

Universal squeezing-enhanced readout approach for bosonic qubits

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In this article, we propose a squeezing-enhanced protocol for the efficient readout of bosonic qubits, whose encoding states can be first mapped onto a multilevel atom and then read out via dispersive coupling. We demonstrate that by simultaneously applying intracavity and external squeezing for the readout of this multilevel atom, the measurement signal-to-noise ratio between adjacent energy levels can be exponentially enhanced. We begin by examining this enhancement in a three-level system and subsequently extend the framework to multilevel systems, proving that the signal-to-noise ratio between any adjacent energy levels can achieve a similar exponential improvement. This dual-squeezing approach leads to an exponential enhancement in both the dispersive coupling strength and the signal-to-noise ratio. As a result, the single-shot readout time is exponentially shortened and the single-shot measurement error is exponentially reduced, ultimately improving the single-shot readout fidelity of bosonic qubits by more than an order of magnitude. This method provides an effective pathway for high-fidelity readout based on bosonic encoding.

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I. INTRODUCTION

High-fidelity qubit readout is an essential component of quantum information technologies, such as quantum computing [1–4], quantum communication [5–7], and quantum precision measurement [8–10]. To achieve high-fidelity qubit readout, the key requirement is that the measurement process must possess the quantum nondemolition (QND) property. The QND measurement refers to a process that avoids the backaction on the physical quantity being observed and does not introduce disturbance, allowing one to obtain consistent results when repeatedly measuring the same physical quantity [11–17]. This property is indispensable for logical qubit readout, as it permits the repeated extraction of error syndromes or state information via ancillary systems while preserving the underlying logical quantum states [18,19].

A prime example of a QND measurement is the dispersive readout [20,21]. In this approach, the qubit is dispersively coupled to a readout cavity so that its state results in a shift of the cavity resonance frequency, and this frequency shift is subsequently exploited for measurement using homodyne detection [22–24]. Such a readout method has become one of the widely applied readout techniques in circuit quantum electrodynamics [23,25–30] and optomechanical systems [31].

Despite its successful implementation across various physical platforms, the ultimate readout fidelity of dispersive readout is still subject to two main limitations: the measurement error due to the finite signal-to-noise ratio (SNR) in a single-shot measurement [32] and qubit-state decay caused by relaxation during the measurement process [33]. Given the standard requirements for speed and fidelity in such a readout protocol to overcome the limitations above, employing squeezed-light states to improve performance is consequently a highly desirable approach. As an effective quantum resource capable of surpassing the standard quantum limit [32,34], squeezed light has been widely applied in light-matter interaction enhancement [35–39], and high-precision measurements [40,41], and can effectively enhance dispersive coupling and the SNR [36,42,43]. In 2024, Qin *et al.* [44] demonstrated that by concurrently employing both injected external squeezing and intracavity squeezing, it is possible to achieve an exponential improvement in the readout SNR for arbitrary measurement times. The idea of this method is to encode the qubit-state information into a Bogoliubov mode of the squeezed cavity. This prevents the state-dependent rotation of the squeezing ellipse, thereby ensuring an exponential enhancement of the SNR. The resulting SNR scales as e^{2r} for short-time measurements, where r is the squeezing parameter. This work greatly advances the QND readout of two-level systems. Beyond two-level architectures, however, multilevel systems are of profound importance in modern quantum information processing, particularly for bosonic encoding that utilizes the infinite-dimensional Hilbert space of a harmonic oscillator to construct logical qubits [45]. Therefore, extending the dual-squeezing enhancement mechanism to multilevel architectures provides a promising way for fast, high-fidelity readout of multilevel bosonic qubits.

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In this article, following Ref. [44], we calculate the exponential enhancement of the SNR in a three-level system. Then, we extend this result to multilevel systems, demonstrating that the SNR between any adjacent energy levels can also be exponentially enhanced. Building on this, we use both injected external squeezing and intracavity squeezing to achieve a more efficient enhancement in the dispersive readout for bosonic qubits, which have been introduced to encode logical qubits in multiphoton Fock states, providing greater fault tolerance [46–48]. Note that a robust readout of bosonic qubits in the dispersive coupling regime was introduced in 2018 [49]. The approach leverages the statistical averaging effect from multiple independent measurements, effectively suppressing the impact of the single-shot measurement error and qubit relaxation, thereby improving the readout fidelity. However, the performance improvement of Ref. [49] relies on the increase of both the encoding level and the number of measurements, which imposes significant demands on coherence time and limits the speed of the readout. By simultaneously applying intracavity and external squeezing to the readout cavity of the ancilla for bosonic qubits, we achieve an exponential enhancement of the dispersive coupling strength and the SNR, thereby exponentially reducing both the single-shot measurement time and the measurement error. Physically, this exponential enhancement arises from the constructive interplay between the two squeezing mechanisms: The intracavity squeezing amplifies the effective light-matter interaction, while the external squeezing reshapes the input noise to be orthogonal to the amplified signal quadrature. As a result, the signal grows while the noise is suppressed, leading to an exponential improvement in the SNR and an exponential reduction in both the measurement time and the error probability.

The rest of this article is structured as follows. In Sec. II, we briefly review the robust readout scheme of Hann *et al.* In Sec. III, we investigate the squeezing enhancement in a three-level system, demonstrating that the simultaneous application of intracavity and external squeezing leads to an exponentially enhanced dispersive coupling strength and SNR, and further analyze the resulting exponential suppression of the single-shot readout time and the measurement error. In Sec. IV, we generalize this dual-squeezing framework to multilevel systems, proving that the dispersive coupling strength and measurement SNR between any adjacent energy levels achieve a similar exponential enhancement. In Sec. V, we demonstrate that compared with the protocol without squeezing, the squeezing-enhanced protocol can improve the readout fidelity by more than an order of magnitude. Finally, Sec. VI provides a conclusion of the entire work.

II. MAPPING A BOSONIC QUBIT ONTO AN ANCILLA

We first review the previous dispersive readout approach for bosonic qubits [49]. As shown in Fig. 1(a), the process of mapping a bosonic qubit onto an ancilla and subsequently reading out the ancilla is referred to as first-level readout, where τ is the duration of a first-level readout (i.e., the single-shot readout time). The readout and reset of the an-

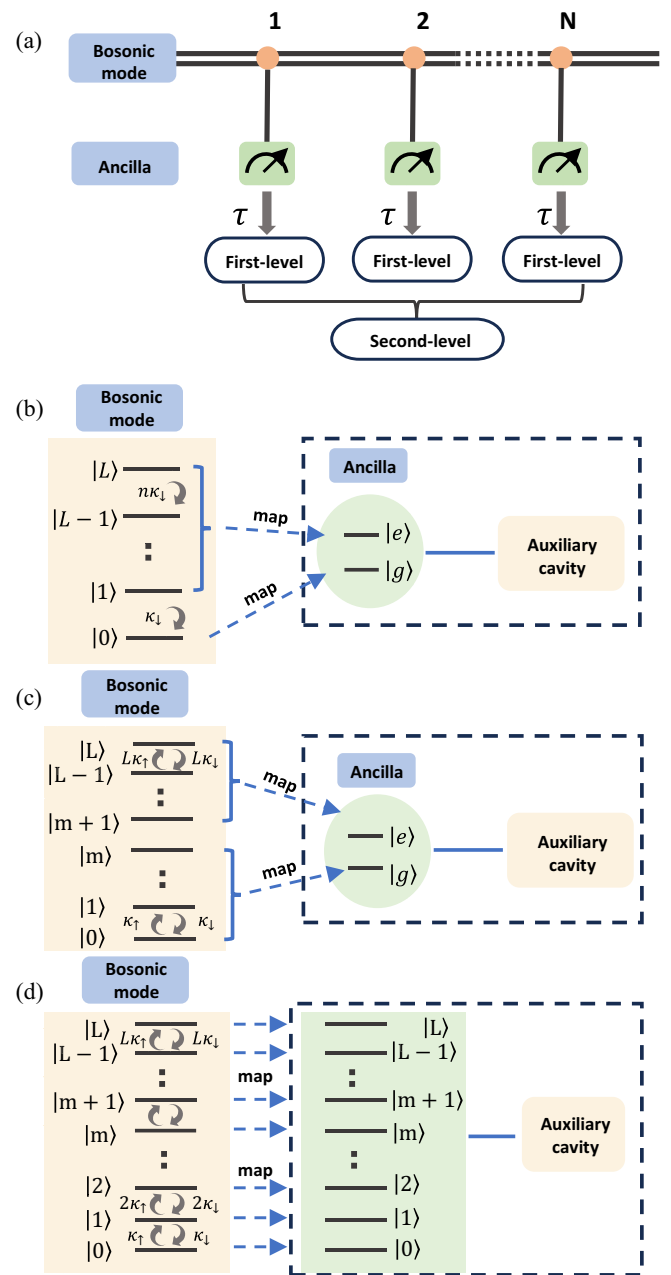


FIG. 1. (a) Quantum circuit for the readout scheme. The bosonic qubit state is read out via repeated QND mappings onto an ancilla. (b) Considering only bosonic mode relaxation. Fock states in the bosonic mode decay with rates proportional to their excitation number. State $|0\rangle$ maps to the ancilla ground state and all excited states map to the ancilla excited state. (c) Including both relaxation and heating. Decays and excitations occur between adjacent Fock states with rates proportional to the excitation number. States $|n < m\rangle$ map to the ancilla ground state and $|n > m\rangle$ map to the ancilla excited state. (d) Readout scheme for multilevel ancilla. Decays and excitations take place only between adjacent bosonic Fock states, with each Fock state being mapped to a corresponding unique ancilla state. In the majority voting, all ancilla readouts of $|n > m\rangle$ are counted as votes for $|L\rangle$, while readouts of $|n \leq m\rangle$ are counted as votes for $|0\rangle$.

cilla still require coupling to an additional ancillary cavity to perform further QND measurements [as shown in the part of the dashed box in Figs. 1(b)–1(d)]. Conventional dispersive readout schemes for the ancilla typically require long measurement times to achieve high readout fidelity [20,50,51]. This problem can be resolved by introducing squeezing (see Secs. III and IV). The process of performing N first-level readouts, analyzing the outcomes, and finally determining the state of the bosonic qubit is referred to as second-level readout.

For instance, the logical states encoding the bosonic qubit are chosen to be the Fock states $|0\rangle_B = |0\rangle$ and $|1\rangle_B = |L\rangle$, where L denotes a positive integer. These logical states can be distinguished using a QND measurement in the dispersive coupling regime. We consider projection measurements, which can be described by a set of measurement operators $\{\mathcal{O}_k\}$ satisfying $\mathcal{O}_k^2 = \mathcal{O}_k$ and $\sum_k \mathcal{O}_k = 1$. When the condition $[\mathcal{O}_k, H(t)] = 0$ (for all k and t) is satisfied, the measurement is QND, where $H(t)$ denotes the Hamiltonian that describes the readout process.

Two scenarios were considered in the protocol in Ref. [49]: (1) accounting only for the relaxation of the bosonic mode, and (2) incorporating both the relaxation and heating of the bosonic mode. For the two-level ancilla, the specific mapping processes are shown in Figs. 1(b) and 1(c). Case (1) is illustrated in Fig. 1(b): The bosonic state $|0\rangle$ is mapped onto the ancilla state $|g\rangle$, while all intermediate Fock states $|n\rangle$ with $0 < n < L$ are mapped onto the ancilla state $|e\rangle$, where $\kappa_\downarrow$ denotes the relaxation rate of the bosonic mode. Case (2) is shown in Fig. 1(c): $|m\rangle$ is a threshold state defined to describe the heating effect, such that the mapping satisfies the following relation:

$$|n\rangle \rightarrow \begin{cases} |g\rangle, & n \leq m, \\ |e\rangle, & n > m. \end{cases} \quad (1)$$

For the multilevel ancilla, the specific mapping process is shown in Fig. 1(d); each Fock state corresponds to a unique auxiliary state.

