

Fluctuation dissipation relations for active field theories

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Breakdown of time-reversal symmetry is a defining property of nonequilibrium systems, such as active matter, which is composed of units that consume energy. We employ a formalism that allows us to derive a class of identities associated with the time-reversal transformation in nonequilibrium field theories, in the spirit of Ward-Takahashi identities. We present a generalization of the fluctuation dissipation theorem valid for active systems as a particular realization of such an identity, and consider its implications and applications for a range of active field theories. The field theoretical toolbox developed here helps to quantify the degree of nonequilibrium activity of complex systems exhibiting collective behavior.

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Introduction. Symmetry with respect to time reversal is a necessary requirement for a system to reach thermodynamic equilibrium. It gives rise to strong constraints on the dynamics, such as the fluctuation dissipation theorem (FDT) that connects the stochastic fluctuations of a thermal system at equilibrium and its response to external perturbations, which can be implemented as a supersymmetry in the field-theoretical context [1]. This powerful relation can, however, no longer be relied upon in nonequilibrium conditions corresponding to externally driven systems [2–12], as well as active matter [13], which comprises synthetic and biological systems that can produce mechanical and chemical activity [14–21]. A question of particular importance is how to quantify the degree of nonequilibrium activity in such active systems [22–27]. Possible scenarios to achieve this goal can build on strategies exploited in existing nonequilibrium generalization of the FDT [6,7,11,24,28–31], which we will hereby generically label as fluctuation dissipation relations (FDRs).

Phenomenological descriptions of active matter systems have provided us with a rich variety of active field theories, including the Toner-Tu equations for polar flocks [32], active nematics [33], field theories describing collective chemotaxis and growth [34–36], the active model B(+) [37,38], active model H [39], active field theories driven by persistent noise [40,41], and the nonreciprocal Cahn-Hilliard (NRCH) model [42,43]. These active field theories exhibit a plethora of features out of reach for equilibrium and externally driven systems, such as the existence of true long-range polar order associated with a continuous symmetry

breaking in two dimensions [32] and emergent long-range polar order in scalar systems even in two dimensions [44], which are in violation of the expectations from the Mermin-Wagner theorem, among others. Active field theories can moreover describe counterintuitive phenomena such as condensation without attractive interactions [45,46], and open biological frontiers by providing spatiotemporal control capabilities in complex metabolically active mixtures [47]. Therefore, active field theories provide a promising and rich setting for the development of techniques for quantitative studies of nonequilibrium behavior at the coarse-grained level.

In this Letter, we consider a general class of active field theories described by $\varphi_a(\mathbf{x}, t)$, representing the physical fields of the system, and $\tilde{\varphi}_a(\mathbf{x}, t)$, representing the corresponding response fields, in the context of the Martin-Siggia-Rose Janssen-De Dominicis response-field formalism [1]. Using the transformation properties of the field theories under time- and parity-reversal operation, we derive exact identities that connect the correlation and response functions of the system to expectation values of quantities that involve the stochastic entropy production of the field theory $S[\varphi]$, by following prescriptions similar to those used in deriving Ward-Takahashi identities associated with continuous symmetries.

To demonstrate applications of the Ward-like identities, we focus on two-point correlations and derive relations concerning the correlation functions, defined in real space as $C_{ab}(\mathbf{x} - \mathbf{x}', t - t') \equiv \langle \varphi_a(\mathbf{x}, t) \varphi_b(\mathbf{x}', t') \rangle$, as well as the linear response of the system to a perturbation field $h(\mathbf{x}, t)$ quantified by the susceptibility χ_{ab} , which is defined in Fourier space via $\chi_{ab}(\mathbf{q}, \omega) \equiv \frac{\delta \langle \varphi_a(\mathbf{q}, \omega) \rangle}{\delta h_b(-\mathbf{q}, -\omega)}$. We show that for an active system these quantities obey the following *exact* identities:

$$\begin{aligned} C_{ab}(\mathbf{x} - \mathbf{x}', t - t') - C_{ba}(\mathbf{x} - \mathbf{x}', t - t') \\ = \langle \varphi_a(\mathbf{x}, t) \varphi_b(\mathbf{x}', t') (e^{-S[\varphi]} - 1) \rangle \end{aligned} \quad (1)$$

