)/\sqrt{2\pi}$ is the Gaussian measure, and the effective free energy g is given by

$$g = \frac{1}{4} x^4 - \frac{1}{2} [a + J^2 \xi^2 \beta (Q - q_1)] x^2 \\ - J \xi (\sqrt{q_0} v + \sqrt{q_1 - q_0} u) x. \quad (10)$$

The expressions of the order parameters are given by the saddle-point equations as derived in Sec. 3 of the SM [35]. In the above equations, the term $\beta(Q - q_1)$ can be interpreted as the susceptibility $\chi \equiv \beta(Q - q_1)$ of x in the presence of an external field in the light of the fluctuation-response relation.

In our analysis so far, we observe that the energy has the dimension of the fourth order of the current x , and the dimensions of other quantities are determined accordingly. Using $J\xi$ as the basic unit, which scales as x^2 , we note that a , Q , q_1 , and q_0 scale as $J\xi$, χ scales as $(J\xi)^{-1}$, and the decoded energy

scales as J . Hereafter, we adopt the convention $J\xi = 1$ and express our analysis in terms of dimensionless variables. All results, including those in Sec. III, are summarized in Sec. 6 of the SM [35].

III. ANALYTICAL TOOLS

A. The spin distributions

We now introduce two powerful tools to analyze the phases of AIMS. First, the (analog) spin distribution is the instantaneous distribution of the individual spins in equilibrium at zero temperature of an AIM, given by $P(x) = [\langle \delta(x_i - x) \rangle]_J$. Here, the angular and square brackets denote thermal and disordered averages, respectively. In the framework of the replica method, the value of a representative spin x is determined by a mean-field effective free energy. For CIM, it is given by

$$g(x) = \frac{1}{4} x^4 - \frac{a_{\text{eff}}}{2} x^2 - wx, \quad (11)$$

where w represents the cavity field, which is the field experienced by the spin x when the system excluding x is in thermodynamic equilibrium. The distribution of the cavity field depends on the level of RSB. Parameter $a_{\text{eff}} = a + \chi$ is the effective gain, where χ is the susceptibility, which is equal to the thermodynamic average of $\partial x / \partial h$ when an infinitesimal external field h is applied to the system. Hence, χ represents the mean-field influence of the spin on its neighbors, which is fed back to the spin itself, commonly referred to as the Onsager reaction [39]. Hence, the dynamics of a node is no longer merely driven by the external term a , but additionally,

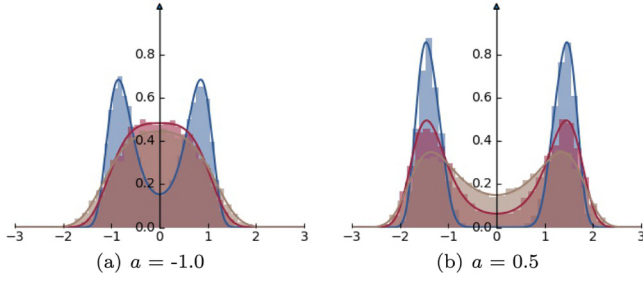


FIG. 2. Probability distribution functions for analog spins. The solid line is the theoretical value based on Eq. (12) and the shaded area is the numerical simulation of the CIM dynamics. The brown, red, and blue colors correspond to the values at temperatures of 1, 0.5, and 0.03, respectively.

the feedback from its neighbors also plays a role. The 1RSB spin distribution of CIM is derived in Sec. 5 of the SM [35] and reads

$$P(x) = \int Dv \frac{\int Du [\int dx' e^{-\beta g(x')}]^{m-1} e^{-\beta g(x)}}{\int Du [\int dx' e^{-\beta g(x')}]^m}. \quad (12)$$

Here, u and v are Gaussian variables forming the cavity field w according to $w = \sqrt{q_0}v + \sqrt{q_1 - q_0}u$. Figure 2 visualizes Eq. (12) under different parameter settings. In the 1RSB

framework, the solution space consists of a collection of clusters, with q_1 and q_0 representing the correlation of solutions in the same and different clusters, respectively. The parameter m characterizes the relative weights of the same and different clusters, and $m\beta$ approaches a constant when $T \rightarrow 0$.

Besides the above *instantaneous* spin distribution, we also derive the *average* distribution in Sec. 4 of the SM [35].

The inner expression of Eq. (12) is the Gibbs distribution shaped by $g(x)$ in Eq. (10). At $T = 0$, the spin distribution is dominated by the values x^* , which are the minimizers of $g(x)$, and $P(x)$ is given by

$$P(x) = \int Dv \frac{\int Du e^{-m\beta g^*} \delta(x - x^*)}{\int Du e^{-m\beta g^*}}, \quad (13)$$

B. The decoded energy

Next, we introduce the decoded energy

$$E_{\text{dec}} = -\frac{1}{2N} \left[\left\langle \sum_{i,j} J_{ij} \text{sgn} x_i \text{sgn} x_j \right\rangle \right], \quad (14)$$

which represents the thermodynamic average of the Ising energy after binarization of the analog spin values of AIM at equilibrium. The decoded energy at $T = 0$ is a lower bound on the accessible energy of the AIM. Using the 1RSB framework, we find the decoded energy of CIM at zero temperature (see Sec. 5 of the SM [35])

$$E_{\text{dec}} = -\frac{2J}{\sqrt{2\pi}(q_1 - q_0)} \left(\int Dv \frac{\int Du e^{-m\beta g^*} |x^*|}{\int Du e^{-m\beta g^*}} \right) \int Dv \frac{\exp\left(-\frac{q_0 v^2}{2(q_1 - q_0)} + \frac{1}{4} m\beta a_{\text{eff}}^2 \Theta(a_{\text{eff}})\right)}{\int Du e^{-m\beta g^*}} \\ - \frac{1}{2} m\beta J \left\{ \left[\int Dv \frac{\int Du e^{-m\beta g^*} |x^*|}{\int Du e^{-m\beta g^*}} \right]^2 - \left[\int Dv \left(\frac{\int Du e^{-m\beta g^*} x^*}{\int Du e^{-m\beta g^*}} \right) \left(\frac{\int Du e^{-m\beta g^*} \text{sgn} x^*}{\int Du e^{-m\beta g^*}} \right) \right]^2 \right\}. \quad (15)$$

Figure 3 displays the decoded energy contours for different parameter pairs using the decoded energy expression. It is apparent that when the energy is low, the rate of energy decrease with the gain and the temperature becomes increasingly slow.

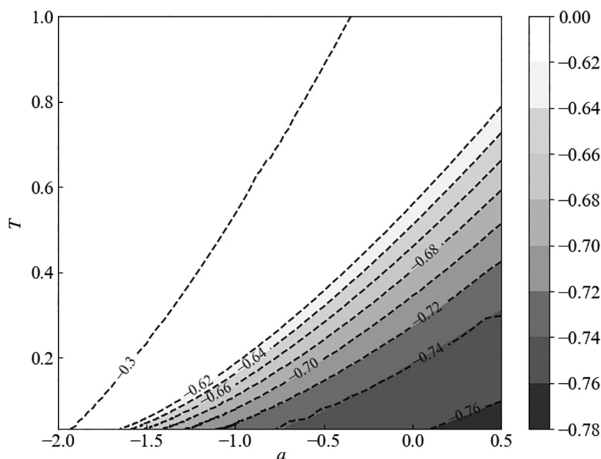


FIG. 3. Decoded energy contour line in the space of a and T .

Moreover, the effect of a on the decoded energy is akin to that of the inverse temperature in this system. Lowering the temperature (noise level) or raising a helps to attain states at lower energy levels. As both a and the inverse temperature increase, the energy reaches approximately -0.765 , which corresponds to the ground state of the SK model under the 1RSB assumption. In practical applications of CIMs, achieving lower energy states quickly may be prioritized over lowest-energy states. However, directly setting low T or high a risks premature local minima trapping. Based on Fig. 3, researchers can choose optimal parameters or annealing schedules for the Ising machine that yield the target decoded energy. We caution that the contour presented has been constructed with the parameter $\beta = 1/T$ ranging from 1 to 30, which implies that the temperature does not approach absolute zero. Consequently, the contour plot in Fig. 3 does not capture the behavior of the energy feature for temperatures in the vicinity of zero.

IV. PHASES OF AIM

Phases of AIM are characterized by spin distribution modes as depicted in Fig. 1, which includes *paramagnetic*

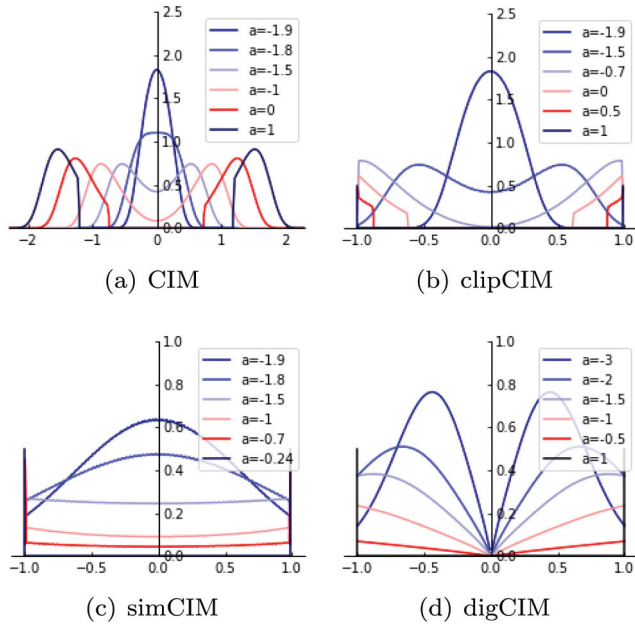


FIG. 4. The distributions $P(x)$ of (a) CIM, (b) clipCIM, (c) simCIM (bSB), and (d) digCIM (dSB) at different values of the gain a . $B = 1$ for the clipCIM, simCIM, and digCIM. Panels (a)–(c) are obtained in the 1RSB ansatz, whereas panel (d) is obtained from the simulated field distribution of the SK model, which will be shown to be equivalent to the full RSB ansatz.

phase (spin distribution is a delta function at $T = 0$), *gapless phase* (single-band distribution), *gap-opened phase* (double-band distribution), *binary phase* (dual delta distribution), and a gapless-binary coexistence scenario associated with the ground state of the SK model [40,41]. Note that the phases characterize the qualitative properties of AIMs at fixed parameters.