The performance of the readout protocol is quantified by the readout fidelity [49]

$$F = 1 - P(0_B|1_B) - P(1_B|0_B). \quad (2)$$

Here, $P(i|j)$ represents the probability that the second-level readout yields outcome state $|i\rangle$, given that the bosonic qubit was initially in state $|j\rangle$. Taking case (1) as an example, we have

$$1 - F = P(0|L) + P(L|0) \approx 2 \binom{N}{\lceil N/2 \rceil}^{\delta^{[N/2]}} + (\lceil N/2 \rceil \kappa_\downarrow \tau)^L, \quad (3)$$

where $\lceil \cdot \rceil$ denotes the ceiling function [49]. It is shown that the probabilities $P(0_B|1_B)$ and $P(1_B|0_B)$ can be suppressed by increasing the encoding level L and the number of measurements N , thus improving the readout fidelity. In practice, however, when the single-shot measurement time is fixed, the total measurement time increases with N , imposing significant demands on coherence time and limits the speed of the readout. In the following, we present our protocol, which ex-

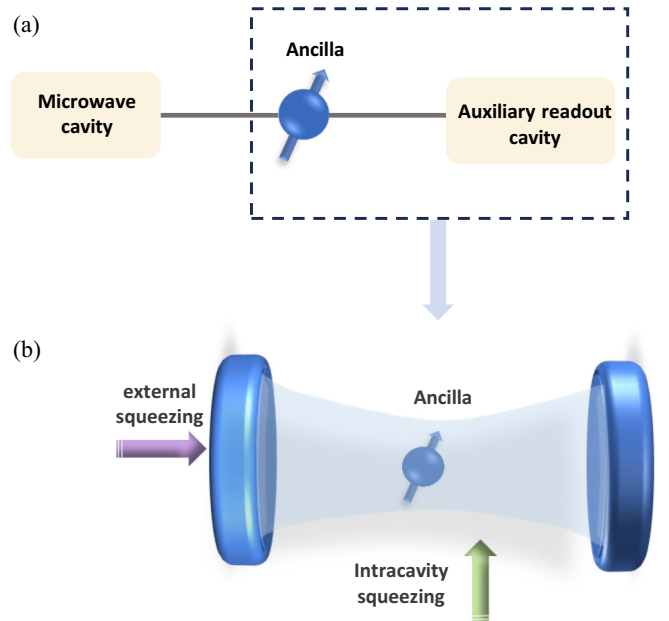


FIG. 2. (a) Schematic of our readout protocol. The state of a microwave cavity mode can be read out by a dispersive coupling to a logical qubit. The logical qubit is measured via its coupling to the readout cavity. (b) Schematic of intracavity squeezing and external squeezing applied to the readout cavity.

ponentially suppresses both the single-shot readout time and the measurement error by enhancing the dispersive coupling and the SNR, resulting in a further exponential improvement of the readout fidelity.

III. SQUEEZING ENHANCEMENT IN THREE-LEVEL SYSTEMS

The model we considered is shown in Fig. 2(a): The state of the microwave cavity mode can be read out via its dispersive coupling to an ancilla, and the measurement of the ancilla is realized through its coupling to a readout cavity.

As shown in Fig. 2(b), the key aspect of our protocol is the simultaneous application of both intracavity squeezing (a two-photon drive with amplitude η , frequency ω_d , and phase θ) and external squeezing (a squeezed vacuum reservoir with squeezing parameter r and reference phase φ) to the readout cavity. By simultaneously applying both intracavity and external squeezing and mapping the qubit-state information onto the Bogoliubov mode of the cavity, the squeezing axis becomes independent of the qubit state. This ensures that the squeezing direction remains optimally aligned with the signal quadrature, effectively preventing detrimental rotations of the squeezing axis and thus leading to a stable enhancement of the signal-to-noise ratio.

For the case of a two-level ancilla, the exponential enhancement of dispersive coupling and SNR has already been derived in Ref. [44] (in subsequent simulations of the two-level case, we directly apply these results). Therefore, we focus on the case of a multilevel ancilla. We first consider the three-level case.

A. Exponentially reduced single-shot readout time

The system Hamiltonian of the qutrit coupled to a cavity subject to dual squeezing in a frame rotating at ω_d is described by

$$H_{\text{qutrit}} = \Delta_c a^\dagger a + \Delta_{ge} |e\rangle\langle e| + (\Delta_{ge} + \Delta_{ef}) |f\rangle\langle f| + g_1 (a^\dagger \sigma_{ge}^- + a \sigma_{ge}^+) + g_2 (a^\dagger \sigma_{ef}^- + a \sigma_{ef}^+) + \lambda (e^{i\theta} a^{\dagger 2} + e^{-i\theta} a^2), \quad (4)$$

where $\Delta_c = \omega_c - \omega_d/2$, $\Delta_{ge} = \omega_{ge} - \omega_d/2$, and $\Delta_{ef} = \omega_{ef} - \omega_d/2$, respectively, $\sigma_{ge}^- = |g\rangle\langle e|$, $\sigma_{ge}^+ = |e\rangle\langle g|$, $\sigma_{ef}^- = |e\rangle\langle f|$, and $\sigma_{ef}^+ = |f\rangle\langle e|$. Here, ω_c is the frequency of the readout cavity, ω_{ge} and ω_{ef} are the transition frequencies from $|g\rangle$ to $|e\rangle$ and from $|e\rangle$ to $|f\rangle$ with coupling strengths of g_1 and g_2 , respectively.

The Langevin equation of the cavity mode a reads [52]

$$\dot{a}(t) = -i(\Delta_c - i\kappa/2)a - i(g_1 \sigma_{ge}^- + g_2 \sigma_{ef}^-) - i2\lambda e^{i\theta} a^\dagger - \sqrt{\kappa} a_{\text{in}}(t), \quad (5)$$

where κ is the cavity-photon loss rate and $a_{\text{in}}(t)$ denotes the cavity input field. To go beyond the standard quantum limit and exponentially enhance the measurement signal-to-noise ratio, we additionally inject an external squeezed vacuum field into the readout cavity [see Fig. 2(b)]. With the external squeezing, the correlations of the input-noise operator $\mathcal{A}_{\text{in}}(t) = a_{\text{in}}(t) - \langle a_{\text{in}}(t) \rangle$ satisfy [52]

$$\begin{pmatrix} \langle \mathcal{A}_{\text{in}}^\dagger(t) \mathcal{A}_{\text{in}}(t') \rangle & \langle \mathcal{A}_{\text{in}}(t) \mathcal{A}_{\text{in}}(t') \rangle \\ \langle \mathcal{A}_{\text{in}}^\dagger(t) \mathcal{A}_{\text{in}}^\dagger(t') \rangle & \langle \mathcal{A}_{\text{in}}(t) \mathcal{A}_{\text{in}}^\dagger(t') \rangle \end{pmatrix} = \begin{pmatrix} \sinh^2 r & \frac{1}{2} \sinh 2r e^{i\varphi} \\ \frac{1}{2} \sinh 2r e^{-i\varphi} & \cosh^2 r \end{pmatrix} \delta(t - t'). \quad (6)$$

We consider the case of $\Delta_c \neq 0$ and introduce the Bogoliubov mode

$$\beta = \cosh(r_c) a + e^{i\theta} \sinh(r_c) a^\dagger, \quad (7)$$

where the squeezing parameter r_c satisfies $\tanh(2r_c) = 2\lambda/\Delta_c$. The Bogoliubov transformation diagonalizes the squeezed cavity Hamiltonian, converting the original mode a into a squeezed-light mode β that experiences an effective vacuum input when the squeezing parameters are optimally matched. By substituting Eq. (7) into Eq. (5), we can obtain the form of the Langevin equation for mode β ,

$$\dot{\beta}(t) = -i\left(\omega_s - \frac{i}{2}\kappa\right)\beta - i(g_1 S_{ge}^- + g_2 S_{ef}^-) - \sqrt{\kappa} \beta_{\text{in}}(t), \quad (8)$$

$$S_{ge(e f)}^- = \cosh(r_c) \sigma_{ge(e f)}^- - e^{i\theta} \sinh(r_c) \sigma_{ge(e f)}^+,$$

where $\omega_s = \sqrt{\Delta_c^2 - 4\lambda^2}$ and $\beta_{\text{in}}(t)$ is the input field of the Bogoliubov mode β . According to

$$\beta_{\text{in}}(t) = \cosh(r_c) a_{\text{in}}(t) + e^{i\theta} \sinh(r_c) a_{\text{in}}^\dagger(t), \quad (9)$$

the quantum fluctuation operator $\mathcal{B}_{\text{in}}(t) = \beta_{\text{in}}(t) - \langle \beta_{\text{in}}(t) \rangle$, which includes both fluctuations and the mean value, can be expressed as

$$\mathcal{B}_{\text{in}}(t) = \cosh(r_c) \mathcal{A}_{\text{in}}(t) + e^{i\theta} \sinh(r_c) \mathcal{A}_{\text{in}}^\dagger(t). \quad (10)$$

Then, the correlations for the operator $\mathcal{B}_{\text{in}}(t)$ are given by

$$\begin{pmatrix} \langle \mathcal{B}_{\text{in}}^\dagger(t) \mathcal{B}_{\text{in}}(t') \rangle & \langle \mathcal{B}_{\text{in}}(t) \mathcal{B}_{\text{in}}(t') \rangle \\ \langle \mathcal{B}_{\text{in}}^\dagger(t) \mathcal{B}_{\text{in}}^\dagger(t') \rangle & \langle \mathcal{B}_{\text{in}}(t) \mathcal{B}_{\text{in}}^\dagger(t') \rangle \end{pmatrix} = \begin{pmatrix} N & M \\ M^* & N+1 \end{pmatrix} \delta(t - t'). \quad (11)$$

Here,

$$\begin{aligned} N &= \cosh^2(r_c) \sinh^2(r) + \sinh^2(r_c) \cosh^2(r) \\ &\quad + \frac{1}{2} \cos(\theta - \varphi) \sinh(2r_c) \sinh(2r), \\ M &= \frac{1}{2} [e^{i\varphi} \cosh^2(r_c) \sinh(2r) + e^{i\theta} \sinh(2r_c) \sinh^2(r) \\ &\quad + e^{i\theta} \sinh(2r_c) \cosh^2(r) \\ &\quad + e^{i(2\theta - \varphi)} \sinh^2(r_c) \sinh(2r)] \end{aligned} \quad (12)$$

represent the thermal noise and two-photon correlation noise, respectively. Physically, N arises from the mismatch between the squeezing axes of the intracavity and external fields, while M originates from the interference between the two-photon drives and the squeezed reservoir. These two types of noise are undesired in our proposal. When we choose $r_c = r$ and $\theta - \varphi = \pi$, these two noises can be eliminated, i.e., $N = M = 0$. Then, according to Eq. (C13), we obtain

$$\langle \mathcal{B}_{\text{in}}(t) \mathcal{B}_{\text{in}}^\dagger(t') \rangle = \delta(t - t'), \quad \langle \mathcal{B}_{\text{in}}(t) \mathcal{B}_{\text{in}}(t') \rangle = 0, \quad (13)$$

meaning that $\mathcal{B}_{\text{in}}(t)$ is vacuum noise.

When applying the dispersive approximation under conditions $\Delta_{ge} - \omega_s \gg g_1 \cosh(r_c)$ and $(\Delta_{ef} + \Delta_{ge}) - \omega_s \gg g_2 \cosh(r_c)$, we obtain the dispersive coupling (see Appendix A for details)

$$G_{\text{qutrit}} = \chi_{ge} \beta^\dagger \beta (|e\rangle\langle e| - |g\rangle\langle g|) + \chi_{ef} \beta^\dagger \beta (|f\rangle\langle f| - |e\rangle\langle e|) \quad (14)$$

and the effective dynamics of the system

$$\begin{aligned} \dot{\beta}(t) &= -i(\chi_{ge} (|e\rangle\langle e| - |g\rangle\langle g|) + \chi_{ef} (|f\rangle\langle f| - |e\rangle\langle e|)) \beta \\ &\quad - i(\omega_s - i\kappa/2) \beta - \sqrt{\kappa} \beta_{\text{in}}. \end{aligned} \quad (15)$$

Here, χ_{ge} and χ_{ef} are the dispersive coupling strengths given by

$$\chi_{ge} = g_1^2 \left[\frac{\cosh^2(r)}{\Delta_{ge} - \omega_s} + \frac{\sinh^2(r)}{\Delta_{ge} + \omega_s} \right] \quad (16)$$

and

$$\chi_{ef} = g_2^2 \left[\frac{\cosh^2(r)}{\Delta_{ef} - \omega_s} + \frac{\sinh^2(r)}{\Delta_{ef} + \omega_s} \right]. \quad (17)$$

We rewrite χ_{ge} as [44]

$$\chi_{ge} = \chi_1 \left[\cosh(r) + \frac{\sinh^2(r)}{\cosh(r) + 2\omega_s \nu_1 / g_1} \right], \quad (18)$$

where we define $\chi_1 = g_1 \nu_1$ with $\nu_1 = g_1 \cosh(r) / (\Delta_{ge} - \omega_s)$. For the three-level Jaynes-Cummings model without squeezing, the dispersive coupling strength is

$$\chi_J = \frac{g^2}{\omega_q - \omega_c} = g \nu_J, \quad (19)$$

with

$$\nu_J = \frac{g}{\omega_q - \omega_c}. \quad (20)$$

The parameters ν_1 and ν_J , respectively, set the accuracy of the dispersive approximations.

Therefore, in order to effectively compare the dispersive coupling, we assume that their accuracies are the same, that is, $\nu_1 = \nu_J$. This leads to $\chi_1 = \chi_J$, so according to Eq. (18), for $(\Delta_{ge} - \omega_s) \gg g \cosh(r)$, i.e., $\nu_1 \ll 1$, we obtain an exponentially enhanced dispersive coupling

$$\chi_{ge} \approx \chi_1 \exp(r). \quad (21)$$

Similarly, for χ_{ef} , we also achieve an exponentially enhanced dispersive coupling strength $\chi_{ef} \approx \chi_2 \exp(r)$. Since the readout time is inversely proportional to the dispersive coupling strength, we can achieve an exponentially reduced single-shot readout time for bosonic qubits,

$$\tau_s \approx \tau \exp(-r). \quad (22)$$

Therefore, the intracavity squeezing effectively amplifies the dispersive coupling strength, while the external squeezing allows us to reshape the input noise and, when properly phased, cancel the added noise. Their combination is key to achieving exponential enhancement.