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in real space, as well as

$$\begin{aligned} \chi_{ab}(\mathbf{q}, \omega) - \chi_{ab}(-\mathbf{q}, -\omega) - i\omega\beta C_{ab}(\mathbf{q}, \omega) \\ = \Gamma \langle \varphi_a(-\mathbf{q}, -\omega) i\tilde{\varphi}_b(\mathbf{q}, \omega) (e^{-S[\varphi]} - 1) \rangle \end{aligned} \quad (2)$$

and

$$\begin{aligned} \chi_{ab}(\mathbf{q}, \omega) - \chi_{ba}(-\mathbf{q}, -\omega) - i\omega\beta C_{ab}(\mathbf{q}, \omega) \\ = \frac{\Gamma^2}{i\omega\beta} \langle i\tilde{\varphi}_a(-\mathbf{q}, -\omega) i\tilde{\varphi}_b(\mathbf{q}, \omega) (e^{-S[\varphi]} - 1) \rangle \end{aligned} \quad (3)$$

in Fourier space, where Γ is the mobility of the system and $\beta = \Gamma/D$, where D is the noise strength. We observe that for equilibrium systems, where $S = 0$, Eq. (1) recalls the symmetry of the correlation functions, $C_{ab}^{\text{eq}}(\mathbf{x}, t) = C_{ba}^{\text{eq}}(\mathbf{x}, t)$, whereas Eqs. (2) and (3) recover the familiar FDT, namely, $2\text{Im}\chi_{ab}^{\text{eq}}(\mathbf{q}, \omega) = \omega\beta C_{ab}^{\text{eq}}(\mathbf{q}, \omega)$, as well as the symmetry properties of the susceptibility, $\chi_{ba}^{\text{eq}}(\mathbf{q}, \omega) = \chi_{ab}^{\text{eq}}(\mathbf{q}, \omega)$. These identities make powerful statements as they quantify the departure from equilibrium of a system due to internal driving that is a hallmark of active systems. This is in contrast to fluctuation theorems where an external force protocol drives the system [48], such as for the related identities derived in Refs. [49,50]. They provide us with a fine-grained measure of deviation from equilibrium at all frequencies and wave numbers. The formalism introduced in this Letter is applicable to a wide range of scenarios and models. We discuss some of the generalizations and avenues of further research, and in a companion paper [51] we showcase its applications in the context of the NRCH.

Deriving the identities. Consider a model governed by a general overdamped Langevin field equation

$$\partial_t \varphi_a(\mathbf{x}, t) = \Gamma K_a[\varphi](\mathbf{x}, t) + \sqrt{2D} \xi_a(\mathbf{x}, t), \quad (4)$$

where ξ_a represents unit white noise. What follows will equally hold for conserved models by mapping $\Gamma \rightarrow -\nabla^2 \Gamma$ and $D \rightarrow -\nabla^2 D$. For now, we consider Γ and D to be constants, and assume that the Einstein relation applies, namely, $\Gamma = \beta D$.

The corresponding response-field action reads

$$A[\varphi, \tilde{\varphi}] = \int_{t,x} \{ i\tilde{\varphi}_a(\partial_t \varphi_a - \Gamma K_a[\varphi]) + \tilde{\varphi}_a D \tilde{\varphi}_a \}, \quad (5)$$

where summation is implied over the index a , and a shorthand is used for integrals $\int_{t,x} \equiv \int dt \int d^d \mathbf{x}$ over all of space and time. The expectation value of a given functional of the field configurations, $\mathcal{O}[\tilde{\varphi}, \varphi]$, is

$$\langle \mathcal{O}[\tilde{\varphi}, \varphi] \rangle = \int \mathcal{D}\tilde{\varphi} \mathcal{D}\varphi \mathcal{O}[\tilde{\varphi}, \varphi] e^{-A[\tilde{\varphi}, \varphi]}. \quad (6)$$

