

Fundamental scaling limit in critical quantum metrology

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Critical quantum metrology aims to harness critical properties near quantum phase transitions to enhance parameter estimation precision. However, critical slowing down inherently limits the achievable precision within a finite evolution time. To address this challenge, we establish a fundamental scaling limit of critical quantum metrology with respect to the total evolution time. We find that the effective winding number of the system's phase-space trajectory determines the scaling bound of quantum Fisher information. Furthermore, we demonstrate that the exponential scaling of the quantum Fisher information can be obtained, and for this, it is necessary to increase the effective winding number by the total evolution time. Based on the topological nature of the effective winding number, we explicitly derive an optimal time-dependent control whose form is, surprisingly, a simple on-off scheme. We highlight that such an exponential scaling of quantum Fisher information remains valid even without reaching the critical point and in the presence of thermal dissipation, albeit with a decreased exponent.

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I. INTRODUCTION

Quantum metrology [1,2] utilizes quantum mechanical principles to achieve sensing tasks beyond the classical limit [3], which can be applied to various tasks, such as detecting gravitational wave [4], biological imaging [5], improving radars [6], and exploring the foundations of physics [7]. A crucial step in quantum metrology is to encode the parameter to be estimated in a quantum state from which nonclassical advantages originate [8–11]. More precisely, the quantum Fisher information (QFI) [12] characterizes the performance of quantum metrology, providing the ultimate precision of parameter estimation, as given by the Cramer-Rao bound.

In conventional quantum metrology, the preparation of a quantum probe state with a large QFI has been widely studied [8,11,13–20]. An alternative direction utilizing critical phenomena in quantum systems, known as critical quantum metrology, has recently been investigated [21–40] with experimental demonstrations [41–43]. The key idea is to harness the diverging susceptibility [25,44,45], accompanied by enhanced nonclassical correlations [21,25,27], to improve metrological precision. In particular, phase transitions occurring in fully connected systems have played an important role in developing ideas for critical quantum metrology [27–34,38], as these

systems encompass a wide range of quantum many-body systems, including the Dicke [46] and the Lipkin-Meshkov-Glick (LMG) model [47], as well as finite-component systems such as the quantum Rabi model [48] and Kerr resonators [43,49]. While these systems have an effective bosonic description in the thermodynamic limit [30,48,50], the finite-component system realizations enable critical sensing using a small-scale quantum system [43,51–56].

In fully connected models, the Heisenberg scaling of the QFI can be achieved by adiabatically driving the system's ground state toward the critical point [27], where the susceptibility diverges. However, this approach suffers from a critical slowing down, as the time required to reach the ground state also diverges, limiting the QFI scaling to T^4 in terms of the total evolution time T . To address this limitation, recent studies have explored various nonadiabatic protocols such as sudden quench [28,30,31] and periodic driving [32,33], revealing that T^6 scaling and even exponential scaling can be achieved, respectively. Nevertheless, the ultimate scaling limit with respect to the total evolution time T remains unclear.

In this work, we establish a fundamental scaling limit in critical quantum metrology. To this end, we introduce the effective winding number n determined by the system's phase-space trajectory, which plays a central role in both bounding the QFI scaling and constructing optimal control. Our scaling bound provides a unified perspective on various scaling behaviors of QFI in critical metrology for fully connected many-body systems, including adiabatic [27] and nonadiabatic protocols [28,30], as well as nonmonotonic protocols [32,33]. More surprisingly, we demonstrate that the exponential scaling of the QFI is achievable by determining the optimal effective winding number for a given time T .

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We also explicitly derive an on-off protocol as an optimal protocol for achieving the fundamental QFI scaling limit, in which the effective winding number determines how many times to turn the control on and off. The optimal control undergoes a radical transition at the point where the effective winding number changes, implying its topological nature.

Finally, we highlight that the QFI scaling with total evolution time is determined solely by the effective winding number rather than the ability to drive the system close to the critical point. As a result, the exponential scaling of the QFI is achievable even without reaching the critical point and remains robust under thermal dissipation and imperfect controls. We present a concrete analysis of parameter estimation using our approach and discuss experimentally feasible parameter regimes.

II. CRITICAL QUANTUM METROLOGY WITH A FULLY CONNECTED SYSTEM

Suppose a time-dependent Hamiltonian of a quantum system,

$$\hat{H}(t) = \omega \hat{H}_0 + \hat{H}_c(t), \quad (1)$$

with $\omega = \omega_0 + \delta\omega$. Here, an unknown parameter $\delta\omega$ is encoded in $\hat{H}_0$ with a reference parameter ω_0 , and $\hat{H}_c(t)$ can be controlled by some tunable parameters. When an initial state $|\psi_0\rangle$ evolves under $\hat{H}(t)$ for total evolution time T , $\delta\omega$ can be estimated from the measurement on the final state $|\psi_T\rangle$. The ultimate precision bound of estimating $\delta\omega$, the quantum Cramér-Rao bound [12], is then given in terms of the QFI,

$$\mathcal{F}_\omega = 4[\langle \partial_\omega \psi_T | \partial_\omega \psi_T \rangle - |\langle \partial_\omega \psi_T | \psi_T \rangle|^2]_{|\omega=\omega_0}. \quad (2)$$

To optimize the performance, the QFI should be maximized for a given total evolution time T , subject to constraints in $\hat{H}_c(t)$ [37,40,57–61]. This framework incorporates an arbitrary time-dependent protocol $\hat{H}_c(t)$ for critical quantum metrology, incorporating conventional quantum metrology as a special case with $\hat{H}_c(t) = 0$.

We focus on the following bosonic Hamiltonian ($\hbar = 1$):

$$\hat{H}(t) = \omega \hat{a}^\dagger \hat{a} - \frac{\epsilon(t)}{4} (\hat{a}^\dagger + \hat{a})^2, \quad (3)$$

which provides an effective description of fully connected models, including the Dicke [50] and the LMG [30] as well as the quantum Rabi [48] and the two-photon driven Kerr [43]. Here, $\hat{a}$ and $\hat{a}^\dagger$ are the bosonic creation and annihilation operators, respectively, and a single control parameter $\epsilon(t)$ induces squeezing. We note that such an effective bosonic Hamiltonian is valid in the thermodynamic limit, where the number of spins or the spin frequency, in the unit of the oscillator frequency, tends to infinity. Such a model can be realized in various physical platforms, including superconducting circuits [43,53,56], cold atoms [62,63], and trapped ions [51,64]. In these physical systems, the parameter ω typically corresponds to detuning of the oscillators, whose precise estimation can be utilized for quantum-enhanced sensing of electric [42,55] and magnetic [41,65] fields. The critical point lies at $\epsilon = \omega$, where the energy spectrum becomes continuous with the closing energy gap. We focus on the symmetric

