

Integer quantum Hall effect in bilayer Dice lattices

Han-Lin Liu^{✉*} and J. Wang^{✉†}*School of Physics, Southeast University, Nanjing 210096, China*

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We investigate the quantum Hall effect in bilayer dice lattices with AA-BB-CC and AB-BC-CA stacking configurations, focusing on the interplay between flat band stability and band topology. For the aligned AA-BB-CC stacking, the system exhibits a narrow zero Hall conductivity plateau at the Dirac point and hybrid integer quantum Hall states ($n = \pm 2, \pm 4$) in some energy regions, originating from the coexistence of spin, valley, and layer degeneracies. Additionally, the flat band generated by AA-BB-CC stacking is more easily destroyed. In contrast, the cyclic AB-BC-CA stacking displays irregular Hall conductivity due to complex band structures lacking complete flat bands. By tuning interlayer coupling strength $t_{\perp}$ and hopping parameter α , the corresponding results show that the position of the Dirac point relative to the high-symmetry point in the Brillouin zone determines the appearance of the conductivity transitions. A mixed integer quantum Hall effect with $n = 2$ and $n = 4$ appears for specific values of α .

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I. INTRODUCTION

The breakthrough in isolating monolayer graphene has sparked a surge of research into relativistic materials [1–3]. Characterized by their remarkable physical behavior, these materials show great potential for applications ranging from thermoelectric devices [4–6] to topological insulators. At low energies, graphene's quasiparticles obey a two-dimensional Dirac equation [1,7,8], a feature shared by the surface states of three-dimensional topological insulators [2,9,10]. Furthermore, three-dimensional linear energy spectra manifest in Weyl and Dirac semimetals [11–15]. However, layered systems, occupying an intermediate regime between two and three dimensions, offer additional opportunities for engineering electronic properties through controlled stacking configurations. The electronic characteristics of few-layer systems can differ substantially from their monolayer counterparts, as exemplified by Bernal-stacked bilayer graphene, which exhibits a quadratic quasiparticle spectrum near band-touching points [8,16,17], leading to distinct quantum Hall effect [16,18] and optical responses [19].

Although flat band research has a long history [20–24], interest in this field has surged recently after the discovery of superconductivity in twisted bilayer graphene with a moiré structure [25,26], where a nearly flat band forms near the so-called magic angle due to the misalignment between the two monolayer graphene layers. The theoretical foundation for flat band physics was established decades earlier through

investigations of kagome [27], Lieb [28], and dice (or T3) lattice [29,30]. The dice lattice, characterized by its hexagonal structure with an additional central site in each hexagon, presents a particularly intriguing platform. This lattice configuration gives rise to three energy bands, including Dirac points intersected by a flat band [31], enabling the description of low-energy excitations in terms of spin-1 fermions—a feature without direct analogs in high-energy physics. The presence of the flat band leads to distinctive physical properties [31,32], such as a paramagnetic response contrasting with graphene's diamagnetism [33].

The exploration of quantum phases arising from engineered lattice geometries continues to reshape our understanding of topological matter. Recent advances in van der Waals heterostructures have demonstrated that stacking orientation and interlayer coupling can fundamentally reconstruct electronic spectra, as evidenced by correlated insulating states in twisted bilayer graphene and Hofstadter butterfly patterns in moiré superlattices [34]. These developments highlight a critical synergy between symmetry-driven band engineering and emergent quantum phenomena. Of particular interest are systems hosting coexisting Dirac cones and dispersionless flat bands, where geometric frustration and kinetic energy quenching create fertile ground for unconventional transport and many-body effects [35]. Within this context, the dice lattice emerges as a prototypical platform, combining hexagonal symmetry with a tripartite sublattice structure to generate spin-1 fermionic excitations—a feature that challenges conventional paradigms of Dirac physics [36].

While monolayer dice lattices have been extensively studied for their anomalous quantum Hall signatures and magnetic responses, the extension to multilayer architectures remains largely uncharted territory. The introduction of interlayer degrees of freedom in bilayer systems introduces dimensions for band structure manipulation, where competing effects of hybridization, symmetry breaking, and moiré potential could give rise to unprecedented topological phases. Previous

*Contact author: hanlinl@seu.edu.cn†Contact author: jwang@seu.edu.cn

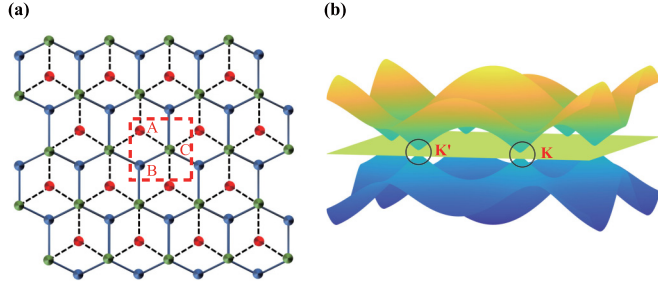


FIG. 1. (a) Schematic of monolayer dice lattices with a primitive cell containing ABC atoms. The BC atoms constitute the basic honeycomb lattice with the A atom residing in the center of each hexagon. (b) The band structure of the monolayer dice lattices. Only one orbital degree of freedom is active in each atom, and there is a nondispersion flat band intersecting the Dirac bands at the band center.

studies on graphene bilayers have established that stacking configuration governs critical properties, such as Berry curvature distribution and Landau level degeneracy [37], yet analogous investigations for dice lattice are conspicuously absent. Our work addresses this gap by systematically analyzing two fundamental stacking modes, AA-BB-CC and AB-BC-CA stacking configurations, which exhibit starkly distinct interlayer coupling symmetries [19]. It is found that, due to the presence of the flat bands on the energy levels of $E = \pm 1t$, the bilayer dice lattices in AA-BB-CC stacking exhibit mixed quantum Hall states, and their flat bands will be more easily destroyed. In contrast, the bilayer dice lattices in AB-BC-CA stacking fail to exhibit the integer quantum Hall effect. However, when hopping parameter α and interlayer coupling strength $t_{\perp}$ are changed, the Hall conductivity σ_{xy} of AA-BB-CC stacking will produce some strange properties, and even mixed Hall states will be produced under some α values.