We derive the Ward-like identity by applying the following transformation:

$$\mathcal{T} : \begin{cases} \tilde{\varphi}_a(\mathbf{x}, t) \rightarrow \tilde{\varphi}_a(-\mathbf{x}, -t) + \frac{i}{D} \partial_t \varphi_a(-\mathbf{x}, -t), \\ \varphi_a(\mathbf{x}, t) \rightarrow \varphi_a(-\mathbf{x}, -t), \end{cases} \quad (7)$$

which has been used in derivations of fluctuation and response relations in glassy [52] and nonequilibrium systems [50], and found applications in other classical and quantum nonequilibrium studies [53–56]. This transformation is an involution; i.e., $\mathcal{T}^2$ is the identity transformation, as is necessary for time reversal.

We consider active forces of the general form $K_a = -(\frac{\delta F}{\delta \varphi_a}) + W_a$, i.e., composed of a part that derives from a free energy F , as well as a nonconservative part W_a . Importantly, the fact that W_a cannot be written as a functional derivative breaks the $\mathcal{T}$ -invariance of A . The difference in action A between a forward path $(\tilde{\varphi}, \varphi)$ and the corresponding transformed path $(\mathcal{T}\tilde{\varphi}, \mathcal{T}\varphi)$ is given by the total entropy production operator

$$S[\varphi] = \beta \int_{t,x} \partial_t \varphi_a(\mathbf{x}, t) W_a[\varphi](\mathbf{x}, t). \quad (8)$$

By applying $\mathcal{T}$ to Eq. (6), we obtain (see the Supplemental Material [57] for full derivation)

$$\langle \mathcal{O}[\mathcal{T}\tilde{\varphi}, \mathcal{T}\varphi] \rangle = \langle \mathcal{O}[\tilde{\varphi}, \varphi] e^{-S[\varphi]} \rangle, \quad (9)$$

which allows us to generate an arbitrary number of Ward-like identities that connect correlations and responses to the entropy production, thereby generalizing the integral fluctuation theorem that corresponds to the specific choice of $\mathcal{O} = 1$.

To proceed further, we introduce a compact notation and define the vector $(\psi_1, \dots) = (i\tilde{\varphi}_1, \dots, \varphi_1, \dots)$ that contains both the response and physical fields. To investigate two-point functions, we choose $\mathcal{O} = \psi_i \psi_j$, and introduce the following definition [58]:

$$\langle \psi_i(\mathbf{q}, \omega) \psi_j(\mathbf{q}', \omega') (e^{-S} - 1) \rangle \equiv \Delta_{ij}(\mathbf{q}, \omega) \delta_{\mathbf{q}+\mathbf{q}'} \delta_{\omega+\omega'}. \quad (10)$$

The above expressions include the response propagator $G_{ab}(\mathbf{q}, \omega) \delta_{\mathbf{q}+\mathbf{q}'} \delta_{\omega+\omega'} = \langle \varphi_a(\mathbf{q}, \omega) i\tilde{\varphi}_b(\mathbf{q}', \omega') \rangle$ (or the Green's tensor), which is related to susceptibility by $\chi_{ab}(\mathbf{q}, \omega) = \Gamma G_{ab}(\mathbf{q}, \omega)$. Following straightforward calculations starting from Eq. (9), we obtain four matrix identities, depending on whether the ψ components are the regular fields φ or the response fields $\tilde{\varphi}$, which can be written in a matrix-block form as follows [57]:

$$\begin{bmatrix} \frac{i\omega}{D}(G_+ - G_+^\dagger - \frac{i\omega}{D}C_+) & G_+^\dagger - G_-^\dagger + \frac{i\omega}{D}C_+^\dagger \\ G_+ - G_- - \frac{i\omega}{D}C_+ & C_+ - C_- \end{bmatrix} = \Delta_{--}. \quad (11)$$