To explore these phases and their impact on optimization, we start with CIM, for which Figs. 1(a)–1(c) appear in order as the control parameter a changes from negative to positive values in Eq. (3).

In the following sections, we will introduce four types of AIMs: CIM (Sec. IV A–IV C), clipCIM (Sec. IV D), simCIM (Sec. V), and digCIM (Sec. VI). As each of them undergoes a unique phase transition trajectory, it is illustrative to show the evolution of their spin distributions in Fig. 4 as the gain a increases. Their features will be discussed subsequently.

The CIM's phase diagram in Fig. 5 illustrates several of the aforementioned phases with their typical spin distributions. The paramagnetic phase with single peaks P_{inst} are indicated by the gray area. The spin glass phase with a broadened P_{inst} is indicated by the blue area. The pseudogap region, spanning across both the paramagnetic and spin glass phases, with a bimodal P_{inst} is partially indicated by the green area.

In the paramagnetic phase, the variance of P_{aver} is $q = 0$ according to the replica symmetric solution, implying that it is a delta function at $\langle x \rangle = 0$. Decoding the ground-state energy using the thermal-averaged spin values is not possible. On the other hand, P_{inst} has a nonzero distribution. As we shall see, the nonzero distribution endows the system with the ability,

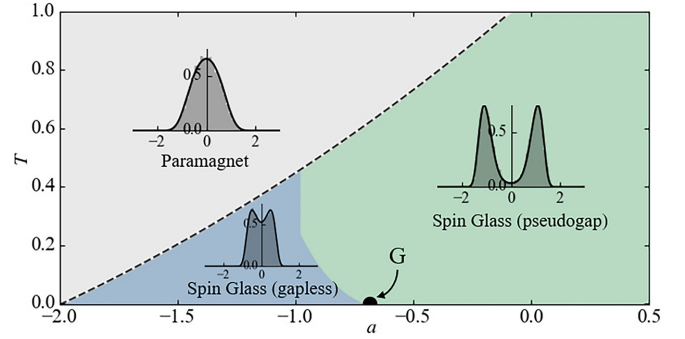


FIG. 5. Phase diagram of CIM in the space of gain a and temperature T with typical analog spin distributions shown in each region, respectively, located at $(a, T) = (-1.5, 0.5)$ (paramagnetic phase, defined by $q_1 = 0$), $(-1.5, 0.03)$ (gapless spin glass phase, defined by $q_1 > 0$), and $(0.5, 0.03)$ (pseudogap spin glass phase, defined by $q_1 > 0$ and $a_{\text{eff}} > 0$). In these distributions, the solid black lines correspond to 1RSB theory, while the gray shadings indicate simulation results. The dashed phase line is obtained in RS. G is the 1RSB gap-opening point at $T = 0$. The paramagnetic phase also consists of a pseudogap region, but for clarity it is not shown in the figure (see Sec. 4 of the SM [35]).

albeit limited, to decode the ground-state energy even in the paramagnetic phase.

In the spin glass phase, P_{inst} is gapless at low gains [the blue curves in Fig. 2(a)]. At higher gains and zero temperature, a gap opens in P_{inst} . At finite temperature, the gap becomes a pseudogap, but P_{inst} remains to have two maxima, facilitating efficient decoding of the minimum energy [the blue curve in Fig. 2(b)]. The pseudogap region is also present in the paramagnetic phase. Furthermore, we found that $P''_{\text{inst}}(0) > \beta a_{\text{eff}} P_{\text{inst}}(0)$. Hence, in the pseudogap region where $a_{\text{eff}} > 0$, P_{inst} always has two maxima.

A. Paramagnetic phase [Fig. 1(a)]

The paramagnetic phase is characterized by the RS order parameter $q = 0$. We first consider the stability of this phase at zero temperature. In this limit, the integrals of x in the saddle-point equation of q are dominated by the minimizer $x^* := \arg \min_x (g)$, that is,

$$x^{*3} - a_{\text{eff}} x^* = \sqrt{q} v. \quad (16)$$

As a result,

$$q = \int Dv \left[\frac{\int dx e^{-\beta g x}}{\int dx e^{-\beta g}} \right]^2 \approx \frac{q}{|a_{\text{eff}}|^2}. \quad (17)$$

This shows that the paramagnetic phase is unstable when $a_{\text{eff}} > -1$.

In the paramagnetic phase, the susceptibility is given by the fluctuation-response relation as $\chi = \beta Q$. In fact, the second moment Q also undergoes a continuous transition from 0 to q at the paramagnet-spin glass transition in the zero-temperature limit (different from the nonzero-temperature case in which Q takes finite values). This simplifies the numerical detection of the transition point as Q is independent of thermal averaging.

The saddle-point equation for Q is given by

$$Q = \int Dv \frac{\int dx e^{-\beta g x^2}}{\int dx e^{-\beta g}} \approx \frac{T}{|a_{\text{eff}}|}. \quad (18)$$

This implies that the paramagnetic phase becomes unstable at $a = a_{\text{eff}} - \chi = -2\xi$. At this point, the zero solution of the Ising machine bifurcates. Note that $a = -2$ is the minimum eigenvalue of the coupling matrix J_{ij}/J , whose eigenvalue distribution obeys Wigner's semicircular law with center 0 and radius 2 [42]. This agrees with the prediction and simulation results of previous studies based on bifurcation analysis [43].

When the temperature is nonzero, Taylor expansion of Eq. (18) to the next order yields

$$Q \approx \frac{T}{|a_{\text{eff}}|} \left(1 - \frac{3T}{|a_{\text{eff}}|^2} \right). \quad (19)$$

Combining with the stability condition $a_{\text{eff}} > -1$, the phase transition line in the space of T versus a in the low-temperature limit becomes

$$T \approx \frac{1}{3}(a + 2), \quad (20)$$

agreeing with the slope of the dashed line in Fig. 5 near $T = 0$.

The thermal averaged spin distribution in the paramagnetic phase is a delta peak centered at zero and has vanishing thermodynamic average $[\langle x_i \rangle^2]_T = 0$. The individual spins do not have a preferred direction to be positive or negative. For $T > 0$, the paramagnetic phase still exists even for higher gains. However, the decoded energy is already negative, indicating the existence of locally optimal clusters in the spin configurations, but due to the inversion symmetry of the Hamiltonian, clusters favoring either orientation remain equally probable and the decoded energy is suboptimal.

B. Gapless phase [Fig. 1(b)]

At fixed temperature T , when the gain a exceeds the paramagnetic threshold, the thermodynamic average $[\langle x_i \rangle^2]_T$ of the spin population becomes nonzero. This is due to the dominance of one of the two inversion-symmetric states at the phase transition line between paramagnet and spin glass, breaking inversion symmetry. Compared with the paramagnetic phase, the spin distribution is broadened and bimodal for sufficiently large gain. The two peaks remain connected in a single energy band.

Figure 6 presents the results of numerical experiments with 10 000 spins ($T = 10^{-4}$) by linearly increasing a from -2 to 1 with simulation time. These results are compared with the theoretical value at zero temperature. The simulation and analytical results are found to be in good agreement. As shown in Fig. 6(a), when the gain is low ($a < -2$), the order parameter is in the paramagnetic phase with $Q = 0$. At $a = -2$, Q starts to undergo a continuous transition. This agrees excellently with the prediction that transition sets in when the gain a becomes equal to the minimum eigenvalue of the coupling matrix [43]. When $a > -2$, Q increases and becomes linear asymptotically. This is expected since for $a \gg 1$, x^* approaches $\text{sgn} w \sqrt{a_{\text{eff}}} \pm O(1)$, such that Q approaches a_{eff} . As the magnitude of the variables x_i becomes uniformly distributed, the system becomes Ising-like. It is recognized that the uniform distribution of the variable magnitudes is

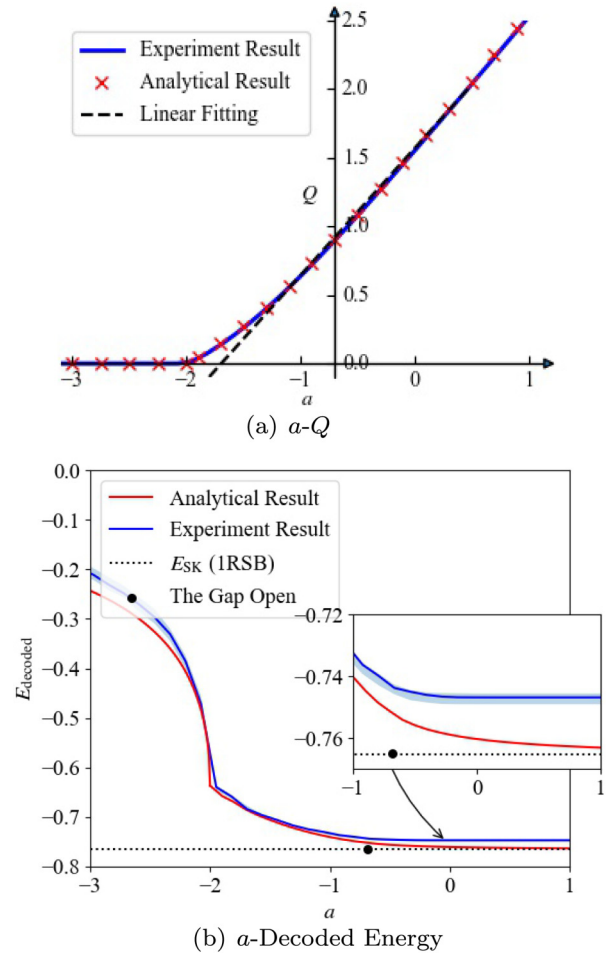


FIG. 6. (a) The dependence of the order parameter Q on the gain a in the zero-temperature limit of CIM. In the simulations, the second moment of CIM's analog spins is obtained from one trial. When a is large, the function approaches a linear relation. (b) The dependence of CIM's decoded energy on the gain a at zero temperature. $E_{\text{SK}}(1\text{RSB})$ is taken to be -0.765 . In the simulations, the CIM's decoded energy is obtained from 20 trials, and the shaded areas are the maximum and minimum values for those experiments.

a prerequisite for the analog system to serve as an accurate optimizer of Ising problems [27,44]. Thus, we expect that accurate optimization can be achieved for sufficiently strong gains.