This work is organized as follows. In Sec. II, we introduce the bilayer dice lattice model and methods to calculate Hall conductivity σ_{xy} . In Sec. III, the calculation results of Hall conductivity and flat bands of bilayer dice lattices in AA-BB-CC stacking are given. In Sec. IV, we give the results of Hall conductivity of bilayer dice lattices in AB-BC-CA stacking. In Sec. V, the relationship between Hall conductivity and energy band is discussed. Finally, a brief conclusion is drawn in the last section.

II. MODEL AND METHOD

We start with the monolayer dice lattices; their structure is illustrated in Fig. 1(a), where each unit cell consists of three atomic sites A, B, and C. Edge sites B and C form a honeycomb lattice, while an additional hub site A is located at the center of each hexagon. This unique configuration gives rise to pseudospin-1 Dirac fermions and generates a dispersionless flat band intersecting with two Dirac points at K and K' points in the Brillouin zone, as shown in Fig. 1(b). The specific layer arrangement restricts electron hopping between the A and C sites as well as between the B and C sites, while prohibiting direct hopping between A and B sites. The hopping energy

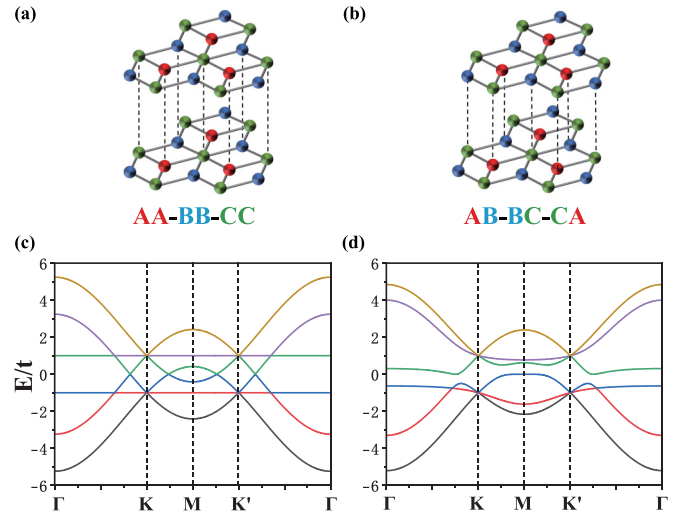


FIG. 2. (a) Schematic diagram of bilayer dice lattices in the aligned AA-BB-CC stacking, where red atom represents atom A, blue atom represents atom B, and green atom represents atom C. (b) Schematic diagram of bilayer dice lattices in the cyclic AB-BC-CA stacking. (c) The energy band of the bilayer dice lattices in the aligned AA-BB-CC stacking. (d) The energy band of the bilayer dice lattices in the cyclic AB-BC-CA stacking.

between the B and C sites is characterized by αt , where t represents the lattice hopping energy. The parameter α typically ranges from 0 to 1, and the system is referred to as the dice lattice when $\alpha = 1$.

For bilayer dice lattices, we primarily investigate two stacking modes that exhibit the most significant numerical differences in their electronic properties: the aligned AA-BB-CC stacking and the cyclic AB-BC-CA stacking configurations [38], as illustrated in Figs. 2(a) and 2(b), respectively. These particular stacking configurations were selected on the basis of their markedly distinct interlayer coupling mechanisms and resulting band structure modifications. As illustrated in Fig. 2(a), the most straightforward bilayer stacking configuration is achieved by aligning identical sublattices between the two layers, which we consequently designate as aligned AA-BB-CC stacking. Notably, we observe that the A and B sublattices exhibit distinct connectivity characteristics compared to the C sublattice. Through rotational or translational manipulation of one layer, we can establish a configuration in which the A, B, and C sublattices of the upper layer align with the C, A, and B sublattices of the lower layer, respectively. This particular arrangement is termed cyclic AB-BC-CA stacking, as shown in Fig. 2(b). The corresponding band structures for both stacking configurations are presented in Figs. 2(c) and 2(d). Interestingly, the AA-BB-CC stacking exhibits two prominent flat dispersionless bands at energy levels of $E = \pm 1t$, suggesting strong localization effects and potential correlation phenomena, and the two Dirac points that were originally located at K and K' points in the monolayer dice lattices split into four Dirac points on either side of K and K' points. However, for the AB-BC-CA stacking, no such flat bands are generated, and the bands are more complex overlapping around $E = \pm 1t$.

The tight-binding Hamiltonian governing the electronic behavior in the bilayer dice lattices can be expressed as [39]

$$H = \sum_{m \in l, u} \left[\sum_{(i, j)} t_{ij} e^{ia_{ij}^m} C_i^{m\dagger} C_j^m + \text{H.c.} \right] + \left[\sum_{(p, q)} t_{\perp p, q} C_p^{\dagger} C_q^u + \text{H.c.} \right] + \left[\sum_i w_i C_i^{\dagger} C_i \right], \quad (1)$$

where m is the layer index, l and u represent the bottom layer and the top layer, respectively, and $C_i^{m\dagger}$ and C_i^m are the creation and annihilation operators of i lattice point in the m th layer, respectively. We define the lattice hopping energy as $t = -2.7$ eV, with the interlayer coupling strength set to $t_{\perp} = 1t$. The on-site energy at all lattice sites is configured to be zero. A uniform perpendicular magnetic field $\mathbf{B}$ is applied to the lattice plane, which can be incorporated into the Hamiltonian through the Peierls substitution. The Peierls phase factor $a_{ij} = \int_i^j e\mathbf{A} \cdot d\mathbf{r}/\hbar$ can be obtained from the formula $\nabla \times \mathbf{A} = \mathbf{B}$ with $\mathbf{B}(0, 0, 1)$, and the magnetic flux $\phi = \Sigma a_{ij} = 2\pi/M$ in each hexagon is proportional to the magnetic field $\mathbf{B}$, where M is assumed to be any integer in principle. In this calculation, we take $\phi = 2\pi/48$. The last term w_i is the random disorder potential uniformly distributed in the interval $w_i \in [-W/2, W/2]$, with W being the disorder strength. We neglect the spin splitting induced by the magnetic field; therefore, based on the assumption of spin degeneracy, all subsequent results should be multiplied by a factor of 2 to account for the spin degree of freedom. We consider a finite-size system $N_s = L_x \times L_y$ with periodic conditions in the lattice plane to calculate the quantum Hall effect [40,41]. The Hall conductivity at 0 K can be calculated by the Kubo formula