Here, the sign in the subscript denotes the sign of the arguments, e.g., $C_{\pm,ab}(\mathbf{q}, \omega) = C_{ab}(\pm\mathbf{q}, \pm\omega)$, and similar for the Green's tensor $G_{ab}(\mathbf{q}, \omega)$. We note that this notation naturally carries over to time domain, as $f(-t)$ is the Fourier transform of $f(-\omega)$. In Fourier space, changing the signs of the arguments amounts to complex conjugation. The dagger denotes Hermitian conjugate, which corresponds to changing the signs of the arguments and taking the transpose, namely, $G_\pm^\dagger \equiv (G_\pm)^\dagger = G_\mp^T$. Equations (1)–(3) can be read from the matrix elements of Eq. (11).

Active forces. The formulas quantifying the deviations (Δ) from the equilibrium FDT, namely, Eq. (11), as well as their further generalizations, are the main results of this Letter. To gain further insight and illustrate how the identities work in practice, we now consider some specific models and applications. For example, nonconservative forces materialize in the action-reaction symmetry-breaking terms of the NRCH model [42,43], which take the form $W_a = f[\varphi] \epsilon_{ab} \varphi_b$, where ϵ is the antisymmetric Levi-Civita tensor and f is a functional of φ . These forces are divergence free (in the space of fields) and cannot be written as gradients of scalar potentials, resulting in nonzero steady-state entropy production [Eq. (8)].

With standard methods from stochastic field theory, we can define an *entropy-consumption vertex* from $-S$ and calculate Eq. (10) perturbatively using Feynman diagrams. We note that such a strategy allows us to eliminate the entropy consumption vertex by combining Eqs. (1)–(3), and obtain FDRs that are valid out of equilibrium, such as

$$G_+ - C_+ C_-^{-1} G_- = \frac{i\omega}{D} C_+ \quad (12)$$

and $\frac{i\omega}{D}(G_+ - G_+^\dagger - \frac{i\omega}{D}C_+) = G_+^\dagger C_-^{-1}(C_+ - C_-)C_-^{-1}G_+$. The details of the derivation are presented in a companion paper [51].

Models with a single scalar field cannot exhibit nonreciprocal interactions, but the gradient structure may break time-reversal symmetry. Examples of such field theories are Kardar-Parisi-Zhang (KPZ) equation [59] and active model B [37], for which the active forces are given by $W = \frac{1}{2}\lambda[\nabla\varphi(\mathbf{x}, t)]^2$, whereas nonlocal generalizations of this term appear in field theories describing collective chemotaxis and growth [34–36]. In the case of the KPZ, the full force term is $K = v\nabla^2\varphi + \frac{1}{2}\lambda(\nabla\varphi)^2$, which is not conserved. The entropy consumption vertex is (see the Supplemental Material [57] for details)

$$\text{---} \text{---} \text{---} \text{---} \equiv -i\omega_3\beta\lambda \mathbf{q}_1 \cdot \mathbf{q}_2, \quad (13)$$

which can be used to perform a one-loop calculation, which yields

$$G_+ - G_- - \frac{i\omega}{D}C_+ = \frac{D}{i\omega}G\Sigma_{22}^*G = -\frac{2i\omega\delta D/D}{\omega^2 + v^2q^4}, \quad (14)$$

where $\delta D \propto \int_k 1/k^2$ is the correction to D and Σ_{22} is the amputated version of Δ_{11} [60]. In one dimension, the KPZ equation has a Gaussian steady state and consequently obeys an FDR (see below) [59,61].

The map $\Gamma \rightarrow -\nabla^2\Gamma$ and $D \rightarrow -\nabla^2D$ yields the conserved KPZ [62]. This, however, breaks the Galilean symmetry that underlies the KPZ universality class [63], opening up the possibility of adding further symmetry-breaking terms to the equation of motion. Active model B adds a “mass-term” $\frac{1}{2}r\varphi^2$ to F [37,63]. This yields a different correction $\delta D'$, which will still satisfy the FDR as $\Sigma_{22} \propto \delta D'$. Due to the conservation law, this correction turns out to be irrelevant (in the RG sense) to all orders in the perturbation theory. This approach can be extended with further terms, as is done for the active model B+ [38].

