

Tractable protocol for detection-loophole-free Bell tests over long distances

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Certifying genuine nonclassical correlations over long distances is essential for device-independent quantum information. In photonic platforms, however, this remains challenging due to photon loss, which opens the detection loophole, rendering violations increasingly difficult for less-efficient detectors. Eberhard showed that using nonmaximally entangled states lowers the detection-efficiency threshold to 66.7%, but existing photonic approaches are restricted to short distances with linear transmittance scaling. Conversely, single-photon event-ready schemes extend the distance with favorable square-root scaling with channel transmittance, yet still demand detection efficiencies above 82.6%. Here, we propose the first all-photonic, heralded entanglement distribution protocol that unifies these two advances: It achieves a postselection-free violation at the Eberhard limit while preserving twin-field-like scaling. We identify the loss independent of the vacuum component amplitude of the prepared state as the source of this enhancement. Our approach addresses both resilience to loss and scalability, providing a practical route toward long-distance, loophole-free Bell tests and device-independent applications with current technology.

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I. INTRODUCTION

A cornerstone of device-independent (DI) quantum information protocols—where neither the source nor the measurement devices are trusted—is the violation of a loophole-free Bell inequality for appropriate variables [1–4]. Among the various loopholes in Bell tests, the detection loophole is perhaps the most challenging to close as it requires a total end-to-end throughput—hereby referred to as the detection efficiency—exceeding 82.6% to ensure unconditional violation when using maximally entangled states [5,6]. Photonics is the only viable platform for distributing entanglement over appreciable distances. However, photon loss can drastically hinder the end-to-end efficiency of a communication channel, even with high-efficiency detectors, thus making the prospect of closing the detection loophole even more daunting in practice.

To date, such tests have only been demonstrated over a few hundred meters [7–10]—the distance required for spacelike separation between two parties, Alice and Bob. To extend these tests to longer distances, several heralded photonic-qubit approaches have been proposed [11–13]. While conceptually appealing, these schemes rely on successful two-photon

interference, which fundamentally limits their success probability and scalability. Recently, there has been growing interest in *event-ready* protocols based on single-photon path-entanglement (SPPE) as a promising approach to mitigate photon loss [14–17]. These protocols are attractive because the entanglement distribution efficiency scales more favorably, following the characteristic twin-field-like scaling [18]. Specifically, the success rate scales proportionally to $\sqrt{\eta_C}$, where η_C denotes the total transmittance of the channel connecting the users. Moreover, these entanglement-based protocols offer the potential to perform long-distance, loophole-free Bell tests, thereby enabling DI operation—a key milestone for secure quantum communication [11,14–17,19].

Eberhard showed that the detection efficiency threshold for violating a Bell inequality can be reduced to 2/3 ($\approx 66.7\%$) by using nonmaximally entangled states [20]. In photonic implementations, this approach has been employed to close the detection loophole in two-photon polarization entanglement experiments [5–7,10,21,22]. For photon-number entanglement up to the single-photon level, the Eberhard limit can be reached with displacement-based measurements, provided the state exhibits a perfect photon-number correlation, namely, the vacuum-two-photon-entangled (VTPE) state [23,24]. This state is often approximated from a two-mode squeezed vacuum (TMSV) source operating in the low-gain regime. In contrast, existing long-distance distribution protocols typically generate the anticorrelated SPPE state, which cannot reach the Eberhard limit and instead requires detection efficiencies above 82.6% to demonstrate post-selection-free Bell violations [15,19].

The fact that these photon-number path-entangled (PNPE) states exhibit different loss tolerance is nuanced and

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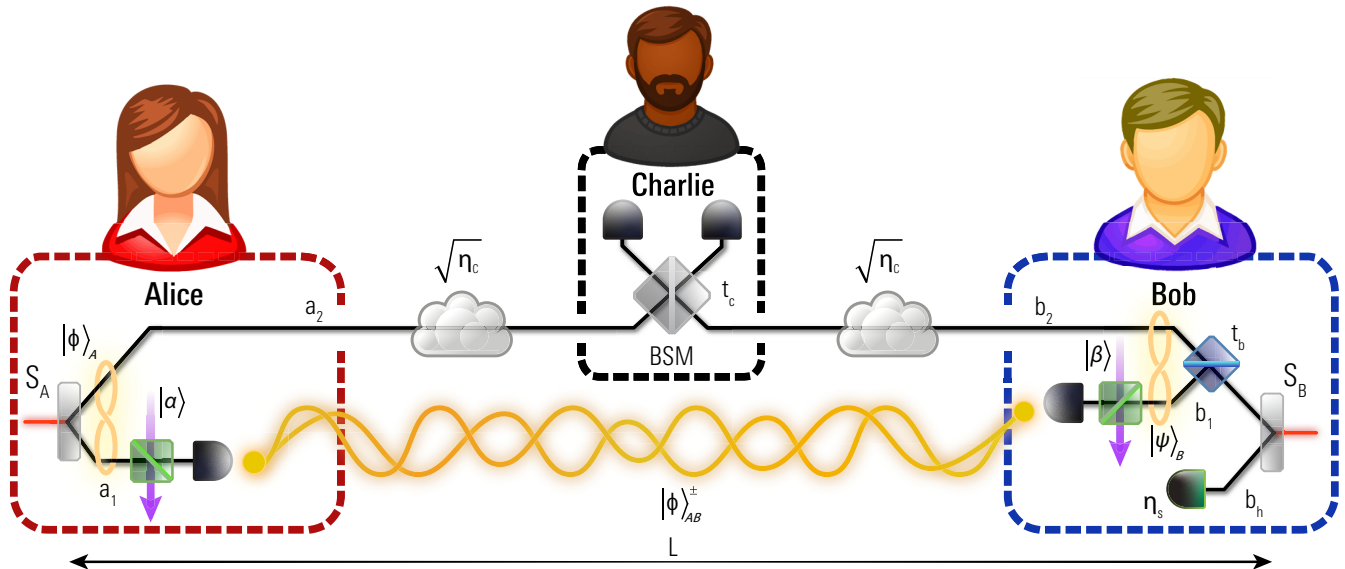


FIG. 1. An illustration of our protocol for the heralded distribution of VTPE. Alice and Bob, separated by a distance L , generate entangled states using sources S_A and S_B , respectively. The modes a_2 and b_2 are directed to a middle station (Charlie), which performs a Bell-state measurement (BSM), thereby heralding entanglement between the modes a_1 and b_1 . The photons traveling to Charlie experience a channel transmittance of $\sqrt{\eta_c}$, and a successful BSM occurs if exactly one photon arrives. The modes a_1 and b_1 are then interfered with weak local oscillators ($|\alpha\rangle$, $|\beta\rangle$) to implement displacement-based measurements for the Bell test. In this illustration, the sources are based on SPDC: Alice prepares a two-mode squeezed vacuum state, while Bob generates a heralded SPPE.

surprising, given that both are Bell-like states and are related by a local unitary transformation. This contrast becomes even more striking when compared to the case of two-photon polarization entanglement, where both correlated and anticorrelated states can, as expected, reach the Eberhard limit. To the best of our knowledge, a clear explanation for this disparity in the photon-number setting has not yet been established.

Here, we introduce the first protocol that distributes the desired VTPE state between distant parties, reducing the detection-efficiency threshold to the Eberhard limit. Crucially, the heralding relies only on single-photon detection at a central station, preserving the favorable $\sqrt{\eta_c}$ scaling [18]. Our approach resolves the long-standing trade-off between closing the detection loophole and enabling long-distance operation. We evaluate its performance against existing schemes based on SPPE and polarization entanglement and provide a theoretical analysis that clarifies why different PNPE states exhibit contrasting loss tolerances in the context of Bell tests.

II. THE PROTOCOL

The notion of heralding entanglement between Alice and Bob, based on a single-photon detection at a central station, has been previously reported, for example, in Refs. [14,15], where both parties use TMSV states, and Ref. [16], where SPPE states are used. In both cases, the heralded state is SPPE, which is highly sensitive to loss.