$$\sigma_{xy} = \frac{ie^2\hbar}{S} \sum_{\alpha\beta} \frac{\langle \psi_{\alpha} | v_x | \psi_{\beta} \rangle \langle \psi_{\beta} | v_y | \psi_{\alpha} \rangle - \text{H.c.}}{(\varepsilon_{\alpha} - \varepsilon_{\beta})^2}, \quad (2)$$

where the eigenstate ψ_{α} and eigenvalue ε_{α} of the system are obtained from Eq. (1), S is the region of the sample, and v_x and v_y are velocity operators in the X and Y directions, given by

$$v = \frac{1}{i\hbar} [r, H], \quad (3)$$

where r is the operator representation of the position of the electron in the lattice. The Hall conductivity exhibits size independence, remaining unaffected by changes in the system dimensions. In this paper, the system size $N_s = 96 \times 96 \times 2$ and consists of a two-dimensional lattice with periodic boundary conditions. Periodic boundary conditions were applied to all boundaries, meaning the left and right boundaries along the x direction are connected, and the top and bottom boundaries along the y direction are connected. Velocity operators under the periodic boundary conditions can be calculated using the standard position operator.

III. QUANTUM HALL EFFECT OF BILAYER DICE LATTICES IN AA-BB-CC STACKING

We first discuss the quantum Hall effect in aligned AA-BB-CC stacking. Figure 3(a) presents the Hall conductivity of AA-BB-CC stacking in the low-energy region near $E_f =$

0. The monolayer dice lattices exhibit conventional quantum Hall effect of Dirac electrons, $\sigma_{xy} = 4ce^2/h$ ($c = 0, \pm 1, \pm 2, \dots$) [42], arising from valley and spin degeneracies, as shown in Fig. 3(b). The AA-BB-CC stacking system maintains the same integer filling factor $n = 4$ in the central energy region. However, the zero Hall conductivity plateau at the Dirac point ($E_f = 0$) becomes significantly narrower compared to the monolayer case. This phenomenon can be attributed to the absence of the flat band crossing the Dirac point at zero energy in the bilayer system, as the flat bands have shifted to $E = \pm 1t$, resulting in the substantial reduction of the central zero-conductivity plateau. In the monolayer α -T3 model, the flat band intersects the Dirac cones exactly at $E = 0$, resulting in a robust and wide zero Hall plateau due to the preserved three-band degeneracy. However, in the bilayer AA-BB-CC stacking, this degeneracy is lifted by interlayer coupling, which pushes the flat bands away from zero energy and splits the Dirac points.

While the monolayer dice lattices exhibit a zero Hall conductivity plateau at the flat band position ($E_f = 0$), far away from low-energy regions ($E_f = 0.75t - 2.25t$), the AA-BB-CC stacking manifests a well-defined integer quantum Hall plateau $\sigma_{xy} = \mp 50e^2/h$ at the flat band positions ($E_f = \pm 1t$). Furthermore, the side plateaus exhibit more complex behavior compared to the central region. The AA-BB-CC stacking displays a mixed state of $n = 2$ and 4 in this energy range, featuring some distinct $n = 4$ Hall steps within a specific energy range, as illustrated in Fig. 3(c), whereas the monolayer system shows edge Hall conductivity plateaus with integer filling factor $n = 2$ in Fig. 3(d). We attribute this phenomenon to several potential mechanisms: the bilayer structure introduces an additional layer degree of freedom, when combined with the existing spin and valley degrees of freedom in the monolayer, which enhances the degeneracy of certain Landau levels, resulting in the mixed $n = 2$ and $n = 4$ states. The monolayer dice lattices bands possess nonzero Chern numbers (e.g., $C = 1$), and the bilayer stacking could lead to a total Chern number of 2. The interlayer coupling effects induce further band splitting or the formation of topological states, potentially generating higher Chern numbers, which could account for the observed $n = 4$ plateaus.

In the above calculation, we examined the Hall conductivity in the absence of disorder strength ($W = 0$). In a clean limit system, the presence of the flat band minimally affects the stability of Hall conductivity, with well-defined integer quantum Hall plateaus observed at the flat band positions ($E = \pm 1t$). However, as disorder strength W increases, the originally dispersionless flat bands become disrupted, leading to fluctuations in the integer quantum Hall effect as illustrated in Fig. 4. We first calculated the results of the monolayer dice lattices for comparison; Figs. 4(a) and 4(b) depict the Hall conductivity of monolayer dice lattices with varying disorder strengths, while in the inset of Fig. 4(b) we plot the Hall conductivity at the flat band in the clean limit. The monolayer system demonstrates remarkable robustness against disorder, with only minor fluctuations observed at the flat band position ($E_f = 0$) for W ranging from $0.04t$ to $0.16t$. Even at $W = 0.16t$, the Hall conductivity σ_{xy} fluctuates within a narrow range of $0.01e^2/h$. Significant fluctuations near the flat band emerge only at $W \approx 0.5t$, with the fluctuations symmetrically