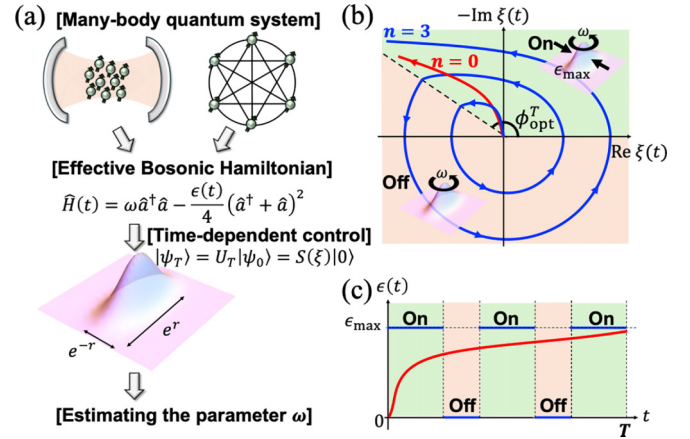


FIG. 1. (a) Schematics of critical quantum metrology and (b) phase-space trajectories with (c) monotonic (red) and on-off (blue) controls. The effective winding number n of the phase-space trajectory governs the scaling limit of QFI.

phase, $0 \leq \epsilon(t) \leq \omega$, where the order parameter remains zero prior to the onset of spontaneous symmetry breaking [48].

We assume that the system is initially prepared in the vacuum state $|\psi_0\rangle = |0\rangle$, which is the ground state of the undriven (i.e., $\epsilon = 0$) Hamiltonian $\hat{H}_0$, independent of ω . As the Hamiltonian is quadratic in $\hat{a}$ and $\hat{a}^\dagger$, the final state becomes a Gaussian state [66], represented as

$$|\psi_t\rangle = \hat{S}(\xi(t)) |0\rangle = e^{\frac{1}{2}(\xi^*(t)\hat{a}^2 - \xi(t)\hat{a}^{\dagger 2})} |0\rangle, \quad (4)$$

where $\hat{S}(\xi(t))$ is the squeeze operator with $\xi(t) = r(t)e^{-i\varphi(t)}$. From the Schrödinger equation, $\frac{d|\psi_t\rangle}{dt} = -i\hat{H}(t)|\psi_t\rangle$, the equation of motion can be written as

$$\begin{aligned} \dot{r}(t) &= \frac{\epsilon(t)}{2} \sin \varphi(t), \\ \dot{\varphi}(t) &= 2\omega - \epsilon(t)(1 - \coth 2r(t) \cos \varphi(t)). \end{aligned} \quad (5)$$

We then obtain the explicit form of QFI from Eq. (2),

$$\mathcal{F}_\omega = 2 \left| \int_0^T dt \sinh 2r(t) e^{i\theta(t)} \right|^2, \quad (6)$$

where $\theta(t) = -\int_0^t \frac{\epsilon(t') \cos \varphi(t')}{\sinh 2r(t')} dt'$ (see Appendix A for details).

III. QFI SCALING IN TERMS OF THE EFFECTIVE WINDING NUMBER

We show that the total phase accumulated during the dynamics, $\Phi = \int_0^T \dot{\varphi}(t) dt$, determines the scaling limit of the QFI. We introduce the effective winding number

$$n \equiv \left\lfloor \frac{\Phi}{2\pi} \right\rfloor, \quad (7)$$

a topological quantity that counts the number of cycles throughout the entire phase-space trajectory, independent of its detailed shape (see Fig. 1). While the winding number is generally defined for a closed trajectory, the definition in Eq. (7) offers an intuitive geometric interpretation of the phase-space trajectory, especially since the total phase

increases monotonically along the trajectory in the large-squeezing limit ($r \gg 1$). We demonstrate the polynomial scaling bound of QFI for a fixed effective winding number n as follows:

Theorem 1. The scaling of QFI with a fixed effective winding number n is bounded by

$$\mathcal{F}_\omega(T) \leq c_n T^{4n+6} + o(T^{4n+6}), \quad (8)$$

where c_n is independent of T and $o(T^{4n+6})$ grows much slower than T^{4n+6} .

An explicit form of c_n can be found in the Supplemental Material [67]. This implies that the effective winding number n determines the order of the polynomial scaling of the QFI with respect to T . Based on this, we show the following no-go theorem on any monotonic control:

Theorem 2. For any control with monotonically increasing $\epsilon(t)$, the QFI scaling cannot exceed T^6 as the effective winding number remains zero, i.e., $n = 0$.

This demonstrates that the QFI in such monotonic protocols, encompassing both adiabatic [27] and quenching protocols [28,30], is fundamentally limited to T^6 scaling, while the sudden quench protocol [28] saturates the optimal scaling.

Theorems 1 and 2 establish a fundamental limitation on the QFI scaling when the effective winding number n is fixed. Surprisingly, however, when the effective winding number n increases with the total evolution time T , this limitation can be overcome. In this case, the exponential QFI scaling is achievable as follows:

Theorem 3. The scaling bound of the QFI for a long-time limit ($T \gg 1$) is given by

$$\mathcal{F}_\omega(T) \propto e^{\Gamma \omega T}, \quad (9)$$

with $\Gamma \approx 0.9745$. This bound can be saturated by taking the effective winding number $n \approx 0.169\omega T$.

Compared to previous results utilizing periodic driving with a fixed time period [33], Theorem 3 improves the exponent by approximately a factor of 7. We also note that this represents the ultimate bound on the exponent among all possible controls on $0 \leq \epsilon(t) \leq \omega$, achievable by optimizing the effective winding number n for a given T .

IV. ON-OFF CONTROL FOR THE OPTIMAL QFI SCALING

Having established the fundamental scaling limit of the QFI, it is important to identify a time-dependent control, $\epsilon(t)$, that realizes the optimal scaling. By explicitly solving the variational calculus over $\epsilon(t)$ under the constraint on the total evolution time T , we derive the optimal protocol, which surprisingly turns out to be a simple on-off scheme based on the phase of the quantum state.

As we are interested in the scaling behavior in the long-time limit $T \gg 1$, we consider a system with sufficiently large squeezing, $|r| \gg 1$, after a finite time. In this limit, Eq. (5) becomes

$$\begin{aligned} \dot{r}(t) &= \epsilon(t) \sin(\varphi/2) \cos(\varphi/2), \\ \dot{\varphi}(t) &\approx 2\omega - 2\epsilon(t) \sin^2(\varphi/2), \end{aligned} \quad (10)$$

and $\theta(t)$ becomes constant over time. The QFI in Eq. (6) is then simplified as $\mathcal{F}_\omega \approx 2(\int_0^T dt \sinh(2r(t)))^2$, solely governed by squeezing.