Figure 6(b) shows the dependence of the decoded energy on the gain. It is interesting to note that the decoded energy is already negative even in the paramagnetic phase. The exact expression at $T = 0$ shows that below the bifurcation threshold,

$$E_{\text{dec}} = -\frac{2\chi}{\pi} = \frac{1}{\pi}(a + \sqrt{a^2 - 4}). \quad (21)$$

We remark that the above expression is applicable in the limit of vanishing temperature. At exactly zero temperature in the paramagnetic phase, the dynamical behavior is dominated by other spurious states.

At $T = 0$, at the bifurcation threshold $a = -2$, the decoded energy is continuous with that of the paramagnetic phase, but a kink exists at the bifurcation point as the square root

behavior below the threshold [see Eq. (21)] changes to a linear behavior above,

$$E_{\text{dec}} = -\frac{2}{\pi} \left(1 + \frac{a+2}{3} \right), \quad a \gtrsim -2. \quad (22)$$

The decoded energy of $-2/\pi$ at this point corresponds to the minimum eigenvalue of the coupling matrix with elements J_{ij}/J [42]. Although the spin values at this point are somehow indicative of the optimal configuration [43], but noting that the decoded energy of -0.64 remains higher than the optimal energy of -0.76 , the configuration may still be considerably remote from the optimal configuration.

When a increases above the bifurcation point, the decoded energy drops rapidly. When it reaches the gap-opening point at $a = -0.70$, the decoded energy already reaches -0.753 , very close to the optimal energy. When a increases further, the decoded energy decreases smoothly to the optimum. The asymptotic theoretical value of the 1RSB decoded energy extrapolated to infinite values of a is -0.766 , which is slightly below the value of -0.765 obtained from simulations around $a \sim O(1)$. For reference, the optimal ground-state energy of the SK model is -0.763 to -0.765 [37,45], and the 1RSB value is known to be slightly below. This shows that the agreement between the CIM and the SK model is excellent.

C. Gap-opened phase [Fig. 1(c)]

When the gain a is further increased above a threshold (e.g., G in Fig. 5 for $T = 0$), the bimodal spin distribution becomes separated by a gap. When the effective gain a_{eff} is negative, the minimizer x^* of Eq. (11) is a single-valued function of the cavity field w . This implies that $P(x)$ is a continuous distribution.

On the other hand, when a_{eff} is positive, there exists a range of w in which multiple values of x^* for each w are possible. In this case, the value of x^* that yields the minimum g^* is dominant. For positive and negative values of w , the range of the minimizers is $\sqrt{a_{\text{eff}}} \leq x^* < \infty$ and $-\infty < x^* \leq -\sqrt{a_{\text{eff}}}$, respectively. As shown in Fig. 7, this leads to the emergence of a gap $(-\sqrt{a_{\text{eff}}}, \sqrt{a_{\text{eff}}})$ in the two-band distribution $P(x)$. Hence, the gap-opening point is given by

$$a_{\text{eff}} = 0. \quad (23)$$

Since $\chi = 0.69$ in 1RSB, the gap-opening point is expected to be $a = -0.69$. On the other hand, Yamamura *et al.* computed the gap-opening point of CIM to be $a = -0.45$ (corresponding to $\chi = 0.45$), and is expected to approximate well the FRSB value [34]. We accept that $a = -0.45$ is a more accurate estimate of the gap-opening point.

At finite temperatures, thermally excited spins are located in the gap states, and the gap becomes a pseudogap (green region in Fig. 5).

The 1RSB decoded energy of the CIM reaches -0.753 at this point, which is significantly closer to the optimal energy compared with the energy in the gapless phase (see the red curve of Fig. 6). Beyond the gap-opening point, the decoded energy slowly decreases toward the optimal value with increasing gain a , but does not reach it.

The double peaks with equal magnitudes of the spin distribution $P(x)$, shown in the green region of Fig. 8, become

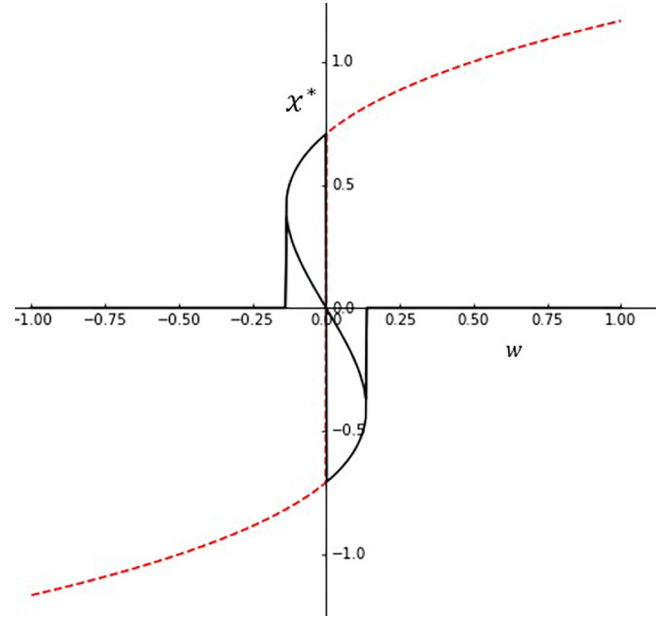


FIG. 7. When a_{eff} is positive, the stationary condition of the effective free energy provides three roots. The optimal root is the red dashed line, which is an odd and discontinuous function of w .

increasingly prominent as a increases, implying that the system becomes more Ising-like. Although it was recognized that the spin magnitudes need to be uniform for the analog system to be an accurate optimizer of Ising problems [27,46], the spin amplitudes in CIM still exhibit a spread even for large values of a . This leads to underperformance of the Ising machine, which has previously motivated the introduction of a clipping function [28,47]. We will show next that the binary phase therein can attain the ground state.

D. Binary phase [Fig. 1(d)]

The binary phase is characterized by a spin distribution that assumes two delta peaks (at values $x = \pm B$), fully emulating the binarized states of an Ising model. This means each spin is frozen at $+B$ or $-B$, and both the order parameters Q and the decoded energy E_{dec} are fixed. However, for CIM, the spin values approach the binary distribution only for infinitely large systems and gain, i.e., $N, a \rightarrow \infty$.

Decimation is a common technique to boost the optimization performance of binary models, leading to enhanced accuracy [48–55] and robustness against hyperparameter uncertainties [56]. When applied to CIMs, a fraction of nodes with the strongest amplitudes of x components are frozen according to their signs. The dynamics of the rest of the nodes then continue, with the prospect of improved convergence under the influence of the frozen nodes. Recent experiments on CIMs with the trap-and-correct technique demonstrate the effectiveness of this method [43].

To theoretically explore the advantage of decimation, we consider the optimization performance of a clipped CIM. We analyze the Hamiltonian of a CIM in which the variables are restricted to within the range $[-B, B]$, thus forcing amplitudes greater than B to be clipped. As we are considering the equilibrium state of the system, we assume that the decimated

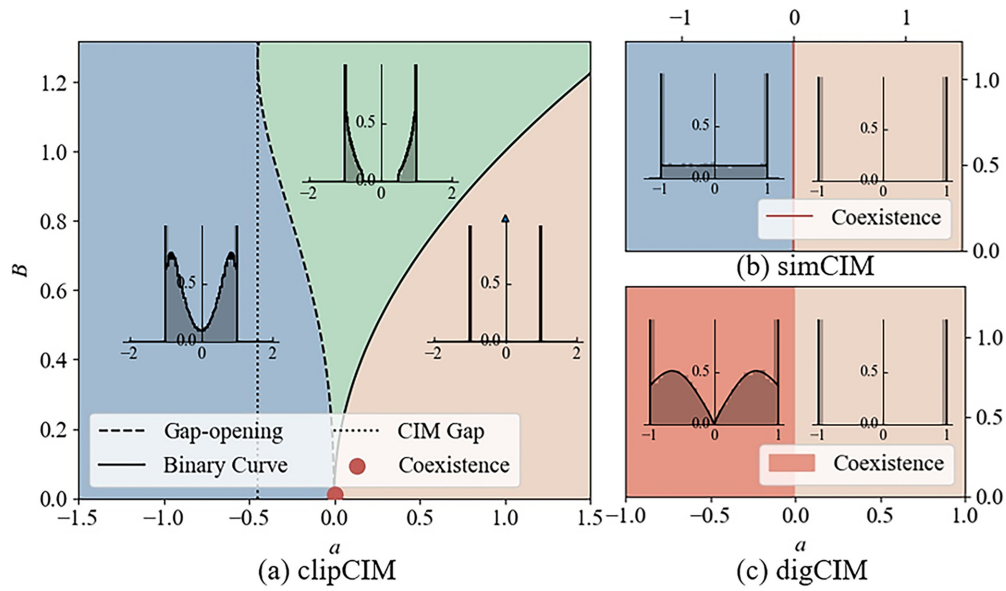


FIG. 8. Phase diagram and the spin distribution for (a) clipCIM ($B = 1$), (b) simCIM, and (c) digCIM in the space of gain a and bound B at zero temperature. In panel (a), the solid and dashed phase lines are obtained by FRSB and simulation, respectively. Distributions are annotated similarly to those in Fig. 5.

nodes have been optimally chosen. Therefore, our analysis provides an upper bound to the performance of practical optimizers and approximates asymptotically those with prudent choice of decimated nodes.

Thus, the clipCIM is characterized by the driving term

$$\mathcal{F}_i^{\text{clipCIM}}(\mathbf{x}, a, \xi) = -x_i^3 + ax_i + \xi \sum_{\substack{j=1, \\ j \neq i}}^N J_{ij}x_j, \quad \text{for } |x_i| \leq B. \quad (24)$$

Carrying out similar calculations as the CIM, we can derive the related quantities for the clipCIM. Results are summarized in Sec. 7 of the SM [35].