Protocol

We propose an event-ready protocol, illustrated in Fig. 1, in which Alice prepares a TMSV state, while Bob generates an SPPE state. This asymmetry in the initial states enables

the heralding of the target entangled state, which significantly improves performance. A related scheme was previously introduced in the context of long-distance continuous-variable-QKD and quantum repeaters [25], whereas here we focus on closing the detection-efficiency loophole for long-distance Bell tests and enabling DI applications. A practical realization uses type-II spontaneous parametric down-conversion (SPDC) in the low-gain regime, which directly produces the required TMSV state for Alice

$$|\phi\rangle_A = \frac{1}{\sqrt{1+\lambda^2}}(|0,0\rangle_{a_1,a_2} + \lambda e^{i\varphi_a}|1,1\rangle_{a_1,a_2}), \quad (1)$$

where $\lambda = \tanh g$, g is the squeezing parameter, and φ_a is the pump phase. Since $\lambda \ll 1$, we ignore higher-order photon-number contributions for this illustration, although they are included in the numerical simulation. Bob prepares his SPPE state by heralding on a single-photon detection and sending the other photon through a beam splitter with transmittance t_b , yielding

$$|\psi\rangle_B = \sqrt{t_b}|0,1\rangle_{b_1,b_2} + \sqrt{1-t_b}e^{i\varphi_b}|1,0\rangle_{b_1,b_2}, \quad (2)$$

where φ_b is the phase difference in the path. Since Bob prepares his state by heralding on a single-photon detection with efficiency η_s , the state $|\psi\rangle_B$ is generated with a success probability $P_s = \eta_s \lambda^2 / (1 + \lambda^2)$. We ignore the probability that a heralding detector response yields more than one photon in the state, which is reasonable for extremely low parametric gains.

Alice and Bob are separated by a distance L , as shown in Fig. 1, and each sends one mode of their bipartite state (a_1 and b_1) to a central station (Charlie) located midway between them, at a distance $L/2$. Charlie performs a partial BSM using a beamsplitter (BS), with transmittance t_c , followed by click

or photon-number-resolving detection. The transmittance of the direct channel between Alice and Bob is given by $\eta_C = 10^{-\gamma L/10}$, where γ is the attenuation coefficient, which implies a transmittance of $\sqrt{\eta_C}$ to the midpoint station. Now, if the modes are indistinguishable after entering Charlie's BS, it becomes impossible to determine the origin of a single detection event at Charlie's detectors. As a result, the shared state between Alice and Bob is projected onto one of the desired (unnormalized) states

$$|\phi\rangle_{AB}^{\pm} = \sqrt{t_b}|0, 0\rangle_{a_1, b_1} \pm \lambda\sqrt{1-t_b}e^{i\varphi}|1, 1\rangle_{a_1, b_1}, \quad (3)$$

where $\varphi = \varphi_a + \varphi_b$, and the exact state depends on which detector clicks at Charlie. The degree of entanglement in these states can be continuously tuned—from a product state to a maximally entangled state—by adjusting the amplitudes of the initial states. While the beam splitter at Charlie's station was assumed to be balanced in Eq. (3), its transmittance can also serve as an additional degree of freedom to tune the entangled state (see Appendix A for details).

The final heralded state, including symmetric losses in the transmitted modes (i.e., $\eta_{a_2} = \eta_{b_2} = \eta_H$), is

$$\rho_{AB}^{(\pm)} = \mathcal{N}^{-1}(|\phi\rangle\langle\phi|_{AB}^{\pm} + \varepsilon|1, 0\rangle\langle 1, 0|_{a_1, b_1}), \quad (4)$$

where $\mathcal{N}$ is the normalization factor, $\varepsilon = 2\lambda^2 t_b(1 - \eta_H)$, and $\eta_H = \eta_D^c \sqrt{\eta_C}$, with η_D^c denoting Charlie's detection efficiency (see Appendix B for the full derivation, including the treatment of asymmetric losses and imperfect spectral overlap). We observe that Eq. (4) is equivalent to the desired state in Eq. (3), up to an additional noise term $|1, 0\rangle\langle 1, 0|$, which can be neglected when $\varepsilon \ll 1$. In this limit, which can be plausibly reached, the shared state is effectively pure, independent of channel losses. However, the overall success probability is naturally reduced as the heralding efficiency appears in the state normalization, trading off the degree of shared entanglement with the distribution rate.

By tuning g of each source and t_b , one can control the degree of entanglement in the shared state. For instance, a maximally entangled state between Alice and Bob is achieved when the two sources have the same gain and the condition $\sqrt{t_b} = \lambda\sqrt{1-t_b}$ is satisfied. To further quantify the degree of entanglement, we compute the log-negativity of the shared state and examine its dependence on the tuning parameters [see Appendix C]. This tunability highlights the versatility of our protocol, enabling it to be adapted to diverse experimental constraints and optimized for various levels of detection efficiency to maximize the Bell violation as will be shown later.

III. PERFORMANCE

Once the entangled state is shared, Alice and Bob interfere their respective modes with weak coherent states $|\alpha\rangle$ and $|\beta\rangle$ to perform displacement-based measurements. By choosing the amplitudes and phases of their local oscillators, they realize two dichotomic measurement settings per party, enabling a post-selection-free Bell test of the form [26]

$$S = \sum_{i,j=0}^1 (-1)^{ij} E(\alpha_i, \beta_j) \leq 2, \quad (5)$$

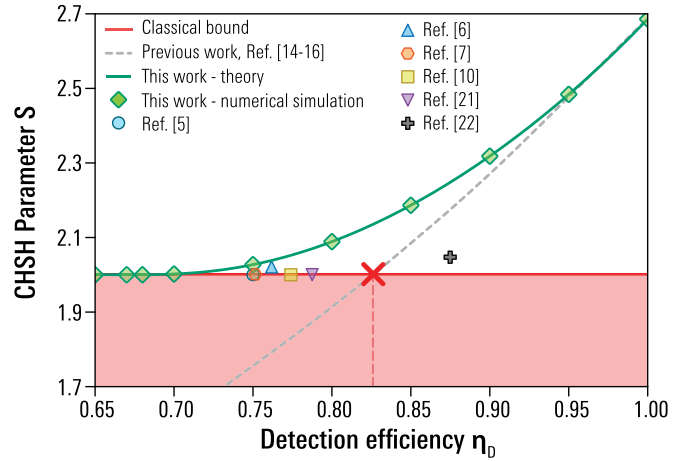


FIG. 2. Optimal CHSH violations as a function of symmetric detection efficiency η_D . The solid green curve shows the theoretical prediction for our protocol, with numerical simulations that account for higher-order photon-number contributions (green diamonds) in excellent agreement. The gray dashed line corresponds to previous photon-number-based protocols, which require detection efficiencies above 82.6% (red cross and dropline) to observe a violation. Colored markers denote detection-loophole-free violations from leading polarization-based experiments. While those can operate at the Eberhard limit (66.7%), the observed violations are marginal and their distance scaling is limited. In contrast, our protocol achieves significantly stronger violations down to the Eberhard limit, demonstrating enhanced robustness to loss and improved overall performance.

where $E(\alpha_i, \beta_j)$ are the measured correlations. A violation $S > 2$ certifies nonlocality of the shared state [27–29]. See Appendix D for a detailed derivation of the inequality in the Clauser-Horne-Shimony-Holt (CHSH) and Clauser-Horne (CH) forms, for both the ideal and realistic cases.

Figure 2 presents the central result of this work, comparing different protocols each optimized for maximum CHSH violation as a function of detection efficiency. Our protocol achieves a violation with total detection efficiency as low as 67%, reaching the Eberhard limit, while existing photon-number-based protocols [14–16] require efficiencies above 82.6% [30]. Although the Eberhard approach enables operation near the threshold, the corresponding CHSH violations are necessarily small. To contextualize this trade-off, the figure also includes results from state-of-the-art polarization-entanglement experiments [5–7, 10, 21, 22]. While these setups can also operate at the Eberhard limit, the observed violations remain lower than those attainable with our scheme. Indeed, it has been shown that the maximum achievable post-selection-free CHSH violation decreases from approximately 2.82 to 2.35 for polarization-based schemes using a single SPDC source [31], with even lower values reported for heralded variants [32, 33]. This is compared to a maximum of 2.69 for PNPE protocols. Remarkably, our protocol not only surpasses existing schemes in the low-efficiency regime, but also yields stronger violations across the practical range of detection efficiencies, while matching their performance in the high-efficiency limit. The excellent agreement between our theoretical model and numerical simulations, which account for

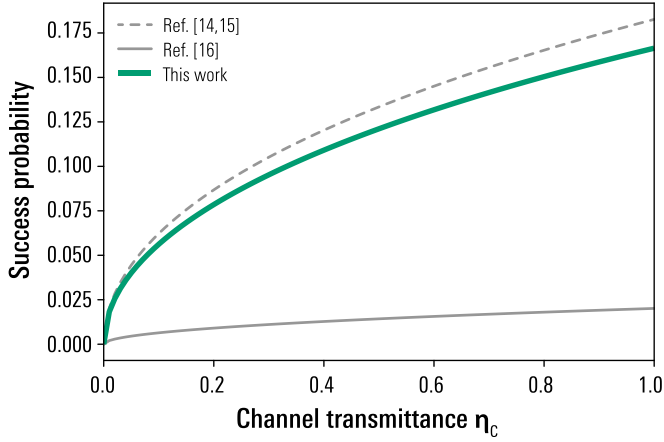


FIG. 3. Success probability of performing the BSM at Charlie and heralding entanglement between Alice and Bob. The squeezing parameter g is fixed at 0.33. For Ref. [16] and for this work, the BS transmittance is set to 0.1 and 0.8, respectively. The parameters are chosen to remain consistent with the operating assumptions and optimal regimes of each protocol.

higher-order photon-number contributions from the TMSV, further validates the assumptions made in our analysis.