B. Exponentially suppressed single-shot measurement error

In dispersive readout, the measurement signal quantifies the distinguishability between two quantum states. Concretely, the difference in the expectation values of a suitably chosen measurement operator $\mathcal{M}$ for two different states $|i\rangle$ and $|j\rangle$ determines how well these states can be separated in a single shot. We define the signal as $|\langle \mathcal{M} \rangle_i - \langle \mathcal{M} \rangle_j|$ ($i, j \in \{g, e, f\}$, $i \neq j$), where $\langle \mathcal{M} \rangle$ is the average of the measurement operator

$$\mathcal{M} = \sqrt{\kappa} \int_0^\tau dt (a_{\text{out}}^\dagger(t) e^{i\phi_m} + a_{\text{out}}(t) e^{-i\phi_m}). \quad (23)$$

Here, ϕ_m is the measurement angle and $a_{\text{out}}(t) = \cosh(r)\beta_{\text{out}}(t) - e^{i\theta} \sinh(r)\beta_{\text{out}}^\dagger(t)$, where $\beta_{\text{out}} = \beta_{\text{in}} + \sqrt{\kappa}\beta$ denotes the cavity output field. We refer to fluctuations by $\sqrt{\langle \mathcal{M}_N^2 \rangle_i + \langle \mathcal{M}_N^2 \rangle_j}$, with $\mathcal{M}_N = \mathcal{M} - \langle \mathcal{M} \rangle$ as the fluctuation-noise operator. The SNR then reads

$$\text{SNR} = \frac{|\langle \mathcal{M} \rangle_i - \langle \mathcal{M} \rangle_j|}{\sqrt{\langle \mathcal{M}_N^2 \rangle_i + \langle \mathcal{M}_N^2 \rangle_j}}. \quad (24)$$

The SNR reflects a worst-case estimate of the measurement uncertainty, ensuring that the error probability is bounded from above in a device-independent manner [53]. We first consider the measurement noise $\langle \mathcal{M}_N^2 \rangle_k$ ($k \in g, e, f$). The cavity output-noise operator

$$\mathcal{A}_{\text{out}}(t) = a_{\text{out}}(t) - \langle a_{\text{out}}(t) \rangle \quad (25)$$

can be expressed in terms of $\mathcal{B}_{\text{out}}(t) = \beta_{\text{out}}(t) - \langle \beta_{\text{out}}(t) \rangle$ as

$$\mathcal{A}_{\text{out}}(t) = \cosh(r)\mathcal{B}_{\text{out}}(t) - e^{i\theta} \sinh(r)\mathcal{B}_{\text{out}}^\dagger(t). \quad (26)$$

From Eqs. (13) and (15), we find

$$\begin{pmatrix} \langle \mathcal{B}_{\text{out}}^\dagger(t) \mathcal{B}_{\text{out}}(t') \rangle & \langle \mathcal{B}_{\text{out}}(t) \mathcal{B}_{\text{out}}(t') \rangle \\ \langle \mathcal{B}_{\text{out}}^\dagger(t) \mathcal{B}_{\text{out}}^\dagger(t') \rangle & \langle \mathcal{B}_{\text{out}}(t) \mathcal{B}_{\text{out}}^\dagger(t') \rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \delta(t - t'). \quad (27)$$

Subsequently, by using the Bogoliubov transformation in Eq. (C14), the correlations for $\mathcal{A}_{\text{out}}(t)$ are given by

$$\begin{aligned} \langle \mathcal{A}_{\text{out}}(t) \mathcal{A}_{\text{out}}^\dagger(t') \rangle &= \cosh^2(r) \delta(t - t'), \\ \langle \mathcal{A}_{\text{out}}(t) \mathcal{A}_{\text{out}}(t') \rangle &= -\frac{1}{2} e^{i\theta} \sinh(2r) \delta(t - t'). \end{aligned} \quad (28)$$

Integrating these correlation functions over the measurement time τ (see Appendix A for details), the total measurement noise is derived as

$$\langle \mathcal{M}_N^2 \rangle_k = \kappa \tau [\cosh(2r) - \cos(2\phi_m - \theta) \sinh(2r)]. \quad (29)$$

Clearly, if $2\phi_m = \theta$ is chosen, this noise can be minimized, and we then obtain

$$\langle \mathcal{M}_N^2 \rangle_k = \kappa \tau \exp(-2r). \quad (30)$$

Having obtained the measurement noise, we now turn to the measurement signal. Given a coherent measurement tone, $\langle a_{\text{in}}(t) \rangle = \alpha_{\text{in}} e^{i\phi_{\text{in}}}$ and according to

$$\beta_{\text{in}}(t) = \cosh(r_c) a_{\text{in}}(t) + e^{i\theta} \sinh(r_c) a_{\text{in}}^\dagger(t), \quad (31)$$

we find the expectation value

$$\langle \beta_{\text{in}}(t) \rangle = \alpha_{\text{in}} e^{i\phi_{\text{in}}} [\cosh(r) + \sinh(r) e^{-i(2\phi_{\text{in}} - \theta)}]. \quad (32)$$

By choosing $2\phi_{\text{in}} = \theta$, we can exponentially enhance $|\langle \beta_{\text{in}}(t) \rangle|$ with the squeezing parameter r , yielding

$$\langle \beta_{\text{in}}(t) \rangle = \alpha_{\text{in}} e^r e^{i\phi_{\text{in}}}. \quad (33)$$

We denote the parallel and perpendicular components of the signal separation as $|\langle \mathcal{M} \rangle_i - \langle \mathcal{M} \rangle_j|_p$ and $|\langle \mathcal{M} \rangle_i - \langle \mathcal{M} \rangle_j|_d$, respectively. Then, we impose $|\langle \mathcal{M} \rangle_i - \langle \mathcal{M} \rangle_j|_d = 0$; therefore, we can maximize $|\langle \mathcal{M} \rangle_i - \langle \mathcal{M} \rangle_j|_p$ to achieve the maximum SNR [44].

In case $\kappa\tau \rightarrow 0$, the SNR reads (see Appendix A for details)

$$\text{SNR} \simeq 0.81 e^{2r} \text{SNR}_{\text{std}}, \quad (34)$$

and in case $\kappa\tau \rightarrow \infty$, the SNR is

$$\text{SNR} \simeq \frac{\sin(\psi_{ge(ef)})}{\sin(2\psi_{1(2)})} \exp(r) \text{SNR}_{\text{std}}. \quad (35)$$

Here, SNR_{std} is the SNR of the standard readout without squeezing.

From the relationship between the measurement error and the SNR, $\delta = \frac{1}{2} [1 - \text{erf}(\text{SNR}/2)]$, we can obtain an exponentially suppressed measurement error δ_s .

IV. SQUEEZING ENHANCEMENT IN MULTILEVEL SYSTEMS

The aforementioned case for three-level systems can be directly generalized to multilevel systems. Consider a multilevel qubit (the energy levels are labeled as $|0\rangle, |1\rangle, \dots, |d-1\rangle$) coupled to a cavity, where transitions are only allowed

between adjacent levels. In the rotating frame ω_d , the Hamiltonian is given by

$$H = \Delta_c a^\dagger a + H_{\text{multi}} + H_{\text{int}} + \lambda(e^{i\theta} a^{\dagger 2} + e^{-i\theta} a^2). \quad (36)$$

Here,

$$H_{\text{multi}} = \sum_{n=0}^{d-1} \left(\sum_{l=0}^n \Delta_{l,l+1} \right) |n\rangle \langle n| \quad (37)$$

and

$$H_{\text{int}} = \sum_{n=0}^{d-2} g_n (a^\dagger |n\rangle \langle n+1| + a |n+1\rangle \langle n|), \quad (38)$$

where $\Delta_{l,l+1} = \omega_{l,l+1} - \omega_d/2$ and g_n represents the coupling strength between energy levels $|n\rangle$ and $|n+1\rangle$. Following the same procedure as in the three-level case, we introduce the Bogoliubov mode β to diagonalize the cavity Hamiltonian. By applying the dispersive approximation, the effective Hamiltonian in terms of mode β reads

$$H_{\text{eff}} = \omega_s \beta^\dagger \beta + \sum_{n=0}^{d-1} \left(\sum_{l=0}^n \Delta_{l,l+1} \right) |n\rangle \langle n| + \beta^\dagger \beta \sum_{n=0}^{d-2} \chi_{n,n+1} (|n+1\rangle \langle n+1| - |n\rangle \langle n|), \quad (39)$$

where

$$\chi_{n,n+1} = g_{n,n+1}^2 \left[\frac{\cosh^2(r)}{\Delta_{l,l+1} - \omega_s} + \frac{\sinh^2(r)}{\Delta_{l,l+1} + \omega_s} \right]. \quad (40)$$

Therefore, similarly to the three-level case,

$$\chi_{n,n+1} \approx \chi_n \exp(r). \quad (41)$$

Since the measurement noise is governed by the input-output relations of the cavity, in a multilevel system with $2\phi_m = \theta$, the measurement noise $\langle \mathcal{M}_N^2 \rangle_{k'}$ remains subject to exponential suppression, namely,

$$\langle \mathcal{M}_N^2 \rangle_{k'} = \kappa \tau \exp(-2r). \quad (42)$$

We then proceed to consider the measurement signal. For a multilevel system, the expression for the measurement signal associated with a specific energy level k' (where $k' \in \{0, 1, \dots, d-1\}$) is written as

$$\begin{aligned} \langle \mathcal{M} \rangle_{k'} &= \frac{2\alpha_{\text{in}} e^r}{\sqrt{\kappa}} \{ [2 - \kappa \tau + 2\cos(2\psi_{k'})] \\ &\times [\cos(\Psi_+) \cosh(r) - \cos(\Psi_- + \theta) \sinh(r)] \\ &- 4e^{-\kappa \tau/2} \cos^2(\psi_{k'}) \cos(\Psi_+ + \omega_{k'} \tau) \cosh(r) \\ &- 4e^{-\kappa \tau/2} \cos^2(\psi_{k'}) \cos(\Psi_- + \theta + \omega_{k'} \tau) \sinh(r) \}, \end{aligned} \quad (43)$$

where $\Psi_\pm = 2\psi_{k'} \pm \phi_m - \phi_{\text{in}}$, $\tan(\psi_{k'}) = 2\omega_{k'}/\kappa$. Unlike the three-level case, here we have $\omega_{k'} = \omega_s + \sum_{n=0}^{d-2} \chi_{n,n+1} (|n+1\rangle \langle n+1| - |n\rangle \langle n|)$. Under the same conditions (see Appendix A for details)

$$2\phi_m - \theta = 0, \quad \phi_{\text{in}} - \phi_m = 0 \quad (44)$$

and

$$\theta - 2\phi_m = \pi, \quad \phi_{\text{in}} - \phi_m = \frac{\pi}{2}, \quad (45)$$

the parallel and perpendicular components of the signal separation between adjacent energy levels can be obtained as

$$\begin{aligned} &|\langle \mathcal{M} \rangle_{n+1} - \langle \mathcal{M} \rangle_n|_p \\ &= \frac{2\alpha_{\text{in}}}{\sqrt{\kappa}} |[2 - \kappa \tau + 2\cos(2\psi_n) + 2\cos(2\psi_{n+1})] \\ &\times [\cos(2\psi_n) - \cos(2\psi_{n+1})] - e^{-\kappa \tau/2} [\cos(\omega_n \tau) \\ &+ 2\cos(2\psi_n + \omega_n \tau) + \cos(4\psi_n + \omega_n \tau) \\ &- 4\cos^2(\psi_{n+1}) \cos(2\psi_{n+1} + \omega_{n+1} \tau)]| \end{aligned} \quad (46)$$

and

$$\begin{aligned} &|\langle \mathcal{M} \rangle_{n+1} - \langle \mathcal{M} \rangle_n|_d \\ &= \frac{2\alpha_{\text{in}} e^{2r}}{\sqrt{\kappa}} |[2 - \kappa \tau + 2\cos(2\psi_n)] \sin(2\psi_n) \\ &- [2 - \kappa \tau + 2\cos(2\psi_{n+1})] \sin(2\psi_{n+1}) \\ &- e^{-\kappa \tau/2} [\sin(\omega_n \tau) + 2\sin(2\psi_n + \omega_n \tau) \\ &+ \sin(4\psi_n + \omega_n \tau) - 4\cos^2(\psi_{n+1}) \\ &\times \sin(2\psi_{n+1} + \omega_{n+1} \tau)]|. \end{aligned} \quad (47)$$

Therefore, when $|\langle \mathcal{M} \rangle_{n+1} - \langle \mathcal{M} \rangle_n|_d = 0$ and in the limit of $\kappa \tau \rightarrow 0$, we have

$$\begin{aligned} |\langle \mathcal{M} \rangle_{n+1} - \langle \mathcal{M} \rangle_n| &= |\langle \mathcal{M} \rangle_{n+1} - \langle \mathcal{M} \rangle_n|_p \\ &\simeq \frac{0.27\alpha_{\text{in}}}{\kappa} \tan(\psi_{n,n+1}) (\kappa \tau)^3, \end{aligned} \quad (48)$$

where $\tan(\psi_{n,n+1}) = (2\chi_{n,n+1} - \chi_{n+1,n+2} - \chi_{n-1,n})/\kappa$. Thus, we obtain

$$\begin{aligned} |\langle \mathcal{M} \rangle_{n+1} - \langle \mathcal{M} \rangle_n| &\simeq \frac{0.27\alpha_{\text{in}}}{\kappa} \tan(\psi_{n,n+1}) (\kappa \tau)^3 \\ &\simeq 0.81 e^r |\langle \mathcal{M} \rangle_{n+1} - \langle \mathcal{M} \rangle_n|_{\text{std}}. \end{aligned} \quad (49)$$

Thus, in the limit $\kappa \tau \rightarrow 0$, the SNR between adjacent energy levels in a multilevel system can be exponentially enhanced, yielding

$$\text{SNR} \simeq 0.81 e^{2r} \text{SNR}_{\text{std}}. \quad (50)$$

And in case $\kappa \tau \rightarrow \infty$, the SNR is

$$\text{SNR} \simeq \frac{\sin(\psi_{n,n+1})}{\sin(2\psi_n)} \exp(r) \text{SNR}_{\text{std}}, \quad (51)$$

where ψ_n satisfies $\tan(\psi_n) = (2\chi_n - \chi_{n+1} - \chi_{n-1})/\kappa$.

