

## Transition to dilatation-dominated isothermal compressible turbulence

Shadab Alam <sup>1</sup>, Christoph Federrath <sup>2</sup>, and Jörg Schumacher <sup>1,\*</sup>

<sup>1</sup>Institut für Thermo- und Fluidodynamik, *Technische Universität Ilmenau*, P.O. Box 100565, D-98684 Ilmenau, Germany

<sup>2</sup>*Research School of Astronomy and Astrophysics, [Australian National University](#), Canberra ACT 2611, Australia*

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The kinetic energy dissipation rate is of central importance for small-scale statistics in turbulent flows. Here, we report a transition to the dilatation-dominated regime of three-dimensional isothermal fully compressible, homogeneous, isotropic turbulence by moments of energy dissipation and its components up to order 4 for root-mean-square (rms) Mach numbers  $0.1 \leq M_{\text{rms}} \leq 10$  and for Reynolds numbers  $100 \leq \text{Re} \leq 2400$ . Our high-resolution numerical simulations show a crossover from incompressible to  $M_{\text{rms}}$ -independent, Burgers turbulencelike scaling of energy dissipation rate moments with respect to Reynolds number  $\text{Re}$ . This confirms the statistical dominance of shocks for rms Mach numbers  $M_{\text{rms}} \gtrsim 1$ .

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## I. INTRODUCTION

Compressible turbulence introduces additional layers of complexity through mechanisms absent in its incompressible counterpart, including shocklets and shocks, dilatational strain, and thermodynamic interactions [1,2]. These mechanisms shape the flow dynamics in many systems ranging from the interstellar medium (ISM) to engineering applications, such as gas turbines and supersonic propulsion. This is done by modifying the energy cascade [3–8] and thus influencing small-scale statistics and dissipation of kinetic energy [9,10]. The root-mean-square (rms) Mach number and large-scale Reynolds number, which are given by

$$M_{\text{rms}} = \frac{u_{\text{rms}}}{c_s} \quad \text{and} \quad \text{Re} = \frac{u_{\text{rms}} L_f}{\nu}, \quad (1)$$

characterize, respectively, the level of compressibility and the vigor of turbulence. Here,  $\nu$  is the kinematic viscosity,  $L_f$  is the forcing length scale,  $c_s$  is the speed of sound,  $u_{\text{rms}} = \sqrt{\langle \mathbf{u}^2 \rangle}$  is rms of velocity field  $\mathbf{u}$ , and  $\langle \cdot \rangle$  denotes the average over volume  $V$  and time  $t$ . The flow is homogeneous and locally isotropic with  $\langle \mathbf{u} \rangle \approx 0$ .

A defining characteristic of fully turbulent flows is spatial intermittency—the emergence of strongly localized bursts, steep fronts, and intense vortices at the small-scale end of the inertial cascade close to the Kolmogorov dissipation length  $\eta$  [11–16]. They give rise to strongly non-Gaussian statistics of turbulence fields and their derivatives, in particular, the kinetic energy dissipation rate  $\epsilon$ , which is given by

$$\epsilon(\mathbf{x}, t) = 2\mu \mathbf{S} : \mathbf{S} - \frac{2\mu}{3} (\nabla \cdot \mathbf{u})^2, \quad (2)$$

\*Contact author: joerg.schumacher@tu-ilmenau.de

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with the dynamic viscosity  $\mu$  and the rate-of-strain tensor  $\mathbf{S} = [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]/2$ , which probes the strength of local shear. In the inertial range for scales  $\ell > \eta$  and at  $M_{\text{rms}} \leq 1$ , the structure functions of the velocity field were found to follow scaling laws with power law exponents  $\zeta_n$  close to incompressible predictions [17,18]. In Lagrangian studies of compressible turbulence, the situation is slightly different. For  $M_{\text{rms}} \lesssim 0.5$ , the statistics remain essentially incompressible [19,20]; however, deviations become evident at higher  $M_{\text{rms}}$  [20,21] as shocklets begin to influence the dynamics. In Ref. [22], these scaling exponents  $\zeta_n$  were shown to fit to a She-L  v  que-type formula for the compressible case, where a single parameter—the Hausdorff dimension (or box counting dimension)  $D$  of the most intense dissipative structures—varies with  $M_{\text{rms}}$ . Supersonic isothermal simulations at  $M_{\text{rms}} = 6$  and 17 [6,23] exhibited, for example, kinetic energy spectra close to the shock-dominated Burgers turbulence-type  $k^{-2}$  scaling, together with larger values of the low-order structure-function exponents [23] compared to incompressible turbulence. The deviations increase when the volume forcing of the compressible flow injects additional velocity field divergence into the flow [6,21,24], i.e., when it becomes increasingly dilatational. These results clearly reflect the growing impact of shocks on the turbulence statistics. As we can see, most analysis of compressible turbulence focused on the statistics in the inertial cascade range, such as scaling laws for energy spectra and structure functions at sufficiently high Reynolds numbers.

Here, we will turn the attention from the velocity to the velocity derivative statistics at somewhat smaller Reynolds numbers. In these cases, a developed inertial cascade range with a power law scaling of structure functions is absent, but we will show that the intermittent statistical properties of derivatives have already developed, e.g., for the kinetic energy dissipation rate and, in compressible flow case, its solenoidal and dilatational components. It is known from incompressible flows that derivative moments transition from Gaussian to intermittent scaling at  $\text{Re} \sim 10^2$  [25–32]. The same holds for Burgers turbulence [33]. Connections between turbulent

statistics at different scales are described by fusion rules. They either connect velocity structure functions at different distances  $R > r$  in multiscale correlators in the inertial range [34] or velocity structure functions to the Reynolds number scaling of dissipation rate moments, which in case of the energy dissipation moments for incompressible turbulence leads to [25]

$$M_n = \frac{\langle \epsilon^n \rangle}{\langle \epsilon \rangle^n} \propto \text{Re}^{\beta_n} \quad \text{with} \quad \beta_n = n + \frac{\zeta_{4n}}{\zeta_{4n} - \zeta_{4n+1} - 1}.$$

The scaling exponents  $\beta_n$  depend nonlinearly on the order  $n$  (via the structure function exponents  $\zeta_n$ ), thus reflecting the intermittent character of the fields by an anomalous moment scaling with respect to  $\text{Re}$  [25].

Here, we conduct a series of direct numerical simulations (DNS) of compressible turbulence at very high resolution. We monitor the Reynolds number scaling of  $M_n(\text{Re})$  for different degrees of compressibility, quantified by the rms Mach number  $M_{\text{rms}}$ . Our main finding is a detection of a transition from a solenoidal statistics of moments  $M_n$  for  $M_{\text{rms}} \lesssim 1$  to a close-to-Burgers-turbulence-type behavior for  $M_{\text{rms}} > 1$ , where the energy dissipation is dominated by its dilatational dynamics.

## II. DIRECT NUMERICAL SIMULATIONS

The high-resolution DNS of three-dimensional homogeneous, isotropic, compressible turbulence solve the following equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (3)$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p + \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}, \quad (4)$$

together with the isothermal equation of state,  $p = \rho c_s^2$ . Here,  $\rho$  is the mass density,  $p$  is the pressure,  $\boldsymbol{\sigma}$  is the viscous stress tensor,  $\mathbf{f}$  is an external forcing, and  $\otimes$  denotes the dyadic product. The isothermal closure corresponds to a barotropic model,  $p = K\rho$ , which is not thermodynamically faithful for isolated or weakly cooled molecular fluids, but can approximate rapidly cooled systems where heat generated by viscous dissipation is removed to keep the temperature nearly constant [8]. Such conditions occur when the cooling time is much shorter than fluid timescales, such as in the dense, star-forming phase of the interstellar medium. The viscous stress tensor is given by

$$\boldsymbol{\sigma} = \mu[\nabla \mathbf{u} + (\nabla \mathbf{u})^T] + \left(\zeta - \frac{2\mu}{3}\right)(\nabla \cdot \mathbf{u})\mathbf{I}, \quad (5)$$

where  $\mathbf{I}$  denotes the identity tensor. We set the bulk viscosity  $\zeta = 0$ , corresponding to a Newtonian fluid under the Stokes hypothesis. The flow is driven by a stochastic forcing  $\mathbf{f}$  implemented in Fourier space using an Ornstein-Uhlenbeck (OU) process [35,36]. The forcing is *purely solenoidal*, applied at the smallest wave numbers,  $1 < kL/2\pi < 3$ , with a parabolic spectral distribution peaking at  $kL/2\pi = 2$ , where  $L$  is the domain size. This corresponds to a characteristic forcing length scale  $L_f \approx L/2$ . The equations are solved in a triply periodic cube of volume  $V = L^3$  using a modified version of the FLASH code [2,37,38]. In each simulation run, the kinematic viscosity  $\nu = \mu/\rho$  is kept constant, while the

dynamic viscosity  $\mu = \rho\nu$  varies with the local density. The sound speed is fixed at  $c_s = 1$ , while the mean kinetic energy injection rate  $\langle \epsilon_{\text{in}} \rangle = \langle \rho \mathbf{f} \cdot \mathbf{u} \rangle$  is varied to attain the target  $M_{\text{rms}}$ . For each fixed  $M_{\text{rms}}$ , different Reynolds numbers  $\text{Re}$  are obtained by adjusting  $\nu$  between runs.