Since our protocol and those in Refs. [14–16] rely on quantum interference at Charlie’s station followed by the detection of a single photon to perform a successful BSM, they exhibit a success probability that scales as $\sqrt{\eta_c}$. The probability of performing a successful BSM at Charlie—and thereby heralding entanglement between Alice and Bob—using this approach can be calculated as

$$P_{\text{succ}} = \sqrt{\eta_c}(P_A(1) + P_B(1)) - 2\eta_c P_A(1)P_B(1). \quad (6)$$

Here, $\sqrt{\eta_c}$ denotes the channel transmittance from Alice or Bob to Charlie, and $P_x(1)$ is the probability that party x prepares a state with one photon in the transmitted mode (see Appendix E for full details). Equation (6) clearly shows that the success probability of these protocols scales as $\mathcal{O}(\sqrt{\eta_c})$, meaning that it grows proportionally to $\sqrt{\eta_c}$ up to constant factors, provided that $P_A(1)P_B(1) \ll 1$. It is straightforward to use Eq. (6) together with Alice and Bob’s states to calculate the success probability for each protocol. Figure 3 shows these probabilities as a function of the channel transmittance when using SPDC sources with equal squeezing parameters. While all protocols exhibit the expected square-root dependence on transmittance, their success probabilities differ. The protocol using two TMSV states [15] achieves the highest success rate due to the relatively high probability of producing single photons in the transmitted modes. In contrast, the protocol employing two heralded single-photon entangled states [16] exhibits a lower success rate due to its operation in the low-transmissivity regime. Our scheme, which combines both resources, achieves an intermediate success probability. It is worth noting that with our protocol, one can trade off some degree of Bell violation to further increase the success probability (see Appendix E).

The event-ready polarization-based protocols with SPDC sources suffer from two significant shortcomings. First, they rely on quantum interference at the heralding station, where

the detection of two photons causes the success probability to scale linearly with the channel transmittance [34]. Second, due to the possibility of both photons coming from the same source, the fidelity of the heralded entangled state is reduced: Only one-third of the successful heralded events correspond to the desired case where one photon originates from each source [32,33]. Table I provides a brief summary of the differences in the heralded states and performance metrics across the protocols we compare [35]. Notably, our protocol is the only one that simultaneously achieves both the optimal square-root scaling and the Eberhard limit.

IV. REALIZATION AND APPLICATIONS

Given that our protocol can operate at the Eberhard limit without requiring postselection of measurement outcomes, it can be readily implemented using current photon sources and detectors. In fact, total detection efficiencies exceeding 67%—typically in the range of 73%–78%—have already been demonstrated by several experimental groups [5–7,10,21,22], making our scheme well-suited for near-term, long-distance, all-photon implementations of DI quantum information tasks such as DI quantum key distribution (DI-QKD), randomness extraction, and certification [4,15,16,19,36]. As concrete use cases, we evaluate two key quantities relevant to device-independent applications: the extractable randomness per round and the Holevo bound. The rate of certifiable randomness R can be lower bounded by the min-entropy via $R \geq rH_{\min}$, where r is the experimental repetition rate, typically limited by the overall detection rate [19]. The second quantity, $\chi_{\max}$, is the Holevo bound, which quantifies the maximum information that an eavesdropper could gain in a DI-QKD protocol with one-way classical communication from Bob to Alice. Both quantities can be expressed as functions of the observed CHSH value S [3,19]:

$$H_{\min} = 1 - \log_2(1 + \sqrt{2 - S^2/4}), \quad (7a)$$

$$\chi_{\max} = h\left(\frac{1 + \sqrt{S^2/4 - 1}}{2}\right), \quad (7b)$$

where $h(x)$ is the binary entropy function.

Figure 4 presents both the min-entropy $H_{\min}$ and the Holevo bound $\chi_{\max}$ as functions of the detection efficiency η_D for the different PNPE-based protocols. As expected, higher CHSH violations yield larger values of $H_{\min}$, indicating greater certifiable randomness, and smaller values of $\chi_{\max}$, reflecting stronger privacy guarantees and a higher-potential secret key rate in DI-QKD. Notably, below the threshold $\eta_D = 82.6\%$, only our protocol provides positive rates for both randomness and key generation. Moreover, it consistently outperforms previous methods across a broad range of detection efficiencies (up to $\sim 95\%$), beyond which all protocols converge. These results demonstrate the enhanced robustness and practical viability of our scheme for both DI randomness generation and QKD.

V. DISCUSSION

The question arises as to why the state $|\phi\rangle = c_0|0,0\rangle + c_1|1,1\rangle$ can reach the Eberhard limit, with our mea-

TABLE I. Comparison of heralded entangled states and key performance metrics for the protocols discussed in the main text.

No.	Protocol	Ideal Heralded state	Scaling	Required detection efficiency
1	SPPE based	$ \psi\rangle_{AB} = \epsilon 0, 1\rangle_{a,b} + \sqrt{1 - \epsilon^2}e^{i\varphi} 1, 0\rangle_{a,b}$	$\mathcal{O}(\sqrt{\eta_C})$	$\geq 82.6\%$
2	Polarization	$ \Phi\rangle_{AB} = \epsilon H, H\rangle_{a,b} + \sqrt{1 - \epsilon^2}e^{i\varphi} V, V\rangle_{a,b}$	$\mathcal{O}(\eta_C)$	$\geq 66.67\%$
3	This work	$ \phi\rangle_{AB} = \epsilon 0, 0\rangle_{a,b} + \sqrt{1 - \epsilon^2}e^{i\varphi} 1, 1\rangle_{a,b}$	$\mathcal{O}(\sqrt{\eta_C})$	$\geq 66.67\%$

surement scheme, while the seemingly equivalent state $|\psi\rangle = c_0|0, 1\rangle + c_1|1, 0\rangle$ cannot. This distinction is, in one sense, counterintuitive: Both states are Bell-like and related by a local unitary, suggesting that they could yield the same violation in principle [37]. In the case of two-photon polarization entanglement, such a difference does not arise; these states can, in principle, achieve the Eberhard limit. It is also reasonable to expect this behavior in dual-rail encoding of other degrees of freedom, such as time-bin and frequency.

The key to understanding this tolerance is to note that loss does not affect the vacuum, but only the components of the state that contain photons. In the state in $|\psi\rangle$, both amplitudes are reduced when there is loss present (and a vacuum component introduced), whereas in the state $|\phi\rangle$, only the second amplitude is reduced.

Perhaps most remarkably, the minimum detection efficiency required to observe a violation with $|\phi\rangle$ approaches the Eberhard limit for two-qubit entanglement. A full quantitative analysis of the CHSH inequality for each state in the presence of loss is presented in Appendix F.