The resulting equations allow us to plot the order parameter and the decoded energy as a function of the gain a as shown in Figs. 9(a) and 9(b). One notices that with the increase of a , the order parameters Q (or q_1) approach B^2 . (The effect of decreasing B is similar.) In these cases, the distribution $P(x)$ approaches one with two delta function peaks, and the decoded energy at zero temperature is equivalent to SK model's ground-state energy.

Thus, a natural question relevant to the search for the optimal solution is, what are the ranges of a or B in which Q or q_1 equals to B^2 . Considering Eq. (11), we know that when $w = 0$, the minimizer of the effective free energy provides the smallest absolute value at $|x^*| = \sqrt{a_{\text{eff}}}$ as shown in Fig. 7. We call this value the half gap width, since it represents the half-width between the two peaks within the distribution $P(x)$, and is nonzero only when gap opening occurs. If the half gap width exceeds B , then $P(x)$ transforms into a dual delta function [as shown in Fig. 4(b)]. Thus, we can derive the following condition for the binary phase:

$$\min |x^*| = \sqrt{a + \chi} \geq B. \quad (25)$$

This equation defines the region in the space of a and B where the ground state of the SK model can be attained by

the clipCIM. We refer to the boundary of this region as the binary curve.

To derive the binary curve in 1RSB, we substitute the solution $x = B \text{sgn} w$ into the saddle-point equations of the clipCIM, where $\sqrt{q_1}w = \sqrt{q_0}v + \sqrt{q_1 - q_0}u$. For $B = 1$, they become

$$q_1 = 1, \quad (26)$$

$$q_0 = \int Dv \left(\frac{\int Du e^{m\beta|w|\text{sgn}w}}{\int Du e^{m\beta|w|}} \right)^2, \quad (27)$$

$$\chi = \frac{2}{\sqrt{2\pi(1-q_0)}} \int Dv \frac{e^{-\frac{q_0 v^2}{2(1-q_0)}}}{\int Du e^{m\beta|w|}}, \quad (28)$$

$$\frac{1}{4}(1-q_0^2) + \frac{1}{(m\beta)^2} \int Dv \ln \left(\int Du e^{m\beta|w|} \right) + \frac{1}{m\beta} \frac{\int Du e^{m\beta|w||w|}}{\int Du e^{m\beta|w|}} = 0, \quad (29)$$

$$E_{\text{dec}} = -\chi - \frac{1}{2}m\beta(1-q_0^2). \quad (30)$$

The solution is $q_0 = 0.477$, $m\beta = 1.361$, and $\chi = 0.239$. The decoded energy per node is -0.765 , agreeing with the 1RSB ground-state value of the SK model. For general values of B , the solution is obtained by multiplying q_0 and $m\beta$ by B^2 and B^{-2} , respectively. Hence, remarkably, the parameters along the entire binary curve are constants after rescaling.

The phase boundary between the gap-opening phase and the binary phase is even simpler in the framework of FRSB. In FRSB, the susceptibility χ of the SK model vanishes [32]. Subsequent calculations by incrementally increasing the replica symmetry-breaking steps showed that the susceptibility approaches zero in the limit of infinite steps [57]. This

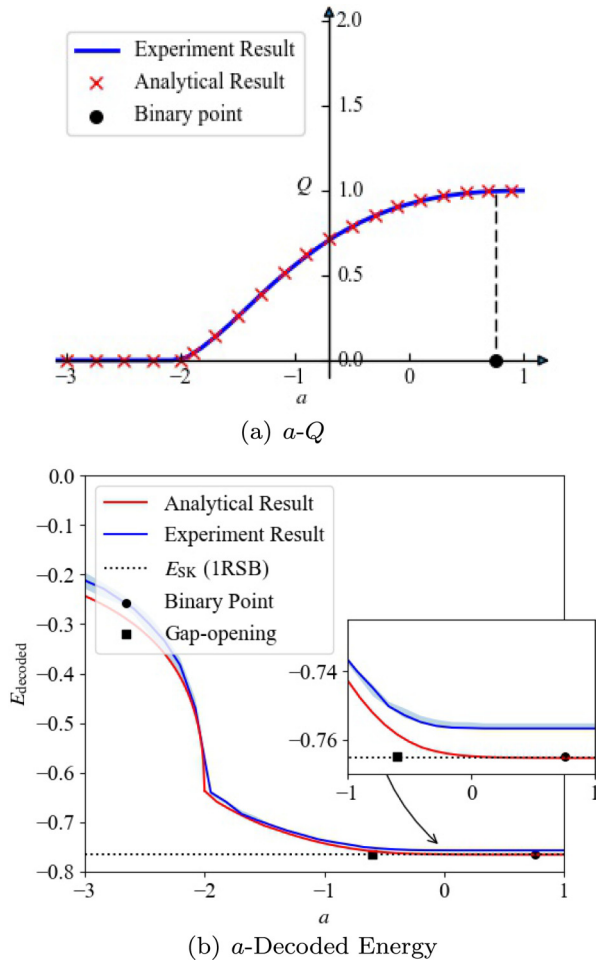


FIG. 9. (a) The second moment (Q) of analog spins (from a single trial) for the clipCIM as a function of a in the zero-temperature limit and $B = 1$. (b) The mean value of the decoded energy of the clipCIM (20 trials), in comparison with the theoretical decoded energy value at zero temperature. One notices that in the gap-opening regime, the simulated decoded energy approaches the ground state of the SK model. In the insets of panel (b), the analytical result does not fully attain the ground state until the binary point is reached.

implies that the phase boundary is given by

$$a = B^2. \quad (31)$$

Furthermore, we derive that the decoded energy in the binary phase is the ground-state energy of the SK model, independent of the level of RSB (see Sec. 5 of the SM [35]) and hence equals -0.763 in FRSB [37,45,58–60].

Typically, at nonzero temperatures, it is preferable to have large values of B and to avoid values of a too deep in the binary phase, as local minima may be prevalent in the deep region of the binary phase. Therefore, the preferred parameters correspond to the case where the equality in Eq. (31), which theoretically attains the ground state with minimal restrictions.

Figure 8(a) shows the phase diagram of clipCIM in the space of a and B at $T = 0$. Here, we have generalized the phase diagram to the case of FRSB. In the blue region of Fig. 8(a), the distribution $P(x)$ is bounded by $\pm B$ with a con-

tinuous distribution in between. The phase boundary between the gapless and gap-opening phases is obtained by numerically detecting the gap of the clipCIM distribution. It starts from $(a, B) = (0, 0)$ and in the limit of large B , it approaches the gap-opening point of CIM with $a = -0.45$, which is the FRSB gap-opening point of CIM obtained in Ref. [34]. When a increases further, the gap of the spin distribution widens, and the continuous component vanishes eventually, leaving behind only the delta components. Remarkably, the shift of a from 1RSB to FRSB is effectively independent of B (≈ 0.24 at both $B = 0$ and ∞).

The phase boundary between the gap-opening phase and the binary phase is given by Eq. (31). The binary curve separates the region with the decoded energy equal to the SK ground-state energy (orange) and that above (blue).

An implication of the above result is that the ground-state energy can be attained at a pump rate $a \sim O(1)$. A recent analysis showed that for the CIM solution to reach the ground state, it is sufficient that the pump rate $a \sim O(\sqrt{N})$ [43]. Here, we demonstrate that in practice, the required pump rate is much lowered by introducing clipping.

Below, we will explore how the driving term in clipCIM can be modified to broaden the range of gain for the binary phase and squeeze the suboptimal gap-opened phase.

V. GAPLESS-BINARY TRANSITION OF DISTRIBUTION

In both CIM and clipCIM, the profile of the spin distribution is influenced by the presence of the cubic saturation term (see Ref. [46]). We now consider an AIM without cubic saturation, referred to as simCIM [28], whose driving term is given by

$$\mathcal{F}_i^{\text{simCIM}} = ax_i + \sum_{\substack{j=1, \\ j \neq i}}^N J_{ij}x_j, \quad \text{for } |x_i| \leq B. \quad (32)$$

The Hamiltonian of this dissipative system is equivalent to the potential energy of another popular algorithm, the ballistic simulated bifurcation (bsB), which is momentum-based and found to have an excellent accuracy and speed performance [29]. Therefore, the results of the analysis can also be applied to it. Using the replica method, the saddle-point equations are provided in Sec. 8 of the SM [35].

Comparing with clipCIM, we notice from Fig. 10(a) that once it bifurcates from $a = -2$, the order parameter Q rises more sharply in simCIM, owing to the absence of confinement due to the saturation term. As shown in Fig. 4(c), the distribution $P(x)$ has a much broader spread than CIM and clipCIM. For the decoded energy shown in Fig. 10(b), it is remarkable to note that when the gain reaches the gap-opening point at $a = -\chi = -0.24$ for 1RSB, the distribution of x already becomes a pair of delta functions at $x = \pm B$. The solution is again given by the binary curve equations (26)–(30), and it reaches the ground state of the SK model. More accurately, the gap-opening point is located at $a = -\chi = 0$ for FRSB. Hence, the gap-opening point of simCIM is also the binary point and the gapless phase is absent in simCIM.