We note that $\varphi(t)$ is independent of r and monotonically increasing by t as $\dot{\varphi}(t) \geq 0$. This allows us to express $\epsilon(t) = \frac{2\omega - \dot{\varphi}}{2 \sin^2(\varphi/2)}$, which leads to an integral expression of the squeezing as

$$r(T, \Phi) = \int_0^\Phi d\varphi \left(\frac{2\omega}{\dot{\varphi}} - 1 \right) \frac{\cot(\varphi/2)}{2}, \quad (11)$$

with two constraints of the total evolution time T and the total phase $\Phi = \int_0^T \dot{\varphi}(t) dt$. The control parameter's range $0 \leq \epsilon(t) \leq \omega$ yields the boundary condition $\frac{1}{2} \leq \frac{\omega}{\dot{\varphi}} \leq \frac{\sec^2(\varphi/2)}{2}$.

Let us first consider the case with a fixed effective winding number n . We show that there is a unique local maximum of $r(T, \Phi)$ within the region $n\pi \leq \Phi < (n+1)\pi$. The corresponding optimal protocol is given by the following on-off scheduling Supplemental Material [67]:

$$\frac{\epsilon}{\omega} = \begin{cases} 1 & (\cot(\varphi/2) \geq \cot(\phi_n^T/2)) \\ 0 & (\cot(\varphi/2) < \cot(\phi_n^T/2)), \end{cases} \quad (12)$$

where ϕ_n^T is determined by T and the effective winding number n . The optimal squeezing behaves as

$$r(T, \Phi_n^T) \equiv \max_{n\pi \leq \Phi \leq (n+1)\pi} r(T, \Phi) \approx (n+1) \ln(\omega T) + d_n, \quad (13)$$

with some constant d_n . Here, Φ_n^T is the total phase within $n \leq \frac{\Phi}{2\pi} < n+1$ that gives the optimal squeezing. This achieves the optimal scaling $\mathcal{F}_\omega \approx 4|\int_0^T dt \sinh^2(2r)| \propto T^{4n+6}$ in Theorem 1 in the Supplemental Material [67].

Next, we find the global optimum of squeezing $r(T, \Phi)$ for a given T , by maximizing over n as

$$r_{\max}(T) = \max_n r(T, \Phi_n^T) = r(T, \Phi_{\text{opt}}^T) \approx \frac{\Gamma}{4} \omega T, \quad (14)$$

where Φ_{opt}^T is the total phase among $\{\Phi_n^T\}$ that gives the optimal squeezing. As $r_{\max}$ linearly scales with T , $\mathcal{F}_\omega \approx 2|\int_0^T dt \sinh^2(\frac{\Gamma}{2} \omega T)| \propto e^{\Gamma \omega T}$ scales exponentially on T , achieving the optimal scaling in Theorem 3. We also note that the optimal effective winding number linearly increases as $n_{\text{opt}}^T \approx 0.169\omega T$ for a sufficiently large T . The numerical simulation in Fig. 2(a) verifies that the on-off controls yield the optimal scaling of QFI for both cases with a fixed n (Theorem 1) and with an increasing n linearly by T (Theorem 3).

Interestingly, upon increasing T , we observe a series of abrupt jumps in the optimal phase Φ_{opt}^T , accompanied by a unit increase in the optimal effective winding number. At each transition point, the maximum squeezing $r_{\max}(T)$ changes its functional form, from $\propto (n+1) \ln(\omega T)$ to $\propto (n+2) \ln(\omega T)$. This discontinuity associated with the optimal effective winding number is a consequence of a first-order phase transition in the squeezing parameter, as illustrated in Figs. 2(b) and 2(c), which reflects the underlying topological nature in the optimal phase-space trajectory.

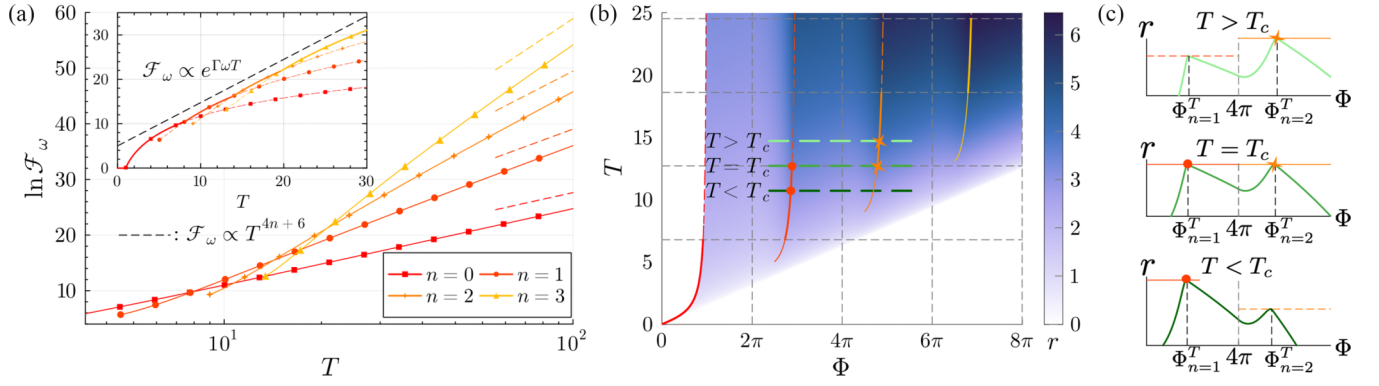


FIG. 2. (a) QFI for a given total evolution time T by adapting the on-off control, where $n = 0$ corresponds to the quenching protocol [28]. The QFI is obtained by exact simulation using Eqs. (5) and (6). The main figure verifies the T^{4n+6} scaling at $T \gg 1$. The inset figure indicates the exponential scaling of QFI by taking the maximum QFI among different effective winding numbers. (b) The optimal squeezing parameter $r(T, \Phi)$ as a function of T and the total phase Φ . The squeezing parameter has a local maximum $r(T, \Phi_n^T)$ for each $n = \lfloor \Phi/(2\pi) \rfloor$, indicated by the dashed lines. The solid line indicates the global maximum $r_{\max}(T)$ for a given T , optimized over all possible Φ . (c) The squeezing parameter and the optimal phase Φ_{opt}^T near the critical time. When T crosses the critical time T_c , the optimal phase undergoes a discontinuous jump as the optimal effective winding number changes from $n = 1$ to $n = 2$. At the critical time $T = T_c$, the optimal squeezing is given by the two effective winding numbers.

V. SCALING OF THE QFI AWAY FROM THE CRITICAL POINT

We further explore a scenario where the system cannot reach the critical point. For a fixed effective winding number, the squeezing saturates at a finite maximum value, which is in stark contrast to the critical case (Theorem 1).

Theorem 4. For $0 \leq \epsilon(t) \leq \epsilon_{\max} < \omega$ and a fixed effective winding number n , the squeezing r is upper bounded as

$$\sinh 2r(T) \leq \frac{1}{(1 - (\epsilon_{\max}/\omega))^{n+1}}, \quad (15)$$

regardless of the total evolution time T . This implies that the QFI eventually saturates to T^2 .