Colored noise and multiplicative noise. The nonintuitive form of $\mathcal{T}$, Eq. (7), is designed to leave the dissipative part of the response-field action, $\int_{t,x} \tilde{\varphi}(D\tilde{\varphi} + i\partial_t\varphi)$, invariant. While we have considered only its simplest form, we note that the noise strength D may be generalized to have a nonlocal kernel, $\gamma(t, t')$, or to be multiplicative and depend on φ , in which case there are subtleties associated with the time discretization in the path-integral formulation [64]. Such generalizations have been investigated in equilibrium cases, where these terms obey the Einstein relation [50].

Multiplicative noise can be incorporated in the response-field formalism by introducing Grassmann variables to handle the resulting Jacobian, in which case $\mathcal{T}$ symmetry includes

these new anticommuting fields [50,65]. As we will show next, the $\mathcal{T}$ transformation can be readily generalized to deal with field-dependent noise strength.

When the Einstein relation is not obeyed, there exists some freedom in how to define $\mathcal{T}$. We can write $\mathcal{T} = \mathcal{M}\mathfrak{T}$, where we defined the operator $\mathfrak{T}$ via $\mathfrak{T}f(\mathbf{x}, t) = f(-\mathbf{x}, -t)$, while $\mathcal{M}$ is a matrix defined as follows:

$$\mathcal{M}_{ij}\psi_j = \begin{pmatrix} \delta_{ab} & -\mathbf{m}_{ab}\partial_t \\ 0 & \delta_{ab} \end{pmatrix} \begin{pmatrix} i\tilde{\varphi}_b \\ \varphi_b \end{pmatrix}. \quad (15)$$

Using this definition, we obtain the second term of the transformed response field in Eq. (7) by setting $\mathbf{m} = D^{-1}I = \beta\Gamma^{-1}I$. If the Einstein relation is broken, we are afforded freedom to choose one of these two transformations. Importantly, we have $\mathcal{M}\mathfrak{T} = \mathfrak{T}\mathcal{M}^{-1}$, $\mathfrak{T}^2 = I$, and $\mathcal{T}^2 = I$, and therefore, we observe that this transformation is an involution for any choice of the submatrix $\mathbf{m}$. In fact, we may even choose $\mathbf{m}$ to be a function of φ (the Jacobian will still be unity and $\mathcal{T}$ will be an involution). This means that we can readily incorporate multiplicative noise, where D depends on φ .

Within our current framework, the kernel γ can be included straightforwardly. A system driven by persistent noise characterized by a finite correlation time will feature activity. The simplest example is the active Ornstein-Uhlenbeck (AOU) model [66,67], which may be generalized to fields yielding an active scalar field theory associated with a system that features effective inertia, white noise, and a nonlinear friction term $\gamma(\varphi)\partial_t\varphi$, which breaks time-reversal symmetry. This leads to a nonzero entropy-consumption operator $\sigma \propto (\frac{\delta^3 F}{\delta\varphi^3})(\partial_t\varphi)^3$ [40,41]. The appropriate form for $\mathcal{T}$ can be obtained within the formalism introduced here by choosing $\mathbf{m} = D^{-1}\gamma(\varphi)$, which enables us to derive the corresponding FDRs.

Nonsymmetric mobilities and parity. Complex environments, such as fluids with odd viscosity [68], can give rise to more general mobilities Γ_{ab} and diffusion matrices D_{ab} . We observe that for the diffusion matrix D only the symmetric part influences the system, as it appears in a quadratic form in the action equation (5), and therefore any antisymmetric component has no physical effect [69]. This is, however, not the case for the mobility matrix Γ . Consequently, we have the possibility to break the Einstein relation and drive the system out of equilibrium via this route. For general (constant) mobilities and diffusion matrices, the two natural generalizations are

$$\mathcal{T}_I : \mathbf{m}_{ab} \equiv D_{ab}^{-1}, \quad \mathcal{T}_{II} : \mathbf{m}_{ab} \equiv \beta\Gamma_{ab}^{-1}, \quad (16)$$

denoted by type I and type II, respectively.