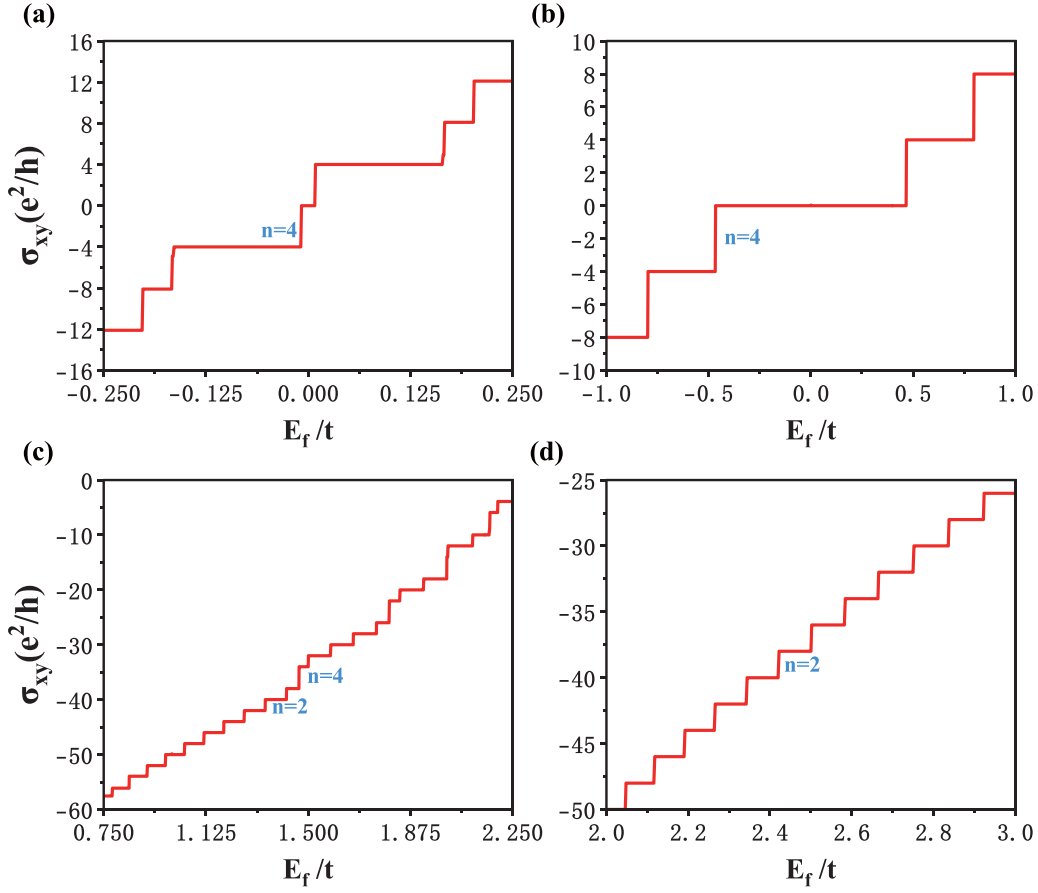


FIG. 3. Hall conductivity σ_{xy} as a function of the Fermi energy E_f for the bilayer dice lattices in AA-BB-CC stacking (a) and monolayer dice lattices (b) in the low-energy region. Hall conductivity σ_{xy} as a function of the Fermi energy for the bilayer dice lattices in AA-BB-CC stacking (c) and monolayer dice lattices (d) far away from low-energy regions. Parameters are $\phi = 2\pi/48$, $\alpha = 1$.

distributed around the Dirac point and reaching an amplitude of approximately $0.12e^2/h$ at $W = 1t$. In contrast, the AA-BB-CC stacking bilayer system exhibits dramatically different behavior under disorder, as shown in Fig. 4(c). Compared to the monolayer, the bilayer system shows substantial sensitivity to disorder, with noticeable fluctuations in σ_{xy} appearing even at $W = 0.04t$.

The fluctuation amplitude far exceeds that of the monolayer, reaching up to $50e^2/h$ at $W = 0.16t$. While the bilayer Hall conductivity displays larger fluctuations, the overall profile remains smoother without pronounced oscillations. At $W = 0.16t$, the flat band disruption becomes significant, causing the Hall conductivity to vary dramatically from $50e^2/h$ to nearly 0. This illustrates that the flat band produced by the AA-BB-CC stacking is more fragile and more easily destroyed by disorder. Unlike the monolayer's symmetric fluctuations around the Dirac point, the bilayer system develops a peaklike structure in its Hall conductivity response. This distinct behavior stems from fundamental differences in their band structures: while the monolayer features a single flat band crossing the Dirac point at K and K' , the flat bands generated by AA-BB-CC stacking are shifted to $E = \pm 1t$ and exhibit additional crossing points at other locations in the Brillouin zone, and the points are very close together, making them more susceptible to disorder. Despite the asymmetric fluctuations near the flat bands, the overall Hall conductivity

maintains symmetry about the zero-energy point, with disorder-induced peaks consistently pointing toward zero conductivity at both positive and negative Fermi energies.

IV. HALL CONDUCTIVITY OF BILAYER DICE LATTICES IN AB-BC-CA STACKING

In addition to the aligned AA-BB-CC stacking configuration, we also investigate the cyclic AB-BC-CA stacking configuration. Unlike the AA-BB-CC stacking, the AB-BC-CA stacking does not exhibit a well-defined integer quantum Hall effect, as shown in Fig. 5(a). While the high-energy regions maintain strict quantum Hall plateaus, the central energy region displays irregular Hall conductivity patterns. In the low-energy regime, several integer quantum Hall plateaus can be identified. However, their extremely narrow widths render them difficult to observe. In particular, well-defined plateaus do exist in certain energy regions where the band structure is relatively simple and less overlapped. Nevertheless, for the majority of the low-energy spectrum, the Hall plateaus remain ambiguous and challenging to observe. It is well known that increasing the magnetic field enlarges the Landau level spacing, thereby broadening the corresponding Hall plateaus. Based on this principle, we calculated the Hall conductivity of the AB-BC-CA stacking configuration under an enhanced magnetic field, as shown in Fig. 5(b). Even when