Nevertheless, the exponential scaling of QFI can still be achieved by allowing the effective winding number to increase with the total evolution time, which leads to the following generalization of Theorem 3:

Theorem 5. With the control parameter in the range $0 \leq \epsilon(t) \leq \epsilon_{\max}$, the fundamental scaling limit is given as

$$\mathcal{F}_\omega(T) \propto e^{\Gamma(\epsilon_{\max})\omega T}. \quad (16)$$

We highlight that $\Gamma(\omega) = \Gamma$ described in Theorem 3, and $\Gamma(\epsilon_{\max})$ smoothly decreases with decreasing $\epsilon_{\max}$ and does not vanish even when $\epsilon_{\max}$ is far from the critical point ω [see Fig. 3(a)]. This observation has interesting implications that implementing a nonmonotonic control to increase the effective winding number is a more crucial factor for enhancing QFI than driving the system near the critical point, which opens a potential to utilize a wide range of physical systems even with weak squeezing. Furthermore, the exponential scaling of QFI away from the critical point is expected to persist under finite-size effects, as the additional gap opening induced by finite size plays a role analogous to moving away from the critical point [48].

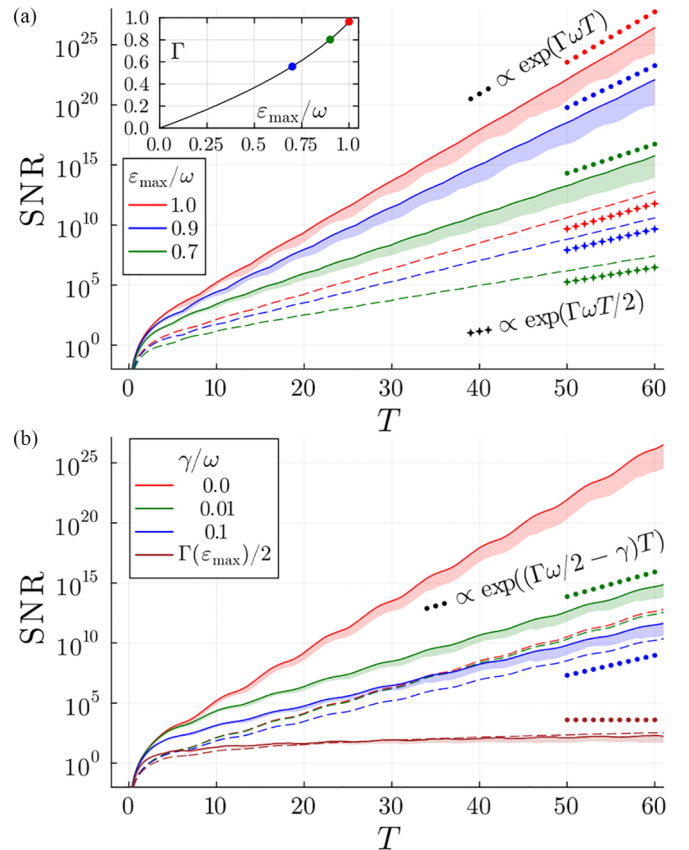


FIG. 3. Exponential scaling of SNR for (a) various maximum control parameters $\epsilon_{\max}/\omega = 0.7, 0.9$, and 1.0 and (b) dissipation rates $\gamma/\omega = 0, 10^{-2}, 10^{-1}$ and $\Gamma/2 \approx 0.487$. The inset figure describes the exponent $\Gamma(\epsilon_{\max})$. The SNR is obtained by the optimized on-off control with squared quadrature measurements, while shaded areas present the results for imperfect time scheduling and ramping (see the Supplemental Material [67] for more details). The dashed lines present the corresponding SQL.

VI. CONCRETE QUANTUM PARAMETER ESTIMATION

We discuss a concrete parameter estimation of ω by performing a measurement on the final state $|\psi_T\rangle$. More explicitly, the signal-to-noise ratio (SNR) of the estimator $\hat{O}$,

$$\text{SNR} = \frac{|\partial_\omega \langle \hat{O} \rangle|^2}{\text{Var}(\hat{O})}, \quad (17)$$

determines the sensitivity of ω as $\geq 1/\sqrt{\text{SNR}}$ [69]. We show that a squared quadrature operator [30,31] $\hat{O} = \hat{X}_\alpha^2$ with $\hat{X}_\alpha = \frac{e^{-i\alpha}\hat{a} + e^{i\alpha}\hat{a}^\dagger}{\sqrt{2}}$ saturates the optimal QFI scaling by taking a proper choice of α . We numerically verify that the exponential scaling of the SNR is robust against imperfect time scheduling and ramping of $\epsilon(t)$ in the on-off protocol [see Fig. 3(a) and Appendix B for details].

We analyze the robustness of the parameter estimation under thermal dissipation [38,39], described by the Lindblad equation,

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}(t), \hat{\rho}] + \gamma(\bar{n} + 1)\mathcal{D}_{\hat{a}}(\hat{\rho}) + \gamma\bar{n}\mathcal{D}_{\hat{a}^\dagger}(\hat{\rho}), \quad (18)$$

with the dissipation rate γ and the mean occupation number of the bath $\bar{n}$. Here, $\mathcal{D}_{\hat{A}}(\hat{\rho}) = \hat{A}\hat{\rho}\hat{A}^\dagger - \frac{1}{2}(\hat{A}^\dagger\hat{A}\hat{\rho} + \hat{\rho}\hat{A}^\dagger\hat{A})$.

Remarkably, the exponential QFI scaling persists under thermal dissipation with a decreased exponent Supplemental Material [67],

$$\mathcal{F}_\omega(T) \propto e^{(\frac{\Gamma(\epsilon_{\max})\omega}{2} - \gamma)T}, \quad (19)$$

where the SNR for the square quadrature achieves the same scaling (see the Supplemental Material [67] for more details). This implies that a sufficiently low thermal dissipation rate, $\gamma < \frac{\Gamma(\epsilon_{\max})\omega}{2}$, is the most important factor for achieving exponential scaling. Such a parameter regime is feasible in superconducting circuits, e.g., $\epsilon_{\max}/2\pi \approx 600$ kHz and $\gamma/2\pi \approx 72$ kHz, with tunable detuning ω in Ref. [43], making the proposed protocol realizable with current technology. In Fig. 3(b), we numerically verify such exponential scaling of the SNR under thermal dissipation and imperfect pulse control, which leads to quantum-enhanced sensitivity beyond the standard quantum limit (SQL).

VII. CONCLUSION

We have established a fundamental scaling limit of critical quantum metrology in fully connected systems for a given total evolution time T . The effective winding number n , capturing the topological properties of phase-space trajectories, provides a unifying principle for bounding the QFI of both polynomial scaling $\mathcal{F}_\omega(T) \propto T^{4n+6}$ for a fixed n and exponential scaling $\mathcal{F}_\omega(T) \propto e^{\Gamma(\omega T)}$ by taking n proportional to T .