The phase diagram in the space of a and B is shown in Fig. 8(b). The gapless-binary boundary is given by $a = 0$. We

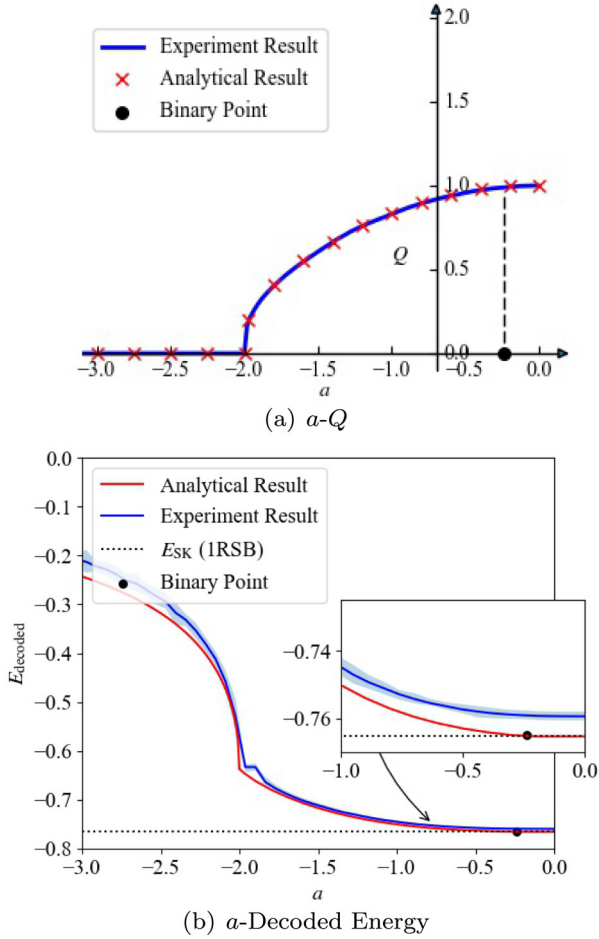


FIG. 10. (a) The second moment (Q) of analog spins (from a single trial) for simCIM as a function of a in the zero-temperature limit and $B = 1$. (b) The mean value of the decoded energy of the simCIM (20 trials), in comparison with the theoretical decoded energy value at zero temperature. One notices that in the gap-opening regime, the simulated decoded energy approaches the ground state of the SK model. In the inset of panel (b), the analytical result does not fully attain the ground state until the binary point is reached.

observe that compared with clipCIM at finite values of B , the SK ground state is achieved at lower values of a , demonstrating the superiority of simCIM over CIM and clipCIM in the search for the ground state. Such superiority was confirmed in the experimental comparison between the versions of CIM and simCIM with ballistic dynamics [29].

Furthermore, the gap-opening and binary lines of simCIM coincide, resulting in a triple point in the parameter space. This means that the gapless phase transits directly to the binary phase without going through the gap-opening phase and is referred to as a *gapless-binary transition*. Such a transition ensures that the probability density at $x = 0$ remains nonzero until the entire system fully transitions to an Ising-like state at zero temperature. The gapless nature of the probability density at $x = 0$ indicates that it is easy for spins to change signs during the optimization dynamics. For other systems such as CIM and clipCIM, the existence of the gapped region prevents the spins to flip easily during the optimization dynamics, and noise (or finite temperature) is required to facilitate the spin

update process. SimCIM, in contrast, does not require such assistance. Numerical experiments reported in Ref. [61] also support this assertion.

Toward further improving the optimization performance, we note that the contributions of the spins to the local field are digitized in the binary phase, where the ground state is reached. Below, we show that the digitization operation is indeed closely related to the ground state and can lead to superior performance when implemented appropriately, leading to a new type of AIM.

VI. DIGITIZATION

Here, we introduce the digitized CIM (digCIM) with gradient descent dynamics and binarized driving term, given by

$$\mathcal{F}_i(t) = ax_i + \sum_{\substack{j=1, \\ j \neq i}}^N J_{ij} \text{sgn} x_j, \quad \text{for } |x_i| \leq B. \quad (33)$$

As shown in Fig. 11(a), Q stabilizes at a constant value when $a > 0$, showing that it enters the binary state. For negative a , the continuous component of the digCIM spin distribution vanishes linearly at $x = 0$, and the positive and negative bands touching only in a single point at origin, resembling the local field distribution of the SK model obtained from solving the TAP equations [32,40,41]. When $a \geq 0$, the spin distribution is binary. Notably, Fig. 11(b) shows that the decoded energy remains independent of a , consistently matching the ground-state energy of the SK model. Indeed, we show below that a steady-state solution of the SK model can be mapped to that of the digCIM family by clipping the fields H_i to within the bounds $\pm B|a|$, and the converse is valid if one notes that $\text{sgn} x$ in the digCIM solution is independent of the magnitude of x .

To demonstrate this superior behavior, we first introduce the auxiliary variables s_i defined by

$$s_i \equiv \text{sgn} x_i = \lim_{A \rightarrow \infty} \tanh(Ax_i). \quad (34)$$

Then, the dynamical equation is transformed to

$$\frac{ds_i}{dt} = A(1 - s_i^2) \left\{ \frac{a}{2A} \frac{\partial}{\partial s_i} [(1 + s_i) \ln(1 + s_i) + (1 - s_i) \ln(1 - s_i)] + \sum_j J_{ij} s_j \right\}, \quad \text{for } |s_i| \leq 1. \quad (35)$$

Hence, the dynamics stops when s_i reaches the bound ± 1 , or when the equilibrium state reaches a stationary point of the Hamiltonian

$$H = -\frac{a}{2A} [(1 + s_i) \ln(1 + s_i) + (1 - s_i) \ln(1 - s_i)] - \sum_j J_{ij} s_i s_j, \quad \text{for } |s_i| \leq 1. \quad (36)$$

Here, we analyze the Hamiltonian as a candidate for describing the system behavior. Applying the replica method, we find that the effective free energy g can be expressed in terms of the

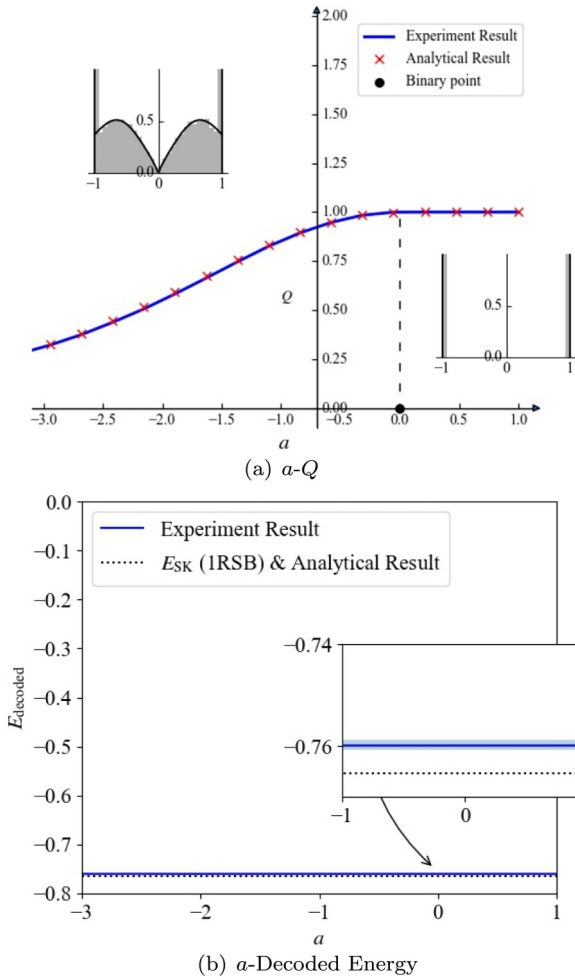


FIG. 11. (a) The second moment (Q or q_1) and distributions of analog spins from a single trial for digCIM as a function of the gain a . In these distributions, the solid black lines correspond to the theoretical values, while the gray shaded regions in the spin distribution indicate the results obtained from simulations. The left distribution is derived from $a = -2$, whereas the right one corresponds to $a = 1$. (b) Mean value of decoded energy of digCIM (20 trials) compared with theoretical decoded energy.

cavity field $\sqrt{q_0}v + \sqrt{q_1 - q_0}u$,

$$g = -\frac{a}{2A}[(1+s)\ln(1+s) + (1-s)\ln(1-s)] - \frac{1}{2}\chi s^2 - (\sqrt{q_0}v + \sqrt{q_1 - q_0}u)s, \quad \text{for } |s| \leq 1. \quad (37)$$

At zero temperature and in the limit $A \rightarrow \infty$, g is dominated by its minimum at the boundaries $s = \pm 1$,

$$s^* = \text{sgn}(\sqrt{q_0}v + \sqrt{q_1 - q_0}u). \quad (38)$$

Not surprisingly, the corresponding saddle-point equations are identical to those of the binary curve in Eqs. (26)–(30), and the decoded energy becomes the ground-state energy of the SK model. Remarkably, this energy is attained for all values of a with no need of any transition from suboptimal states to the SK ground state at the binary point.

However, whereas the distribution of analog spins is well described by the double delta peak when a is positive, it is

very different from simulation results when a is negative, and further analysis is required. In this respect, we consider the steady-state equations of the local fields in the SK model, commonly known as the TAP equations [32]. This set of equations is based on mean-field arguments of statistical physics and its validity is independent of the replica method [32]. It is applicable to extensively connected networks such as the SK model. As will be seen below, its only connection with the replica method is the value of the susceptibility. In the TAP approach, the local field H_i of node i is defined by $\langle \sigma_i \rangle \equiv \tanh(\beta H_i)$. At $T = 0$, the local field at node i of the SK model is given by

$$H_i = \sum_j J_{ij} \text{sgn} h_j, \quad (39)$$

where h_j is the cavity field at node j . The cavity field of a node is the field experienced by the node when its impact on the rest of the network is neglected. This impact, commonly known as the Onsager reaction, is expected to modify the cavity field into the true local field of the node. Due to the fully connected structure of the SK model, the Onsager reaction term can be calculated, yielding

$$H_i = \sum_j J_{ij} \text{sgn} H_j - \chi \text{sgn} H_i. \quad (40)$$

Comparing with the steady-state solution of Eq. (33), we note that the last term is extra in the TAP equation. However, it is important to note that in the full RSB solution, χ vanishes at $T = 0$ [32,57]. Consequently, the TAP equation reduces to

$$H_i = \sum_j J_{ij} \text{sgn} H_j. \quad (41)$$

We propose that given a solution $\{H_i\}$ of the TAP equation, we may map it to a stationary state of the digCIM dynamics with negative gain a by setting

$$x_i = \min\left(\frac{|H_i|}{|a|}, B\right) \text{sgn} H_i. \quad (42)$$

To see this, we consider the cases with $|x_i| \leq B$ and $|x_i| > B$ separately. In the former case, the steady-state equation of digCIM yields

$$|a|x_i = \sum_j J_{ij} \text{sgn} x_j. \quad (43)$$

Substituting Eq. (42) into the right-hand side of the digCIM equation, we see that

$$\begin{aligned} \sum_j J_{ij} \text{sgn} x_j &= \sum_{|H_j| \leq |a|} J_{ij} \text{sgn} \left(\frac{H_j}{a} \text{sgn} H_j \right) \\ &\quad + \sum_{|H_j| > |a|} J_{ij} \text{sgn} (B \text{sgn} H_j). \end{aligned} \quad (44)$$

Thus, the expression in the above equation is equal to $H_i = |a|x_i$ according to the TAP equations, consistent with the steady state of digCIM.