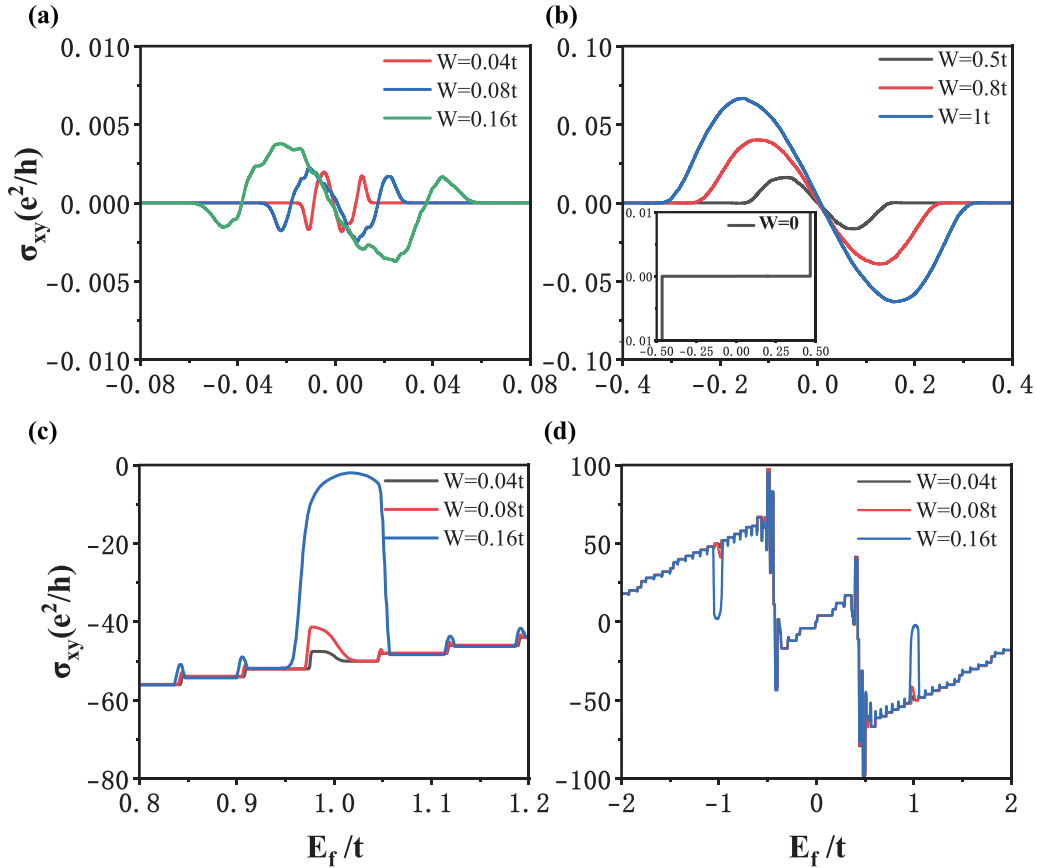


FIG. 4. Hall conductivity σ_{xy} vs the Fermi energy E_f for the monolayer dice lattices with different disorder strengths in panels (a) and (b). The results of the bilayer dice lattices in AA-BB-CC stacking are plotted in panels (c) and (d), $\alpha = 1$.

the magnetic field is increased in an attempt to broaden the plateaus, clear observation and distinction of these features remain difficult. This phenomenon stems from the more complex band structure of AB-BC-CA stacking near $E = \pm 1t$, as shown in Fig. 2(d). The AB-BC-CA stacking produces multiple energy band systems in the low-energy region. Due to the complexity of the band structure and the dense distribution of Landau levels, the system fails to form well-defined plateaus, resulting in an irregular overall pattern. As energy increases,

the band structure becomes simpler and more well defined, allowing the system to recover conventional integer quantum Hall behavior.

V. EFFECT OF α STRENGTH AND INTERLAYER COUPLING STRENGTH $t_\perp$

The distinct transport properties of AB-BC-CA stacking can be attributed to its unique interlayer coupling geometry,

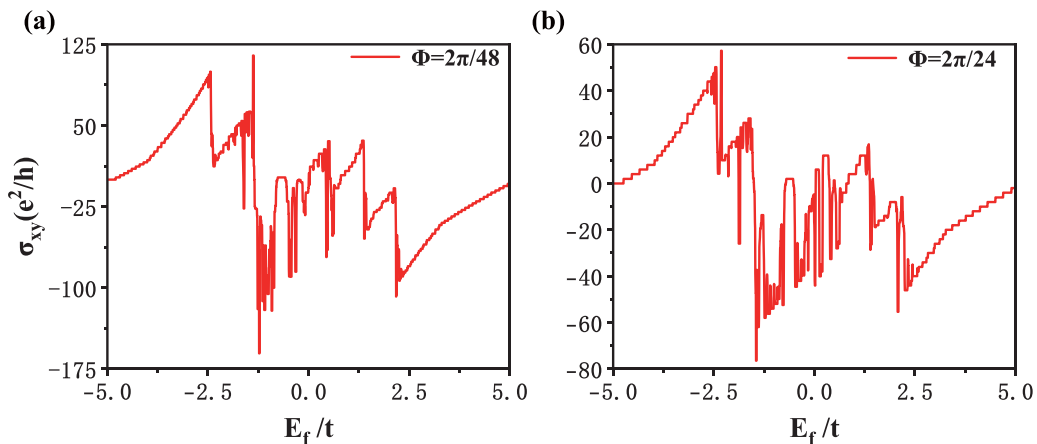


FIG. 5. (a), (b) The relationship between Hall conductivity σ_{xy} and energy of the cyclic AB-BC-CA stacking for different magnetic flux ϕ (ϕ is described in each panel, $\alpha = 1$).

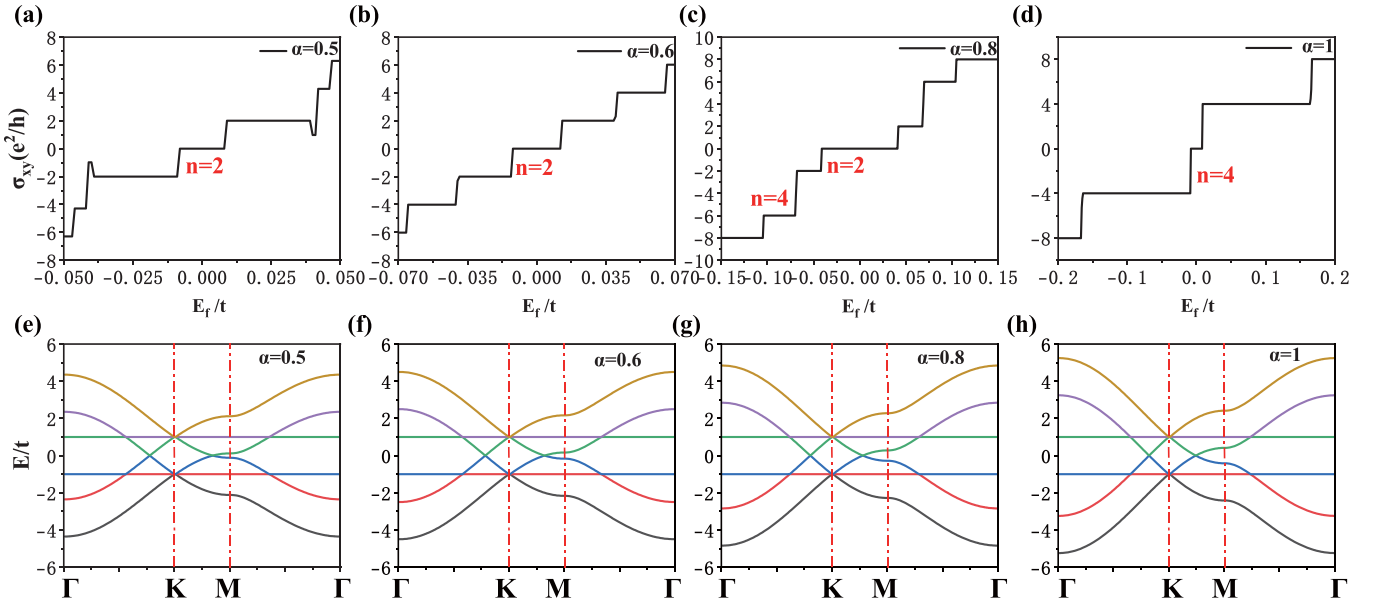


FIG. 6. (a)–(d) The relationship between Hall conductivity and α change in aligned AA-BB-CC stacking. (e)–(h) The relationship between energy bands and α change in aligned AA-BB-CC stacking.