In addition to the effect of $\mathcal{T}$ on the fields, the parameters v that appear in the equation of motion, for example, Γ_{ab} , D , or λ , might have nontrivial time-reversal parity; i.e., some might change sign for a physical time-reversal transformation due to their microphysical origins. We denote this action by $Pv_i = \varepsilon_i v_i$ (no summation), where $\varepsilon_i = \pm 1$ is the parity of the parameter v_i . Defining

$$\mathcal{T}_P \equiv P\mathcal{M}\mathfrak{T}, \quad (17)$$

we have the full, general time-reversal transformation, which is still an involution. By generalizing the derivation of the

Ward identity [Eq. (9)] using the $\mathcal{T}_P$ transform, we obtain

$$\langle \mathcal{O}[\mathcal{T}\tilde{\varphi}, \mathcal{T}\varphi] \rangle_P = \langle \mathcal{O}[\tilde{\varphi}, \varphi] e^{-S_{\mathcal{T}_P}[\varphi]} \rangle. \quad (18)$$

$$\begin{bmatrix} i\omega(mG_{P,+} - G_{P,+}^\dagger m^T - i\omega m C_{P,+} m^T) \\ G_{P,+} - G_- - i\omega C_{P,+} m^T \end{bmatrix}$$

where the P subscript indicates that the correlations are in an ensemble with the P -transformed action, $\Delta_{\mathcal{T}_P, ij}(\mathbf{q}, \omega) \delta_{\mathbf{q}+\mathbf{q}'} \delta_{\omega+\omega'} = \langle \psi_i(\mathbf{q}, \omega) \psi_j(\mathbf{q}', \omega) (e^{-S_{\mathcal{T}_P}} - 1) \rangle$, and $S_{\mathcal{T}_P} \equiv \mathcal{T}_P A - A$.

Let us first consider the case with only even parity for the parameters. For type-I and type-II cases, we obtain

$$S_{\mathcal{T}_I} = \beta \int_{t,x} \partial_t \varphi_a (W_a + \delta \Gamma_{ab} K_b), \quad (20)$$

$$S_{\mathcal{T}_{II}} = \beta \int_{t,x} \partial_t \varphi_a (W_a + \sigma_{\Gamma, ab} i \tilde{\varphi}_b). \quad (21)$$

Here, we have defined $\delta \Gamma \equiv (\beta D)^{-1} \Gamma^A$ and $\sigma_{\Gamma} \equiv 2(\beta \Gamma^T)^{-1} \Gamma^A$, and assumed a generalized Einstein relation, $\Gamma^S = \beta D$, where the symmetric and antisymmetric parts of the mobility are given as $\Gamma^S = \frac{1}{2}(\Gamma + \Gamma^T)$ and $\Gamma^A = \frac{1}{2}(\Gamma - \Gamma^T)$.

Both choices have their advantages. If we consider time reversal in the Onsager-Machlup formalism, where the action $\mathcal{G}$ is independent of $\tilde{\varphi}$, and thus also independent of the choice of m , we obtain $\mathcal{T}\mathcal{G}[\varphi] - \mathcal{G}[\varphi] = S_{\mathcal{T}_I}[\varphi]$. On the other hand, if we choose type-II transformation, we obtain FDRs of the form of Eqs. (2) and (3), as the physical response function is related to the response propagator in the form of $\chi = G\Gamma^T$, and *not* via D . In fact, in the case where $W_a = 0$, entropy production is given by only $\partial_t \varphi_a \sigma_{\Gamma} i \tilde{\varphi}$, and causality implies that $\langle \tilde{\varphi}_a \tilde{\varphi}_b e^{-S_{\mathcal{T}_{II}}} \rangle = 0$. As a consequence, we have the *exact* generalization of the FDT as follows:

$$\chi_{ab}(\mathbf{q}, \omega) - \chi_{ab}^\dagger(\mathbf{q}, \omega) = i\omega \beta C_{ab}(\mathbf{q}, \omega). \quad (22)$$