Our DNS cover two orders of magnitude of rms Mach numbers,  $0.1 \leq M_{\text{rms}} \leq 10$ , thus spanning subsonic and supersonic regimes. DNS are performed at very high spatial resolutions with  $k_{\text{max}}\eta > 10$ , where  $k_{\text{max}} = \pi N/L$  and the Kolmogorov length scale  $\eta = (\langle \rho \rangle \nu^3 / \langle \epsilon \rangle)^{1/4}$ . We conducted 56 simulations on uniform meshes with up to  $N^3 = 2048^3$  grid points. The attainable Reynolds numbers are  $100 \lesssim \text{Re} \lesssim 2400$  [39], with corresponding Taylor microscale Reynolds numbers  $15 \lesssim \text{Re}_\lambda = \sqrt{(5\langle \rho \rangle / (3\nu \langle \epsilon \rangle))} u_{\text{rms}}^2 \lesssim 117$ . All DNS thus start beyond the transition from Gaussian to non-Gaussian derivative statistics [29]. For each simulation,  $N_s$  snapshots of the statistically stationary state are taken at intervals of one quarter of the large-scale eddy turnover time  $L_f/u_{\text{rms}}$  for statistical analysis. Details of the DNS parameters, characteristic statistical quantities, and numerical method are provided in Table I and Appendix A. We also give detailed validations of our statistical results by comparison with other numerical methods for the lowest [1,27,28,40] and largest  $M_{\text{rms}}$  [9] in Appendix B. There, we show that the present numerical scheme is not significantly affected by numerical artifacts for  $M_{\text{rms}} \geq 0.1$  (see Refs. [41–43]).

Similar to the Helmholtz decomposition of the velocity field into solenoidal and dilatational parts,  $\mathbf{u} = \mathbf{u}_s + \mathbf{u}_d$ , with  $\nabla \cdot \mathbf{u}_s = 0$ , and  $\nabla \times \mathbf{u}_d = 0$ , we can decompose the kinetic energy dissipation rate field (2) into solenoidal, dilatational, and inhomogeneous components [44–46]. To derive this decomposition, we first write Eq. (2) in index notation as

$$\epsilon(\mathbf{x}, t) = 2\mu S_{ij}S_{ij} - \frac{2\mu}{3} \left( \frac{\partial u_k}{\partial x_k} \right)^2. \quad (6)$$

The first term on the right-hand side of Eq. (6) can be expanded as follows:

$$2\mu S_{ij}S_{ij} = 2\mu \Omega_{ij}\Omega_{ij} + 2\mu \left[ \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \right], \quad (7)$$

where  $\Omega_{ij} = (\partial u_i / \partial x_j - \partial u_j / \partial x_i) / 2$  is the antisymmetric vorticity tensor and  $S_{ij}\Omega_{ij} = 0$ . Using the product rule, we express the second term on the right-hand side of Eq. (7) as

$$\begin{aligned} 2\mu \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} &= 2\mu \left[ \frac{\partial}{\partial x_j} \left( u_i \frac{\partial u_j}{\partial x_i} \right) - u_i \frac{\partial}{\partial x_j} \left( \frac{\partial u_j}{\partial x_i} \right) \right] \\ &= 2\mu \frac{\partial}{\partial x_j} \left[ \frac{\partial}{\partial x_i} (u_i u_j) - u_j \frac{\partial u_i}{\partial x_i} \right] \\ &\quad - 2\mu \left[ \frac{\partial}{\partial x_i} \left( u_i \frac{\partial u_j}{\partial x_j} \right) - \left( \frac{\partial u_j}{\partial x_j} \right)^2 \right] \\ &= 2\mu \frac{\partial^2}{\partial x_i \partial x_j} (u_i u_j) - 4\mu \frac{\partial}{\partial x_i} \left( u_i \frac{\partial u_k}{\partial x_k} \right) \\ &\quad + 2\mu \left( \frac{\partial u_k}{\partial x_k} \right)^2. \end{aligned} \quad (8)$$

TABLE I. DNS parameters of solenoidally forced compressible isotropic turbulence. The run number, rms Mach number  $M_{\text{rms}}$ , large-scale Reynolds number  $\text{Re}$ , Taylor microscale Reynolds number  $\text{Re}_\lambda$ , statistically independent snapshots  $N_s$ , minima and maxima of  $M_4$ , and  $k_{\text{max}}\eta$ .