We perform numerical simulations of this result. Taking the ground state and the first excited state as an example, Fig. 3 shows the SNR between the ground state and the first excited state versus $\kappa \tau$. As expected, in the short-time regime, the SNR of the dual-squeezing protocol diverges from the standard no-squeezing case and approaches the theoretically predicted bound of $e^{2r} \text{SNR}_{\text{std}}$ (indicated by the upper dashed line). Furthermore, the numerical simulation confirms that the exponential enhancement of the SNR still persists as $\kappa \tau$ approaches 1.

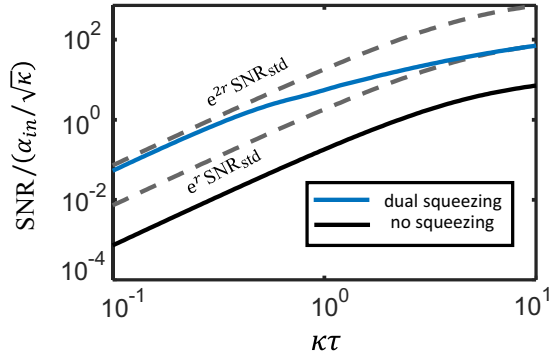


FIG. 3. SNR between the ground state and the first excited state as a function of $\kappa\tau$. The solid blue and black lines represent the cases with and without dual squeezing, respectively. The dashed lines represent $e^r \text{SNR}_{\text{std}}$ and $e^{2r} \text{SNR}_{\text{std}}$, respectively. We have set $\nu_1 = 1/20$, $e^r = 10$, and $\alpha_{\text{in}}/\sqrt{\kappa} = 1$.

Therefore, in a multilevel system, we can similarly achieve an exponentially suppressed readout time as well as an exponentially suppressed measurement error.

V. EXPONENTIALLY ENHANCED READOUT FIDELITY

In this section, leveraging the theoretical results derived in the previous sections [the exponentially suppressed readout time (τ_s) and measurement error (δ_s)], we demonstrate a significant enhancement in the readout fidelity of the dispersive readout scheme presented in Sec. II.

The squeezing-enhanced dispersive readout protocol is modeled using a hidden Markov model (HMM) (see Appendix C for details). Then, the infidelity $1 - F$ of this protocol can be expressed through the parameters δ , $\kappa_\downarrow \tau$, and $\kappa_\uparrow \tau$. This infidelity is related to the determination strategy of the second-level readout results. We focus on comparing two strategies: majority voting and the maximum likelihood estimate (MLE).

The majority voting determines the second-level readout result by aggregating the outcomes of N first-level readouts, thereby identifying whether the initial state is $|0\rangle$ or $|L\rangle$. Unlike the majority voting method, the MLE utilizes the known transition and emission probabilities to calculate the likelihood values of the initial states under a given observation sequence. Then, the state with the higher likelihood is selected as the final result. To be specific, the likelihood $\lambda_{\mathbf{a}}(i)$ of the ancilla's readout set $\mathbf{a}_n \in \{g, e\}$ (for $n \in \{1, \dots, N\}$) conditioned on the initial state $|i\rangle$ is given by [49]

$$\lambda_{\mathbf{a}}(i) = \sum_{j_1, \dots, j_N} T_{i,j_1} E_{j_1, a_1} \cdots T_{j_{N-1}, j_N} E_{j_N, a_N}. \quad (52)$$

Here, T_{ij} and E_{ij} are the transition probabilities and the emission probabilities, respectively. Then, the outcome of a second-level measurement is decided by comparing $\lambda_{\mathbf{a}}(0)$ and $\lambda_{\mathbf{a}}(L)$.

The infidelity of majority voting and the MLE can be expressed as a function of the likelihood [49],

$$1 - F = \sum_{\mathbf{a} \in A_0} \lambda_{\mathbf{a}}(L) + \sum_{\mathbf{a}' \in A_L} \lambda_{\mathbf{a}'}(0), \quad (53)$$

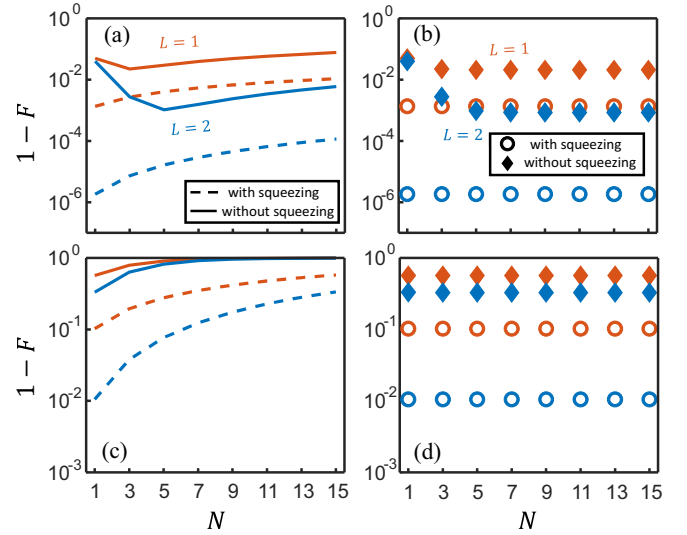


FIG. 4. Infidelities of majority voting [panels (a) and (c)] and MLE [panels (b) and (d)] as a function of the number of measurements N for $L = 1$ and $L = 2$ with two-level ancilla. The first row [panels (a) and (b)] corresponds to the short-time limit (we set $\kappa_\downarrow \tau = 0.01$). The second row [panels (c) and (d)] corresponds to the finite- $\kappa\tau$ regime (we set $\kappa_\downarrow \tau = 0.8$). Other parameters are $\delta = 0.02$, and the squeezing parameter $r = 2$.

where A_0 (A_L) denotes the set of vectors $\mathbf{a}$ corresponding to ancilla readout outcomes that are classified as initial state $|0\rangle$ ($|L\rangle$).

As shown in Fig. 4, we simulate the infidelity $1 - F$ as a function of the number of measurements N , considering only spontaneous relaxation of the bosonic mode. We compare the readout protocol without squeezing [49] with our squeezing-enhanced protocol for two cases, $\kappa\tau \rightarrow 0$ and $\kappa\tau \rightarrow \infty$, using two distinct classification strategies: majority voting [Figs. 4(a) and 4(c)] and the MLE [Figs. 4(b) and 4(d)] at encoding levels $L = 1$ and $L = 2$.

In Fig. 4(a), using the majority voting strategy, we observe that for both protocols, the infidelity eventually rises as N becomes large. This occurs because a larger number of measurements prolongs the total readout time, which consequently increases the probability of relaxation for the qubit in the excited state, leading to a higher infidelity. At small numbers of measurements, the readout infidelity is dominated by the single-shot measurement error and the single-shot relaxation. Since our squeezing-enhanced scheme achieves exponential suppression of the single-shot measurement error and an exponential reduction in the single-shot readout time, the infidelity remains low from the very beginning. Furthermore, our squeezing-enhanced protocol (dashed lines) consistently outperforms that without squeezing (solid lines), lowering the infidelity by more than an order of magnitude for the same L . It is worth noting that with an encoding level of $L = 1$, our protocol attains the infidelity of $\sim 10^{-3}$ at $N = 1$, which is the lowest infidelity achieved when the encoding level $L = 2$ in the protocol without a squeezing-enhanced cavity (with $N = 5$). This indicates that our protocol can effectively enhance the readout fidelity even when the number of measurements is relatively small. For the MLE strategy

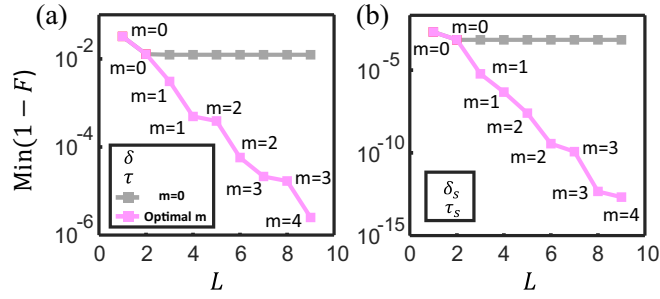


FIG. 5. Minimum attainable infidelity as a function of L for $m = 0$ (pink) and optimal m (gray). (a) Minimum attainable infidelity of the readout protocol without squeezing. (b) Minimum attainable infidelity of the squeezing-enhanced dispersive readout protocol. We set $\delta = 0.02$, $\kappa_{\downarrow}\tau = 0.01$, $\kappa_{\uparrow}\tau = 0.005$ and the squeezing parameter $r = 2$.

[Fig. 4(b)], our squeezing-enhanced protocol also exhibits a significant improvement over the previous readout protocol [49]. For the encoding level $L = 2$, our protocol reduces the infidelity by approximately 3 orders of magnitude compared to that without squeezing.

Figures 4(c) and 4(d) present the infidelity results for finite $\kappa\tau$. It can be seen that, under both classification strategies, our dual-squeezing protocol consistently outperforms the protocol without squeezing. This further confirms the feasibility of our protocol.

In the subsequent simulations, the results for the case of finite $\kappa\tau$ are the same as those in Fig. 4, with our dual-squeezing protocol always outperforming the without squeezing one. Therefore, in the following, we only present the simulations under the same short-time condition as that used in Ref. [49] and no longer include the plots for finite $\kappa\tau$.

We then consider the case where the bosonic mode is subject to both spontaneous relaxation and heating. In Fig. 5, we plot the minimum infidelity as a function of the encoding level L for the two cases: with a fixed threshold $m = 0$ and with an optimally chosen m . It can be observed that the optimal threshold is $m = 0$ when $L \leq 2$, while it becomes $m = 1$ for $L = 3$ and $L = 4$.

Correspondingly, Fig. 6 displays the infidelity as a function of the number of measurements N for these two protocols, at encoding levels L and their respective optimal thresholds m . The results show that without increasing the threshold m , the infidelity is affected by the first-order heating. When the threshold m is increased, the heating effect can be suppressed to a higher order, leading to a significant reduction in the infidelity. Furthermore, for the majority voting [Fig. 6(a)] and MLE [Fig. 6(b)] strategies, our protocol achieves an infidelity reduction of more than an order of magnitude compared to that without squeezing [49], under the same encoding levels and thresholds.