This does not hold for the other relations, as Eqs. (1) and (2) will in general have nonvanishing right-hand sides, which can thus provide an experimental protocol for detecting odd mobilities. More specifically, if the measured susceptibility obeys Eq. (22) but *not* $\chi_+ - \chi_- = i\omega \beta C_+$, then we can conclude that the out-of-equilibrium behavior stems from odd mobilities *only*. Furthermore, exact relations such as these are of great use in analytical calculations, e.g., as a tool for constraining the flow under renormalization group [70].

Equation (1) immediately implies that the diagonal elements of C_{ab} are time reversible. This is a consequence of the translation symmetry in space and time, i.e., $\langle \varphi_a(t) \varphi_b(t') \rangle = \langle \varphi_a(t - (t + t')) \varphi_b(t' - (t + t')) \rangle$ (and similarly for space), which together yield $C_{+,aa} = C_{-,aa}^T$, where there is no summation implied. This property may, however, be lost in the case of parity transformations, as the lower right quadrant of Eq. (19) is in general nonzero also on the diagonal.

If we consider the type-I transformation, and assume that the odd mobility has odd parity, such that $P\Gamma^T = \Gamma$, the

Here, $\langle \cdot \rangle_P$ indicates averages using the action with P -transformed parameters, $PA[\psi; v_i] = A[\psi; \varepsilon_i v_i]$.

With a straightforward generalization of the above derivation, we obtain

$$\begin{bmatrix} G_{P,+}^\dagger - G_-^\dagger + i\omega(C_{P,+} m^T)^\dagger \\ C_{P,+} - C_- \end{bmatrix} = \Delta_{\mathcal{T}_P}, \quad (19)$$

entropy production is found as

$$S_{\mathcal{T}_P} = \beta \int_{t,x} \{ \partial_t \varphi_a W_a + (2D_{ab} i \tilde{\varphi}_b - \partial_t \varphi_a) \delta \Gamma_{ac} K_c \}. \quad (23)$$

While the first term is the same as before, the $\delta \Gamma$ term is different and now has a dependence on $\tilde{\varphi}$. The two terms are independently odd under $\mathcal{T}_P$.

We now address the FDR that holds in one dimension for KPZ field theory, which arises due to a combination of three factors. First, the equilibrium force, i.e., the diffusion term $v \nabla^2 \varphi$, is linear. Second, in one dimension, this equilibrium term is orthogonal to the nonequilibrium drive $\frac{1}{2} \lambda (\nabla \varphi)^2$, in the sense that the integral of their product gives a boundary term. Finally, it is customary to assume that $P\lambda = -\lambda$ and thus consider the $\mathcal{T}_P$ choice, which results in a “class B FDT” relating C and χ [61]. This differs from the relations discussed in this Letter. As a consequence, the Onsager-Machlup entropy $S_{OM} = \mathcal{T}_P \mathcal{G} - \mathcal{G}$ vanishes, since the velocity $\times$ force term $\frac{1}{2} \lambda \partial_t \varphi (\nabla \varphi)^2$ is even under $\mathcal{T}_P$. This is, however, in many instances not physical, as the KPZ is an effective theory for nonequilibrium models such as nonreciprocal active matter [44] or driven quantum systems [71], which dissipate energy in steady state. Note that the class B FDT is still valid. In the response-field formalism presented here, the entropy is the integral of $(2D i \tilde{\varphi} - \partial_t \varphi) \frac{1}{2} \lambda (\nabla \varphi)^2$, which is odd under $\mathcal{T}_P$. Our formalism thus quantifies the out-of-equilibrium nature of fluctuations in a seemingly equilibrium steady state.