| Run | $N^3$             | $M_{\text{rms}}$ | $\text{Re}$ | $\text{Re}_\lambda$ | $N_s$ | $M_4$                 | $\min[M_4]$        | $\max[M_4]$           | $k_{\text{max}}\eta$ |
|-----|-------------------|------------------|-------------|---------------------|-------|-----------------------|--------------------|-----------------------|----------------------|
| 1   | 512 <sup>3</sup>  | 0.10             | 100         | 17                  | 100   | $1.58 \times 10^1$    | $1.01 \times 10^1$ | $3.77 \times 10^1$    | 29.17                |
| 2   | 512 <sup>3</sup>  | 0.10             | 219         | 29                  | 100   | $2.64 \times 10^1$    | $1.51 \times 10^1$ | $6.53 \times 10^1$    | 17.45                |
| 3   | 512 <sup>3</sup>  | 0.10             | 399         | 43                  | 100   | $4.12 \times 10^1$    | $2.29 \times 10^1$ | $9.36 \times 10^1$    | 11.58                |
| 4   | 1024 <sup>3</sup> | 0.10             | 520         | 50                  | 100   | $5.33 \times 10^1$    | $3.23 \times 10^1$ | $1.25 \times 10^2$    | 19.24                |
| 5   | 1024 <sup>3</sup> | 0.10             | 709         | 58                  | 100   | $6.41 \times 10^1$    | $3.90 \times 10^1$ | $1.29 \times 10^2$    | 15.26                |
| 6   | 1024 <sup>3</sup> | 0.10             | 932         | 67                  | 100   | $8.50 \times 10^1$    | $4.46 \times 10^1$ | $2.63 \times 10^2$    | 12.40                |
| 7   | 2048 <sup>3</sup> | 0.10             | 2469        | 117                 | 100   | $2.11 \times 10^2$    | $9.15 \times 10^1$ | $7.69 \times 10^2$    | 12.40                |
| 8   | 512 <sup>3</sup>  | 0.34             | 102         | 17                  | 100   | $1.59 \times 10^1$    | $8.29 \times 10^0$ | $2.94 \times 10^1$    | 29.06                |
| 9   | 512 <sup>3</sup>  | 0.33             | 222         | 29                  | 100   | $2.52 \times 10^1$    | $1.63 \times 10^1$ | $6.00 \times 10^1$    | 17.29                |
| 10  | 512 <sup>3</sup>  | 0.33             | 391         | 42                  | 100   | $4.51 \times 10^1$    | $2.35 \times 10^1$ | $1.49 \times 10^2$    | 11.74                |
| 11  | 1024 <sup>3</sup> | 0.33             | 511         | 49                  | 100   | $5.31 \times 10^1$    | $3.21 \times 10^1$ | $1.60 \times 10^2$    | 19.46                |
| 12  | 1024 <sup>3</sup> | 0.33             | 715         | 59                  | 100   | $6.37 \times 10^1$    | $3.16 \times 10^1$ | $1.59 \times 10^2$    | 15.26                |
| 13  | 1024 <sup>3</sup> | 0.33             | 956         | 70                  | 100   | $8.85 \times 10^1$    | $4.86 \times 10^1$ | $6.75 \times 10^2$    | 12.39                |
| 14  | 2048 <sup>3</sup> | 0.33             | 2411        | 116                 | 100   | $1.69 \times 10^2$    | $1.02 \times 10^2$ | $3.25 \times 10^2$    | 12.64                |
| 15  | 512 <sup>3</sup>  | 0.55             | 96          | 16                  | 200   | $1.80 \times 10^1$    | $1.12 \times 10^1$ | $3.43 \times 10^1$    | 29.88                |
| 16  | 512 <sup>3</sup>  | 0.54             | 213         | 30                  | 200   | $2.61 \times 10^1$    | $1.36 \times 10^1$ | $6.58 \times 10^1$    | 18.33                |
| 17  | 512 <sup>3</sup>  | 0.54             | 378         | 41                  | 200   | $5.49 \times 10^1$    | $2.29 \times 10^1$ | $9.53 \times 10^2$    | 12.05                |
| 18  | 1024 <sup>3</sup> | 0.55             | 497         | 48                  | 200   | $1.61 \times 10^2$    | $2.98 \times 10^1$ | $1.81 \times 10^4$    | 19.75                |
| 19  | 1024 <sup>3</sup> | 0.56             | 712         | 61                  | 100   | $1.89 \times 10^3$    | $3.95 \times 10^1$ | $1.76 \times 10^5$    | 15.51                |
| 20  | 1024 <sup>3</sup> | 0.56             | 955         | 71                  | 100   | $6.11 \times 10^3$    | $6.57 \times 10^1$ | $3.90 \times 10^5$    | 12.46                |
| 21  | 2048 <sup>3</sup> | 0.56             | 2405        | 115                 | 100   | $5.74 \times 10^3$    | $9.87 \times 10^1$ | $1.57 \times 10^5$    | 12.60                |
| 22  | 512 <sup>3</sup>  | 0.78             | 97          | 16                  | 200   | $5.36 \times 10^1$    | $1.11 \times 10^1$ | $2.78 \times 10^3$    | 29.55                |
| 23  | 512 <sup>3</sup>  | 0.77             | 212         | 29                  | 200   | $2.91 \times 10^2$    | $2.28 \times 10^1$ | $1.34 \times 10^4$    | 17.91                |
| 24  | 512 <sup>3</sup>  | 0.77             | 385         | 42                  | 200   | $5.50 \times 10^3$    | $4.24 \times 10^1$ | $2.09 \times 10^5$    | 11.91                |
| 25  | 1024 <sup>3</sup> | 0.78             | 505         | 49                  | 200   | $3.14 \times 10^4$    | $9.99 \times 10^1$ | $1.11 \times 10^6$    | 19.68                |
| 26  | 1024 <sup>3</sup> | 0.77             | 695         | 59                  | 100   | $4.58 \times 10^4$    | $4.63 \times 10^2$ | $6.12 \times 10^5$    | 15.68                |
| 27  | 1024 <sup>3</sup> | 0.78             | 947         | 70                  | 100   | $7.28 \times 10^4$    | $2.91 \times 10^3$ | $4.88 \times 10^5$    | 12.51                |
| 28  | 2048 <sup>3</sup> | 0.78             | 2414        | 116                 | 100   | $1.05 \times 10^6$    | $2.64 \times 10^4$ | $9.76 \times 10^6$    | 12.62                |
| 29  | 512 <sup>3</sup>  | 1.0              | 95          | 16                  | 200   | $1.77 \times 10^2$    | $1.56 \times 10^1$ | $2.44 \times 10^3$    | 29.60                |
| 30  | 512 <sup>3</sup>  | 1.0              | 218         | 29                  | 200   | $4.10 \times 10^3$    | $8.09 \times 10^1$ | $5.23 \times 10^4$    | 17.40                |
| 31  | 512 <sup>3</sup>  | 1.0              | 386         | 42                  | 200   | $3.60 \times 10^4$    | $5.66 \times 10^2$ | $8.06 \times 10^5$    | 11.88                |
| 32  | 1024 <sup>3</sup> | 1.0              | 499         | 49                  | 200   | $1.35 \times 10^5$    | $2.91 \times 10^3$ | $1.51 \times 10^6$    | 19.78                |
| 33  | 1024 <sup>3</sup> | 1.0              | 686         | 58                  | 100   | $3.39 \times 10^5$    | $1.03 \times 10^4$ | $4.07 \times 10^6$    | 15.73                |
| 34  | 1024 <sup>3</sup> | 1.0              | 927         | 69                  | 100   | $4.95 \times 10^5$    | $2.14 \times 10^4$ | $5.72 \times 10^6$    | 12.64                |
| 35  | 2048 <sup>3</sup> | 1.0              | 2389        | 115                 | 100   | $4.95 \times 10^6$    | $1.10 \times 10^6$ | $2.08 \times 10^7$    | 12.69                |
| 36  | 512 <sup>3</sup>  | 3.0              | 95          | 15                  | 200   | $1.70 \times 10^4$    | $6.05 \times 10^2$ | $3.45 \times 10^5$    | 28.32                |
| 37  | 512 <sup>3</sup>  | 3.0              | 217         | 25                  | 200   | $1.11 \times 10^5$    | $9.77 \times 10^3$ | $2.25 \times 10^6$    | 16.23                |
| 38  | 512 <sup>3</sup>  | 3.1              | 390         | 35                  | 200   | $4.40 \times 10^5$    | $2.93 \times 10^4$ | $4.64 \times 10^6$    | 10.76                |
| 39  | 1024 <sup>3</sup> | 3.0              | 507         | 40                  | 200   | $1.28 \times 10^6$    | $9.30 \times 10^4$ | $1.47 \times 10^7$    | 17.72                |
| 40  | 1024 <sup>3</sup> | 3.0              | 721         | 49                  | 200   | $3.27 \times 10^6$    | $1.85 \times 10^5$ | $8.42 \times 10^7$    | 13.92                |
| 41  | 1024 <sup>3</sup> | 3.1              | 944         | 59                  | 200   | $4.44 \times 10^6$    | $5.51 \times 10^5$ | $6.69 \times 10^7$    | 11.31                |
| 42  | 2048 <sup>3</sup> | 3.0              | 2360        | 96                  | 132   | $6.01 \times 10^7$    | $9.46 \times 10^6$ | $5.75 \times 10^8$    | 11.74                |
| 43  | 512 <sup>3</sup>  | 6.2              | 104         | 16                  | 200   | $5.45 \times 10^5$    | $9.00 \times 10^3$ | $3.89 \times 10^7$    | 26.94                |
| 44  | 512 <sup>3</sup>  | 6.2              | 207         | 24                  | 200   | $1.72 \times 10^6$    | $2.30 \times 10^4$ | $4.47 \times 10^7$    | 16.77                |
| 45  | 512 <sup>3</sup>  | 6.2              | 416         | 36                  | 200   | $1.35 \times 10^7$    | $1.98 \times 10^5$ | $6.04 \times 10^8$    | 10.22                |
| 46  | 1024 <sup>3</sup> | 6.2              | 519         | 41                  | 200   | $3.27 \times 10^7$    | $6.76 \times 10^5$ | $4.77 \times 10^8$    | 17.35                |
| 47  | 1024 <sup>3</sup> | 6.2              | 724         | 49                  | 200   | $8.33 \times 10^7$    | $1.49 \times 10^6$ | $2.65 \times 10^9$    | 13.64                |
| 48  | 1024 <sup>3</sup> | 6.2              | 1034        | 59                  | 200   | $1.54 \times 10^8$    | $2.73 \times 10^6$ | $8.70 \times 10^9$    | 10.49                |
| 49  | 2048 <sup>3</sup> | 6.3              | 2631        | 97                  | 150   | $4.88 \times 10^9$    | $4.17 \times 10^7$ | $4.26 \times 10^{11}$ | 10.49                |
| 50  | 512 <sup>3</sup>  | 10.2             | 98          | 15                  | 200   | $1.50 \times 10^6$    | $2.05 \times 10^4$ | $8.71 \times 10^7$    | 28.42                |
| 51  | 512 <sup>3</sup>  | 10.2             | 222         | 25                  | 200   | $1.55 \times 10^7$    | $2.21 \times 10^5$ | $3.62 \times 10^8$    | 16.01                |
| 52  | 512 <sup>3</sup>  | 10.3             | 396         | 36                  | 200   | $5.90 \times 10^7$    | $1.29 \times 10^6$ | $1.24 \times 10^9$    | 10.64                |
| 53  | 1024 <sup>3</sup> | 10.2             | 510         | 40                  | 200   | $2.66 \times 10^8$    | $3.45 \times 10^6$ | $4.56 \times 10^9$    | 17.56                |
| 54  | 1024 <sup>3</sup> | 10.2             | 714         | 48                  | 200   | $6.93 \times 10^8$    | $5.16 \times 10^6$ | $2.35 \times 10^{10}$ | 13.69                |
| 55  | 1024 <sup>3</sup> | 10.3             | 958         | 56                  | 200   | $1.82 \times 10^9$    | $1.40 \times 10^7$ | $1.69 \times 10^{11}$ | 11.08                |
| 56  | 2048 <sup>3</sup> | 10.4             | 2445        | 92                  | 146   | $1.10 \times 10^{10}$ | $2.83 \times 10^8$ | $1.93 \times 10^{11}$ | 11.12                |