For $|H_i| > |a|$, substituting Eq. (42) into the dynamical equation of digCIM, we have

$$\frac{dx_i}{dt} = -|a|B\text{sgn}H_i + \sum_j J_{ij}\text{sgn}H_j = (|H_i| - |a|B)\text{sgn}H_i. \quad (45)$$

This shows that the rate of change of $|x_i|$ is directed along the direction of magnitude increase. However, since $|x_i| = B$ already lies on the boundary, it has also reached the steady state of the digCIM dynamics.

It is important to note that in practical applications, if $a < 0$, then the solution $x_i = \frac{H_i}{|a|}$ is both confined and stable. This implies that the clipping function, which prevents the analog spins from increasing indefinitely, is not required to constrain the magnitudes of the spin values in digCIM when $a < 0$.

In summary, we have verified that the existence of an SK stationary state is a sufficient condition for the existence of the stationary state of the digCIM dynamics described by Eq. (42).

The converse can be verified similarly, namely, if $\{x_i\}$ is a stationary state of the digCIM dynamics, then the corresponding stationary state of the SK model is given by

$$H_i = \sum_j J_{ij}\text{sgn}x_j. \quad (46)$$

To verify this, we look for solutions of the TAP equation with $\text{sgn}H_i = \text{sgn}x_i$. Then, for $|\sum_j J_{ij}x_j| \leq B/|a|$, the steady state of x_i lies in the continuous range of the analog spin distribution, and $H_i = |a|x_i$ directly. On the other hand, for $|\sum_j J_{ij}\text{sgn}x_j| > B/|a|$, the steady state of x_i is constrained by B , but nevertheless, $H_i = \sum_j J_{ij}\text{sgn}x_j$ remains valid.

Hence, we have verified the correspondence between the stationary states of the digCIM dynamics and the SK model. The distribution of the analog spins can then be determined by the field distribution $P(H)$. For $a < 0$, $P(x)$ has a continuous component given by $|a|P(H = |a|x)$ for $|x| < B$, and delta functions at $x = \pm B$ with respective statistical weights of $\int_{|a|B}^{\infty} dH P(H)$ and $\int_{-\infty}^{-|a|B} dH P(H)$. For $a \geq 0$, $P(x)$ consists of delta function peaks at $x = \pm B$.

The local field distribution of the SK model, $P(H)$, has been extensively studied both analytically and numerically. It can be obtained through multistep RSB or by generating an empirical distribution from samples [40,62]. A key finding from these studies is that the probability density at $H = 0$ is nil and increases linearly as $|H|$ in the neighborhood, implying that the probability density of x at zero also vanishes. Thus, the probability density has a profile intermediate between gap-opening and gapless distributions. Despite the absence of a gap in the distribution of x (when $a < 0$), this feature effectively plays the role of gap opening, thus providing the SK model's ground-state energy across all ranges of a .

The vanishing of the gap in the spin distribution of digCIM is a consequence of the vanishing susceptibility χ . For nonvanishing values of χ , Eq. (40) becomes

$$H_i = F\left(\sum_j J_{ij}H_j\right), \quad (47)$$

where $F(x) = x + \chi\text{sgn}x$, implying that a gap of width 2χ exists in the spin distribution at $x = 0$. As our digCIM simulation shows that the gap shrinks to 0, we have demonstrated that digCIM yields solutions consistent with the FRSB case.

The vanishing of the probability density at $x = 0$ further confirms that the digCIM dynamics is stable at the interior points with $|x| < B$ when a is negative. Consider the dynamics of perturbations δx_i when the digCIM dynamics has reached a stationary state in Eq. (33). We see that the second term has contributions dependent on $\delta(x_j)$. Since $P(0) = 0$, the perturbation dynamics only depends on the first term, resulting in $d(\delta x_i)/dt = a\delta x_i$. Thus, stationary states with interior x_i are possible. On the other hand, for $a > 0$, the interior points are not stable, and the analog spin distribution becomes an Ising-like dual-delta function, as depicted in Fig. 4(d). Thus, $a = 0$ is the binary point of digCIM system.

Contrary to simCIM, our simulations show that annealing the temperature is crucial to the quality of the decoding performance of digCIM. Although the ground-state energy exists in the full range of a , its attainment is influenced by the choice of the dependence of a on T , which affects the magnitude of the analog spin during temperature annealing. In general, a larger a necessitates a larger T . This is consistent with the implementation of the dSB system, in which the momentum plays the role of noise.

Especially, one should not anneal a for digCIM at low temperatures, since the probability distribution of $x = 0$ almost vanishes. Such annealing is merely a transition from one frozen system to another, with the sign of the soft spins remaining unchanged.

We note that dSB [29], which uses the same driving term but employed momentum dynamics, thus belongs to the same family as digCIM with the same steady-state behavior. Previous efforts, such as a variant of HTNN [63] and the momentum-based algorithm dSB [29], have experimentally explored related concepts. Some of these approaches have achieved excellent performance. However, these studies did not identify the theoretical optimality of the digCIM family. In this work, we demonstrate that this optimality emerges from the superior performance observed in the coexistence region of the phase diagram, as well as from the steady-state properties characterized by ground-state equations such as the TAP equations. The success probability of digCIM and dSB scales similarly with problem size (Sec. 11 of the SM [35]). For the GSET benchmark, the time to solution of digCIM is comparable to dSB on smaller datasets with advantages, and longer for large instances (Sec. 11 of the SM [35]).

We close this section by comparing the performances of the decoded energies of CIM, clipCIM, simCIM, and digCIM as summarized in Table I. A detailed comparison of the decoded energy and transition points for these four systems under the 1RSB framework is illustrated in Fig. 12. The original CIM manages to approach the ground state of the SK model asymptotically, but there is no binary point beyond which it collapses to the ground state. We found numerical evidence that the required value of a to reach the ground state scales as $\sqrt{N}$, roughly of the same order of magnitude predicted by the sufficient condition proposed in Ref. [43]. On the other hand, by introducing clipping, the decoded energy of the clipCIM is much closer to the SK ground state, and it collapses to the

TABLE I. The analog Ising machines discussed in this paper.

AIMs	CIM	clipCIM	simCIM	digCIM
$\mathcal{F}_i$	$-x_i^3 + ax_i + \sum_j J_{ij}x_j$	$-x_i^3 + ax_i + \sum_j J_{ij}x_j$	$ax_i + \sum_j J_{ij}x_j$	$ax_i + \sum_j J_{ij}\text{sgn}x_j$
$\phi(x)$	x	$\max(-B, \min(B, x))$	$\max(-B, \min(B, x))$	$\max(-B, \min(B, x))$
Gap opening (1RSB)	$a = -0.69$	$a \in [-0.69, -0.24]$	$a = -0.24$	$a = 0$
Gap opening (FRSB)	$a = -0.45$	$a \in [-0.45, 0]$	$a = 0$	$a = 0$
Binary point (1RSB)	$a = +\infty$	$a = B^2 - 0.24$	$a = -0.24$	$a = 0$
Binary point (FRSB)	$a = +\infty$	$a = B^2$	$a = 0$	$a = 0$
Coexistence region	N/A	$(a, B) = (0, 0)$	$a = 0$	$a < 0$

ground state beyond the binary point. The simCIM, with the nonlinear decay term eliminated, has an even lower decoded energy, and the SK ground state sets in at an even lower binary point. The digCIM, with the local fields computed using binarized variables, has the decoded energy identical to the SK ground state for all values of a , even when its binary point of $a = 0$. These results agree with the experimental observation that the ballistic simulated bifurcation machine outperforms the CIM [29].

VII. IMPLICATIONS TO DYNAMICS

A. The gapless-binary coexistence region

The two-phase transition points, namely, the gap-opening point and the binary point, have important implications to the AIM dynamics. Gap opening is a significant milestone in the dynamical process whereby target systems undergo a transition into an Ising system. Each analog spin adopts a meaningful configuration without ambiguity. During the dynamical process, the system typically starts with an initial gapless spin distribution and a decreasing decoded energy. At the instant the gap opens, further decrease of the decoded energy slows down significantly. Specifically, at zero temper-

ature, after the gap opens, the system's decoded energy will no longer change even as a continues to rise, since the gap prevents the spins from changing signs.

The binary point is the end point of the process through which target systems evolve into an Ising system during an annealing process in the space of a and T to search for the ground state. This highlights the importance of ensuring that the annealing schedule for a does not exceed the binary point, as doing so offers no advantage in reaching a better final state and merely increases the chances of being trapped in metastable states [34].

For CIM, the binary point is theoretically located at infinity, indicating that in the thermodynamic limit, the energy landscape of the CIM cannot become equivalent to that of the Ising model. Consequently, it is impossible for the CIM to achieve a ground-state configuration identical to the Ising model at finite values of the gain a .

The clipCIM is an intermediate case between CIM in the large B limit and simCIM in the small B limit. When B increases to infinity, the effect of the clipping boundary is negligible, and the behavior of the clipCIM approaches that of CIM. Thus, the gap-opening point and binary point approaches -0.45 and $+\infty$, respectively. On the other hand, as B approaches 0, the effects of the cubic saturation term of clipCIM are restricted by the narrow range of the spin values, and its behavior approaches that of simCIM. Hence, both the gap-opening point and binary point approaches 0. Detailed illustrations are provided in Fig. 8(a).

For simCIM, the gap-opening point and binary point are unaffected by the choice of the clipping boundary. This is due to the fact that the clipping boundary acts merely as a scaling parameter within the simCIM framework and can thus be selected arbitrarily. Additionally, since the full range of the continuous component of the spin distribution of simCIM vanishes concurrently at the gap-opening point, simCIM experiences a gapless-binary transition. This implies that as long as the system has not undergone transition to the Ising state, the probability density at $x = 0$ remains nonzero. The existence of spin values in the neighborhood of $x = 0$ indicates that it is easy for spin values to switch sign during the optimization dynamics, facilitating more efficient convergence to the final steady state. In contrast, for other systems exhibiting nonvanishing gap regimes, it is difficult for spins to flip across the gap during the optimization dynamics. In such cases, noise (or finite temperature) is required to facilitate the spin update process. SimCIM, in contrast, does not require such assistance.