We now consider the case of a multilevel ancilla. As illustrated in Fig. 7, we simulate the infidelity as a function of the number of measurements N under various encoding levels L and their corresponding optimal thresholds m . To visually evaluate the performance, we also compare the performance of the readout protocol without squeezing [49] and the squeezing-enhanced protocol under majority voting

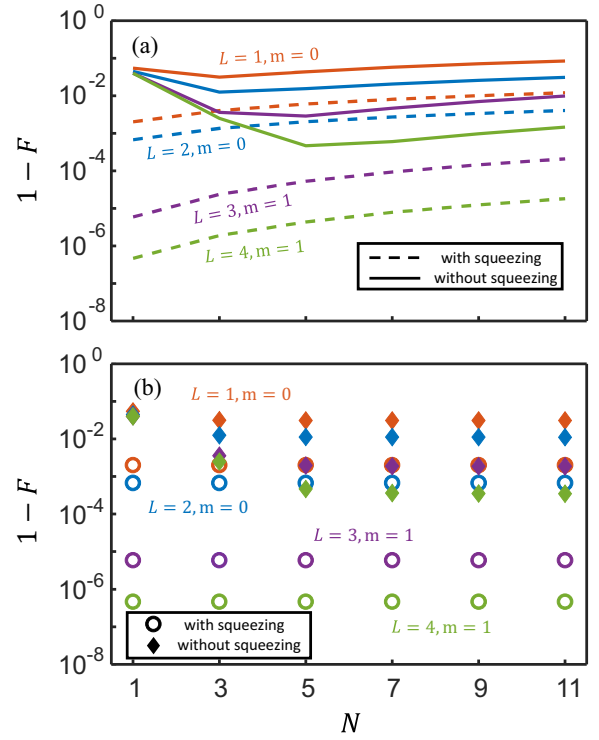


FIG. 6. Infidelity of majority voting and MLE as a function of the number of measurements N for $L = 1, 2, 3$, and 4 as well as the corresponding m with two-level ancilla. (a) Infidelity of majority voting for the readout protocols with and without squeezing. (b) Infidelity of MLE for the readout protocols with and without squeezing. The parameters are the same as those in Fig. 5.

[Fig. 7(a)] and MLE [Fig. 7(b)] strategies. A comparison between the two protocols reveals that the dual-squeezing protocol significantly reduces the infidelity under all encoding and threshold settings. Notably, when $L = 3$, $m = 1$, the minimum infidelity decreases from $\sim 10^{-3}$ to below $\sim 10^{-5}$, exhibiting a performance enhancement of over 2 orders of magnitude.

VI. CONCLUSION

In summary, we have proposed a squeezing-enhanced dispersive readout protocol that significantly improves the measurement performance for bosonic qubits. This can be considered as a universal dispersive readout protocol for bosonic qubits. We demonstrated that the simultaneous application of intracavity and external squeezing in three-level and multilevel systems leads to an exponential enhancement of both the dispersive coupling strength and the measurement SNR between adjacent energy levels. This exponential enhancement originates from the simultaneous increase of the dispersive coupling strength and the suppression of measurement noise, both achieved by the dual-squeezing configuration. Building upon this fundamental mechanism, we applied this squeezing-enhanced approach to the readout cavity of the ancilla in a bosonic qubit architecture. This results in an exponentially shortened single-shot readout time and an exponentially suppressed measurement error. When it is applied to a repeated quantum nondemolition mapping and

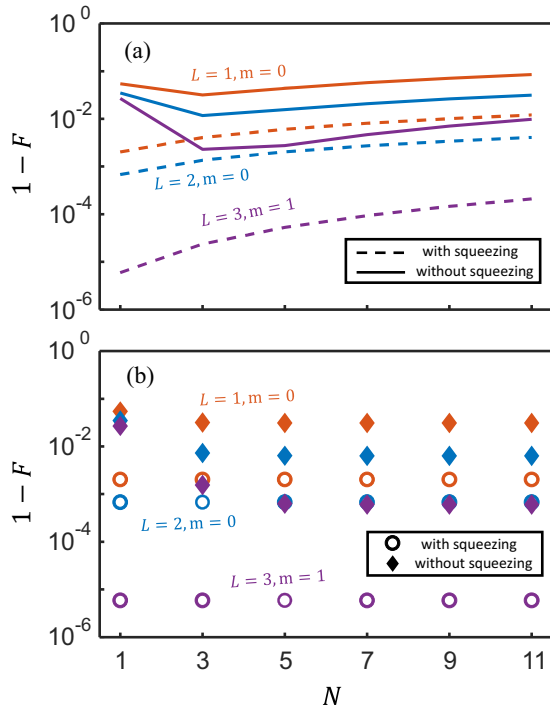


FIG. 7. Infidelity of majority voting and MLE as a function of the number of measurements N for $L = 1, 2$, and 3 , as well as the corresponding m with multilevel ancilla. (a) Infidelity of majority voting for the readout protocols with and without squeezing. (b) Infidelity of MLE for the readout protocols with and without squeezing. The parameters are the same as those in Fig. 5.

readout framework, our protocol reduces the readout infidelity by more than one order of magnitude compared to the previous readout scheme [49], even with lower encoding levels and fewer measurement repetitions. This method provides an effective route toward high-fidelity qubit readout.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: DERIVATION OF THE SNR

We begin with the full Hamiltonian in Eq. (4). Upon introducing a Bogoliubov mode

$$\beta = \cosh(r_c)a + e^{i\theta} \sinh(r_c)a^\dagger, \quad (\text{A1})$$

the cavity Hamiltonian is diagonalized, yielding

$$\Delta_c a^\dagger a + \lambda(e^{i\theta} a^{\dagger 2} + e^{-i\theta} a^2) = \omega_s \beta^\dagger \beta. \quad (\text{A2})$$

Then, the Hamiltonian for mode β takes the form

$$\begin{aligned} H_{\text{qutrit}} = & \omega_s \beta^\dagger \beta + \Delta_{ge} |e\rangle \langle e| + (\Delta_{ge} + \Delta_{ef}) |f\rangle \langle f| \\ & + g_1 \cosh(r_c) (\beta^\dagger \sigma_{ge}^- + \beta \sigma_{ge}^+) \\ & - g_1 \sinh(r_c) (e^{-i\theta} \beta \sigma_{ge}^- + e^{i\theta} \beta^\dagger \sigma_{ge}^+) \\ & + g_2 \cosh(r_c) (\beta^\dagger \sigma_{ef}^- + \beta \sigma_{ef}^+) \\ & - g_2 \sinh(r_c) (e^{-i\theta} \beta \sigma_{ef}^- + e^{i\theta} \beta^\dagger \sigma_{ef}^+). \end{aligned} \quad (\text{A3})$$

Here, in order to improve the SNR, we have set $r_c = r$. We obtain the dispersive coupling

$$\begin{aligned} G_{\text{qutrit}} = & \chi_{ge} \beta^\dagger \beta (|e\rangle \langle e| - |g\rangle \langle g|) \\ & + \chi_{ef} \beta^\dagger \beta (|f\rangle \langle f| - |e\rangle \langle e|), \end{aligned} \quad (\text{A4})$$

when applying the dispersive approximation under conditions $\Delta_{ge} - \omega_s \gg g_1 \cosh(r_c)$ and $(\Delta_{ef} + \Delta_{ge}) - \omega_s \gg g_2 \cosh(r_c)$. Note that conditions $\Delta_{ge} + \omega_s \gg g_1 \sinh(r_c)$ and $(\Delta_{ef} + \Delta_{ge}) + \omega_s \gg g_2 \sinh(r_c)$ are necessarily satisfied once conditions $\Delta_{ge} - \omega_s \gg g_1 \cosh(r_c)$ and $(\Delta_{ef} + \Delta_{ge}) - \omega_s \gg g_2 \cosh(r_c)$ hold. Therefore, only $\Delta_{ge} - \omega_s \gg g_1 \cosh(r_c)$ and $(\Delta_{ef} + \Delta_{ge}) - \omega_s \gg g_2 \cosh(r_c)$ are considered in applying the dispersive approximation. Then, the effective dynamics of the system is described by

$$\begin{aligned} \dot{\beta}(t) = & -i(\chi_{ge}(|e\rangle \langle e| - |g\rangle \langle g|) + \chi_{ef}(|f\rangle \langle f| - |e\rangle \langle e|))\beta \\ & - i(\omega_s - i\kappa/2)\beta - \sqrt{\kappa}\beta_{\text{in}}. \end{aligned} \quad (\text{A5})$$

Here, χ_{ge} and χ_{ef} are the dispersive coupling strengths given by

$$\chi_{ge} = g_1^2 \left[\frac{\cosh^2(r)}{\Delta_{ge} - \omega_s} + \frac{\sinh^2(r)}{\Delta_{ge} + \omega_s} \right] \quad (\text{A6})$$

and

$$\chi_{ef} = g_2^2 \left[\frac{\cosh^2(r)}{\Delta_{ef} - \omega_s} + \frac{\sinh^2(r)}{\Delta_{ef} + \omega_s} \right]. \quad (\text{A7})$$

We rewrite χ_{ge} as

$$\chi_{ge} = \chi_1 \left[\cosh(r) + \frac{\sinh^2(r)}{\cosh(r) + 2\omega_s v_1 / g_1} \right], \quad (\text{A8})$$

where we define $\chi_1 = g_1 v_1$, with $v_1 = g_1 \cosh(r) / (\Delta_{ge} - \omega_s)$. For the three-level Jaynes-Cummings model without squeezing, the dispersive coupling strength is

$$\chi_J = \frac{g^2}{\omega_q - \omega_c} = g v_J, \quad (\text{A9})$$

with

$$v_J = \frac{g}{\omega_q - \omega_c}. \quad (\text{A10})$$

The parameters v_1 and v_J , respectively, set the accuracy of the dispersive approximations.

Therefore, in order to effectively compare the dispersive coupling, we assume that their accuracies are the same, that

is, $v_1 = v_J$. This leads to $\chi_1 = \chi_J$, so according to Eq. (A8), for $(\Delta_{ge} - \omega_s) \gg g \cosh(r)$, i.e., $v_1 \ll 1$, we obtain an exponentially enhanced dispersive coupling

$$\chi_{ge} \approx \chi_1 \exp(r). \quad (\text{A11})$$

Similarly, for χ_{ef} , we also achieve an exponentially enhanced dispersive coupling strength $\chi_{ef} \approx \chi_2 \exp(r)$.

Based on the dispersive coupling given in Eq. (14), the effective dynamics of the system can be described by

$$\dot{\beta}(t) = -i(\omega_k - i\kappa/2)\beta - \sqrt{\kappa}\beta_{\text{in}}, \quad (\text{A12})$$

and integrating it gives

$$\beta(t) = \beta(0) e^{[-i(\omega_k - i\kappa/2)t]} - \sqrt{\kappa} \int_0^t ds e^{[-i(\omega_k - i\kappa/2)(t-s)]} \beta_{\text{in}}(s), \quad (\text{A13})$$

where $\omega_k = \omega_s + \chi_{ge}(|e\rangle\langle e| - |g\rangle\langle g|) + \chi_{ef}(|f\rangle\langle f| - |e\rangle\langle e|)$. Accordingly, the number of photons in the mode β is

$$n(t) = \langle \beta^\dagger(t) \beta(t) \rangle = 4|\langle \beta_{\text{in}}(t) \rangle|^2 \cos^2(\psi_k) \times [1 + e^{-\kappa t} - 2e^{-\kappa t/2} \cos(\omega_k t)], \quad (\text{A14})$$

where $\tan(\psi_k) = 2\omega_k/\kappa$. Then, according to the input-output relation $\beta_{\text{out}} = \beta_{\text{in}} + \sqrt{\kappa}\beta(t)$, we have

$$\beta_{\text{out}}(t) = \beta_{\text{in}}(t) + \sqrt{\kappa}\beta(0) e^{[-i(\omega_k - i\kappa/2)t]} - \kappa \int_0^t ds e^{[-i(\omega_k - i\kappa/2)(t-s)]} \beta_{\text{in}}(s). \quad (\text{A15})$$

Consequently,

$$\begin{aligned} \mathcal{B}_{\text{out}}(t) &= \beta_{\text{out}}(t) - \langle \beta_{\text{out}}(t) \rangle \\ &= \mathcal{B}_{\text{in}}(t) + \sqrt{\kappa}\mathcal{B}(0) e^{[-i(\omega_k - i\kappa/2)t]} \\ &\quad - \kappa \int_0^t ds e^{[-i(\omega_k - i\kappa/2)(t-s)]} \mathcal{B}_{\text{in}}(s). \end{aligned} \quad (\text{A16})$$

The cavity output-noise operator, $\mathcal{A}_{\text{out}}(t) = a_{\text{out}}(t) - \langle a_{\text{out}}(t) \rangle$, can be expressed in terms of $\mathcal{B}_{\text{out}}(t) = \beta_{\text{out}}(t) - \langle \beta_{\text{out}}(t) \rangle$ as $\mathcal{A}_{\text{out}}(t) = \cosh(r)\mathcal{B}_{\text{out}}(t) - e^{i\theta} \sinh(r)\mathcal{B}_{\text{out}}^\dagger(t)$. Based on fluctuation-noise operator $\mathcal{M}_N = \mathcal{M} - \langle \mathcal{M} \rangle$ and measurement operator given in Eq. (23), the measurement noise is given by

$$\begin{aligned} \langle \mathcal{M}_N^2 \rangle_k &= \kappa \int_0^\tau \int_0^\tau dt_1 dt_2 \{ \langle \mathcal{A}_{\text{out}}(t_1) \mathcal{A}_{\text{out}}(t_2) \rangle e^{-i2\phi_m} \\ &\quad + \langle \mathcal{A}_{\text{out}}^\dagger(t_1) \mathcal{A}_{\text{out}}(t_2) \rangle + \langle \mathcal{A}_{\text{out}}(t_1) \mathcal{A}_{\text{out}}^\dagger(t_2) \rangle \\ &\quad + \langle \mathcal{A}_{\text{out}}^\dagger(t_1) \mathcal{A}_{\text{out}}^\dagger(t_2) \rangle e^{i2\phi_m} \}. \end{aligned} \quad (\text{A17})$$