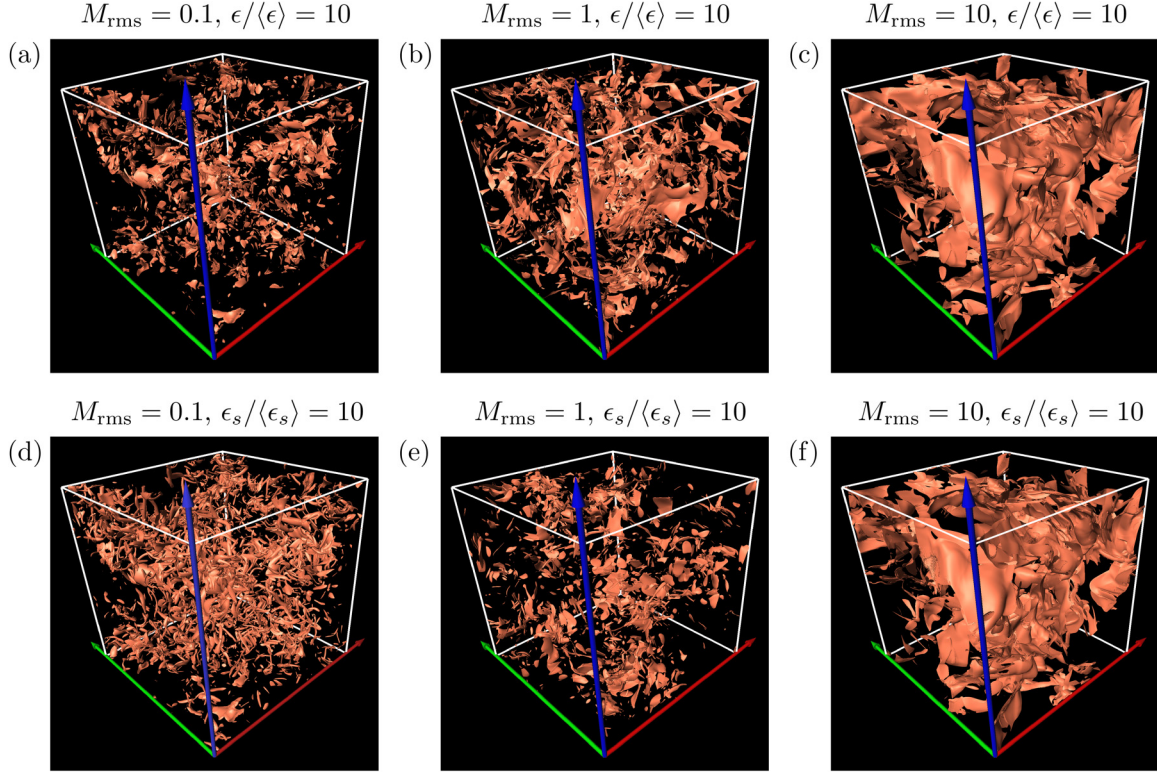


FIG. 1. Structure of the kinetic energy dissipation rate field. Isosurface snapshots of total energy dissipation  $\epsilon(\mathbf{x}, t_0)$  at the level  $\epsilon = 10\langle\epsilon\rangle$  (top row) and its solenoidal component  $\epsilon_s(\mathbf{x}, t_0)$  at  $\epsilon_s = 10\langle\epsilon_s\rangle$  (bottom row) for rms Mach numbers  $M_{\text{rms}} = 0.1$  [panels (a) and (d)],  $M_{\text{rms}} = 1$  [panels (b) and (e)], and  $M_{\text{rms}} = 10$  [panels (c) and (f)]. All data are obtained at the highest Reynolds number  $\text{Re} \approx 2400$ ; isosurfaces are displayed for the decadic logarithm of the fields.

Combining Eqs. (6)–(8) yields the decomposition

$$\epsilon = \epsilon_s + \epsilon_d + \epsilon_I, \quad (9)$$

with the solenoidal ( $s$ ), dilatational ( $d$ ), and inhomogeneous ( $I$ ) components, which follow in vector notation to

$$\begin{aligned} \epsilon_s(\mathbf{x}, t) &= \mu(\nabla \times \mathbf{u})^2, & \epsilon_d(\mathbf{x}, t) &= \frac{4}{3}\mu(\nabla \cdot \mathbf{u})^2, \\ \epsilon_I(\mathbf{x}, t) &= 2\mu[(\nabla \otimes \nabla) : (\mathbf{u} \otimes \mathbf{u}) - 2\nabla \cdot (\mathbf{u}(\nabla \cdot \mathbf{u}))]. \end{aligned}$$

Figures 1(a)–1(c) display isosurfaces of snapshots at  $\epsilon = 10\langle\epsilon\rangle$  of the simulations with  $M_{\text{rms}} = 0.1, 1$ , and  $10$  and the highest  $\text{Re}$ . We see that  $\epsilon(\mathbf{x}, t)$  crosses over from local small shear layers [panel (a)], which are caused by the self-induced strain due to vortex stretching [47], to extended, stacked shock-generated sheets [panel (c)] for the highly supersonic case. Figures 1(d)–1(f) display isosurfaces of  $\epsilon_s = 10\langle\epsilon_s\rangle$  at the same  $M_{\text{rms}}$  and  $\text{Re}$ . We see that the solenoidal part  $\epsilon_s$  changes from tubelike patterns in panel (d), which are typical for the local enstrophy in the incompressible case [12], to shock-surface patterns, almost identical to those of the total dissipation  $\epsilon$  at  $M_{\text{rms}} = 10$  [cf. panels (c) and (f)]. This qualitative transition of the highest-amplitude dissipation structures from vortex filaments with box counting dimension  $D = 1$  for  $M_{\text{rms}} < 1$  to sheetlike structures with  $D = 2$  at  $M_{\text{rms}} \gg 1$  is consistent with earlier studies based on velocity structure function scaling [22,24].

### III. OPERATING POINTS IN PARAMETER SPACE

We first determine the operating points of our DNS runs in the parameter space of compressible turbulence. In addition to the two central parameters,  $\text{Re}$  and  $M_{\text{rms}}$ , a third relevant parameter is the dilatational parameter

$$\delta = \frac{u_{d,\text{rms}}}{u_{s,\text{rms}}}, \quad (10)$$

which quantifies the ratio of dilatational to solenoidal motions in compressible flow [48]. Figure 2(a) shows our DNS series with solenoidal forcing in the  $\delta$ – $M_{\text{rms}}$  plane. We see an asymptotic behavior for the supersonic DNS to reach approximately  $\delta \simeq 0.5$ . This finding is consistent with an earlier study [49], which yielded an asymptotic ratio of  $E_{\text{sol}}/E_{\text{tot}} = u_{s,\text{rms}}^2/(u_{s,\text{rms}}^2 + u_{d,\text{rms}}^2) \simeq 0.8 \Rightarrow \delta \simeq 1/2$  for purely solenoidal forcing in the limit  $M_{\text{rms}} \rightarrow \infty$ . Furthermore, it is consistent with a renormalization group analysis, which gave a ratio  $u_{s,\text{rms}}^2/u_{d,\text{rms}}^2 = 3$ , corresponding to  $\delta = 1/\sqrt{3} \approx 0.58$  [50].

We show the ratios of  $\langle\epsilon_d\rangle/\langle\epsilon\rangle$  and  $\langle\epsilon_d\rangle/\langle\epsilon_s\rangle$  versus  $M_{\text{rms}}$  in Figs. 2(b) and 2(c). Panel (b) shows that the dilatational fraction of dissipation is rising from almost 0 to  $\simeq 0.5$ , underlining the increasing role of compressive motions and shock-driven dissipation with increasing Mach number. At the highest Mach number,  $M_{\text{rms}} = 10$ , the dilatational-to-solenoidal mean dissipation ratio [panel (c)] is  $\lesssim 1$ . The inhomogeneous contribution to the total mean dissipation rate,  $\langle\epsilon_I\rangle/\langle\epsilon\rangle$ , remains relatively small in magnitude as shown in

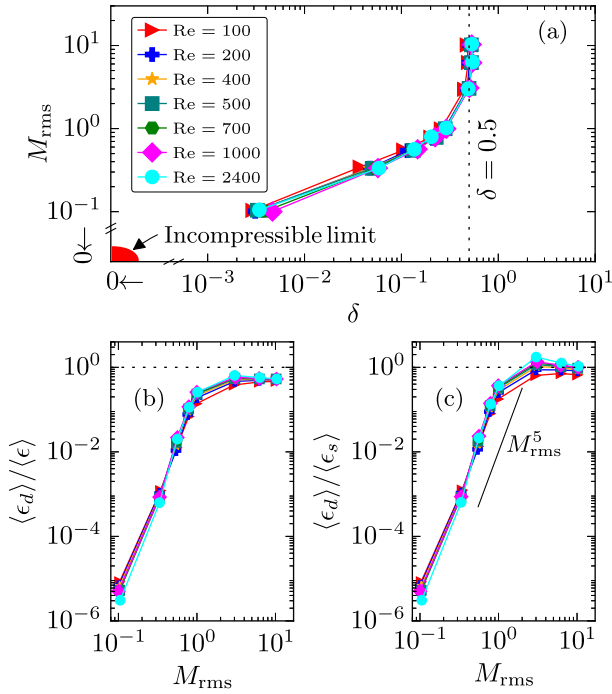


FIG. 2. (a) Operating points of the solenoidally forced compressible turbulence simulations in the  $\delta$ - $M_{\text{rms}}$  parameter plane [panel (a)]. The incompressible limit and the  $\delta$  asymptote are indicated, which is consistently not exceeded by our simulations. The ratios of the mean dissipation rates vs the rms Mach number  $M_{\text{rms}}$ : (b)  $\langle \epsilon_d \rangle / \langle \epsilon \rangle$  and (c)  $\langle \epsilon_d \rangle / \langle \epsilon_s \rangle$ . The dotted horizontal lines indicate asymptotic behavior toward a 1:1 ratio.