For digCIM, our analysis suggests that its decoded energy is independent not only of the clipping boundary but

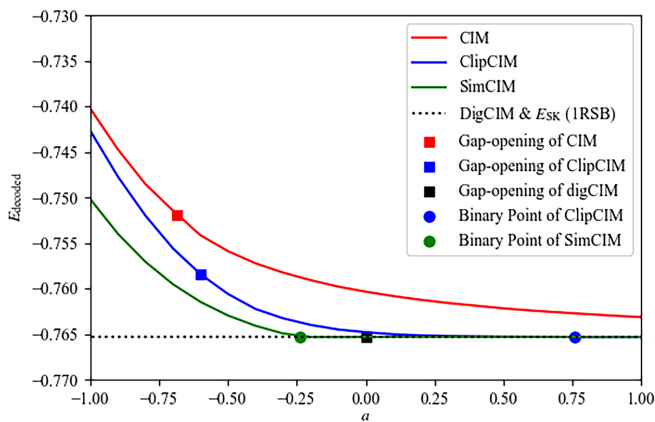


FIG. 12. Theoretical decoded energy of CIM, clipCIM, simCIM, and digCIM at zero temperature as a function of the gain a with constant threshold $B = 1$ in clipCIM, simCIM, and digCIM. Circles and squares symbolize binary points for clipCIM, simCIM, and digCIM, and gap-opening points for CIM and clipCIM under 1RSB assumption, respectively. The decoded energy in digCIM consistently coincides with the SK model's ground state across all a ranges, thereby sharing a common dotted line.

also of the parameter a , and digCIM can attain the lowest decoded energy throughout the entire phase space. As the gain approaches $a = 0$ from the negative side, the continuous component of the spin distribution becomes increasingly flat and finally disappears completely at $a = 0$, leaving behind the delta function distributions at the clipping thresholds. This means that the binary point and the gap-opening point also overlap in digCIM system, and the transition at $a = 0$ can also be considered as a gapless-binary transition. This also implies that assistance from noise (or temperature) is required across the entire phase space during the annealing process.

Based on the above discussions of the phase behaviors, we have gleaned insights into the system dynamics that enable us to offer more promising recommendations for practical experimental setups. In the gap-opening phase, spin flipping ceases in the absence of noise. Hence, it is likely to exhibit poor convergence, unable to spontaneously reduce the decoded energy effectively.

In the binary phase, the analog spins become distinctly Ising-like. However, even though the system shares the same ground state as the Ising model in the binary phase, the dynamics of the system are nearly frozen, preventing any further evolution of either the spin's sign or amplitude. Even with noise intervention, the energy barriers in this phase are significantly higher compared to those in the gapless phase, necessitating much longer dynamics for effective searching. Furthermore, when one operates the dynamics deep inside the binary phase, simulations show that the approach to the ground state is hindered by the existence of metastable states [34].

Thus, from the dynamics point of view, the best strategy to attain the ground state of the Ising model is to operate the dynamics at the point where the spin distribution remains gapless (or approach it in an annealing schedule), such that spin flipping to lower the decoded energy remains feasible. At the same time, the system lies on the binary curve. This ideal operation point (or region) is the gapless-binary transition point (or region).

Drawing from the above understanding of phase properties, we propose that the gapless-binary coexistence region—defined as the intersection of the gap-opening and binary curves, or equivalently the overlap of the gapless and binary (and gap opening, if one takes the point of view that the gap must open during the transition from the gapless phase to the binary phase) phases—represents an optimal location that shares the advantages of both the optimal steady-state decoded energy and the efficient gapless dynamics. Consequently, this region is an ideal parameter endpoint for AIM annealing operations.

A parameter annealing schedule for AIMs that begins in the gapless phase and concludes at the coexistence region can ensure that dynamics spontaneously reduce decoded energy throughout the entire process, thereby preventing the system from becoming prematurely frozen due to the gap.

B. Coexistence point/curve in clipCIM

As a demonstration of its advantage, we apply the notion of the coexistence region to the clipCIM. In the $a - B$ parameter space, the gapless-binary transition point is located at

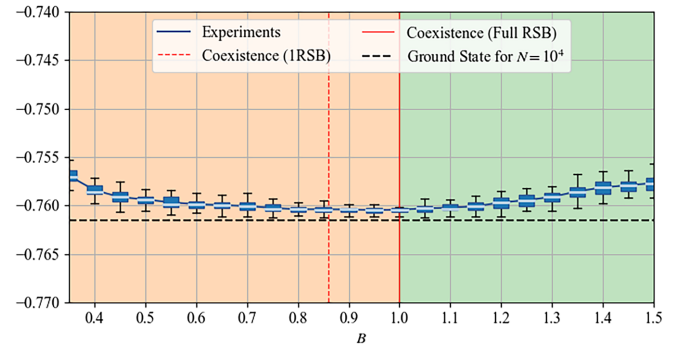


FIG. 13. Decoded energy of clipCIM vs clipping threshold at $a = 1$. The blue boxes are the results of numerical experiments on an SK model with 10 000 spins, running dynamics 50 times through the Euler-Maruyama method for 100 000 steps with a step size of 0.1. On the left-hand side (binary phase), the increase in decoded energy is due to poor dynamical properties. On the right-hand side (gap-opening phase), the increase results from lower accuracy. The mean-field ground-state line corresponds to the full RSB result, including finite-size effects. The lowest decoded energy is achieved using parameters within the coexistence region.

(a, B) = (0, 0). In practical terms, however, this intersection is challenging to exploit since a vanishing clipping threshold results in an undefined spin configuration. The introduction of temperature can extend the coexistence point to the three-dimensional space of a , B , and T . It is worth noting that gap-opened phases are only clearly defined at $T = 0$. The presence of noise at nonzero temperatures can fill in the gap (creating a pseudogap) and induce a degeneration of the gap-opened phase into the gapless phase. Thus, within the expanded parameter space of a , T , and B , the binary curve in Fig. 8(a) also acts as the gap-opening point, and thus, a coexistence curve.

Based on this theoretical prediction, we simulate the clipCIM starting at a higher T , and gradually anneal it until it reaches the vicinity of the coexistence curve in the parameter space. This provides a superior solution in terms of accuracy compared to other parameter schedules aimed at noncoexistence regions, as shown in Fig. 13. Additionally, we have applied the full RSB ansatz to the coexistence curve, as indicated by the red dotted line on the graph. It is noteworthy that the experimental decoded energy lies within the range from the 1RSB to the full RSB coexistence points remains fairly consistent, showing that the 1RSB solution is a reasonable approximation.

C. Coexistence line in simCIM

The coexistence region of SimCIM is the binary line in the $a - B$ parameter space. As described in Sec. V, the gap-opening line coincides with the binary line in the simCIM. Hence, the coexistence region is the entire gapless-binary transition line. A numerical experiment conducted on the simCIM system aligns with our theoretical predictions. Starting from the same a value in the gapless phase and stopping at various points along the coexistence curve, systems targeting the coexistence curve consistently achieved better results compared to others, as illustrated in Fig. 14.

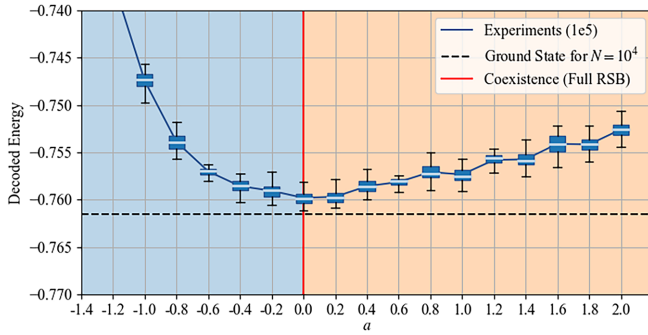


FIG. 14. Decoded energy of simCIM vs gain a at $B = 1$. The blue boxes are the results of numerical experiments on an SK model with 10 000 spins, running dynamics 40 times through the Euler-Maruyama method for 100 000 steps with a step size of 0.1. Temperature annealing from 1 to 0 for each data point. The lowest decoded energy is achieved using parameters within the coexistence region. The mean-field ground-state line corresponds to the full RSB result, including finite-size effects.

D. Coexistence region in digCIM

For digCIM, the binary curve also coincides with the gap-opening curve, similar to simCIM. However, a significant distinction arises because the steady states of the digCIM at negative values of a are isomorphic through rescaling x_i and B by $|a|$. To see this, we multiply both sides of Eq. (42) by a factor λ ,

$$\lambda x_i = \min \left(\frac{|H_i|}{|a|/\lambda}, \lambda B \right) \text{sgn} H_i. \quad (48)$$

This means that if $|a|$ is divided by a factor λ , then after B is multiplied by the same factor, $\{\lambda x_i\}$ will become the solution of digCIM with the rescaled parameters. The profile of $P(x)$ is the same, except that the variable axis is extended by λ . Thus, all models with the same value of $B|a|$ are isomorphic. Consequently, this rescaling expands the coexistence curve to encompass the entire subspace of negative a , forming a comprehensive coexistence region. The entire region exhibits a spin distribution with the dual delta component and dual bands touching at $x = 0$. Experimentally, this configuration provides superior dynamics, leading to better results comparing to other systems. Additionally, due to its wide range of optimal parameter settings, selecting an annealing path for the digCIM is more convenient. Unlike the clipCIM and simCIM, adjusting the parameters a and B in the digCIM does not require meticulous attention, simplifying the parameter selection process.

Comparing the dynamics of digCIM and simCIM, a further advantage of digCIM lies in its band-touching distribution, which vanishes linearly near a as predicted by TAP equations in the coexistence phase. This feature enables soft spins to flip signs more readily during optimization. In contrast, simCIM operating near the coexistence phase concentrates most spins at the boundaries ($\pm B$), leaving only a small fraction of continuous spins available for dynamic sign changes. This fundamental difference in spin distribution explains digCIM's superior exploration capability and faster convergence to optimal solutions.