From Eqs. (13) and (15), we find $\langle \mathcal{B}_{\text{out}}(t) \mathcal{B}_{\text{out}}^\dagger(t') \rangle = \delta(t - t')$ and $\langle \mathcal{B}_{\text{out}}(t) \mathcal{B}_{\text{out}}(t') \rangle = \langle \mathcal{B}_{\text{out}}^\dagger(t) \mathcal{B}_{\text{out}}^\dagger(t') \rangle = \langle \mathcal{B}_{\text{out}}^\dagger(t) \mathcal{B}_{\text{out}}(t') \rangle = 0$. Here, we have assumed that the cavity (subject to a two-photon driving and a squeezed reservoir) is already in a steady state at the initial measurement time $t = 0$. Subsequently,

$$\begin{aligned} \langle \mathcal{A}_{\text{out}}(t) \mathcal{A}_{\text{out}}^\dagger(t') \rangle &= \cosh^2(r) \delta(t - t'), \\ \langle \mathcal{A}_{\text{out}}(t) \mathcal{A}_{\text{out}}(t') \rangle &= -\frac{1}{2} e^{i\theta} \sinh(2r) \delta(t - t'), \end{aligned} \quad (\text{A18})$$

it follows that

$$\langle \mathcal{M}_N^2 \rangle_k = \kappa \tau [\cosh(2r) - \cos(2\phi_m - \theta) \sinh(2r)]. \quad (\text{A19})$$

Clearly, by choosing $2\phi_m = \theta$, we obtain

$$\langle \mathcal{M}_N^2 \rangle_k = \kappa \tau \exp(-2r). \quad (\text{A20})$$

Then, we consider the measurement signal. Given a coherent measurement tone, $\langle a_{\text{in}}(t) \rangle = \alpha_{\text{in}} e^{i\phi_{\text{in}}}$ and according to $\beta_{\text{in}}(t) = \cosh(r_c) a_{\text{in}}(t) + e^{i\theta} \sinh(r_c) a_{\text{in}}^\dagger(t)$, we find

$$\langle \beta_{\text{in}}(t) \rangle = \alpha_{\text{in}} e^{i\phi_{\text{in}}} [\cosh(r) + \sinh(r) e^{-i(2\phi_{\text{in}} - \theta)}]. \quad (\text{A21})$$

When we choose $2\phi_{\text{in}} - \theta = 0$, we can exponentially enhance $|\langle \beta_{\text{in}}(t) \rangle|$ with the squeezing parameter r , yielding

$$\langle \beta_{\text{in}}(t) \rangle = \alpha_{\text{in}} e^r e^{i\phi_{\text{in}}}. \quad (\text{A22})$$

From Eq. (A15), the expectation value of the output field is found to be

$$\langle \beta_{\text{out}}(t) \rangle = \alpha_{\text{in}} e^r e^{i\phi_{\text{in}}} \left[1 + \frac{i\kappa}{\omega_k - i\kappa/2} (1 - e^{-i(\omega_k - i\kappa/2)t}) \right], \quad (\text{A23})$$

under the initial condition of $\langle \beta(0) \rangle = 0$. Therefore, the measurement signal $\langle \mathcal{M} \rangle$, as defined in Eq. (23), is given by

$$\begin{aligned} \langle \mathcal{M} \rangle_k &= \frac{2\alpha_{\text{in}} e^r}{\sqrt{\kappa}} \{ [2 - \kappa\tau + 2\cos(2\psi_k)] \\ &\quad \times [\cos(\Psi_+) \cosh(r) - \cos(\Psi_- + \theta) \sinh(r)] \\ &\quad - 4e^{-\kappa\tau/2} \cos^2(\psi_k) \cos(\Psi_+ + \omega_k\tau) \cosh(r) \\ &\quad - 4e^{-\kappa\tau/2} \cos^2(\psi_k) \cos(\Psi_- + \theta + \omega_k\tau) \sinh(r) \}, \end{aligned} \quad (\text{A24})$$

where $\Psi_\pm = 2\psi_k \pm \phi_m - \phi_{\text{in}}$. The signal separation $|\langle \mathcal{M} \rangle_i - \langle \mathcal{M} \rangle_j|$ can be divided into two components. One is along the squeezing direction, which we denote as $|\langle \mathcal{M} \rangle_i - \langle \mathcal{M} \rangle_j|_p$, and the other is along the antisqueezing direction, which we denote as $|\langle \mathcal{M} \rangle_i - \langle \mathcal{M} \rangle_j|_d$. Taking $|g\rangle$ and $|e\rangle$ as an example, by setting $2\phi_m - \theta = 0$, $\phi_{\text{in}} - \phi_m = 0$, and $\theta - 2\phi_m = \pi$, $\phi_{\text{in}} - \phi_m = \frac{\pi}{2}$, we can obtain the expressions of $|\langle \mathcal{M} \rangle_e - \langle \mathcal{M} \rangle_g|_p$ and $|\langle \mathcal{M} \rangle_e - \langle \mathcal{M} \rangle_g|_d$, respectively, as

$$\begin{aligned} |\langle \mathcal{M} \rangle_e - \langle \mathcal{M} \rangle_g|_p &= \frac{2\alpha_{\text{in}}}{\sqrt{\kappa}} \{ [2 - \kappa\tau + 2\cos(2\psi_g) + 2\cos(2\psi_e)] \\ &\quad \times [\cos(2\psi_g) - \cos(2\psi_e)] - e^{-\kappa\tau/2} [\cos(\omega_g\tau) \\ &\quad + 2\cos(2\psi_g + \omega_g\tau) + \cos(4\psi_g + \omega_g\tau) \\ &\quad - 4\cos^2(\psi_e) \cos(2\psi_e + \omega_e\tau)] \}, \end{aligned} \quad (\text{A25})$$

$$\begin{aligned} |\langle \mathcal{M} \rangle_e - \langle \mathcal{M} \rangle_g|_d &= \frac{2\alpha_{\text{in}} e^{2r}}{\sqrt{\kappa}} \{ [2 - \kappa\tau + 2\cos(2\psi_g)] \sin(2\psi_g) \\ &\quad - [2 - \kappa\tau + 2\cos(2\psi_e)] \sin(2\psi_e) \\ &\quad - e^{-\kappa\tau/2} [\sin(\omega_g\tau) + 2\sin(2\psi_g + \omega_g\tau) \\ &\quad + \sin(4\psi_g + \omega_g\tau) - 4\cos^2(\psi_e) \sin(2\psi_e + \omega_e\tau)] \}. \end{aligned} \quad (\text{A26})$$

We require

$$|\langle \mathcal{M} \rangle_e - \langle \mathcal{M} \rangle_g|_d = 0, \quad (\text{A27})$$

and then, we can maximize $|\langle \mathcal{M} \rangle_e - \langle \mathcal{M} \rangle_g|_p$ so as to maximize SNR. Under the limit $\kappa\tau \rightarrow 0$, Eq. (A27) leads to $\omega_s \simeq 2.58/\tau$. Consequently, we obtain

$$|\langle \mathcal{M} \rangle_e - \langle \mathcal{M} \rangle_g| = |\langle \mathcal{M} \rangle_e - \langle \mathcal{M} \rangle_g|_p \simeq \frac{0.27\alpha_{\text{in}}}{\kappa} \tan(\psi_{ge})(\kappa\tau)^3, \quad (\text{A28})$$

where $\tan(\psi_{ge}) = (2\chi_{ge} - \chi_{ef})/\kappa$. So, we obtain

$$|\langle \mathcal{M} \rangle_e - \langle \mathcal{M} \rangle_g| \simeq \frac{0.27\alpha_{\text{in}}}{\kappa} \tan(\psi_{ge})(\kappa\tau)^3 \simeq 0.81e^r |\langle \mathcal{M} \rangle_e - \langle \mathcal{M} \rangle_g|_{\text{std}}. \quad (\text{A29})$$

Similarly, for $|e\rangle$ and $|f\rangle$, we have

$$|\langle \mathcal{M} \rangle_f - \langle \mathcal{M} \rangle_e| \simeq \frac{0.27\alpha_{\text{in}}}{\kappa} \tan(\psi_{ef})(\kappa\tau)^3 \simeq 0.81e^r |\langle \mathcal{M} \rangle_f - \langle \mathcal{M} \rangle_e|_{\text{std}}, \quad (\text{A30})$$

where $\tan(\psi_{ef}) = (2\chi_{ef} - \chi_{ge})/\kappa$. Therefore, in a three-level system, the SNR between adjacent energy levels in the limit of $\kappa\tau \rightarrow 0$ is given by

$$\text{SNR} \simeq 0.81e^{2r} \text{SNR}_{\text{std}}. \quad (\text{A31})$$

Furthermore, as $\kappa\tau \rightarrow \infty$, Eq. (A27) leads to $\omega_s \simeq \frac{\kappa}{2} \text{sec}(\psi_{ge})$. This yields

$$|\langle \mathcal{M} \rangle_e - \langle \mathcal{M} \rangle_g| \simeq \frac{4\alpha_{\text{in}}}{\kappa} \sin(\psi_{ge}) \simeq \frac{\sin(\psi_{ge})}{\sin(2\psi_1)} |\langle \mathcal{M} \rangle_e - \langle \mathcal{M} \rangle_g|_{\text{std}}, \quad (\text{A32})$$

where ψ_1 satisfies $\tan(\psi_1) = (2\chi_1 - \chi_2)/\kappa$. Hence, the SNR reads

$$\text{SNR} \simeq \frac{\sin(\psi_{ge})}{\sin(2\psi_1)} \exp(r) \text{SNR}_{\text{std}}. \quad (\text{A33})$$

Analogous to the case of $\kappa\tau \rightarrow 0$, when $\kappa\tau \rightarrow \infty$, we can likewise obtain

$$\text{SNR} \simeq \frac{\sin(\psi_{ef})}{\sin(2\psi_2)} \exp(r) \text{SNR}_{\text{std}} \quad (\text{A34})$$

for $|e\rangle$ and $|f\rangle$, where ψ_2 satisfies $\tan(\psi_2) = (2\chi_2 - \chi_1)/\kappa$.

APPENDIX B: HMM

An HMM is a discrete-time stochastic process in which the hidden system states evolve according to the Markov chain, and the output signals that depend on the system states are observed [54,55].

Specifically, we adopt the discrete model illustrated in Fig. 8, where each first-level readout process is modeled as a possible quantum state transition (denoting the bosonic qubit decays), followed by noise emission (denoting the mapping and the readout of the ancilla). This model is parametrized through the transition probability T_{ij} and the emission probability E_{ij} . The transition probability T_{ij} represents the probability of transitioning from bosonic state $|i\rangle$ to state $|j\rangle$ during the first-level readout process, where $i, j \in \{0, 1, \dots, L\}$. The emission probability E_{ij} represents the probability of being read out as the ancillary state $|g\rangle$

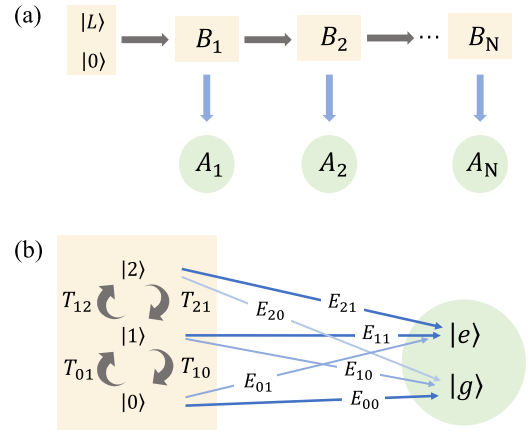


FIG. 8. Hidden Markov model. (a) Markov chain and emissions. Each step of the HMM involves the bosonic system transitioning to a new state and releasing an emission. B_n and A_n represent the bosonic mode state after step n and the n th ancilla readout outcome, respectively. (b) Transition and emission probabilities.

($j = 0$) or $|e\rangle$ ($j = 1$), given that the bosonic system has already transitioned to state $|i\rangle$.