Fig. 7 in Appendix A. We have seen that the isosurfaces of  $\epsilon_s$  at this Mach number also form sharp, sheetlike patterns aligned with shocks, likely indicating vorticity amplification in shock-turbulence interactions; this enhanced vorticity increases solenoidal dissipation and may contribute to the observed magnitude of the ratio of about unity for  $M_{\text{rms}} > 1$ . In the subsonic regime for  $0.3 \leq M_{\text{rms}} \leq 1$ , the ratio follows an approximate  $M_{\text{rms}}^5$  scaling, which is consistent with Ref. [51]; a similar but slightly shallower  $M_{\text{rms}}^{4.1}$  scaling was reported in Ref. [40].

The rate-of-strain tensor can also be Helmholtz decomposed into two parts,  $\mathbf{S} = \mathbf{S}_s + \mathbf{S}_d$  with  $\mathbf{S}_s = [\nabla \mathbf{u}_s + (\nabla \mathbf{u}_s)^T]/2$  and  $\mathbf{S}_d = [\nabla \mathbf{u}_d + (\nabla \mathbf{u}_d)^T]/2$ . Thus, Eq. (2) can be alternatively rewritten as

$$\epsilon(\mathbf{x}, t) = 2\mu(\mathbf{S}_s + \mathbf{S}_d) : (\mathbf{S}_s + \mathbf{S}_d) - \frac{2\mu}{3}(\nabla \cdot \mathbf{u}_d)^2. \quad (11)$$

We denote  $\epsilon_{\text{inc}} = 2\mu \mathbf{S}_s : \mathbf{S}_s$ , the pointwise dissipation rate solely arising from the incompressible motion in the flow. Consequently,  $\epsilon = \epsilon_{\text{inc}} + \epsilon_{\text{com}}$  with a compressible dissipation field that collects the remaining terms in Eq. (11). We find that the mean values  $\langle \epsilon_{\text{inc}} \rangle$  and  $\langle \epsilon_{\text{com}} \rangle$  behave similarly to  $\langle \epsilon_s \rangle$  and  $\langle \epsilon_d \rangle$  in Fig. 2, respectively. Although  $\epsilon_{\text{inc}}$  and  $\epsilon_s$  both arise from solenoidal motions, they are conceptually distinct in their local structure: The former is associated with local shear layers, whereas the latter with the vortical structures in the fluid.

#### IV. STATISTICS AND SCALING OF DISSIPATION MOMENTS

Figure 3 illustrates the convergence of the fourth-order moments of the total energy dissipation rate  $\epsilon$ . We thus plot  $\tilde{\epsilon}^4 \text{pdf}(\tilde{\epsilon})$  versus  $\tilde{\epsilon} = \epsilon / \langle \epsilon \rangle$  for all Mach numbers  $M_{\text{rms}}$  at the highest Reynolds number of  $\text{Re} \approx 2400$ . The panels show that all cases are converged, except for the data at  $M_{\text{rms}} = 0.55$ . This series is characterized by a few extreme fluctuation events, which alter the overall statistics. Figure 4 displays the normalized fourth-order moments  $M_4(t)$  of total dissipation rate obtained from the different snapshots at  $M_{\text{rms}} = 0.55$  and compares it with those at  $M_{\text{rms}} = 0.1$  and  $M_{\text{rms}} = 3$ . First evolving dominant preshocks mature to an increasing number of shock fronts with increasing  $M_{\text{rms}}$ . In Table I, we provide minima and maxima of  $M_4$  for all data series.

Figure 5 summarizes the central result of this work. We show the scaling exponents  $\beta_{k,n}$  versus  $M_{\text{rms}}$  for the time-averaged normalized energy dissipation moments  $M_{k,n}(\text{Re})$  of orders  $n = 2, 3$ , and 4. Here,  $k = \{s, d, \text{inc}\}$  in panels (b)–(d); panel (a) shows the total dissipation for completeness. In the low-Mach-number regime for  $M_{\text{rms}} \leq 0.33$ , the total dissipation exponents  $\beta_n$  in panel (a) align well with theory [25,52] and DNS [27,29] for incompressible limit. Preshocks are not yet present [54] and moments of density  $\langle \rho^n \rangle \sim \text{Re}^0$ , as seen in Fig. 5(e). In the latter panel, we added the scaling exponents  $\beta_n^\rho$  of normalized density moments  $M_n^\rho = \langle \rho^n \rangle / \langle \rho \rangle^n \sim \text{Re}^{\beta_n^\rho}$  for  $n \leq 4$ . The data points for  $\beta_n$  and  $\beta_{k,n}$  can be connected by fits or straight lines in Figs. 5(a), 5(b), and 5(d), which correspond to the incompressible exponents for  $M_{\text{rms}} \ll 1$ .

For  $M_{\text{rms}} = 0.55$ , the density varies already by  $\pm 0.5 \langle \rho \rangle$  for the highest  $\text{Re}$  still leaving  $\langle \rho^n \rangle \sim \text{Re}^0$  up to  $M_{\text{rms}} \lesssim 1$ , as visible in Fig. 5(e). For this  $M_{\text{rms}}$ , we quantify high magnitudes of  $-(\nabla \cdot \mathbf{u})\tau_\eta \sim 10$  that increased by 2 orders of magnitude compared to  $M_{\text{rms}} = 0.33$  (all measured in units of Kolmogorov time  $\tau_\eta$ ). This indicates a beginning of the transition in scaling, seen first for  $\beta_3$  and  $\beta_4$ , which are now larger than their incompressible counterparts, caused by locally strongly enhanced dilatational motion.

For the subsonic case of  $M_{\text{rms}} = 0.77$ , these dilatation amplitudes  $-(\nabla \cdot \mathbf{u})\tau_\eta$  have been grown to those of the supersonic cases, such that  $\beta_n$  and particularly  $\beta_{d,n}$  reach a supersonic plateau for the highest orders [see Figs. 5(a) and 5(c) and 6(a) and 6(c)]. In contrast, the solenoidal and incompressible exponents in Figs. 5(b) and 5(d) still remain relatively unchanged up to  $M_{\text{rms}} \approx 1$ , with  $\beta_{s,n}$  exceeding  $\beta_{\text{inc},n}$ —consistent with findings that  $\epsilon_s$  is more intermittent than  $\epsilon_{\text{inc}}$  for  $\text{Re}_\lambda \lesssim 400$  [55,56]. To conclude this discussion, our DNS suggest that an intermittently emerging dilatational motion for  $0.5 \lesssim M_{\text{rms}} \lesssim 1$  causes the transition in the small-scale derivative statistics. It is in line with Fig. 2(b), which shows  $\langle \epsilon_d \rangle / \langle \epsilon \rangle \rightarrow \mathcal{O}(1)$ . For  $M_{\text{rms}} \gtrsim 3$ , the density moments  $M_n^\rho$  transition to an anomalous  $\text{Re}$  scaling [see Fig. 5(e)]. Mature shocks in  $\rho$  and steepest velocity gradients are then aligned in a shock- or dilatation-dominated turbulence [see Figs. 6(b) and 6(d)].