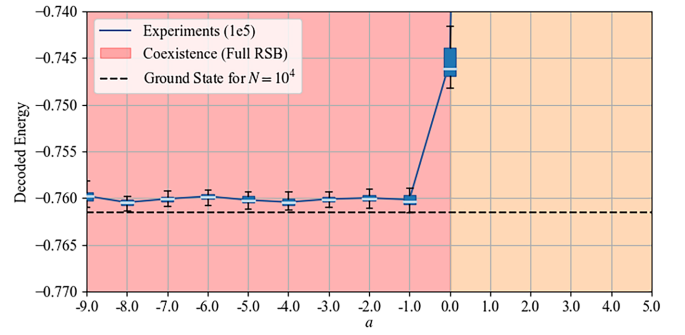


FIG. 15. Decoded energy of digCIM vs gain a at $B = 4$. The blue boxes are the results of numerical experiments on an SK model with 10 000 spins, running dynamics 20 times through the Euler-Maruyama method for 100 000 steps with a step size of 0.1. Temperature annealing from 1 to 0 for each data point. It is possible to achieve optimal decoded energy throughout the entire phase space. However, within the orange area (binary phase), the distribution of metastable states hinders the dynamics from freely exploring the solution space, resulting in suboptimal outcomes. The mean-field ground-state line corresponds to the full RSB result, including finite-size effects.

Numerical results of digCIM dynamics are shown in Fig. 15. We conducted the dynamics starting from the same initial temperature while varying the constant gain a for different data points. The temperature was then gradually annealed to zero following the evolution of the dynamics. The coexistence region consistently yields better results compared to setups within the binary phase.

It is important to note that we intentionally chose a wide range of a in Fig. 15 for demonstration. The distribution of analog spins scales wider by $1/|a|$. With a small B and a small negative a , although the distribution still exhibits band touching, the continuous component of the spin distribution is flattened. Consequently, the distribution will resemble a dual delta function, which slows down the dynamics. For a given finite run-time, this leads to higher decoded energy compared to other settings within the same phase. Figures 13–15 provide evidence for the correctness of the coexistence theory and the associated changes in the decoded energy. We further introduce Fig. 16 to illustrate that, given sufficient time, the coexistence phase indeed enables the system to reach the theoretical ground state. Moreover, the increase in decoded energy near $a = 0$ is due to the broadening and concentration of the distribution, which can be mitigated by increasing the run-time.

The coexistence region for the four systems is summarized in Table I.

VIII. GENERALIZATION TO PRACTICAL PROBLEMS

To validate our results beyond the SK model, we used the MAX-CUT dataset Gset [64], which includes random graphs of varying sizes and topologies, as a benchmark. As an example, the dataset G1 comprises a network of 800 nodes and 19 176 edges, with each edge randomly assigned a value of either 1 or 0 (corresponding to $J_{ij} = -1$ or 0, rendering the problem a frustrated network analogous to a spin glass). Figure 17

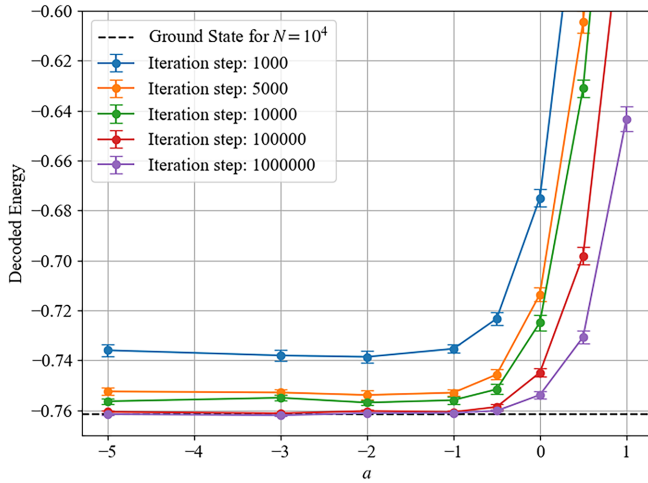


FIG. 16. Decoded energy of digCIM as a function of gain a at $B = 4$ for different iteration steps. All numerical experiments are performed on different SK models with 10^4 spins, running the dynamics 20 times using the Euler-Maruyama method at various iteration steps with a step size of 0.1. The dashed line indicates the mean-field SK ground-state energy with finite-size corrections.

displays strong agreement with our theoretical predictions: CIM exhibits the highest decoded energy, followed by clipCIM and simCIM. These latter models are capable of reaching the ground state provided that the parameter a is sufficiently large, and their performance progressively converges to that of digCIM. The mean output of digCIM consistently remains at a low-energy level, close to the minimum, demonstrating a persistent capability to achieve the ground state throughout the entire phase space.

Section 9 of the SM [35] presents further details confirming the similarity of the optimal solutions of Gset with theoretical predictions. The spin distributions of the four AIMs exhibit patterns similar to the theoretical cases. In particular, gap-opening and binary transitions are observed.

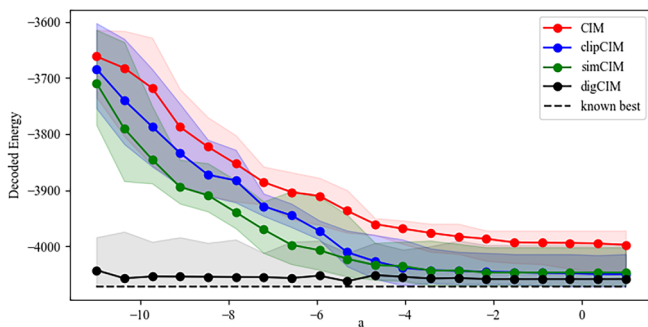


FIG. 17. The experimental decoded energy curve for CIM, clipCIM, simCIM, and digCIM applied to dataset G1. The solid lines represent the mean values obtained over 20 trials, while the shaded areas are bounded by the maximum and minimum values observed. In each trial, the parameter a was divided into 20 equidistant ascending steps ranging from -11 to 1 . At each step, the temperature was individually annealed from a high value down to 0 . The dashed line corresponds to the best-known result for G1, with a value of -4072 .

TABLE II. Performance comparison between DigCIM and conventional solvers on QUBO problems from the QPLIB benchmark [65,66].

Solver name	Number of solved problems [65]
BARON	12
SCIP	7
MCSPARSE	12
GUROBI	13
QUBOWL	15
BIQBIN	9
SHOT	12
DigCIM	19

We also find that the performances of AIMs are enhanced in the coexistence region. We argue that the vanishing of χ and the expressions of the coexistence regions can be extended from the SK model to practical problems, since we expect that an infinitesimal magnetic field h applied to frozen discrete Ising spins should not cause any change in the spin state. This facilitates the parameter search for optimal performance as the coexistence regions of clipCIM and simCIM are $a = B^2$ and $a = 0$, respectively. This proposal aligns well with experimental results.

The performance superiority of the digCIM family is demonstrated not only through comparisons with other AIMs but also against popular state-of-the-art commercial solvers. We evaluate our machine using QUBO problems from the open benchmark library quadratic programming instances, QPLIB [65]. The problems selected from this library originated from both theoretical studies and real-world applications. We compare our results with other optimization software data sourced from [65], adhering to its format and the constraints pertaining to time and hardware.

Remarkably, digCIM delivers the best performance for the QUBO-QPLIB suite [66]. As shown in Table II, the performance not only surpasses the other Ising machine systems discussed in this paper, but also solves 19 out of 23 problems in the suite, significantly exceeding all current state-of-the-art mainstream commercial solvers under the same hardware and time constraints. Details on implementation time, hardware specifications, and the performance of other Ising machines are provided in Sec. 10 of the SM [35].

IX. CONCLUSION

We have studied the spin distribution and the decoded energy in a family of AIMs, optimizing the energy of the SK model. Their behaviors can be classified into several phases: paramagnetic, gapless, gapped, and binary. Based on digCIM's ability to solve the TAP equations, we propose that AIMs with the digitized driving term, such as digCIM and dSB, are the preferred choice. Further tests reported in Sec. 11 of the SM [35] demonstrate that the digCIM family is advantageous when the problem size scales up.

We elucidated the optimal regimes of the dynamics for attaining ground-state solutions. The static analysis of the

phases predicts that the decoded energy attains the ground state of the binary system *beyond* the binary point, but simulation dynamics yield decoded energies higher than the minimum. The situation is analogous to the geometric landscape at high gain of the CIM studied in Ref. [34], where the authors computed the distribution of local minima, showing that local minima prevent the convergence to the ground state. Besides, to prevent the dynamics from freezing prematurely into suboptimal states, it is favorable to work in regimes where the positive and negative bands of the spin distribution are not separated by gaps. Hence, the gapless-binary coexistence region is the most favorable working region for solution accessibility and accuracy, as illustrated in the coexistence point in clipCIM, the coexistence curve in simCIM, and the coexistence region in digCIM.

In practice, the operation of AIMs requires a good annealing path with appropriate control of the system parameters. Including temperature annealing, we found that annealing terminating in the coexistence regions leads to much higher decoding accuracy than otherwise. For digCIM, the system depends on the temperature alone as the sole parameter, resulting in a path with strongly reduced complexity. Together with the two-dimensional coexistence region, the accessibility of the ground state is significantly enhanced.

Indeed, we discussed in a separate paper the advantage of temperature annealing in the planning of optimal anneal-

ing pathway [61]. Gapped distributions at $T = 0$ degenerate into distributions with pseudogaps at finite temperatures. This changes effectively the phase diagram: For instance, in clipCIM, the binary curve becomes the coexisting curve [see Fig. 8(a)]. Such changes can guide the trajectory in parameter space to find optimal decoding schemes. In Sec. VII, we illustrate for various AIMs how good trajectories can lead to superior performance. An analysis based on dynamical mean-field theory is provided in Ref. [61].

This theoretical framework readily generalizes to the steady-state behavior of momentum-based systems, including aSB, bSB, and dSB [29]. Optimal AIM performance requires further understanding nonequilibrium spin distribution dynamics to design parameter schedules that reach coexistence regions, ultimately guiding future system development.

ACKNOWLEDGMENT

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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