which disrupts the formation of well-defined flat bands and introduces additional band complexities near the Dirac points. This contrasts sharply with the AA-BB-CC stacking, where the symmetric interlayer coupling preserves the flat band characteristics and maintains robust integer quantum Hall plateaus throughout the energy spectrum. The above calculation shows that the flat bands produced by monolayer and aligned AA-BB-CC stacking have different responses to disorder, and the flat bands produced by AA-BB-CC stacking are more fragile and easily destroyed. In addition to the flat bands mentioned above, there is also the property of extremely narrow zero platforms. In the monolayer dice lattices, the change of α intensity will greatly affect the formation and value of Hall conductivity [42]. In order to further reveal the unique phenomena and principles generated by AA-BB-CC stacking, we investigate the Hall conductivity plateau in AA-BB-CC stacking, observed at zero disorder strength, and whether it can be changed by other factors.

As illustrated in Fig. 6(a), the Hall conductivity near zero energy demonstrates remarkable evolution with changing α values. At lower α values (e.g., $\alpha = 0.5, 0.6$), the central region develops a well-defined integer quantum Hall plateau with $n = 2$. However, as α increases to 0.8, a mixed state emerges in the central region, characterized by transitions between $n = 2$ and $n = 4$, resulting in a Hall conductivity sequence of 0-2-6-8. This indicates a transition in the central energy region from a pure $n = 2$ state to a mixed $n = 2$ and $n = 4$ configuration. Ultimately, at $\alpha = 1$, all Hall plateaus transition to integer multiples of $n = 4$, eliminating the mixed state behavior. It comes from that at lower α , due to the addition of strong magnetic field, the valley degeneracy breaks down, resulting in $n=2$. As α increases, the valley degeneracy gradually emerges, so n gradually changes from 2 to 4.

Figures 6(e)–6(h) present the band structure of aligned AA-BB-CC stacking in the absence of magnetic field. Notably, the energy band from Γ to K point remains relatively

unchanged with varying α . In the band structure plot of Figs. 6(e)–6(h), some changes occur in the band structure within the region between the K and M points. As α increases, the Dirac point between K and M points gradually shifts toward the K point. This suggests a correlation between the Hall plateau values and the electronic properties in this region. When the Dirac point is closer to M , the system tends to exhibit $n = 2$ Hall plateaus, indicating non-Dirac electronic behavior. It can be observed that as the second Dirac point approaches the M point, the shape of the corresponding energy valley becomes increasingly irregular, leading to a reduced valley degeneracy. As the Dirac point moves away from M point toward K point, the shape of the energy valley located between points K and M gradually becomes clear, thereby increasing the valley degeneracy. A mixed state emerges, characterized by both $n = 2$ and $n = 4$ plateaus, reflecting a coexistence of Dirac and non-Dirac electronic properties. Finally, at $\alpha = 1$, when the Dirac point approaches K , the system transitions to pure $n = 4$ Hall plateaus, suggesting complete dominance of Dirac-like electronic behavior.

In addition to varying the α parameter, modifying the interlayer coupling strength $t_{\perp}$ induces similar effects on the quantum Hall response. As shown in Fig. 7(a), as $t_{\perp}$ increases from $0.5t$ to $1t$, the width of the central zero Hall conductivity plateau initially narrows and then expands, while maintaining strict integer quantum Hall plateaus of $n = 4$. With further increase in $t_{\perp}$, significant changes occur near zero energy: at $t_{\perp} = 1.3t$ and $1.5t$, the central region exhibits Hall plateau inversion, which disappears at $t_{\perp} = 2t$, where all plateaus transition to integer multiples of 2.

Figures 7(e)–7(h) present the zero magnetic field band structures at different interlayer coupling strengths. Analysis reveals that for $t_{\perp} \leq 1t$, the right Dirac point resides to the left of the M point. As $t_{\perp}$ increases, this crossing point progressively shifts rightward, passing through and eventually moving beyond the M point at $t_{\perp} = 1.3t$ and $1.5t$. When