What determines or bounds the plateau value for the  $\beta$ 's in all panels? In one-dimensional, stochastically forced Burgers turbulence, it was shown by Friedrich *et al.* [33] that the normalized moments  $\tilde{M}_n$  of the derivative  $\partial_x u$  satisfy an

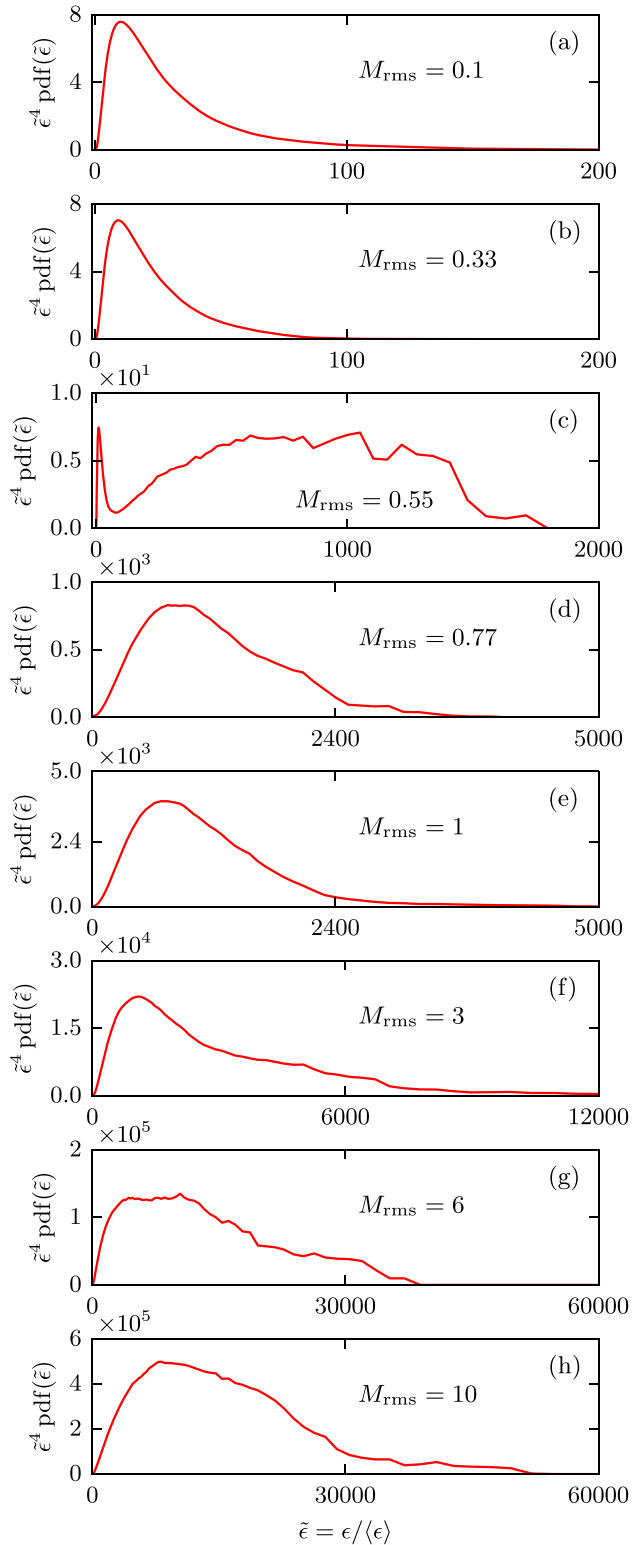


FIG. 3. Statistical convergence test of the normalized fourth-order moments of the total energy dissipation rate field for the highest Reynolds number runs ( $\text{Re} \approx 2400$ ) at rms Mach numbers  $M_{\text{rms}} = 0.1$  [panel (a)],  $M_{\text{rms}} = 0.33$  [panel (b)],  $M_{\text{rms}} = 0.55$  [panel (c)],  $M_{\text{rms}} = 0.77$  [panel (d)],  $M_{\text{rms}} = 1$  [panel (e)],  $M_{\text{rms}} = 3$  [panel (f)],  $M_{\text{rms}} = 6$  [panel (g)], and  $M_{\text{rms}} = 10$  [panel (h)].

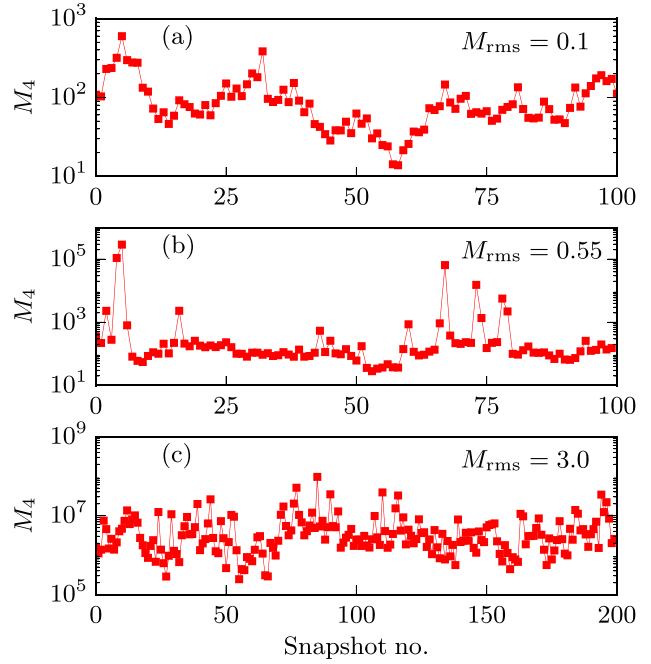


FIG. 4. Fourth-order moments of the total dissipation rate at rms Mach numbers  $M_{\text{rms}} = 0.1$  [panel (a)],  $M_{\text{rms}} = 0.55$  [panel (b)], and  $M_{\text{rms}} = 3.0$  [panel (c)].

anomalous scaling relation for  $\text{Re} \gg 1$ , which is given by and can be translated to

$$\tilde{M}_n = \frac{\langle (\partial_x u)^n \rangle}{\langle (\partial_x u)^2 \rangle^{n/2}} \sim \text{Re}^{n/2-1} \Rightarrow M_n \sim \text{Re}^{n-1}, \quad (12)$$

using  $\epsilon \sim (\partial_x u)^2$ . The plateaus of  $\beta_{k,n}$  in Fig. 5 fall consistently below 1, 2, and 3 for  $M_2$ ,  $M_3$ , and  $M_4$ , respectively. This clearly supports the physical picture that the anomalous exponents of shock-dominated Burgers turbulence (with pressure being absent) bound those of fully compressible turbulence from above.

## V. CONCLUSION

Our numerical simulations of compressible turbulence report a transition of the velocity derivative statistics, here in the form of scaling laws for the kinetic energy dissipation and its components with  $\text{Re}$ , from the scaling for the low-Mach-number limit ( $M_{\text{rms}} \ll 1$ ) to a Burgers turbulence-type scaling for  $M_{\text{rms}} \gg 1$ , dominated by shocks and dilatation sheets for the density and energy dissipation fields. This relatively sharp transition is initiated by increasing dilatational motion for  $0.5 \lesssim M_{\text{rms}} \lesssim 1$ . Both components of dissipation,  $\epsilon_s$  and  $\epsilon_d$ , scale similarly for  $M_{\text{rms}} > 1$ , since steepest shear and dilatation gradients nearly perfectly coincide, which is shown in Figs. 1(c) and 1(f) and 6(b) and 6(d). The present analysis is based on an isothermal formulation; therefore, flows of molecular fluids that are thermodynamically isolated or subject to weak cooling only cannot be fully represented. Thus, DNS that include the full thermodynamic evolution will be studied

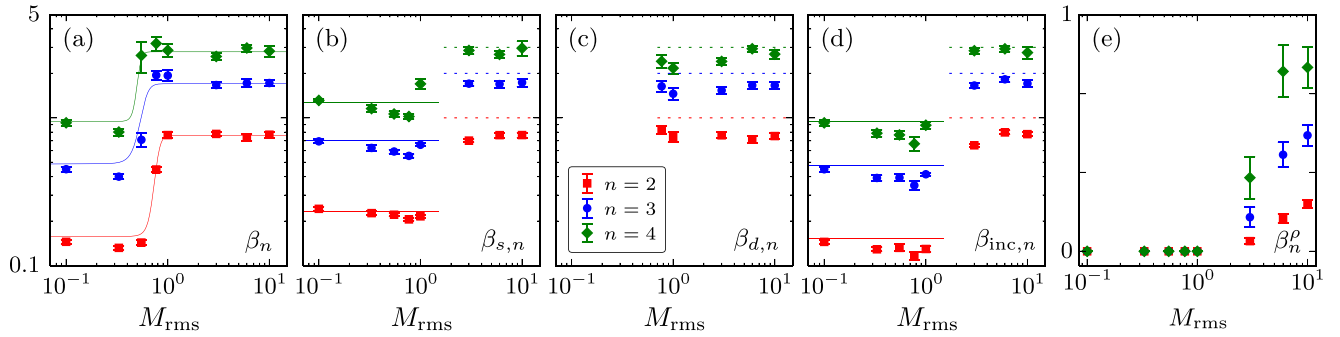


FIG. 5. Reynolds number-scaling exponents of the second-, third- and fourth-order normalized moments vs rms Mach number  $M_{\text{rms}}$ . (a) Total energy dissipation, (b) solenoidal, (c) dilatational, and (d) incompressible components of energy dissipation, and (e) density. In the incompressible limit  $M_{\text{rms}} \ll 1$ ,  $\beta_{\text{inc},n} \approx \beta_n$  and are consistent with those reported for incompressible isotropic turbulence [25,27,52];  $\beta_2 = 0.157$ ,  $\beta_3 = 0.489$ , and  $\beta_4 = 0.944$ . In this limit, the solenoidal exponents [panel (b)] take the values  $\beta_{s,2} = 0.242$ ,  $\beta_{s,3} = 0.693$ , and  $\beta_{s,4} = 1.31$ , in close agreement with Elsinga *et al.* [53] after necessary adjustment (see Appendix C). For  $M_{\text{rms}} > 1$ , the exponents of the total dissipation rate field and its components are very close to the Burgers' scaling exponents. In panel (a), the data points are connected by a tanh fitting curve. The solid and dotted reference lines in panels (b)–(d) are for incompressible and Burgers turbulence scaling limits, respectively. We exclude all data at  $\text{Re} = 100$  from the fits due to their closeness to the Gaussian to non-Gaussian transition region of derivative statistics [29].

in future work. It remains to be seen if this relatively sharp transition stays intact.