In conclusion, we have proposed an event-ready protocol that heralds a tunable PNPE state upon single-photon detection. Our scheme combines the favorable twin-field-like scaling, $\mathcal{O}(\sqrt{\eta_C})$, with resilience to detection inefficiencies. Our analysis shows that this robustness arises from the coherent superposition of the biphoton state with vacuum, enabling Bell violations even under high-loss conditions. Notably, our protocol achieves a violation at detection efficiencies as low as 66.7%, corresponding to the Eberhard limit for two-qubit en-

tanglement. Given that current experimental platforms already exceed this threshold, our results open a practical route toward near-term, long-distance, all-photonic implementations of device-independent quantum technologies. Moreover, the heralded VTPE state can be tuned to the canonical Bell state $|\phi\rangle$, making it broadly useful for a wide range of photonic quantum information tasks beyond Bell tests.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: IDEAL HERALDED STATE

The idealized states for evaluating the protocol for preparing entanglement between Alice and Bob are

$$|\phi\rangle_A = a_0|0, 0\rangle_{a_1, a_2} + a_1|1, 1\rangle_{a_1, a_2}, \quad (\text{A1a})$$

$$|\psi\rangle_B = b_0|0, 1\rangle_{b_1, b_2} + b_1|1, 0\rangle_{b_1, b_2}, \quad (\text{A1b})$$

where $|a_0|^2 + |a_1|^2 = 1$ and $|b_0|^2 + |b_1|^2 = 1$. Then, the initial joint state is $|\Psi\rangle_i = |\phi\rangle_A \otimes |\psi\rangle_B$

$$\begin{aligned} |\Psi\rangle_i &= |\phi\rangle_A \otimes |\psi\rangle_B \\ &= a_0b_0|0, 0\rangle_{a_1, a_2}|0, 1\rangle_{b_1, b_2} + a_0b_1|0, 0\rangle_{a_1, a_2}|1, 0\rangle_{b_1, b_2} \\ &\quad + a_1b_0|1, 1\rangle_{a_1, a_2}|0, 1\rangle_{b_1, b_2} + a_1b_1|1, 1\rangle_{a_1, a_2}|1, 0\rangle_{b_1, b_2}. \end{aligned} \quad (\text{A2})$$

After the beamsplitter interaction at Charlie, keeping only single-photon outcomes, the state becomes

$$\begin{aligned} |\Psi\rangle_{ABC} &= BS_{a_2, b_2}|\Psi\rangle_i \\ &\propto |1, 0\rangle_{a_2, b_2} [a_0b_0\sqrt{1-t_c}|0, 0\rangle_{a_1, b_1} + a_1b_1\sqrt{t_c}|1, 1\rangle_{a_1, b_1}] \\ &\quad - |0, 1\rangle_{a_2, b_2} [a_0b_0\sqrt{t_c}|0, 0\rangle_{a_1, b_1} - a_1b_1\sqrt{1-t_c}|1, 1\rangle_{a_1, b_1}], \end{aligned} \quad (\text{A3})$$

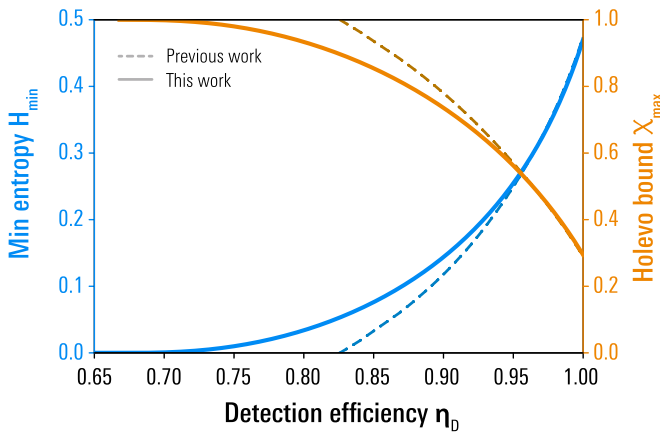


FIG. 4. Min-entropy $H_{\min}$ (left axis, blue) and Holevo bound $\chi_{\max}$ (right axis, orange) as functions of detection efficiency η_D for the protocols compared. Solid lines correspond to the present work; dashed lines represent previous approaches (Refs. [14–17]). Increasing $H_{\min}$ indicates higher certifiable randomness, while decreasing $\chi_{\max}$ reflects reduced eavesdropper information in DI-QKD.

where BS_{a_2, b_2} is the unitary transformation of the beamsplitter interaction at Charlie and t_c is its transmittance. Equation (A3) shows that a measurement at Charlie will project the shared state between Alice and Bob into the desired state. The exact form depends on which detector clicked at Charlie. Specifically, if

(1) Charlie's D_{c_2} detector clicks, then

$$|\phi\rangle_{AB}^+ \propto a_0 b_0 \sqrt{1-t_c} |0, 0\rangle_{a_1, b_1} + a_1 b_1 \sqrt{t_c} |1, 1\rangle_{a_1, b_1}. \quad (\text{A4})$$

(2) Charlie's D_{c_1} detector clicks, then

$$|\phi\rangle_{AB}^- \propto a_0 b_0 \sqrt{t_c} |0, 0\rangle_{a_1, b_1} - a_1 b_1 \sqrt{1-t_c} |1, 1\rangle_{a_1, b_1}. \quad (\text{A5})$$

Using the proposed realization (after normalization), $a_0 = \frac{1}{\sqrt{1+\lambda^2}}$, $a_1 = \frac{\lambda}{\sqrt{1+\lambda^2}}$, $b_0 = \sqrt{t_b}$, and $b_1 = \sqrt{1-t_b}$.

APPENDIX B: GENERAL HERALDED STATE

1. Effect of losses

Here, we derive the heralded state when using the proposed realization of the protocol, by including the channel losses in the transmitted modes (a_2 and b_2), and Charlie's detectors efficiency, although we retain the maximum photon number in any mode at one. We start from the state $|\Psi\rangle_i = |\phi\rangle_A \otimes |\psi\rangle_B$, where $|\phi\rangle_A$ and $|\psi\rangle_B$ are defined in Eqs. (1) and (2), respectively. We model the total loss between Charlie and Alice (Bob) with a beamsplitter of transmittance η_{a_2} (η_{b_2}). This includes the channel losses and Charlie's detector efficiencies, as the loss associated with Charlie's detector commutes with his Bell-state measurement beamsplitter. Then, the initial state after the transmitted modes of Alice and Bob interact with the environment modes e_a and e_b , respectively, becomes

$$\begin{aligned} |\Psi\rangle_I = \frac{1}{\sqrt{1+\lambda_a^2}} & [|0, 0\rangle_{e_a, e_b} (\sqrt{t_b} \sqrt{\eta_{b_2}} |0, 0\rangle_{a_1, b_1} |0, 1\rangle_{a_2, b_2} + \sqrt{1-t_b} e^{i\varphi_b} |0, 1\rangle_{a_1, b_1} |0, 0\rangle_{a_2, b_2}) \\ & + \lambda_a e^{i\varphi_a} \sqrt{t_b} \sqrt{\eta_{a_2}} \sqrt{\eta_{b_2}} |1, 0\rangle_{a_1, b_1} |1, 1\rangle_{a_2, b_2} + \lambda_a e^{i\varphi_a} \sqrt{1-t_b} e^{i\varphi_b} \sqrt{\eta_{a_2}} |1, 1\rangle_{a_1, b_1} |1, 0\rangle_{a_2, b_2}) \\ & + |0, 1\rangle_{e_a, e_b} (\sqrt{t_b} \sqrt{1-\eta_{b_2}} |0, 0\rangle_{a_1, b_1} |0, 0\rangle_{a_2, b_2} + \lambda_a e^{i\varphi_a} \sqrt{t_b} \sqrt{\eta_{a_2}} \sqrt{1-\eta_{b_2}} |1, 0\rangle_{a_1, b_1} |1, 0\rangle_{a_2, b_2}) \\ & + |1, 0\rangle_{e_a, e_b} (\lambda_a e^{i\varphi_a} \sqrt{t_b} \sqrt{1-\eta_{a_2}} \sqrt{\eta_{b_2}} |1, 0\rangle_{a_1, b_1} |0, 1\rangle_{a_2, b_2} + \lambda_a e^{i\varphi_a} \sqrt{1-t_b} e^{i\varphi_b} \sqrt{1-\eta_{a_2}} |1, 1\rangle_{a_1, b_1} |0, 0\rangle_{a_2, b_2}) \\ & + |1, 1\rangle_{e_a, e_b} (\lambda_a e^{i\varphi_a} \sqrt{t_b} \sqrt{1-\eta_{a_2}} \sqrt{1-\eta_{b_2}} |1, 0\rangle_{a_1, b_1} |0, 0\rangle_{a_2, b_2})]. \end{aligned} \quad (\text{B1})$$

The state entering Charlie's station is found by tracing out the environment modes. That is $\tilde{\rho} = \text{Tr}[|\Psi\rangle_I \langle \Psi|_I]_{e_a, e_b}$. Charlie conducts the projective measurement described by

$$\Pi_C^{(\pm)} = \frac{1}{2} (|0, 1\rangle \langle 0, 1| \pm |0, 1\rangle \langle 1, 0| \pm |1, 0\rangle \langle 0, 1| + |1, 0\rangle \langle 1, 0|)_{a_2, b_2}. \quad (\text{B2})$$

Finally, the heralded state between Alice and Bob is given by

$$\begin{aligned} \rho_{AB}^{(\pm)} &= \mathcal{N}^{-1} \text{Tr}[\Pi_C^{(\pm)} \tilde{\rho} \Pi_C^{(\pm)}]_{a_2, b_2} \\ &= \mathcal{N}^{-1} [t_b \eta_{b_2} |0, 0\rangle \langle 0, 0| \pm \lambda \sqrt{\eta_{a_2} \eta_{b_2} t_b (1-t_b)} e^{-i(\varphi_a + \varphi_b)} |0, 0\rangle \langle 1, 1| \\ &\quad \pm \lambda \sqrt{\eta_{a_2} \eta_{b_2} t_b (1-t_b)} e^{i(\varphi_a + \varphi_b)} |1, 1\rangle \langle 0, 0| + \lambda^2 (1-t_b) \eta_{a_2} |1, 1\rangle \langle 1, 1| + \lambda^2 t_b (\eta_{a_2} + \eta_{b_2} - 2\eta_{a_2} \eta_{b_2}) |1, 0\rangle \langle 1, 0|]_{a_1, b_1}, \end{aligned} \quad (\text{B3})$$

where $\mathcal{N} = \text{Tr}[\Pi_C^{(\pm)} \tilde{\rho} \Pi_C^{(\pm)}]$ is the normalization.