For the two-level ancilla: In the case of considering only the relaxation of the bosonic mode, the transition probabilities are derived from the bosonic decay rates. Consider a population of quantum harmonic oscillators, in which $p_i(t)$ oscillators are in the Fock state $|i\rangle$ at time t . The system of differential equations describing the time evolution of the populations is

$$\dot{p}_i(t) = \sum_{j=0}^L (K_{\downarrow})_{ij} p_j(t), \quad (\text{B1})$$

where $(K_{\downarrow})_{ij}$ is the transition rate from state $|j\rangle$ to $|i\rangle$. In bosonic systems, the specific forms of $(K_{\downarrow})_{ij}$ are

$$(K_{\downarrow})_{ij} = \begin{cases} -j\kappa_{\downarrow}, & i = j, \\ j\kappa_{\downarrow}, & i = j - 1, \\ 0, & \text{otherwise.} \end{cases} \quad (\text{B2})$$

The solution of this system is $\mathbf{p}(t) = e^{K_{\downarrow}t} \mathbf{p}(0)$. Therefore, for a first-level readout process that requires time τ , the transition probability can be obtained by computing the matrix elements of $e^{K_{\downarrow}\tau}$,

$$T_{ij}(\tau) = (e^{K_{\downarrow}\tau})_{ji} = \binom{i}{j} (e^{\kappa_{\downarrow}\tau} - 1)^{i-j} e^{-i\kappa_{\downarrow}\tau}. \quad (\text{B3})$$

The definition of the emission probability E_{ij} is based on the error δ that occurs during the mapping and readout processes,

$$E_{ij} = \begin{cases} 1 - \delta, & \text{if } i = j = 0 \text{ or } i > 0, \quad j = 1, \\ \delta, & \text{otherwise.} \end{cases} \quad (\text{B4})$$

When both bosonic mode relaxation and heating are considered, the transition probabilities are calculated from the decay and excitation rates. The system of differential equations describing the time evolution of the Fock state

populations is

$$\dot{p}_i(t) = \sum_{j=0}^L (K_{\downarrow} + K_{\uparrow})_{ij} p_j(t), \quad (\text{B5})$$

where $K_{\uparrow}$ has matrix elements defined as

$$(K_{\uparrow})_{ij} = \begin{cases} -(j+1)\kappa_{\uparrow}, & i = j < L, \\ (j+1)\kappa_{\uparrow}, & i = j+1, \\ 0, & \text{otherwise.} \end{cases} \quad (\text{B6})$$

The transition probabilities for a first-level readout process taking time τ are given by

$$T_{ij}(\tau) = [e^{(K_{\downarrow} + K_{\uparrow})\tau}]_{ji}. \quad (\text{B7})$$

The emission probabilities E_{ij} are similarly defined in terms of the first-level readout error δ as

$$E_{ij} = \begin{cases} 1 - \delta, & \text{if } i \leq m, j = 0 \text{ or } i > m, j = 1, \\ \delta, & \text{otherwise.} \end{cases} \quad (\text{B8})$$

For the multilevel ancilla: The HMM transition probabilities T_{ij} are the same as those defined in the case of two-level ancilla in Eq. (B3). In order to include $L+1$ possible readout outcomes, E_{ij} is defined as

$$E_{ij} = \begin{cases} 1 - \delta, & \text{for } i = j, \\ \delta/L, & \text{otherwise.} \end{cases} \quad (\text{B9})$$

Here, we assume that the probability of an incorrect readout δ is uniformly distributed among the other L incorrect ancilla states.

APPENDIX C: DISCUSSION OF EXPERIMENTAL IMPERFECTIONS

1. Parameter mismatches

In realistic experiments, there are always some parameter mismatches. In the case of parameter mismatch, we assume that

$$r_c = r + \delta_r \quad \text{and} \quad \theta - \varphi = \pi + \delta_a, \quad (\text{C1})$$

where δ_r and δ_a are the squeezing degree and direction mismatches, respectively. In this case, N and M in

$$\begin{pmatrix} \langle \mathcal{B}_{\text{in}}^{\dagger}(t) \mathcal{B}_{\text{in}}(t') \rangle & \langle \mathcal{B}_{\text{in}}(t) \mathcal{B}_{\text{in}}(t') \rangle \\ \langle \mathcal{B}_{\text{in}}^{\dagger}(t) \mathcal{B}_{\text{in}}^{\dagger}(t') \rangle & \langle \mathcal{B}_{\text{in}}(t) \mathcal{B}_{\text{in}}^{\dagger}(t') \rangle \end{pmatrix} = \begin{pmatrix} N & M \\ M^* & N+1 \end{pmatrix} \delta(t-t') \quad (\text{C2})$$

are no longer zero. With the parameter mismatches $r_c = r + \delta_r$ and $\theta - \varphi = \pi + \delta_a$, the input state of the Bogoliubov mode remains a pure squeezed state, since the Bogoliubov transformation is unitary. Therefore, the parameters N and M satisfy the relation $|M|^2 = N(N+1)$. This allows us to parametrize N and M using a single equivalent mismatch squeezing parameter r_0 and an equivalent phase ϕ_0 as [44]

$$N = \sinh^2(r_0) \quad \text{and} \quad M = \frac{1}{2} e^{i\phi_0} \sinh(2r_0). \quad (\text{C3})$$

Under such parameter mismatches, the correlations for the output-noise operator $\mathcal{B}_{\text{out}}(t) = \beta_{\text{out}}(t) - \langle \beta_{\text{out}}(t) \rangle$ become

$$\langle \hat{\mathcal{B}}_{\text{out}}(t) \hat{\mathcal{B}}_{\text{out}}^{\dagger}(t') \rangle = (N+1)\delta(t-t') - \kappa N \exp\left[-\frac{1}{2}(\kappa_k t + \kappa_k^* t')\right], \quad (\text{C4})$$

$$\begin{aligned} \langle \hat{\mathcal{B}}_{\text{out}}(t) \hat{\mathcal{B}}_{\text{out}}(t') \rangle &= M\delta(t-t') - \frac{\kappa}{\kappa_k} M e^{-\frac{1}{2}(t+t')\kappa_k} \\ &\times \begin{cases} (\kappa + i2\omega_{\sigma} e^{\kappa_k t'}), & \text{if } t \geq t', \\ (\kappa + i2\omega_{\sigma} e^{\kappa_k t}), & \text{if } t < t', \end{cases} \end{aligned} \quad (\text{C5})$$

where $\kappa_k = \kappa + i2\omega_k$. Therefore, the measurement of noise is given by

$$\langle \mathcal{M}_N^2 \rangle = \mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3, \quad (\text{C6})$$

where

$$\mathcal{O}_1 = \kappa \tau [\cosh(2r) + \cos(\varphi - \theta) \sinh(2r)], \quad (\text{C7})$$

$$\begin{aligned} \mathcal{O}_2 &= 8e^{-2r_c - \kappa\tau/2} \cos^2(\psi_k) \left[\cosh\left(\frac{\kappa\tau}{2}\right) - \cos(\omega_k \tau) \right] \\ &\times [1 - \cosh(2r_c) \cosh(2r) \\ &- \cosh(\theta - \varphi) \sinh(2r_c) \sinh(2r)], \end{aligned} \quad (\text{C8})$$

$$\begin{aligned} \mathcal{O}_3 &= e^{-2r_c - \kappa\tau} \sinh(2r_0) \cos(\psi_k) \{ e^{\kappa\tau} [(1 - 2\kappa\tau) \cos(\Psi_1) \\ &- 2(1 - \kappa\tau) \cos(\Psi_3) - 3 \cos(\Psi_5)] \\ &+ 8e^{\kappa\tau/2} \cos(\psi_k) \cos(\Psi_4 + \omega_k \tau) \\ &- 4 \cos^2(\psi_k) \cos(\Psi_3 + 2\omega_k \tau) \}. \end{aligned} \quad (\text{C9})$$

Here, $\Psi_n = n\psi_k + \theta - \phi_0$, and we have set $2\phi_h = \theta$. The measured signal $\langle \mathcal{M} \rangle_k$ is the same as the signal in the ideal case ($\delta_r = \delta_a = 0$), but r is replaced by r_c .

After obtaining the measurement noise and signal, we simulated the SNR in the presence of these parameter mismatches, as shown in Fig. 9. It can be seen that even with finite parameter mismatches, an exponential enhancement of the SNR can still be achieved, indicating that the dual-squeezing readout protocol is experimentally feasible. We further simulate the infidelities of the majority voting readout scheme under parameter mismatches, taking a two-level qubit with only relaxation considered as an example.

The results show that the infidelities under our dual-squeezing protocol are almost unaffected. The reason is that the exponential SNR enhancement can still keep the single-shot measurement error at an extremely low level.

2. Impure external squeezed vacuum

To describe the impurity of the external squeezed vacuum, we introduce η_e as the input efficiency of the external squeezed vacuum, where $0 \leq \eta_e \leq 1$, with $\eta_e = 1$ representing the ideal lossless case and $1 - \eta_e$ representing the loss.

We first consider the measurement noise. Taking into account the external input efficiency η_e , the lossy input $\mathcal{A}'_{\text{in}}(t)$ is

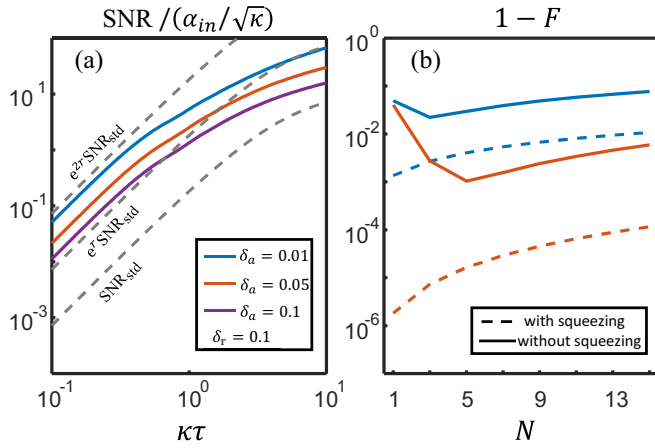


FIG. 9. (a) SNR in the presence of parameter mismatches as a function of the measurement time $\kappa\tau$ for $\delta_a = 0.01, 0.05, 0.1$ and for $\delta_r = 0.1$. Other parameters are the same as in Fig. 3. (b) Infidelities of majority voting as the function of the number of measurements N for $L = 1$ and $L = 2$ with two-level ancilla, where $\delta_a = \delta_r = 0.1$. Other parameters are the same as in Fig. 4.

expressed as

$$\mathcal{A}'_{\text{in}}(t) = \sqrt{\eta_e} \mathcal{A}_{\text{in}}(t) + \sqrt{1 - \eta_e} \mathcal{V}(t). \quad (\text{C10})$$

Here, $\mathcal{A}_{\text{in}}(t)$ is the noise operator of the ideal, pure squeezed vacuum, whose correlation function satisfies Eq. (6) and $\mathcal{V}(t)$ is the uncorrelated vacuum environmental noise, with its nonzero correlation function given by

$$\langle \mathcal{V}(t) \mathcal{V}^\dagger(t') \rangle = \delta(t - t'). \quad (\text{C11})$$

Thus, the correlations of the lossy external input field $\mathcal{A}'_{\text{in}}(t)$ satisfy

$$\begin{pmatrix} \langle \mathcal{A}'_{\text{in}}(t) \mathcal{A}'_{\text{in}}(t') \rangle & \langle \mathcal{A}'_{\text{in}}(t) \mathcal{A}'_{\text{in}}(t') \rangle \\ \langle \mathcal{A}'_{\text{in}}(t) \mathcal{A}'_{\text{in}}(t') \rangle & \langle \mathcal{A}'_{\text{in}}(t) \mathcal{A}'_{\text{in}}(t') \rangle \end{pmatrix} = \begin{pmatrix} \eta_e \sinh^2 r & \eta_e \frac{1}{2} \sinh 2r e^{i\varphi} \\ \eta_e \frac{1}{2} \sinh 2r e^{-i\varphi} & \eta_e \cosh^2 r + (1 - \eta_e) \end{pmatrix} \delta(t - t'). \quad (\text{C12})$$

Consequently, according to Eq. (10), the correlations of the quantum fluctuation operator $\mathcal{B}'_{\text{in}}(t)$ in the presence of loss are given by

$$\begin{pmatrix} \langle \mathcal{B}'_{\text{in}}(t) \mathcal{B}'_{\text{in}}(t') \rangle & \langle \mathcal{B}'_{\text{in}}(t) \mathcal{B}'_{\text{in}}(t') \rangle \\ \langle \mathcal{B}'_{\text{in}}(t) \mathcal{B}'_{\text{in}}(t') \rangle & \langle \mathcal{B}'_{\text{in}}(t) \mathcal{B}'_{\text{in}}(t') \rangle \end{pmatrix} = \begin{pmatrix} N' & M' \\ M'^* & N' + 1 \end{pmatrix} \delta(t - t') \quad (\text{C13})$$

where $N' = (1 - \eta_e) \sinh^2 r \delta(t - t')$ and $M' = \frac{1}{2} (1 - \eta_e) e^{i\theta} \sinh(2r)$ (we have set $r_c = r$ and $\theta - \varphi = \pi$). Since the correlations of $\mathcal{B}'_{\text{out}}(t)$ and $\mathcal{B}'_{\text{in}}(t)$ are the same, it then follows, by using the Bogoliubov transformation

$$\mathcal{A}_{\text{out}}(t) = \cosh(r) \mathcal{B}_{\text{out}}(t) - e^{i\theta} \sinh(r) \mathcal{B}_{\text{out}}^\dagger(t), \quad (\text{C14})$$

that the correlations for $\mathcal{A}'_{\text{out}}(t)$ are

$$\begin{pmatrix} \langle \mathcal{A}'_{\text{out}}(t) \mathcal{A}'_{\text{out}}(t') \rangle & \langle \mathcal{A}'_{\text{out}}(t) \mathcal{A}'_{\text{out}}(t') \rangle \\ \langle \mathcal{A}'_{\text{out}}(t) \mathcal{A}'_{\text{out}}(t') \rangle & \langle \mathcal{A}'_{\text{out}}(t) \mathcal{A}'_{\text{out}}(t') \rangle \end{pmatrix} = \begin{pmatrix} \eta_e \sinh^2 r & -\frac{1}{2} \eta_e \sinh 2r e^{i\theta} \\ -\frac{1}{2} \eta_e \sinh 2r e^{-i\theta} & \eta_e \sinh^2 r + 1 \end{pmatrix} \delta(t - t'). \quad (\text{C15})$$