We have also explicitly derived an optimal protocol, which turns out to be by a simple on-off scheduling, that saturates the fundamental scaling limit of the QFI. This provides an experimentally feasible approach to high-precision metrology through a simple dynamic control of quantum systems. We emphasize that such dynamic control becomes even more crucial in realistic scenarios with a weak squeezing away from the critical point. In such a case, any monotonic protocol is limited to quadratic QFI scaling, whereas the proposed

nonmonotonic on-off scheduling attains exponential scaling. We have also shown the robustness of the exponential scaling of QFI and SNR under imperfect pulse control and thermal dissipation, indicating that the proposed scheme could be experimentally realizable with current technology.

The theoretical framework in our work offers a potential avenue for exploring optimal control beyond critical quantum metrology, for example, enhancing squeezing [32,70] or charging quantum batteries [32,71]. An intriguing future research direction would be extending our formalism beyond the Gaussian limit and into regimes where the system enters the broken-symmetry phase [31,65].

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: DYNAMICS OF SQUEEZED STATES AND THE QUANTUM FISHER INFORMATION

1. Dynamics of fully connected system

We derive the equations of motion under the following effective time-dependent Hamiltonian of the fully connected system in Eq. (3). Let us express the quantum state at time t as $|\psi_t\rangle = \hat{U}_t|\psi_0\rangle$, where the dynamics of U_t is given by the Schrödinger equation

$$\frac{d\hat{U}_t}{dt} = -i\hat{H}(t)\hat{U}_t. \quad (A1)$$

By noting that Eq. (3) only contains quadratic terms in $\hat{a}$ and $\hat{a}^\dagger$, the evolution unitary U_t can be fully characterized by the following Gaussian unitary operations without displacement:

$$\hat{U}_t = e^{i\chi} \hat{R}\left(\frac{\varphi}{2}\right) \hat{S}(r) \hat{R}\left(\frac{\theta}{2}\right), \quad (A2)$$

where $\hat{R}(\phi) = e^{-i\phi\hat{a}^\dagger\hat{a}}$ is the rotation operator and $\hat{S}(r) = e^{\frac{r}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})}$ is the squeezing operator with a real-valued squeezing r . We note that when $\hat{U}_t$ is acting on the vacuum state $|\psi_0\rangle = |0\rangle$, the resulting state becomes a squeezed vacuum state,

$$|\psi_t\rangle = \hat{U}_t|0\rangle = \hat{S}(\xi)|0\rangle, \quad (A3)$$

where $\hat{S}(\xi) = e^{\frac{1}{2}(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2})}$ with $\xi = re^{-i\varphi}$. We note that the state is independent of θ and the overall phase χ does not have any physical significance. From Eq. (A1), we obtain

$$\begin{aligned}
\frac{d\hat{U}_t}{dt} &= \frac{de^{i\chi}}{dt} \hat{R}\left(\frac{\varphi}{2}\right) \hat{S}(r) \hat{R}\left(\frac{\theta}{2}\right) + e^{i\chi} \left[\left(\frac{d\hat{R}\left(\frac{\varphi}{2}\right)}{dt} \right) \hat{S}(r) \hat{R}\left(\frac{\theta}{2}\right) + \hat{R}\left(\frac{\varphi}{2}\right) \left(\frac{d\hat{S}(r)}{dt} \right) \hat{R}\left(\frac{\theta}{2}\right) + \hat{R}\left(\frac{\varphi}{2}\right) \hat{S}(r) \left(\frac{d\hat{R}\left(\frac{\theta}{2}\right)}{dt} \right) \right] \\
&= -\frac{i}{2} e^{i\chi} \left[-2\dot{\chi} + \dot{\varphi}(\hat{a}^\dagger \hat{a}) \hat{R}\left(\frac{\varphi}{2}\right) \hat{S}(r) \hat{R}\left(\frac{\theta}{2}\right) + i\dot{r} \hat{R}\left(\frac{\varphi}{2}\right) (\hat{a}^2 - \hat{a}^{\dagger 2}) \hat{S}(r) \hat{R}\left(\frac{\theta}{2}\right) + \dot{\theta} \hat{R}\left(\frac{\varphi}{2}\right) \hat{S}(r) \hat{R}\left(\frac{\theta}{2}\right) (\hat{a}^\dagger \hat{a}) \right] \\
&= -\frac{i}{2} e^{i\chi} \left[-2\dot{\chi} + \dot{\varphi}(\hat{a}^\dagger \hat{a}) + i\dot{r} \hat{R}\left(\frac{\varphi}{2}\right) (\hat{a}^2 - \hat{a}^{\dagger 2}) \hat{R}^\dagger\left(\frac{\varphi}{2}\right) + \dot{\theta} \hat{R}\left(\frac{\varphi}{2}\right) \hat{S}(r) (\hat{a}^\dagger \hat{a}) \hat{S}^\dagger(r) \hat{R}^\dagger\left(\frac{\varphi}{2}\right) \right] \hat{R}\left(\frac{\varphi}{2}\right) \hat{S}(r) \hat{R}\left(\frac{\theta}{2}\right) \\
&= -\frac{i}{2} e^{i\chi} \left[-2\dot{\chi} + \dot{\varphi}(\hat{a}^\dagger \hat{a}) + i\dot{r} \hat{R}\left(\frac{\varphi}{2}\right) (\hat{a}^2 - \hat{a}^{\dagger 2}) \hat{R}^\dagger\left(\frac{\varphi}{2}\right) + \dot{\theta} \hat{R}\left(\frac{\varphi}{2}\right) \hat{S}(r) (\hat{a} \hat{a}^\dagger) \hat{S}^\dagger(r) \hat{R}^\dagger\left(\frac{\varphi}{2}\right) \right] \hat{U}_t \\
&= (-i) \hat{H}(t) \hat{U}_t.
\end{aligned} \tag{A4}$$

This leads to the following expression:

$$\frac{1}{2} \left[-2\dot{\chi} + \dot{\varphi}(\hat{a}^\dagger \hat{a}) + i\dot{r} \hat{R}\left(\frac{\varphi}{2}\right) (\hat{a}^2 - \hat{a}^{\dagger 2}) \hat{R}^\dagger\left(\frac{\varphi}{2}\right) + \dot{\theta} \hat{R}\left(\frac{\varphi}{2}\right) \hat{S}(r) (\hat{a}^\dagger \hat{a}) \hat{S}^\dagger(r) \hat{R}^\dagger\left(\frac{\varphi}{2}\right) \right] = \hat{H}(t) = \omega \hat{a}^\dagger \hat{a} - \frac{\epsilon(t)}{4} (\hat{a}^\dagger + \hat{a})^2. \tag{A5}$$