In the case of symmetric loss and equal detector efficiencies, $\eta_{a_1} = \eta_{b_1} = \eta_H$, this becomes

$$\begin{aligned} \rho_{AB}^{(\pm)} &= \eta_H \mathcal{N}^{-1} [t_b |0, 0\rangle \langle 0, 0| \pm \lambda \sqrt{t_b (1-t_b)} e^{-i(\varphi_a + \varphi_b)} |0, 0\rangle \langle 1, 1| \\ &\quad \pm \lambda \sqrt{t_b (1-t_b)} e^{i(\varphi_a + \varphi_b)} |1, 1\rangle \langle 0, 0| + \lambda^2 (1-t_b) |1, 1\rangle \langle 1, 1| + 2\lambda^2 t_b (1-\eta_H) |1, 0\rangle \langle 1, 0|]_{a_1, b_1} \\ &= \mathcal{N}'^{-1} ([|\phi\rangle \langle \phi|]_{AB}^{\pm} + 2\lambda^2 t_b (1-\eta_H) |1, 0\rangle \langle 1, 0|_{a_1, b_1}), \end{aligned} \quad (\text{B4})$$

where $|\phi\rangle_{AB}^{\pm}$ is Eq. (3) and $\mathcal{N}'^{-1} = \eta_H \mathcal{N}^{-1}$.

the spectral amplitudes of the single photons in the states of Alice and Bob:

$$\begin{aligned} |\phi\rangle_A &= \alpha_0 |0, 0\rangle_{a_1, a_2} + \alpha_1 \iint d\omega_{a_1} d\omega_{a_2} f(\omega_{a_1}) g(\omega_{a_2}) \\ &\quad \times a_1^\dagger(\omega_{a_1}) a_2^\dagger(\omega_{a_2}) |0, 0\rangle_{a_1, a_2}, \end{aligned} \quad (\text{B5a})$$

$$|\psi\rangle_B = \int d\omega_b h(\omega_b) [\beta_1 b_1^\dagger(\omega_b) + \beta_0 b_2^\dagger(\omega_b)] |0, 0\rangle_{b_1, b_2}, \quad (\text{B5b})$$

2. Impact of reduced visibility at Charlie

The previous derivation assumed an ideal BSM where the interfering photons at Charlie were perfectly indistinguishable. In practice, reduced interference visibility introduces partial distinguishability, which suppresses the coherence of the heralded state. To account for this, we explicitly include

where $f(\cdot)$, $g(\cdot)$, and $h(\cdot)$ are the spectral amplitude functions of photons in modes a_1 , a_2 , and b_2 , respectively. These are normalized such that $\int d\omega |f(\omega)|^2 = \int d\omega |g(\omega)|^2 = \int d\omega |h(\omega)|^2 = 1$. We assume a separable joint spectral amplitude for the down-converted photons, since here we are concerned only with the impact of imperfect interference at Charlie. The initial joint state is then

$$\begin{aligned} |\Psi\rangle_i &= |\phi\rangle_A \otimes |\psi\rangle_B \\ &= \alpha_0 \left[\int d\omega_b h(\omega_b) [\beta_1 b_1^\dagger(\omega_b) + \beta_0 b_2^\dagger(\omega_b)] \right] |\mathbf{0}\rangle_{a_1, b_1, a_2, b_2} \\ &\quad + \alpha_1 \left[\iiint d\omega_{a_1} d\omega_{a_2} d\omega_b f(\omega_{a_1}) g(\omega_{a_2}) h(\omega_b) a_1^\dagger(\omega_{a_1}) \right. \\ &\quad \left. \times a_2^\dagger(\omega_{a_2}) [\beta_1 b_1^\dagger(\omega_b) + \beta_0 b_2^\dagger(\omega_b)] \right] |\mathbf{0}\rangle_{a_1, b_1, a_2, b_2}. \end{aligned} \quad (\text{B6})$$

$$\begin{aligned} \rho_f^{(\pm)} &= \mathcal{N}^{-1} [t_b |0, 0\rangle \langle 0, 0| \pm \mu^* \lambda \sqrt{t_b(1-t_b)} e^{-i(\varphi_a + \varphi_b)} |0, 0\rangle \langle 1, 1| \\ &\quad \pm \mu \lambda \sqrt{t_b(1-t_b)} e^{i(\varphi_a + \varphi_b)} |1, 1\rangle \langle 0, 0| + \lambda^2(1-t_b) |1, 1\rangle \langle 1, 1| + 2\lambda^2 t_b(1-\eta_H) |1, 0\rangle \langle 1, 0|]_{a_1, b_1}. \end{aligned} \quad (\text{B9})$$

APPENDIX C: LOG-NEGATIVITY

To quantify the degree of entanglement in the heralded state, we compute the log-negativity of the shared state, defined as $\mathcal{E}_N(\rho) = \log_2 \|\rho^T\|_1$, where ρ^T is the partially transposed density matrix and $\|\cdot\|_1$ denotes the trace norm. In the limit $\lambda^2 t_b \ll 1$ and perfect interference at Charlie, the partial transpose of the heralded density matrix with respect to B is

$$\begin{aligned} \rho_{\pm}^{T_B} &= \mathcal{N}^{-1} [t_b |0, 0\rangle \langle 0, 0| \pm \lambda \sqrt{t_b(1-t_b)} \\ &\quad \times (e^{-i(\varphi_a + \varphi_b)} |0, 1\rangle \langle 1, 0| + e^{i(\varphi_a + \varphi_b)} |1, 0\rangle \langle 0, 1|) \\ &\quad + \lambda^2(1-t_b) |1, 1\rangle \langle 1, 1|]_{a_1, b_1}. \end{aligned} \quad (\text{C1})$$

The eigenvalues of this density matrix can be easily read as $\mathcal{N}^{-1} \{t_b, \lambda^2(1-t_b), +\lambda\sqrt{t_b(1-t_b)}, -\lambda\sqrt{t_b(1-t_b)}\}$. Hence, its trace norm is the sum of the absolute values of those eigenvalues

$$\begin{aligned} \|\rho^{T_B}\|_1 &= \mathcal{N}^{-1} [t_b + \lambda^2(1-t_b) + 2\lambda\sqrt{t_b(1-t_b)}] \\ &= 1 + 2 \frac{\lambda\sqrt{t_b(1-t_b)}}{\lambda^2(1-t_b) + t_b}. \end{aligned} \quad (\text{C2})$$

The log-negativity is then

$$\mathcal{E}_N = \log_2 \left(1 + 2 \frac{\lambda\sqrt{t_b(1-t_b)}}{\lambda^2(1-t_b) + t_b} \right). \quad (\text{C3})$$

$\mathcal{E}_N$ is an entanglement monotone that vanishes if and only if the state is separable and provides an upper bound on the distillable entanglement [39,40]. This behavior is illustrated in Fig. 5, where the log-negativity is plotted as a function of the tuning parameters g and t_b , showing a smooth transition

The BSM projector [Eq. (B2)] generalizes to

$$\begin{aligned} \Pi_C^{(\pm)} &= \frac{1}{2} \int d\omega (a_2^\dagger(\omega) \pm b_2^\dagger(\omega)) |0, 0\rangle \langle 0, 0| \\ &\quad \times (a_2(\omega) \pm b_2(\omega))_{a_2, b_2}. \end{aligned} \quad (\text{B7})$$