#### ACKNOWLEDGMENTS

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Views and opinions expressed, however, are those of the authors only and do not necessarily reflect those of the European Union or the European Research Council.

#### DATA AVAILABILITY

The FLASH source code used to generate the simulation data can be obtained upon reasonable request through the FLASH Center [59]. The raw simulation data (full three-dimensional fields) are available from the corresponding author upon reasonable request. All post-processed data and Python scripts used to generate the figures (except Fig. 1) are publicly available on Zenodo [60].

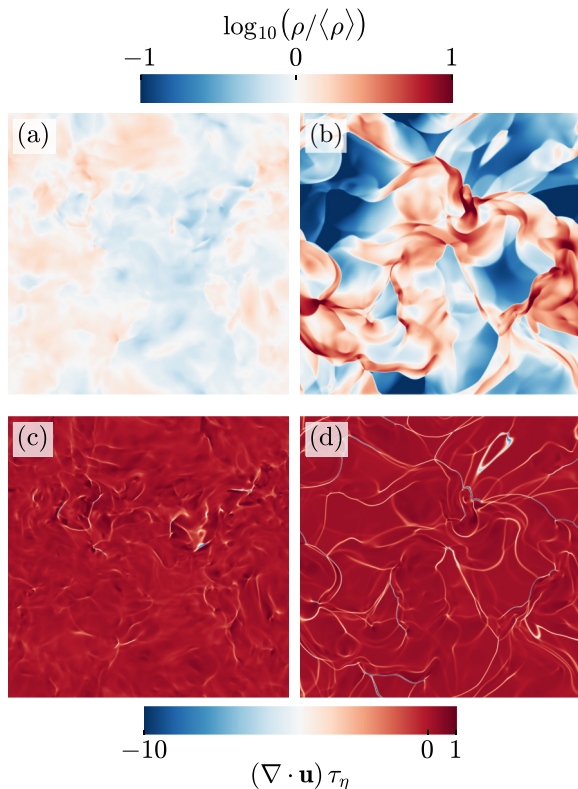


FIG. 6. Contour slice plots of normalized density and dilatation in the top and bottom row, respectively. (a), (c)  $M_{\text{rms}} = 0.77$  and (b), (d)  $M_{\text{rms}} = 3$ .

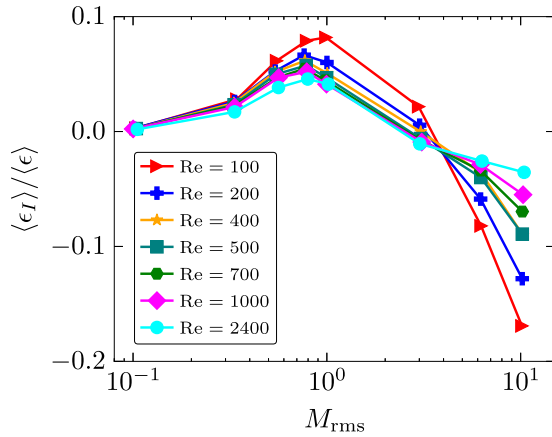


FIG. 7. Inhomogeneous contribution to the total mean dissipation rate as a function of the rms Mach number  $M_{\text{rms}}$ .

## APPENDIX A: NUMERICAL METHOD AND SIMULATION PARAMETERS

Detailed simulation parameters, information about the resolution ( $k_{\text{max}}\eta$ ), and maxima and minima of the normalized fourth-order moments  $M_4$  of the total energy dissipation rate are listed in Table I. We define therefore  $M_4(t) = \langle \epsilon^4 \rangle_V / \langle \epsilon \rangle_V^4$ , where  $\langle \cdot \rangle_V$  denotes a volume average. Furthermore, we provide the number of statistically independent snapshots,  $N_s$ . Successive snapshots are separated by a quarter large-scale eddy-turnover time,  $T_e/4$  with  $T_e = L_f/u_{\text{rms}}$ .

We have already discussed the contributions of the solenoidal and dilatational dissipation components to the total dissipation rate at different Mach numbers in Fig. 2. For completeness, we present the inhomogeneous contribution in Fig. 7. The figure shows that its magnitude remains relatively small for all Mach numbers. This contribution arises solely from spatial variations of the dynamic viscosity,  $\mu = \rho\nu$ . We confirm that the periodicity of the domain yields  $\langle \epsilon_I / \mu \rangle \approx 0$ . Since  $\epsilon_I$  is not positive definite, its contribution may be either positive or negative depending on the local flow structure as seen in the figure.

We perform the direct numerical simulations using a modified version of the FLASH code [37,38], based on release 4.0.1, which employs the MUSCL–Hancock HLL5R scheme [61,62].

(1) The MUSCL–Hancock reconstruction provides second-order accuracy in space and time. Cell-averaged primitive variables (density, pressure, and velocity components) are reconstructed with piecewise-linear slopes computed from neighboring cells and limited with the MC (monotonized central) slope limiter. The limiter suppresses spurious oscillations near discontinuities while retaining second-order accuracy in smooth regions.

(2) The resulting states are used by the HLL5R Riemann solver to compute the fluxes at the cell interfaces. The scheme is applied in a directionally split manner, where each time step consists of successive sweeps along the coordinate directions with the fluxes computed using a one-dimensional Riemann solver. This solver belongs to the relaxation-based HLL family of approximate Riemann solvers, ensuring positivity of density and pressure [61,62]. The viscous terms in

the Navier-Stokes equations are discretized using central finite differences.

(3) Time integration is explicit, and the time step is determined by the Courant-Friedrichs-Lewy and diffusive stability conditions, whichever is more restrictive. The present version of FLASH [63,64] is optimized for large-scale simulations by applying the hybrid-precision scheme [2]: Fluid variables are stored in single precision, while critical operations are performed in double precision. This approach preserves the accuracy of double-precision computations, while reducing computational cost, communication overhead, and memory usage [2,64].

(4) Like all Godunov-type upwind finite-volume methods, the scheme introduces numerical dissipation near discontinuities. However, all simulations are highly resolved and satisfy  $k_{\text{max}}\eta > 10$ . Thus, numerical dissipation remains minimal.

(5) Another known limitation concerns the asymptotic low-Mach-number limit ( $M \rightarrow 0$ ), where Godunov-type upwind schemes yield incorrect scaling of density and pressure fluctuations. In this limit, the continuous governing equations yield pressure fluctuations that scale as  $O(M^2)$ , whereas standard Godunov-type discretizations produce larger fluctuations of order  $O(M)$ . This issue has been discussed extensively (see Refs. [41–43,65–68] and references therein), where several techniques, such as preconditioning or modified dissipation, have been proposed and shown to recover the correct  $O(M^2)$  scaling. These studies further show that the discrepancy in pressure fluctuations between standard Godunov schemes and their corrected variants becomes negligible at  $M = 0.1$ . In particular, Ref. [66] examined several Godunov-type schemes, including Roe, HLL, HLLC, Rusanov, and AUSMPW+, and found that the pressure fluctuations predicted by these schemes become nearly indistinguishable at  $M = 0.1$  and very close to those obtained with low-Mach-corrected formulations (see Fig. 3 of Ref. [66]). Increasing spatial accuracy further alleviates this issue; for example, Ref. [65] shows that second-order MUSCL reconstruction allows standard Godunov-type schemes to capture correct pressure fluctuations down to  $M = 10^{-2}$  (see Fig. 7 of Ref. [65]). Since the present simulations also employ MUSCL reconstruction, the lowest Mach number case considered here ( $M_{\text{rms}} = 0.1$ ) lies outside the regime where the low-Mach pressure scaling problem becomes significant.

Still, we performed several checks to verify that the present results are not affected by numerical artifacts in the subsonic regime and also in the supersonic regime. The validation results are discussed in the next Appendix.