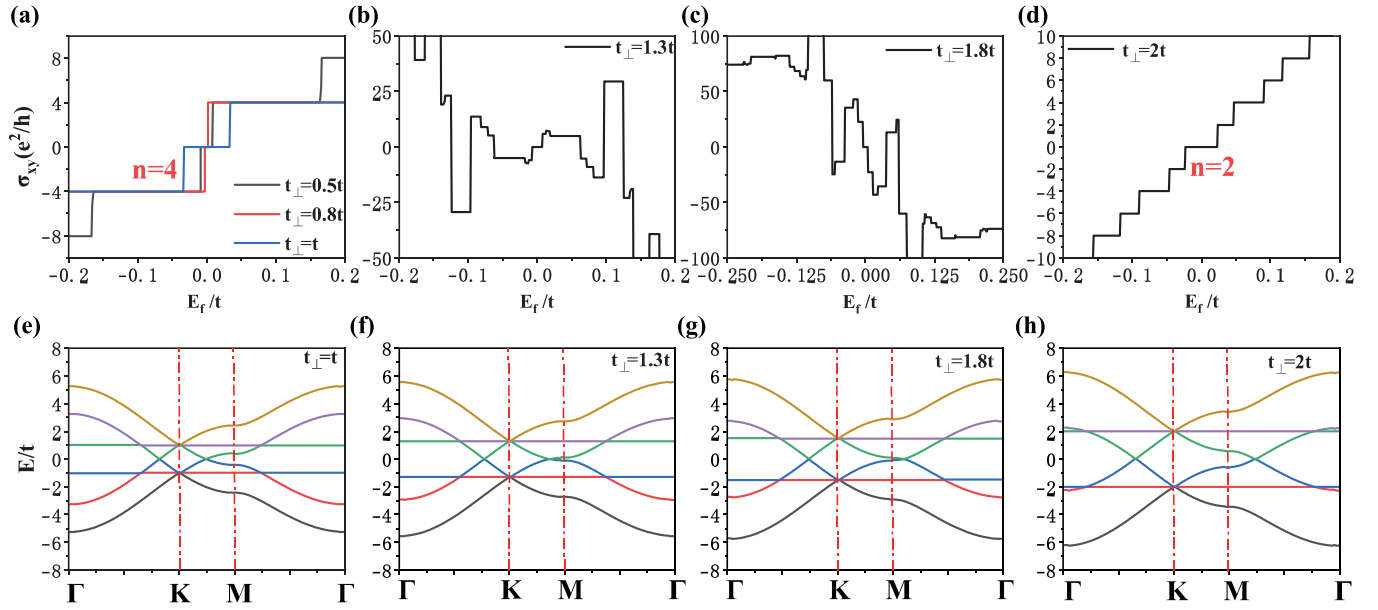


FIG. 7. (a)–(d) The relationship between Hall conductivity of aligned AA-BB-CC stacking and the change of coupling strength $t_\perp$. (e)–(h) The energy bands of aligned AA-BB-CC stacking and the changes of coupling strength $t_\perp$, $\alpha = 1$.

the Dirac point approaches the M point, the zero-energy region develops complex Hall plateau transitions reminiscent of conduction-valence band inversions. However, once the Dirac point moves to the right of M point, the system enters into a pure $n = 2$ quantum Hall state. These observations also confirm that the three-band-crossing point distinctly present in the monolayer system at $E = 0$ is absent in the bilayer when interlayer coupling is present. As shown in Figs. 7(e)–7(h), the interlayer coupling $t_\perp$ modifies the band topology near the original Dirac points, breaking the degeneracy and pushing flat bands to higher energies. The resulting evolution of Hall plateaus—from a broadened zero plateau in the monolayer to split or narrow plateaus in the bilayer—demonstrates that interlayer coupling plays an important role in removing the three-band degeneracy.

This behavior can be understood through the evolution of the band topology. Specifically, the position of the second Dirac point relative to the M point serves as a critical indicator of the system's topological phase. The plateau inversion observed during the transition period ($t_\perp = 1.3t$ to $1.5t$) suggests the formation of intermediate topological states, while the final $n = 2$ state at $t_\perp = 2t$ indicates a complete topological phase transition.

The validity of our conclusions extends to the regime where $\alpha < 0.5$, as demonstrated in Figs. 8(a)–8(d). At $\alpha = 0$, we observe significant Hall conductivity transitions at zero energy, resulting from conduction-valence band inversions. The bilayer dice lattice produces valley splits above and below the Dirac point, thus splitting two positive and negative platforms with opposite signs at Fermi energy 0. In the previous study, we found through the analytical solution of the energy levels that there are two energy levels with opposite upper and lower symbols above and below the Dirac point, which causes the splitting into positive and negative platforms at the Dirac point [42]. In this configuration, the right Dirac point is precisely located at the M point, exhibiting effects analogous to those

previously discussed. As α increases, this Dirac point systematically shifts toward the K point, facilitating the formation of the zero-conductivity plateau and reproducing the effects shown in Fig. 6(a). These observations enable us to conclude that the position of the non-Dirac electronic crossing point in the right half of the Brillouin zone plays a crucial role in determining the low-energy Hall plateaus and quantum Hall response.

Specifically, when the right Dirac point resides near the M point, the system exhibits characteristics typical of non-Dirac electronic behavior, manifested through distinct integer quantum Hall plateaus. As the crossing point shifts toward the K point, the system undergoes a gradual transition, eventually displaying pure Dirac-like behavior. This position-dependent response highlights the intricate relationship between the band structure topology and quantum transport properties in the bilayer dice lattice system.

VI. SUMMARY

In summary, we have investigated the quantum Hall effect in bilayer dice lattices, with a focus on the aligned AA-BB-CC stacking and the cyclic AB-BC-CA stacking structures. The AA-BB-CC stacking demonstrates unique transport characteristics, including a narrow zero Hall conductivity plateau and mixed integer quantum Hall states of $n = 2$ and $n = 4$, attributed to the interplay of spin, valley, and layer degrees of freedom. In contrast, the AB-BC-CA stacking exhibits irregular Hall conductivity owing to its complex band structure, which arises from the breaking of the originally dispersionless band by strong or effective coupling. The study also examines the impact of disorder, interlayer coupling strength, and parameter variations on the quantum Hall response. Flat bands produced by the aligned AA-BB-CC stacking bilayer dice lattices are more unstable and more easily broken by disorder. The reduction of α will cause the breaking of

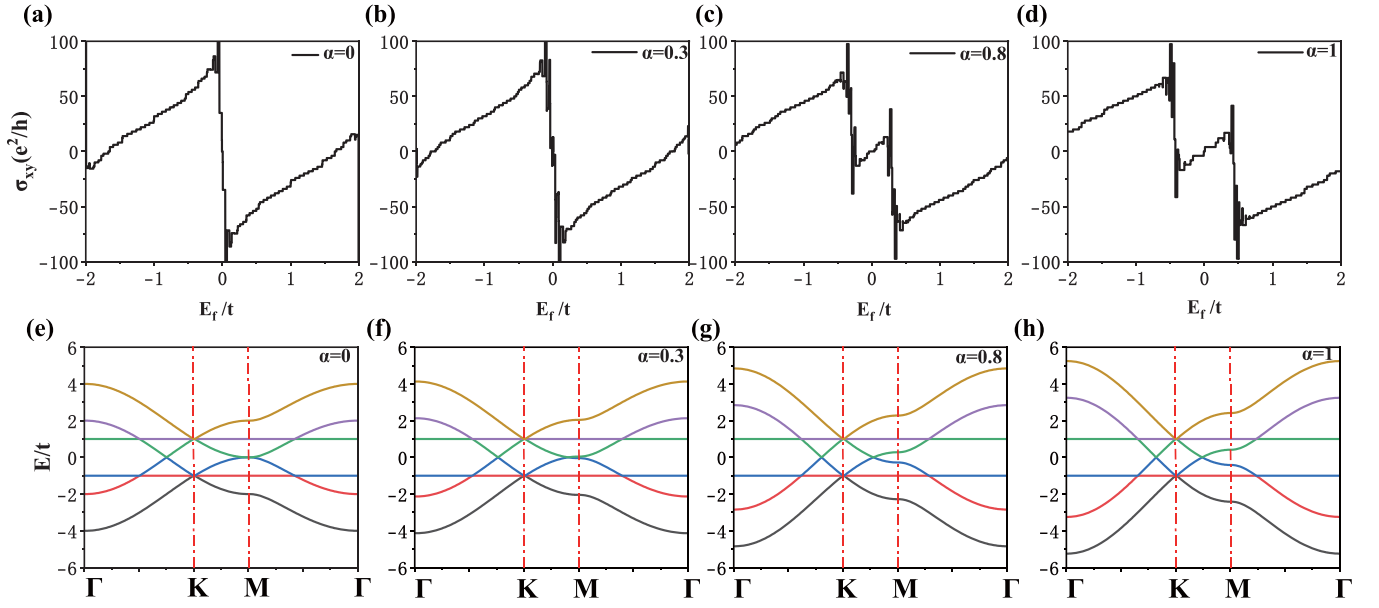


FIG. 8. (a)–(d) The relationship between Hall conductivity and α change in aligned AA-BB-CC stacking. (e)–(h) The relationship between energy bands and α change in aligned AA-BB-CC stacking.