The heralded state is obtained as $\tilde{\rho}_{AB}^{(\pm)} = \mathcal{N}^{-1} \text{Tr}_{a_2, b_2} [\Pi_C^{(\pm)} |\Psi\rangle \langle \Psi|_i \Pi_C^{(\pm)}]$. After lengthy but straightforward calculations, this evaluates to

$$\begin{aligned} \tilde{\rho}_{AB}^{(\pm)} &= \mathcal{N}^{-1} [t_b |0, 0\rangle \langle 0, 0| \pm \lambda \sqrt{t_b(1-t_b)} \\ &\quad \times (\mu^* e^{-i(\varphi_a + \varphi_b)} |0, 0\rangle \langle 1, 1| + \mu e^{i(\varphi_a + \varphi_b)} |1, 1\rangle \langle 0, 0|) \\ &\quad + \lambda^2(1-t_b) |1, 1\rangle \langle 1, 1|]_{a_1, b_1}, \end{aligned} \quad (\text{B8})$$

where we define $\mu = \int d\omega h^*(\omega)g(\omega)$. The spectral mode overlap, and hence the interference visibility at Charlie (assuming indistinguishability in all other degrees of freedom), is quantified by $V = |\mu|^2$ [38]. This shows that the distinguishability reduces the coherent or off-diagonal terms of the heralded state only. Thus, including symmetric channel and detector losses, the final heralded state becomes

from $\mathcal{E}_N = 1$ (maximally entangled) down to $\mathcal{E}_N = 0$ (fully separable).

APPENDIX D: NONLOCALITY TEST FOR PHOTON-NUMBER PATH-ENTANGLEMENT

1. Ideal case

Here, we derive the CHSH and CH inequalities for PNPE states probed using displacement (or weak-field homodyne)

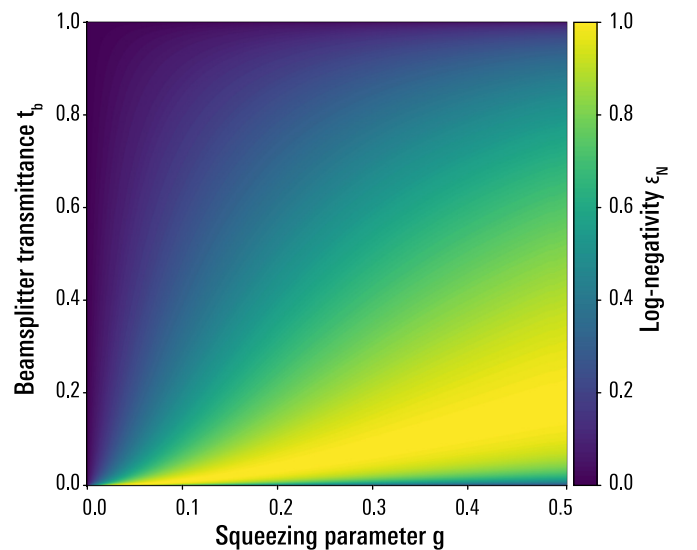


FIG. 5. Log-negativity $\mathcal{E}_N$ of the heralded state as a function of the squeezing parameter g and Bob's BS transmittance t_b . The color scale ranges from $\mathcal{E}_N = 0$ (separable) to 1 (maximally entangled), demonstrating the protocol's tunable entanglement.

measurements, assuming the ideal case with no loss. In practice, the displacement is implemented by mixing the mode with a weak local oscillator on a highly transmissive beamsplitter—an approximation valid in the limit of large coherent state amplitude and vanishing beamsplitter reflectivity [27]. After Alice and Bob interfere their local modes (a_1 and b_1)—hereafter denoted simply as a and b for convenience—with their respective local oscillators ($|\alpha'\rangle$ and $|\beta'\rangle$), the joint state becomes

$$|\psi\rangle_f = \hat{D}_a(-\alpha)\hat{D}_b(-\beta)|\psi\rangle_{AB}, \quad (\text{D1})$$

where $\hat{D}(\cdot)$ denotes the displacement operator, $\alpha = \alpha'\sqrt{r_a}$ and $\beta = \beta'\sqrt{r_b}$, and r_a (r_b) is the reflectance of Alice's (Bob's) beam splitter. Since we are working with PNPE states, photon counting must be dichotomized into binary outcomes. Here, we group the results into zero and nonzero clicks. This type of measurement encompasses all possible outcomes, including losses, allowing both CH and CHSH inequalities to be applied without relying on assumptions such as fair sampling. In what follows, we focus on the CHSH inequality, defined as [26]

$$S = E(\alpha_0, \beta_0) + E(\alpha_0, \beta_1) + E(\alpha_1, \beta_0) - E(\alpha_1, \beta_1). \quad (\text{D2})$$

This means we need to calculate the correlation coefficients, $E(\alpha, \beta)$, for different settings of α & β to calculate the CHSH parameter S . The correlation coefficient for click or no-click outcomes is given by

$$E(\alpha, \beta) = P(i=0, j=0|\alpha, \beta) + P(i \neq 0, j \neq 0|\alpha, \beta) - P(i=0, j \neq 0|\alpha, \beta) - P(i \neq 0, j=0|\alpha, \beta). \quad (\text{D3})$$

To calculate the probabilities—and consequently the correlation coefficients—we first define the measurement operators, or positive-operator valued measure (POVM) elements. Since we are interested in a binary-outcome measurement (zero versus nonzero photon counts), this can be described by the following pair of orthogonal projection operators [27]:

$$\hat{\Pi}_0 = |0\rangle\langle 0|, \quad \hat{\Pi}_n = \hat{I} - \hat{\Pi}_0. \quad (\text{D4})$$

The joint probabilities in Eq. (D3) follow directly from the expectation values of the relevant POVM elements. The joint no-click probability is therefore

$$\begin{aligned} P(i=0, j=0|\alpha, \beta) &= \langle \psi_f | \hat{\Pi}_{0a} \otimes \hat{\Pi}_{0b} | \psi_f \rangle \\ &= \langle \psi_{AB} | \hat{D}_b^\dagger(-\beta) \hat{D}_a^\dagger(-\alpha) | 0, 0 \rangle \\ &\quad \times \langle 0, 0 | \hat{D}_a(-\alpha) \hat{D}_b(-\beta) | \psi_{AB} \rangle \\ &= \langle \psi_{AB} | \alpha, \beta \rangle \langle \alpha, \beta | \psi_{AB} \rangle \\ &= |\langle \alpha, \beta \rangle \psi_{AB}|^2, \end{aligned} \quad (\text{D5})$$

where we used the fact that $\hat{D}^\dagger(-\alpha) = \hat{D}(\alpha)$ and $\hat{D}(\alpha)|0\rangle = |\alpha\rangle$.

Note that Eq. (D5) corresponds to projecting the state $|\psi\rangle_{AB}$ onto the coherent state $|\alpha, \beta\rangle$, and is, up to a normalization factor, equivalent to the joint Husimi Q-function of the state [27]. Therefore, we write $P(i=0, j=0|\alpha, \beta) = Q_{ab}(\alpha, \beta)$, and the marginal probability of observing no clicks at one side is given by the marginal Q-function, i.e., $P(i=0|\alpha) = Q(\alpha)$.

The joint probability of detecting a click at both Alice and Bob is more involved, and is obtained by projecting the state onto the complementary operator $\hat{\Pi}_{na} \otimes \hat{\Pi}_{nb} = \hat{I}_a \otimes \hat{I}_b - \hat{\Pi}_{0a} \otimes \hat{I}_b - \hat{I}_a \otimes \hat{\Pi}_{0b} + \hat{\Pi}_{0a} \otimes \hat{\Pi}_{0b}$. Hence,

$$\begin{aligned} P(i \neq 0, j \neq 0|\alpha, \beta) &= 1 - \langle \hat{\Pi}_{0a} \otimes \hat{I}_b \rangle - \langle \hat{I}_a \otimes \hat{\Pi}_{0b} \rangle \\ &\quad + \langle \hat{\Pi}_{0a} \otimes \hat{\Pi}_{0b} \rangle = 1 - Q_a(\alpha) - Q_b(\beta) + Q_{ab}(\alpha, \beta). \end{aligned} \quad (\text{D6})$$

The joint probability of no-click at Alice and any-clicks at Bob is

$$\begin{aligned} P(i=0, j \neq 0|\alpha, \beta) &= \langle \hat{\Pi}_{0a} \otimes \hat{\Pi}_{nb} \rangle \\ &= \langle \hat{\Pi}_{0a} \otimes \hat{I}_b - \hat{\Pi}_{0a} \otimes \hat{\Pi}_{0b} \rangle \\ &= Q_a(\alpha) - Q_{ab}(\alpha, \beta). \end{aligned} \quad (\text{D7})$$

Similarly, one can easily find the probability

$$P(i \neq 0, j=0|\alpha, \beta) = Q_b(\beta) - Q_{ab}(\alpha, \beta).$$

Substituting these joint probabilities into Eqs. (D2) and (D3), we express the CHSH parameter S in terms of the Q functions:

$$\begin{aligned} S &= 2 + 4[Q_{ab}(\alpha_0, \beta_0) + Q_{ab}(\alpha_0, \beta_1) + Q_{ab}(\alpha_1, \beta_0) \\ &\quad - Q_{ab}(\alpha_1, \beta_1) - Q_a(\alpha_0) - Q_b(\beta_0)]. \end{aligned} \quad (\text{D8})$$

Finally, it is worth mentioning that the CH parameter is related to S via $S = 2 + 4\text{CH}$ [19].