Therefore, substituting correlations for $\mathcal{A}'_{\text{out}}(t)$ into Eq. (A17) of the measurement noise formula and choosing $2\phi_m = \theta$, we obtain

$$\langle \mathcal{M}_N'^2 \rangle_k = \kappa \tau (1 - \eta_e + \eta_e e^{-2r}). \quad (\text{C16})$$

Then, we consider the measurement signal. With the introduction of the input efficiency η_e , the coherent measurement tone becomes $\langle a'_{\text{in}}(t) \rangle = \sqrt{\eta_e} \alpha_{\text{in}} e^{i\phi_{\text{in}}}$. Then, according to $\beta_{\text{in}}(t) = \cosh(r_c) a_{\text{in}}(t) + e^{i\theta} \sinh(r_c) a_{\text{in}}^\dagger(t)$, and choosing $2\phi_{\text{in}} - \theta = 0$, we find

$$\langle \beta'_{\text{in}}(t) \rangle = \sqrt{\eta_e} \alpha_{\text{in}} e^r e^{i\phi_{\text{in}}}. \quad (\text{C17})$$

Thus, the expectation value of the output field is found to be

$$\langle \beta'_{\text{out}}(t) \rangle = \sqrt{\eta_e} \alpha_{\text{in}} e^r e^{i\phi_{\text{in}}} \left[1 + \frac{i\kappa}{\omega_k - i\kappa/2} (1 - e^{-i(\omega_k - i\kappa/2)t}) \right]. \quad (\text{C18})$$

Therefore, based on the measurement operator in Eq. (23), we obtain

$$\langle \mathcal{M}' \rangle_k = \sqrt{\eta_e} \langle \mathcal{M} \rangle_k. \quad (\text{C19})$$

Hence, in the presence of a lossy external squeezed vacuum, taking the ground state and the first excited state as an example, the final SNR in the limit $\kappa\tau \rightarrow 0$ is given by

$$\text{SNR}(\eta_e) \simeq \frac{0.81 \sqrt{\eta_e} e^r}{\sqrt{1 - \eta_e + \eta_e e^{-2r}}} \text{SNR}_{\text{std}}. \quad (\text{C20})$$

As shown in Fig. 10, our numerical simulation of the above result demonstrates that, even with an impure external squeezed vacuum, the SNR can still be exponentially enhanced, and the impact on the infidelitis is negligible.

3. Imperfect quantum efficiency

Here, we set the quantum efficiency of homodyne detection as η_d , so the output field operator of the cavity can be expressed as

$$a'_{\text{out}}(t) = \sqrt{\eta_d} a_{\text{out}}(t) + \sqrt{1 - \eta_d} \mathcal{V}(t), \quad (\text{C21})$$

where $a_{\text{out}}(t)$ is the ideal output field operator and $\mathcal{V}(t)$ is the uncorrelated vacuum noise operator.

We first consider the measurement signal. According to the measurement operator $\mathcal{M}$ in Eq. (23), we obtain

$$\mathcal{M}' = \sqrt{\eta_d} \mathcal{M} + \sqrt{1 - \eta_d} \mathcal{V}_{\text{vac}}, \quad (\text{C22})$$

where we define $\mathcal{V}_{\text{vac}} = \sqrt{\kappa} \int_0^\tau dt (\mathcal{V}^\dagger(t) e^{i\phi_m} + \mathcal{V}(t) e^{-i\phi_m})$ and $\langle \mathcal{V}_{\text{vac}} \rangle = 0$. Thus, the expectation value of the measurement operator satisfies

$$\langle \mathcal{M}' \rangle_k = \sqrt{\eta_d} \langle \mathcal{M} \rangle_k, \quad (\text{C23})$$

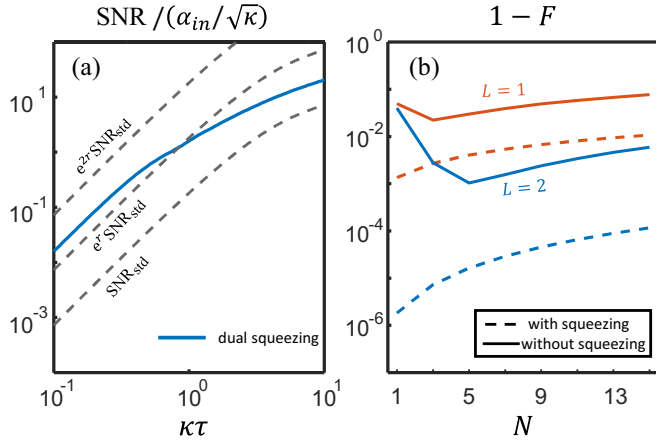


FIG. 10. (a) SNR in the presence of lossy external squeezed vacuum as a function of the measurement time $\kappa\tau$ for $\eta_e = 0.9$. Other parameters are the same as in Fig. 3. (b) Infidelities of majority voting as the function of the number of measurements N for $L = 1$ and $L = 2$ with two-level ancilla. Other parameters are the same as in Fig. 4.

and this yields

$$|\langle \mathcal{M} \rangle_e - \langle \mathcal{M} \rangle_g| \simeq 0.81 \sqrt{\eta_d} e^r |\langle \mathcal{M} \rangle_e - \langle \mathcal{M} \rangle_g|_{\text{std}}. \quad (\text{C24})$$

Then, we consider the measurement noise. Based on fluctuation-noise operator $\mathcal{M}_N = \mathcal{M} - \langle \mathcal{M} \rangle$ and Eq. (C22), we obtain

$$\mathcal{M}'_N = \sqrt{\eta_d} \mathcal{M} + \sqrt{1 - \eta_d} \mathcal{V}_{\text{vac}}. \quad (\text{C25})$$

It follows that

$$\langle \mathcal{M}'^2_N \rangle_k = \eta_d \langle \mathcal{M}^2_N \rangle_k + (1 - \eta_d) \langle \mathcal{V}_{\text{vac}}^2 \rangle, \quad (\text{C26})$$

where vacuum noise $\langle \mathcal{V}_{\text{vac}}^2 \rangle = \kappa\tau$. Therefore, taking the ground and excited states as an example, the final SNR in the presence of imperfect quantum efficiency in the limit $\kappa\tau \rightarrow 0$ reads

$$\text{SNR}(\eta_d) \simeq \frac{0.81 \sqrt{\eta_d} e^r}{\sqrt{1 - \eta_d + \eta_d e^{-2r}}} \text{SNR}_{\text{std}}. \quad (\text{C27})$$

This result is consistent with that of the impure external squeezed vacuum, therefore, we do not repeat the simulation here.

4. Phase errors

We assume a small phase fluctuation $\delta\phi_m$, which follows a Gaussian distribution with zero mean and standard deviation σ_m [56]. Therefore, the cosine term $\cos(2\phi_m - \theta)$ in Eq. (29) contains a phase error $2\delta\phi_m$. Then, using the properties of the Gaussian random variable $\mathbf{X} \sim \mathcal{N}(0, \sigma_X^2)$ [52], the expected value of the cosine is found to be

$$\langle \cos(2\phi_m - \theta) \rangle = e^{-2\sigma_m^2}. \quad (\text{C28})$$

Therefore, the average measurement noise is given by

$$\langle \langle \mathcal{M}'^2_N \rangle_k \rangle = \kappa\tau [\cosh(2r) - e^{-2\sigma_m^2} \sinh(2r)]. \quad (\text{C29})$$

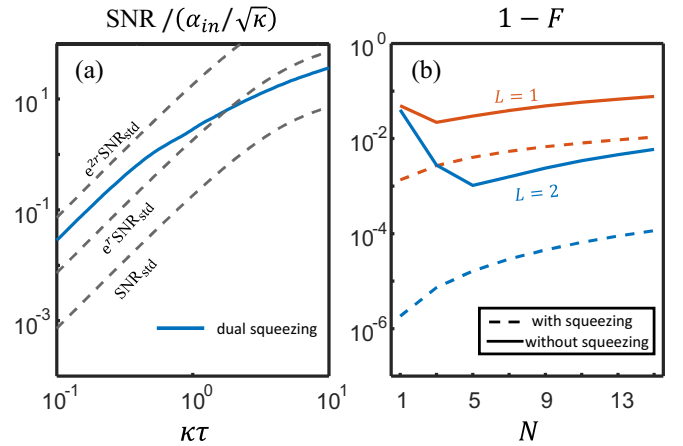


FIG. 11. (a) SNR in the presence of phase noise as a function of the measurement time $\kappa\tau$ for $\sigma_m = 0.0175$. Other parameters are the same as in Fig. 3. (b) Infidelities of majority voting as the function of the number of measurements N for $L = 1$ and $L = 2$ with two-level ancilla. Other parameters are the same as in Fig. 4.

For a small phase noise $\sigma_m \ll 1$, employing the Taylor expansion, the average measurement noise can be approximated as

$$\langle \langle \mathcal{M}'^2_N \rangle_k \rangle = \kappa\tau [e^{-2r} + \sigma_m^2 (e^{2r} - e^{-2r})]. \quad (\text{C30})$$

Then, we consider the measurement signal. In the ideal case $2\phi_m = \theta$, the measurement signal in Eq. (A24) can be simplified to

$$\begin{aligned} \langle \mathcal{M} \rangle_k &= \frac{2\alpha_{\text{in}}}{\sqrt{\kappa}} \{ [2 - \kappa\tau + 2\cos(2\psi_k)] \cos(\Psi_+) \\ &\quad - 4e^{-\kappa\tau/2} \cos^2(\psi_k) \cos(\Psi_+ + \omega_k\tau) \}. \end{aligned} \quad (\text{C31})$$

When considering the phase error, i.e., $2\phi_m - \theta = 2\delta\phi_m$, the measurement signal becomes

$$\begin{aligned} \langle \mathcal{M}' \rangle_k &= \frac{2\alpha_{\text{in}} e^r}{\sqrt{\kappa}} \{ [2 - \kappa\tau + 2\cos(2\psi_k)] [\cos(\Psi_+) \cosh(r) \\ &\quad - \cos(\Psi_+ - 2\delta\phi_m) \sinh(r)] \\ &\quad - 4e^{-\kappa\tau/2} \cos^2(\psi_k) \cos(\Psi_+ + \omega_k\tau) \cosh(r) \\ &\quad - \cos(\Psi_+ + \omega_k\tau - 2\delta\phi_m) \sinh(r) \}. \end{aligned} \quad (\text{C32})$$

By averaging the measurement signal $\langle \mathcal{M}' \rangle_k$, the average measurement signal takes the form

$$\begin{aligned} \langle \langle \mathcal{M}' \rangle_k \rangle &= \frac{2\alpha_{\text{in}} e^r}{\sqrt{\kappa}} [\cosh(r) - e^{-2\sigma_m^2} \sinh(r)] \\ &\quad \times \{ [2 - \kappa\tau + 2\cos(2\psi_k)] \cos(\Psi_+) \\ &\quad - 4e^{-\kappa\tau/2} \cos^2(\psi_k) \cos(\Psi_+ + \omega_k\tau) \}. \end{aligned} \quad (\text{C33})$$

Comparing Eq. (C31) with Eq. (C33), we find that the mean measurement signal contains an additional factor $e^r [\cosh(r) - e^{-2\sigma_m^2} \sinh(r)]$. For a small phase noise $\sigma_m \ll 1$, employing the Taylor expansion, it can be approximated as $[e^{-r} + \sigma_m^2 (e^r - e^{-r})]$.

Subsequently, taking the ground and excited states as an example, the final SNR in the presence of imperfect quantum efficiency in the limit $\kappa\tau \rightarrow 0$ reads

$$\text{SNR}(\sigma_m) \simeq \frac{0.81e^{2r}[e^{-r} + \sigma_m^2(e^r - e^{-r})]}{\sqrt{e^{-2r} + \sigma_m^2(e^{2r} - e^{-2r})}} \text{SNR}_{\text{std}}. \quad (\text{C34})$$

We also performed numerical simulations of the above results. As illustrated in Fig. 11, even with imperfect quantum efficiency, the exponential enhancement of the SNR is preserved, with virtually no impact on the infidelities.

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