valley degeneracy, resulting in the Hall plateaus n changing from 4 to 2, and will generate a mixed quantum Hall effect of $n = 2$ and 4 under appropriate α . Combined with the

energy band, it is revealed that the Hall conductivity σ_{xy} near zero energy can change with α and interlayer coupling strength $t_{\perp}$.

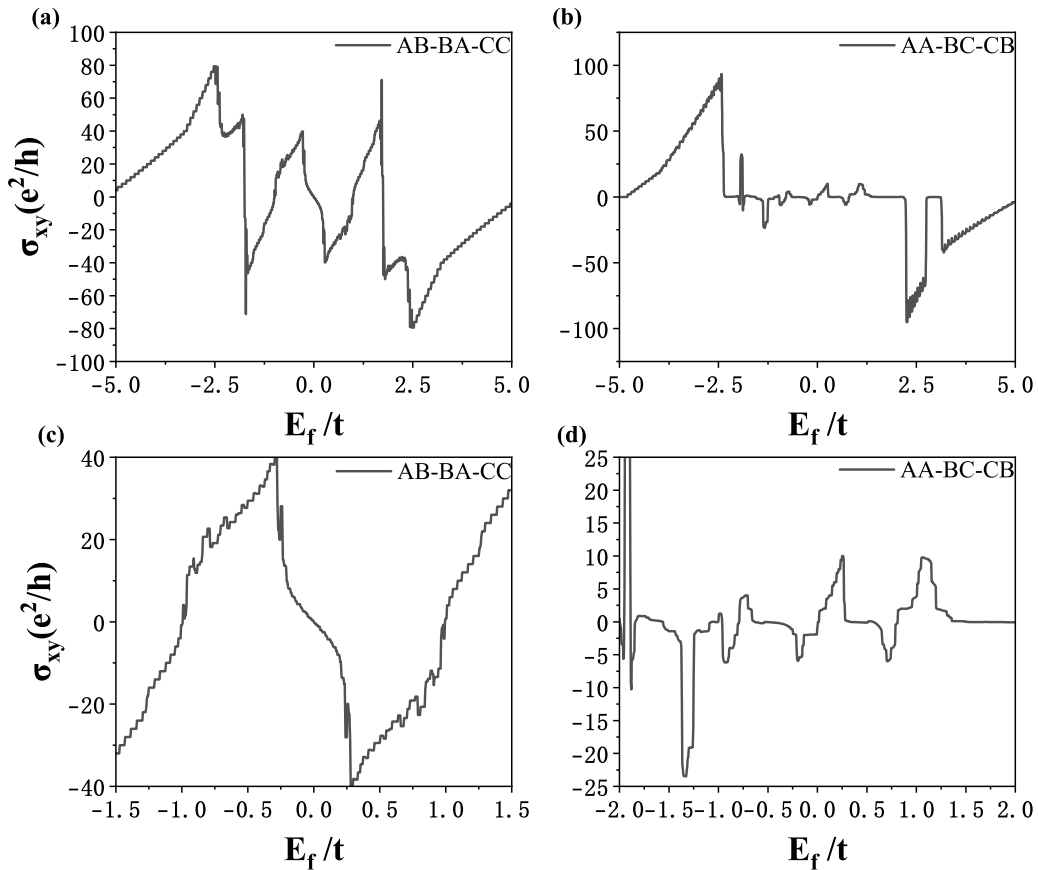


FIG. 9. Hall conductivity σ_{xy} as a function of the Fermi energy E_f for the bilayer dice lattices in AB-BA-CC stacking (a) and AA-BC-CB stacking (b) in the full-energy region. Hall conductivity σ_{xy} as a function of the Fermi energy E_f for the bilayer dice lattices in AB-BA-CC stacking (c) and AA-BC-CB stacking (d) in the low-energy region. Parameters are $\phi = 2\pi/48$ and $\alpha = 1$.

ACKNOWLEDGMENT

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: THE HALL CONDUCTIVITY OF THE BILAYER DICE LATTICES IN AB-BA-CC STACKING AND AA-BC-CB STACKING

In addition to the AA-BB-CC and AB-BC-CA stacking configurations presented in the main text, we also examined several other stacking sequences, including AB-BA-CC stacking and AA-BC-CB stacking. Although the band structures among these configurations exhibit some differences, we found that the AB-BA-CC stacking shows electronic

properties similar to those of the AA-BC-CC stacking, while the AA-BC-CB and AB-BC-CA configurations yield comparable behaviors, as shown in Fig. 9.

In particular, the Hall conductivity calculations for the AB-CA-CC and AA-BC-CB stacking (under interlayer coupling strength $t_{\perp} = 1t$) indicate that their quantized Hall responses closely resemble those obtained from the configurations discussed in the main text. Specifically, as shown in Figs. 9(a) and 9(c), the Hall conductivity of the AB-BA-CC stacking is similar to that of the AA-BB-CC stacking. However, due to the absence of a completely flat band at the Fermi level ($E = \pm 1t$), it is difficult to observe a well-defined integer quantum Hall effect in the low-energy region, particularly near the Dirac point. In the higher-energy region, a quantized Hall response analogous to that of the AA-BB-CC configuration can still emerge. For the AA-BC-CB stacking, the band structure is similar to that of the AB-BC-CA stacking—both lack flat bands and exhibit complex band overlaps. Consequently, as shown in Figs. 9(b) and 9(d), these systems generate indistinct Hall plateaus and exhibit only partial integer quantum Hall effects in specific energy ranges.

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