From the action of the rotation and squeezing operations to the bosonic operators, $\hat{R}\left(\frac{\varphi}{2}\right) \hat{a} \hat{R}^\dagger\left(\frac{\varphi}{2}\right) = e^{i\frac{\varphi}{2}} \hat{a}$ and $\hat{S}(r) \hat{a} \hat{S}^\dagger(r) = \cosh(r) \hat{a} + \sinh(r) \hat{a}^\dagger$, Eq. (A5) can be simplified as

$$\begin{aligned}
&\frac{1}{2} \left[-2\dot{\chi} + \dot{\varphi}(\hat{a}^\dagger \hat{a}) + i\dot{r} (e^{i\varphi} \hat{a}^2 - e^{-i\varphi} \hat{a}^{\dagger 2}) + \dot{\theta} \left(\cosh(2r) \hat{a}^\dagger \hat{a} + \frac{e^{i\varphi} \hat{a}^2 + e^{-i\varphi} \hat{a}^{\dagger 2}}{2} \sinh(2r) + \sinh^2(r) \mathbb{I} \right) \right] \\
&= \omega \hat{a}^\dagger \hat{a} - \frac{\epsilon(t)}{4} (\hat{a}^\dagger + \hat{a})^2.
\end{aligned} \tag{A6}$$

By collecting each order of $\mathbb{I}$, $\hat{a}^\dagger \hat{a}$, $\hat{a}^2$, and $\hat{a}^{\dagger 2}$, we obtain

$$-2\dot{\chi} + \dot{\theta} \sinh^2(r) = -\frac{\epsilon(t)}{2}, \tag{A7}$$

$$\dot{\varphi} + \dot{\theta} \cosh(2r) = 2\omega - \epsilon(t), \tag{A8}$$

$$e^{i\varphi} \left[i\dot{r} + \frac{\dot{\theta}}{2} \sinh(2r) \right] = -\frac{\epsilon(t)}{2}, \tag{A9}$$

$$e^{-i\varphi} \left[-i\dot{r} + \frac{\dot{\theta}}{2} \sinh(2r) \right] = -\frac{\epsilon(t)}{2}. \tag{A10}$$

From Eqs. (A9) and (A10), we obtain $\dot{\theta} = -\frac{\epsilon(t) \cos \varphi}{\sinh(2r)}$. Finally, substituting the above expressions and rearranging the equations leads to the desired equation of motion,

$$\begin{aligned}
\dot{r} &= \frac{\epsilon(t)}{2} \sin \varphi, \\
\dot{\varphi} &= 2\omega - \epsilon(t)(1 - \coth(2r) \cos \varphi), \\
\dot{\chi} &= \frac{\epsilon(t)}{4} \left(1 - \frac{2 \sinh^2(r) \cos \varphi}{\sinh(2r)} \right),
\end{aligned} \tag{A11}$$

where the overall phase χ does not have any physical significance.

2. Quantum Fisher information for dynamical encoding

We provide the closed form of the QFI when estimating the system parameter $\omega = \omega_0 + \delta\omega$ encoded in the evolved state after time T , $|\psi_T\rangle = \hat{U}_T |\psi_0\rangle$. The explicit form of the QFI is given as

$$\mathcal{F}_\omega = 4(\langle \partial_\omega \psi_T | \partial_\omega \psi_T \rangle - |\langle \psi_T | \partial_\omega \psi_T \rangle|^2)_{|\omega=\omega_0}, \tag{A12}$$

where $|\partial_\omega \psi_T\rangle = \frac{\partial |\psi_T\rangle}{\partial \omega} = \frac{\partial \hat{U}_T}{\partial \omega} |\psi_0\rangle + \hat{U}_T \frac{\partial |\psi_0\rangle}{\partial \omega} = \frac{\partial \hat{U}_T}{\partial \omega} |\psi_0\rangle$, by assuming that $|\psi_0\rangle$ is independent of ω . For our case in Eq. (3), the vacuum state $|\psi_0\rangle = |0\rangle$ becomes the ground state of $\hat{H}(t)$ with $\epsilon(t) = 0$, regardless of the value of ω . We note that by introducing the following Hermitian operator,

$$\hat{G}_T = i\hat{U}_T^\dagger \left(\frac{\partial \hat{U}_T}{\partial \omega} \right)_{|\omega=\omega_0}, \tag{A13}$$

the QFI can be expressed as the expectation value over the initial state $|\psi_0\rangle$ as

$$\begin{aligned}
\mathcal{F}_\omega &= 4 \left(\langle \psi_0 | \left(\frac{\partial \hat{U}_T^\dagger}{\partial \omega} \right) \left(\frac{\partial \hat{U}_T}{\partial \omega} \right) | \psi_0 \rangle \right. \\
&\quad \left. - \left| \langle \psi_0 | i\hat{U}_T^\dagger \left(\frac{\partial \hat{U}_T}{\partial \omega} \right) | \psi_0 \rangle \right|^2 \right)_{|\omega=\omega_0} \\
&= 4 \left(\langle \psi_0 | \left[(-i) \left(\frac{\partial \hat{U}_T^\dagger}{\partial \omega} \right) \hat{U}_T \right] \left[i\hat{U}_T^\dagger \left(\frac{\partial \hat{U}_T}{\partial \omega} \right) \right] | \psi_0 \rangle \right. \\
&\quad \left. - \left| \langle \psi_0 | i\hat{U}_T^\dagger \left(\frac{\partial \hat{U}_T}{\partial \omega} \right) | \psi_0 \rangle \right|^2 \right)_{|\omega=\omega_0} \\
&= 4(\langle \psi_0 | \hat{G}_T^2 | \psi_0 \rangle - \langle \psi_0 | \hat{G}_T | \psi_0 \rangle^2) \\
&= 4\text{Var}_{|\psi_0\rangle}(\hat{G}_T),
\end{aligned} \tag{A14}$$

where $\hat{G}_T = \hat{G}_T^\dagger = (-i) \left(\frac{\partial \hat{U}_T^\dagger}{\partial \omega} \right) \hat{U}_T|_{\omega=\omega_0}$ is a Hermitian operator from the fact that $\hat{U}_T^\dagger \hat{U}_T = \mathbb{I} \Leftrightarrow \left(\frac{\partial \hat{U}_T^\dagger}{\partial \omega} \right) \hat{U}_T + \hat{U}_T^\dagger \left(\frac{\partial \hat{U}_T}{\partial \omega} \right) = 0$. This is consistent with the conventional form of the QFI, given by the variance of the generator.