Discussion. Quantifying entropy production in field theories comes with inherent subtleties associated with UV divergences (in the continuum limit) not found in the underlying microscopic theory [72]. Depending on the regularization scheme used to control the effective field theory, one will have access to a coarse-grained picture of dissipation, and not all of it. Indeed, an understanding of the microscopic physics is necessary to obtain the physical dissipation. For example, the nonequilibrium character of active matter systems such as microswimmers should be quantified via the amount of hydrodynamic dissipation, which is inherently related to the faithful implementation of momentum conservation, from the nanoscale [73,74] to the microscale [75–77]. We observe furthermore that by exploiting the existing freedoms in defining the time-reversal transformation, such as the parity of the parameters, one obtained generalizations of entropy that might not directly relate to the physical entropy of the systems. This notion has also been investigated in the context of the stochastic thermodynamics of single particles, where the choice of time-reversed “conjugate dynamics” can give rise to different results for the entropy production [48]. We note, however, that even an unphysical choice of $\mathcal{T}$ can yield useful results, as the

derived identities between observables will be valid regardless of the nature of S .

An earlier generalization of the FDT is the Harada-Sasa relation [6,7], which relates total entropy $\langle S \rangle$ to the functional trace (integration and summation over the indices) of the deviation from the equilibrium FDT, Δ , as defined in Eq. (10). This has been applied to the field theories of active model B and H [24], and may be generalized to models in the form of Eq. (4); see the Supplemental Material [57] for details. Taking the functional trace of Eqs. (1)–(3) gives similar “sum-rules” [57]. The approach laid out here can thus complement the Harada-Sasa relation, giving the relation

$$\int_{\mathbf{q}, \omega} i\omega \Delta_{\tilde{a}\tilde{a}}^*(\mathbf{q}, \omega) = \int_{\mathbf{q}, \omega} D\Delta_{\tilde{a}\tilde{a}}^*(\mathbf{q}, \omega) = -\frac{\langle S \rangle}{TL^d}, \quad (24)$$

where L^d is the volume and T the total time. Here, the tilde on the index, such as $\Delta_{\tilde{a}\tilde{a}}$, is a shorthand indicating that it ranges only over the response fields $\tilde{\varphi}_a$, while indices without tilde range over the physical fields φ_a (see also Refs. [51,57]).

Although $\langle S \rangle$ is a physically relevant quantity, in field theories it is strongly divergent in both the continuum limit and the thermodynamic limit [78]. Spectrally resolved quantities such as $\chi(\mathbf{q}, \omega)$ and $C(\mathbf{q}, \omega)$ are, in contrast, finite and easily accessible, both experimentally and theoretically. The spectral identities for Δ_{ij} , as well as other relations introduced in this Letter, give access to more fine-grained and well-defined measures of the departure from equilibrium. The framework developed in this Letter and its companion paper, Ref. [51], provides us with tools to derive exact or approximate results, and to more generally investigate the phenomenology arising from activity. We would also like to highlight that dealing with the stochastic entropy operator can present technical

challenges for field theories, and this is why we regard Eqs. (12) and (22) as presenting particularly important opportunities to probe FDRs without having to deal with such subtlety. Finally, we note that the framework can be extended to implement the weak noise limit, which has been recently employed to obtain analytical results on time-reversal symmetry breaking and entropy production in active field theories [79–81], by appropriately extending the diagrammatic expansion developed in Ref. [51].

In summary, we have introduced a framework to quantify the nonequilibrium character of field theories, by, for example, probing deviations from the FDT, and applied it to examples from the literature of active field theories. We have shown that systems with odd mobility can obey a restricted FDR for some components, while others are broken. We have also exploited the freedoms that exist in defining time-reversal transformations toward the derivation of alternative classes of FDRs. In the companion paper [51], we develop a diagrammatic formalism for perturbative field theoretical verification of the FDRs. Our framework can be readily generalized in a variety of ways to serve other types of applications in stochastic thermodynamics of fields, e.g., by putting bounds on quantities such as the antisymmetric part of the correlation function in active field theories [82].

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