2. Lossy case

In Appendix D1, we considered the ideal case in which the shared state between Alice and Bob is pure and no losses are present. However, in our proposed implementation, the shared state is described by the mixed density matrix given in Eq. (A1), and both local losses and nonideal detectors must be taken into account. We model local losses on Alice and Bob's sides by introducing fictitious beamsplitters with transmission efficiency η_D in front of each detector. In the presence of loss, the POVM elements corresponding to displacement-based click/no-click measurements are given by

$$\hat{\Pi}_0^{(\alpha, \eta_D)} = D^\dagger(\alpha)(1 - \eta_D)^{a^\dagger a} D(\alpha), \quad (\text{D9a})$$

$$\hat{\Pi}_n^{(\alpha, \eta_D)} = \hat{I} - \hat{\Pi}_0^{(\alpha, \eta_D)}, \quad (\text{D9b})$$

where η_D denotes the total detection efficiency, incorporating transmission, coupling, and detector losses. In the following, we consider the symmetric case where both Alice and Bob have the same detection efficiency.

TABLE II. Optimal settings for each detection efficiency to maximize the CHSH violation in strawberry fields simulations.

No.	η_D	t_b	g	α_0	α_1	β_0	β_1	CHSH
1	0.65	0.999	0.090	0.000	-0.013	0.000	0.013	2.000 000
2	0.67	0.945	0.048	0.001	-0.128	-0.001	0.128	2.000 002
3	0.68	0.905	0.132	0.010	-0.241	-0.010	0.241	2.000 133
4	0.70	0.823	0.226	0.035	-0.364	-0.035	0.364	2.002 067
5	0.75	0.614	0.279	0.097	-0.501	-0.097	0.501	2.027 202
6	0.80	0.181	0.094	0.139	-0.554	-0.139	0.554	2.088 839
7	0.85	0.435	0.320	0.162	-0.574	-0.162	0.574	2.186 472
8	0.90	0.414	0.364	0.172	-0.579	-0.172	0.579	2.318 223
9	0.95	0.296	0.284	0.172	-0.573	-0.172	0.573	2.483 976
10	1.00	0.280	0.300	0.165	-0.560	-0.165	0.560	2.685 871

The joint probability of no detections is now giving by

$$Q_{ab}^{\eta_D}(\alpha, \beta) = \text{Tr}[\rho_{AB}(\hat{\Pi}_0^{(\alpha, \eta_D)} \otimes \hat{\Pi}_0^{(\beta, \eta_D)})] = e^{-(\alpha^2 + \beta^2)\eta_D} \times \left[\frac{t_b + \lambda^2(1 - t_b)(1 - \eta_D + \eta_D^2|\alpha|^2)(1 - \eta_D + \eta_D^2|\beta|^2) + 2\lambda\sqrt{(1 - t_b)t_b}\alpha\beta\eta_D^2\cos(\varphi_a + \varphi_b - \varphi_\alpha - \varphi_\beta)}{t_b + \lambda^2(1 - t_b)} \right]. \quad (\text{D10})$$

Similarly, the marginal probabilities can be obtained by tracing over the measurement outcomes of the other party. That is

$$Q_a^{\eta_D}(\alpha) = \text{Tr}[\rho_{AB}(\hat{\Pi}_0^{(\alpha, \eta_D)} \otimes \hat{\mathbb{I}}_b)] = \left[\frac{t_b + \lambda^2(1 - t_b)(1 - \eta_D + \alpha^2\eta_D^2)}{t_b + \lambda^2(1 - t_b)} \right] e^{-\alpha^2\eta_D}, \quad (\text{D11})$$

$$Q_b^{\eta_D}(\beta) = \text{Tr}[\rho_{AB}(\hat{\mathbb{I}}_a \otimes \hat{\Pi}_0^{(\beta, \eta_D)})] = \left[\frac{t_b + \lambda^2(1 - t_b)(1 - \eta_D + \beta^2\eta_D^2)}{t_b + \lambda^2(1 - t_b)} \right] e^{-\beta^2\eta_D}. \quad (\text{D12})$$

Substituting these expressions into Eq. (D8), we obtain the final form of the CHSH parameter, which we then maximize numerically for each detection efficiency to produce the analytical results shown in the main text.

For the simulation results, the entire experimental layout was modeled using STRAWBERRY FIELDS [41]. Numerical optimization was performed to determine the optimal state preparation and measurement settings for each detection efficiency. The corresponding CHSH values, shown in Fig. 2 in the main text, were obtained using these optimal settings, which are listed in Table II. In the numerical model, the preparation of Alice's and Bob's local state includes higher photon number components, up to a cutoff of the Fock state basis at $n = 6$. This introduces a maximum truncation error of approximately 10^{-5} , given that we operate in the low-gain regime, with the squeezing parameter limited to a maximum of 0.5. Since all gain values in Table II are below 0.5, the truncation error is even smaller than the conservative estimate above, which explains the excellent agreement observed between our numerical simulations and the analytical model.

3. Impact of reduced visibility at Charlie on the Bell test

Since reduced visibility affects only the off-diagonal elements of the heralded state, it modifies the joint no-click probability while leaving the marginal probabilities unchanged. The joint probability of no detections is then

$$Q_{ab}^{\eta_D}(\alpha, \beta) = e^{-(\alpha^2 + \beta^2)\eta_D} \left[\frac{t_b + \lambda^2(1 - t_b)(1 - \eta_D + \eta_D^2|\alpha|^2)(1 - \eta_D + \eta_D^2|\beta|^2) + 2\lambda\sqrt{(1 - t_b)t_b}\mu\alpha\beta\eta_D^2\cos(\varphi_a + \varphi_b - \varphi_\alpha - \varphi_\beta)}{t_b + \lambda^2(1 - t_b)} \right]. \quad (\text{D13})$$

Reduced visibility suppresses the interference term, directly lowering both the maximum achievable CHSH violation and the overall loss tolerance, as illustrated in

Fig. 6. In particular, decreasing visibility raises the threshold detection efficiency required for violation while simultaneously reducing the maximum CHSH value. Nevertheless,

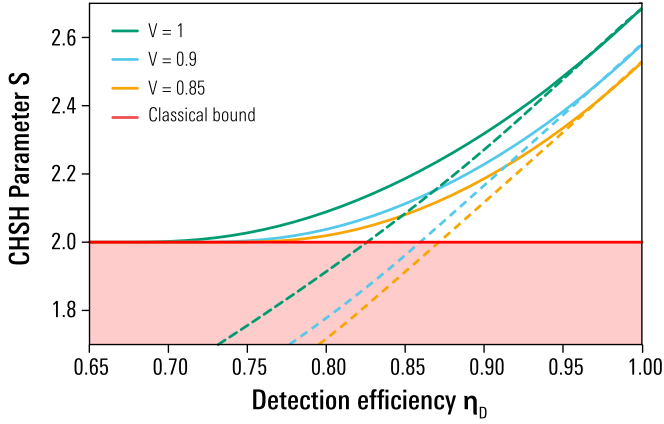


FIG. 6. Optimal CHSH violations as a function of symmetric detection efficiency η_D for different interference visibilities at Charlie, computed for our protocol (solid lines) and SPPE-based protocols (dashed lines).

the figure demonstrates that our protocol is not only more resilient to loss but also less sensitive to imperfect interference between independent sources. Moreover, visibilities exceeding 90% between independent SPDC sources are routinely achieved with state-of-the-art technology [42].