Now, we evaluate the explicit form of $\hat{G}_T$ for general time-dependent dynamics with $\hat{U}_{t \rightarrow t'} = \mathcal{T} e^{-i \int_t^{t'} dt \hat{H}(t)}$, where $\hat{U}_T = \hat{U}_{0 \rightarrow T}$ is the special case with $t = 0$ and $t' = T$. We note that

$$\begin{aligned} \hat{G}_T &= i \hat{U}_T^\dagger \left(\frac{\partial \hat{U}_T}{\partial \omega} \right) \Big|_{\omega=\omega_0} \\ &= i \hat{U}_T^\dagger \int_0^T d\tau \mathcal{T} e^{-i \int_\tau^T \hat{H}(t) dt} (-i) \left(\frac{\partial \hat{H}(\tau)}{\partial \omega} \right) \Big|_{\omega=\omega_0} \\ &\quad \times \mathcal{T} e^{-i \int_0^\tau \hat{H}(t) dt} \\ &= \int_0^T d\tau \hat{U}_{0 \rightarrow \tau}^\dagger \hat{U}_{\tau \rightarrow T} \left(\frac{\partial \hat{H}(\tau)}{\partial \omega} \right) \Big|_{\omega=\omega_0} \hat{U}_{0 \rightarrow \tau} \\ &= \int_0^T d\tau \hat{U}_{0 \rightarrow \tau}^\dagger \left(\frac{\partial \hat{H}(\tau)}{\partial \omega} \right) \Big|_{\omega=\omega_0} \hat{U}_{0 \rightarrow \tau}. \end{aligned} \quad (\text{A15})$$

For the Hamiltonian in Eq. (3), we have

$$\frac{\partial \hat{H}(t)}{\partial \omega} = \hat{a}^\dagger \hat{a}, \quad (\text{A16})$$

which is time independent. We also note that the time-evolution operator given by Eq. (A2) reads

$$\hat{U}_{0 \rightarrow \tau} = e^{i\chi(\tau)} \hat{R}\left(\frac{\varphi(\tau)}{2}\right) \hat{S}(r(\tau)) \hat{R}\left(\frac{\theta(\tau)}{2}\right), \quad (\text{A17})$$

at time τ . From this, we obtain

$$\begin{aligned} \hat{G}_T &= \int_0^T d\tau \hat{U}_{0 \rightarrow \tau}^\dagger (\hat{a}^\dagger \hat{a}) \hat{U}_{0 \rightarrow \tau} \\ &= \int_0^T d\tau \left(\cosh 2r(\tau) \hat{a}^\dagger \hat{a} - \frac{e^{-i\theta(\tau)} \hat{a}^2 + e^{i\theta(\tau)} \hat{a}^{\dagger 2}}{2} \right. \\ &\quad \left. \times \sinh 2r(\tau) \right). \end{aligned} \quad (\text{A18})$$

By taking the initial state $|\psi_0\rangle = |0\rangle$, we note that

$$\hat{G}_T |0\rangle = - \int_0^T d\tau e^{i\theta(\tau)} \sinh(2r(\tau)) \left(\frac{\hat{a}^{\dagger 2}}{2} \right) |0\rangle, \quad (\text{A19})$$

which leads to

$$\langle 0 | \hat{G}_T | 0 \rangle = 0, \quad \langle 0 | \hat{G}_T^2 | 0 \rangle = \frac{1}{2} \left| \int_0^T d\tau \sinh(2r(\tau)) e^{i\theta(\tau)} \right|^2. \quad (\text{A20})$$

Finally, we obtain the closed form of the QFI,

$$\begin{aligned} \mathcal{F}_\omega &= 4(\langle \psi_0 | \hat{G}_T^2 | \psi_0 \rangle - \langle \psi_0 | \hat{G}_T | \psi_0 \rangle^2) \\ &= 2 \left| \int_0^T d\tau \sinh(2r(\tau)) e^{i\theta(\tau)} \right|^2, \end{aligned} \quad (\text{A21})$$

which corresponds to Eq. (6).

3. Standard quantum limit

The standard quantum limit serves as a benchmark of the quantum advantage. The standard quantum limit is written as $\langle \hat{N} \rangle T$ for time-independent dynamics with a fixed mean photon number $\langle \hat{N} \rangle$ over the total evolution time T . Meanwhile, we consider time-dependent dynamics, in which the mean photon number increases over time. To consider both effects, we define the standard quantum limit as a time integral of the mean photon number as

$$\text{SQL} = \int_0^T dt \langle \hat{N}(t) \rangle. \quad (\text{A22})$$

Equation (A22) is a general expression valid for arbitrary dynamics, and for the special case of a squeezed vacuum state, it can be rewritten as

$$\text{SQL} = \int_0^T dt \frac{1}{2} (\cosh 2r - 1). \quad (\text{A23})$$

APPENDIX B: CONCRETE PARAMETER ESTIMATION WITH QUADRATURE MEASUREMENTS

In this Appendix, we consider a concrete parameter estimator for ω from Gaussian measurements. We show that by taking the squared quadrature, $\hat{O} = \hat{X}_\alpha^2$ with $\hat{X}_\alpha = \frac{e^{-i\alpha} \hat{a} + e^{i\alpha} \hat{a}^\dagger}{\sqrt{2}} = \hat{x} \cos \alpha + \hat{p} \sin \alpha$, as an estimator of ω , the signal-to-noise ratio saturates the scaling of QFI. To this end, let us first rewrite the QFI in Eq. (A21) in terms of r and φ as

$$\mathcal{F}_\omega = 4 \text{Var}_{|\psi_0\rangle}(\hat{G}_T) = 2(\partial_\omega r)^2 + \frac{1}{2}(\partial_\omega \varphi)^2 \sinh^2 2r, \quad (\text{B1})$$

where $\hat{G}_T = (-i)(\frac{\partial \hat{U}_T}{\partial \omega}) \hat{U}_T|_{\omega=\omega_0}$. This can be shown by bringing Eq. (A2), to represent the unitary operator at time T ,

$$\hat{U}_T = e^{i\chi(T)} \hat{R}\left(\frac{\varphi(T)}{2}\right) \hat{S}(r(T)) \hat{R}\left(\frac{\theta(T)}{2}\right), \quad (\text{B2})$$

and directly evaluating its derivative with respect to ω as

$$\begin{aligned} \frac{\partial \hat{U}_T}{\partial \omega} &= i(\partial_\omega \chi) e^{i\chi} \hat{R}\left(\frac{\varphi}{2}\right) \hat{S}(r) \hat{R}\left(\frac{\theta}{2}\right) + e^{i\chi} \hat{R}\left(\frac{\varphi}{2}\right) \left(-i \frac{\partial_\omega \varphi}{2} \hat{a}^\dagger \hat{a} \right) \hat{S}(r) \hat{R}\left(\frac{\theta}{2}\right) \\ &\quad + e^{i\chi} \hat{R}\left(\frac{\varphi}{2}\right) \hat{S}(r) \left(\frac{\partial_\omega r}{2} (\hat{a}^2 - \hat{a}^{\dagger 2}) \right) \hat{R}\left(\frac{\theta}{2}\right) + e^{i\chi} \hat{R}\left(\frac{\varphi}{2}\right) \hat{S}(r) \hat{R}\left(\frac{\theta}{2}\right) \left(-i \frac{\partial_\omega \theta}{2} \hat{a}^\dagger \hat{a} \right) \\ &= e^{i\chi} \hat{R}\left(\frac{\varphi}{2}\right) \hat{S}(r) \left(-i \frac{\partial_\omega \varphi}{2} \left(\cosh 2r \hat{a}^\dagger \hat{a} - \sinh^2 r + \sinh 2r \left(\frac{\hat{a}^2 + \hat{a}^{\dagger 2}}{2} \right) \right) + \frac{\partial_\omega r}{2} (\hat{a}^2 - \hat{a}^{\dagger 2}) \right) \hat{R}\left(\frac{\theta}{2}\right) \\ &\quad + e^{i\chi} \hat{R}\left(\frac{\varphi}{2}\right) \hat{S}(r) \hat{R}\left(\frac{\theta}{2}\right) \left(i(\partial_\omega \chi) - i \frac{\partial_\omega \theta}{2} \hat{a}^\dagger \hat{a} \right). \end{aligned} \quad (\text{B3})$$