4. Resilience to losses

Figure 7 shows the CHSH violation as a function of the local detection efficiencies at Alice and Bob, and the symmetric heralding efficiency. We see that the violation is almost independent of the losses in the transmitted modes to Charlie, but strongly dependent on the losses in the local mode, which are the losses we wish to minimize to achieve a violation and close the detection loophole.

APPENDIX E: SUCCESS RATE

Here, we derive the success probability for each protocol. In all cases, a successful BSM occurs when Charlie receives

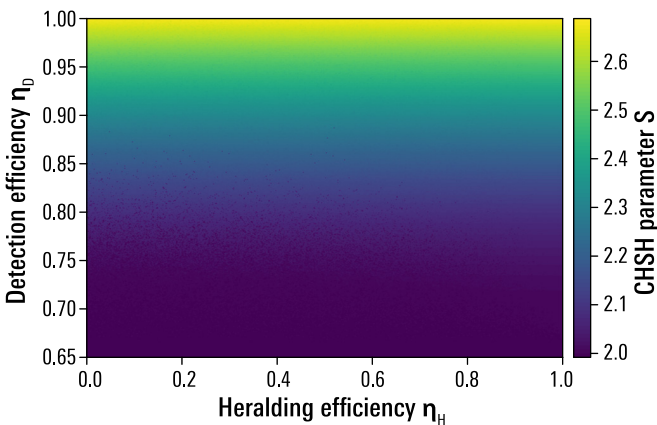


FIG. 7. CHSH as a function of heralding η_H and detection η_D efficiencies. We see that our protocol is independent of η_H .

exactly one photon. The probability of this event is given by

$$\begin{aligned} P_{\text{succ}} &= \sqrt{\eta_C}(P_A(0)P_B(1) + P_A(1)P_B(0)) \\ &= (1 - P_A(1)\sqrt{\eta_C})(P_B(1)\sqrt{\eta_C}) \\ &\quad + (1 - P_B(1)\sqrt{\eta_C})(P_A(1)\sqrt{\eta_C}) \\ &= \sqrt{\eta_C}(P_A(1) + P_B(1)) - 2\eta_C P_A(1)P_B(1), \end{aligned} \quad (\text{E1})$$

where $\sqrt{\eta_C}$ is the channel transmittance from Alice or Bob to Charlie and $P_x(y)$ is the probability that the initial state prepared by party x contains y photons in the transmitted mode.

For the protocol proposed in Ref. [15], we have $P_A(1) = P_B(1) = \frac{\lambda^2}{1+\lambda^2}$. In the protocol of Ref. [16], assuming ideal heralding detectors, the corresponding probabilities are $P_A(1) = P_B(1) = \frac{\lambda^2}{1+\lambda^2}t$, where t is the transmission probability through the beamsplitters used to generate the single-photon entangled states. Following the condition $t \ll 1$ assumed in the original work, we set $t = 0.1$.

For our protocol, we have $P_A(1) = \frac{\lambda^2}{1+\lambda^2}$ and $P_B(1) = \frac{\lambda^2}{1+\lambda^2}t_b$, where we set $t_b = 0.8$ to maximize the CHSH violation at detection efficiency $\eta_D = 0.75$. We note that in our scheme, one can trade off some Bell violation to increase the success probability. In the limit $t_b \rightarrow 1$, our protocol's success probability becomes identical to that of the TMSV-based protocol in Ref. [15].

APPENDIX F: SOURCE OF IMPROVEMENT WITH EXACT MODEL

In this Appendix, we investigate the origin of the performance advantage in our protocol and the reason for the differing behavior exhibited by the states used in various schemes. To this end, we derive an analytical expression for the CHSH inequality as a function of the measurement settings and detection efficiency, assuming systematic loss, for the states

$$|\psi\rangle = c_0|0, 1\rangle_{a,b} + c_1e^{i\varphi}|1, 0\rangle_{a,b}, \quad (\text{F1a})$$

$$|\phi\rangle = c_0|0, 0\rangle_{a,b} + c_1e^{i\varphi}|1, 1\rangle_{a,b}. \quad (\text{F1b})$$

Without loss of generality, the states are expressed in this form to simplify the analysis and to isolate the effect of loss on each Fock component individually. Since the Hilbert spaces for these states are spanned by vacuum and single-photon only (i.e., $\{|0\rangle, |1\rangle\}$), then the matrix form of the no-click POVM is given by [19]

$$\hat{\Pi}_0^{(\alpha, \eta)} = \begin{bmatrix} e^{-\eta|\alpha|^2} & -\eta\alpha^*e^{-\eta|\alpha|^2} \\ -\eta\alpha e^{-\eta|\alpha|^2} & (1 - \eta + \eta^2|\alpha|^2)e^{-\eta|\alpha|^2} \end{bmatrix}. \quad (\text{F2})$$

We follow the same procedure outlined in Sec. V to compute the exact Q-functions for both states and evaluate the CHSH inequality as a function of loss. This analysis reveals the fundamental difference between the two states in terms of their ability to reach the Eberhard limit.

1. Exact probabilities for $|\psi\rangle$ with loss

The single probabilities of no-detection are given by [27]

$$Q_a^\psi(\alpha) = \langle \hat{\Pi}_0^{(\alpha,\eta)} \otimes \hat{\mathbb{I}}_b \rangle = [c_1^2(1 - \eta + \eta^2\alpha^2) + c_0^2]e^{-\eta|\alpha|^2}, \quad (\text{F3a})$$

$$Q_b^\psi(\beta) = \langle \hat{\mathbb{I}}_a \otimes \hat{\Pi}_0^{(\beta,\eta)} \rangle = [c_0^2(1 - \eta + \eta^2\beta^2) + c_1^2]e^{-\eta|\beta|^2}. \quad (\text{F3b})$$

The joint probability of no-detection is

$$\begin{aligned} Q_{ab}^\psi(\alpha, \beta) &= \langle \hat{\Pi}_0^{(\alpha,\eta)} \otimes \hat{\Pi}_0^{(\beta,\eta)} \rangle \\ &= [c_0^2(1 - \eta + \eta^2|\beta|^2) + c_1^2(1 - \eta + \eta^2|\alpha|^2) \\ &\quad + 2\alpha\beta c_0 c_1 \eta^2 \cos(\varphi + \varphi_\alpha - \varphi_\beta)]e^{-\eta(|\alpha|^2 + |\beta|^2)}. \end{aligned} \quad (\text{F4})$$

2. Exact probabilities for $|\phi\rangle$ with loss

The single probabilities are given by

$$Q_a^\phi(\alpha) = \langle \hat{\Pi}_0^{(\alpha,\eta)} \otimes \hat{\mathbb{I}}_b \rangle = [c_0^2 + c_1^2(1 - \eta + \eta^2\alpha^2)]e^{-\eta|\alpha|^2}, \quad (\text{F5a})$$

$$Q_b^\phi(\beta) = \langle \hat{\mathbb{I}}_a \otimes \hat{\Pi}_0^{(\beta,\eta)} \rangle = [c_0^2 + c_1^2(1 - \eta + \eta^2\beta^2)]e^{-\eta|\beta|^2}. \quad (\text{F5b})$$

The joint probability of no-detection is

$$\begin{aligned} Q_{ab}^\phi(\alpha, \beta) &= \langle \hat{\Pi}_0^{(\alpha,\eta)} \otimes \hat{\Pi}_0^{(\beta,\eta)} \rangle \\ &= [c_0^2 + c_1^2(1 - \eta + \eta^2|\alpha|^2)(1 - \eta + \eta^2|\beta|^2) \\ &\quad + 2\alpha\beta c_0 c_1 \eta^2 \cos(\varphi - \varphi_\alpha - \varphi_\beta)]e^{-\eta(|\alpha|^2 + |\beta|^2)}. \end{aligned} \quad (\text{F6})$$

3. Observations

Finally, this analysis reveals the underlying reason for the contrast in performance between the two states. In the Q -function expressions, we observe that loss affects only the coefficient c_1 in the state $|\phi\rangle$, whereas in the state $|\psi\rangle$, both c_0 and c_1 are impacted. This is intuitive: Photon loss does not affect the vacuum, but it does degrade the single-photon components of the state. In $|\phi\rangle$, only the $|1, 1\rangle$ term is susceptible to loss, while in $|\psi\rangle$, both terms involve single-photon contributions. As a result, minimizing c_1 in $|\phi\rangle$ can suppress the effect of loss, enabling a Bell violation even under high-loss conditions when optimizing the measurement settings, as demonstrated in the main text. This highlights the vacuum as a valuable resource in our protocol.

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