Equation (B3) directly yields the following:

$$\hat{G}_T = \hat{R}^\dagger \left(\frac{\theta}{2} \right) \left(\frac{\partial_\omega \varphi}{2} \left(\cosh 2r \hat{a}^\dagger \hat{a} - \sinh^2 r + \sinh 2r \left(\frac{\hat{a}^2 + \hat{a}^{\dagger 2}}{2} \right) \right) + i \frac{\partial_\omega r}{2} (\hat{a}^2 - \hat{a}^{\dagger 2}) \right) \hat{R} \left(\frac{\theta}{2} \right) - (\partial_\omega \chi) + \frac{\partial_\omega \theta}{2} \hat{a}^\dagger \hat{a}. \quad (\text{B4})$$

As we are considering the vacuum state as the initial state, i.e., $|\psi_0\rangle = |0\rangle$, we obtain

$$\begin{aligned} \langle \psi_0 | \hat{G}_T^2 | \psi_0 \rangle &= \left(\frac{\partial_\omega \varphi}{2} \right)^2 \left[\sinh^4 r + \frac{\sinh^2 2r}{2} \right] + \frac{1}{2} (\partial_\omega r)^2 + (\partial_\omega \chi)(\partial_\omega \varphi) \sinh^2 r + (\partial_\omega \chi)^2, \\ \langle \psi_0 | \hat{G}_T | \psi_0 \rangle &= -\frac{\partial_\omega \varphi}{2} \sinh^2 r - \partial_\omega \chi, \end{aligned} \quad (\text{B5})$$

which leads to Eq. (B1).

Now, let us evaluate the SNR for the estimator $\hat{O} = \hat{X}_\alpha^2$, given in Eq. (17). By taking $\hat{O} = \hat{X}_\alpha^2$, we have

$$\begin{aligned} \langle \hat{O} \rangle &= \langle \psi_T | \hat{O} | \psi_T \rangle = \langle 0 | \hat{U}_T^\dagger \hat{X}_\alpha^2 \hat{U}_T | 0 \rangle = \cosh 2r - \sinh 2r \cos(\varphi + 2\alpha), \\ \text{Var}(\hat{O}) &= \langle \psi_T | \hat{O}^2 | \psi_T \rangle - \langle \psi_T | \hat{O} | \psi_T \rangle^2 = \langle \psi_0 | \hat{U}_T^\dagger \hat{X}_\alpha^4 \hat{U}_T | 0 \rangle - \langle 0 | \hat{U}_T^\dagger \hat{X}_\alpha^2 \hat{U}_T | 0 \rangle^2 = 2(\cosh 2r - \sinh 2r \cos(\varphi + 2\alpha))^2. \end{aligned} \quad (\text{B6})$$

This leads to

$$\text{SNR} = \frac{|\partial_\omega (\cosh 2r - \sinh 2r \cos(\varphi + 2\alpha))|^2}{2(\cosh 2r - \sinh 2r \cos(\varphi + 2\alpha))^2} = \frac{1}{2} \left(\frac{2\partial_\omega r (\sinh 2r - \cosh 2r \cos(\varphi + 2\alpha)) + \partial_\omega \varphi \sinh 2r \sin(\varphi + 2\alpha)}{\cosh 2r - \sinh 2r \cos(\varphi + 2\alpha)} \right)^2, \quad (\text{B7})$$

where r and φ are the squeezing parameter and the squeezing angle, respectively, of the final state $|\psi_T\rangle = \hat{U}_T|0\rangle$.

While the SNR is, in general, no larger than the QFI, i.e., $\text{SNR} \leq \mathcal{F}_\omega$, we highlight that after optimizing α of the quadrature operator $\hat{X}_\alpha$, one can saturate the QFI as

$$\text{SNR}(\hat{X}_{\alpha_{\text{opt}}}^2) = 2(\partial_\omega r)^2 + \frac{1}{2}(\partial_\omega \varphi \sinh 2r)^2 = \mathcal{F}_\omega, \quad (\text{B8})$$

where the optimal choice of α_{opt} is given by

$$\begin{aligned} \cos(\varphi + 2\alpha_{\text{opt}}) &= \frac{\sqrt{(2\partial_\omega r)^2 + (\partial_\omega \varphi \sinh 2r)^2} \sinh 2r - 2\partial_\omega r \cosh 2r}{\sqrt{(2\partial_\omega r)^2 + (\partial_\omega \varphi \sinh 2r)^2} \cosh 2r - 2\partial_\omega r \sinh 2r}, \\ \sin(\varphi + 2\alpha_{\text{opt}}) &= \frac{-\partial_\omega \varphi \sinh 2r}{\sqrt{(2\partial_\omega r)^2 + (\partial_\omega \varphi \sinh 2r)^2} \cosh 2r - 2\partial_\omega r \sinh 2r}. \end{aligned} \quad (\text{B9})$$

On the other hand, the optimal α depends on not only r and φ but also $\partial_\omega r$ and $\partial_\omega \varphi$, which makes it challenging to determine the optimal α in actual experiments. Nevertheless, we overcome this challenge by finding a nearly optimal estimator of ω from the square quadrature $\hat{X}_{\alpha^*}$ such that

$$\cos(\varphi + 2\alpha^*) = \tanh 2r, \quad (\text{B10})$$

where the SNR is given as

$$\text{SNR}(\hat{X}_{\alpha^*}^2) = \frac{1}{2}(\sinh^2 2r)(\partial_\omega \varphi)^2. \quad (\text{B11})$$

As the value of α^* can be determined only from the squeezing $r(T)$ and the phase $\varphi(T)$ of the final state, this provides a more practically accessible estimator in realistic experiments. Despite being a suboptimal estimator that does not exactly saturate the QFI, this choice of SNR is proportional to $\sinh^2(2r)$, still maintaining the same exponential scaling as in the QFI. For this reason, we analyze the SNR with $\hat{X}_{\alpha^*}$ in the main text.

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