## APPENDIX B: VALIDATION OF TURBULENCE STATISTICS WITH OTHER BENCHMARK DNS

The results of the present simulations are not significantly affected by numerical artifacts, as verified by comparison with well-established results from previous studies in Fig. 8. Panel (a) compares the scaling of the moments of the total energy dissipation rate  $M_n = \langle \epsilon^n \rangle / \langle \epsilon \rangle^n$  at  $M_{\text{rms}} = 0.1$  with incompressible homogeneous isotropic turbulence obtained using pseudospectral DNS [28]. The scaling of the present results agrees very well with the incompressible reference data. Panel (b) shows the probability density function (pdf)

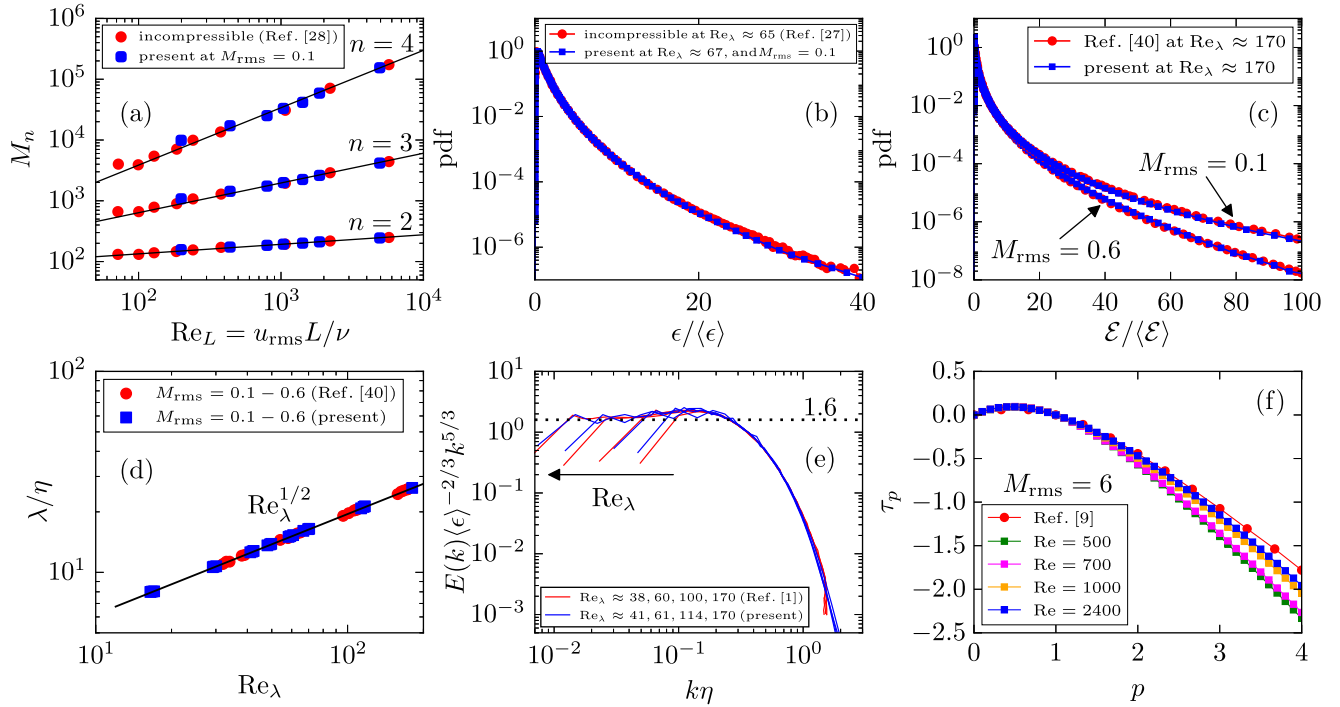


FIG. 8. Validation of the present simulations. Panel (a) compares the Reynolds number dependence of moments of the total energy dissipation rate  $M_n$  for orders  $n = 2, 3$ , and  $4$  at  $M_{rms} = 0.1$  with incompressible isotropic turbulence from Ref. [28]. Both datasets are vertically shifted by constant factors. Panel (b) shows the pdf of the normalized dissipation rate  $\epsilon / \langle \epsilon \rangle$  at  $M_{rms} = 0.1$  compared with Ref. [27]. Panel (c) shows the pdf of normalized enstrophy  $\mathcal{E} / \langle \mathcal{E} \rangle$  at  $M_{rms} = 0.1$  and  $0.6$  compared with Ref. [40]. Panel (d) compares the dependence of  $\lambda / \eta$  on  $Re_\lambda$  with Ref. [40]. Panel (e) shows the normalized energy spectra at several  $Re_\lambda$ , compared with Ref. [1]. Panel (f) compares the scaling exponents  $\tau_p$  of the coarse-grained dissipation moments  $\langle \epsilon_l^p \rangle \sim l^{\tau_p}$  at  $M_{rms} = 6$  with Ref. [9].

of the normalized dissipation rate  $\epsilon / \langle \epsilon \rangle$  at  $M_{rms} = 0.1$  and  $Re_\lambda \approx 65$ , compared with incompressible DNS results [27] at the same  $Re_\lambda$ , again showing excellent agreement.

Panels (c)–(e) provide further validation in the subsonic regime  $0.1 \leq M_{rms} \leq 0.6$  through comparison with compressible turbulence simulations of Refs. [1,40], which employed a tenth-order compact finite-difference scheme. Panel (c) shows pdf of the normalized enstrophy  $\mathcal{E} / \langle \mathcal{E} \rangle$  for  $M_{rms} = 0.1$  and  $0.6$  at  $Re_\lambda \approx 170$ . For this validation, we carried out two additional simulations at these Mach numbers with  $Re_\lambda \approx 170$ . The present results agree very well with the reference data. Panel (d) compares the ratio of the Taylor microscale to the Kolmogorov length scale  $\lambda / \eta$  as a function of the Taylor microscale Reynolds number  $Re_\lambda$ . The results follow the expected scaling  $\lambda / \eta \sim Re_\lambda^{1/2}$  and agree with the data reported in Ref. [40]. The normalized energy spectra at several  $Re_\lambda$  are displayed in panel (e). The spectra agree closely with the compact-scheme results of Ref. [40]. The dependence of the spectra on Mach number in the range  $0.1 \leq M_{rms} \leq 0.6$  is negligible, apart from minor differences near the forcing wave number (not shown).

Panel (f) compares the scaling exponents  $\tau_p$  of the moments of the coarse-grained total dissipation rate  $\langle \epsilon_l^p \rangle \sim l^{\tau_p}$  at  $M_{rms} = 6$  with results obtained from simulations of the Euler hydrodynamic equations reported in Ref. [9] at the same Mach number. The analysis is carried out in the same manner as in Ref. [9]. The scaling exponents obtained in the present simulations approach the reference data as the Reynolds number increases and are already very close to the reference

values at  $Re \approx 2400$ . In Ref. [9], the Reynolds number is controlled by numerical dissipation and is therefore not known explicitly.

Furthermore, we analyze the pressure spectrum  $E_p(k)$ , which is theoretically predicted to exhibit a  $k^{-7/3}$  scaling in the inertial range of weakly compressible flows [69]. The spectrum is defined as

$$E_p(k) dk = \sum_{k \leq |\mathbf{k}'| < k+dk} |\hat{p}(\mathbf{k}')|^2, \quad (B1)$$

where  $\mathbf{k}'$  is the wave number vector and  $\hat{p}(\mathbf{k}')$  is the Fourier coefficient of the pressure field. At a Reynolds number of  $Re_\lambda \approx 350$ , Ref. [51] reported a  $k^{-7/3}$  scaling range for  $0.015 \lesssim k\eta \lesssim 0.04$  at low Mach numbers ( $0.1 \leq M_{rms} \leq 0.3$ ). For higher Mach numbers of  $0.5 \leq M_{rms} \leq 1$ , the pressure spectrum approached a  $k^{-5/3}$  scaling. Also at lower Taylor microscale Reynolds numbers  $Re_\lambda \lesssim 100$ , Ref. [1] observed a narrow plateau in the compensated spectrum  $k^{7/3} E_p(k)$  for the wave number range  $0.1 \lesssim k\eta \lesssim 0.3$  at rms Mach numbers  $M_{rms} = 0.1$  and  $0.6$ .

Figure 9 shows our own results. We plot the compensated pressure spectrum  $k^{7/3} E_p(k)$  for our highest Reynolds number simulations at  $Re_\lambda \approx 100$ . For  $M_{rms} \leq 1$ , the compensated spectra likewise exhibit a plateau over  $0.1 \lesssim k\eta \lesssim 0.3$ , consistent with the findings of Ref. [1]. In comparison with the  $k^{-7/3}$  scaling range reported in Ref. [51], the plateau appears at larger values of  $k\eta$ , possibly due to the lower Reynolds numbers considered here. Simulations at higher Reynolds

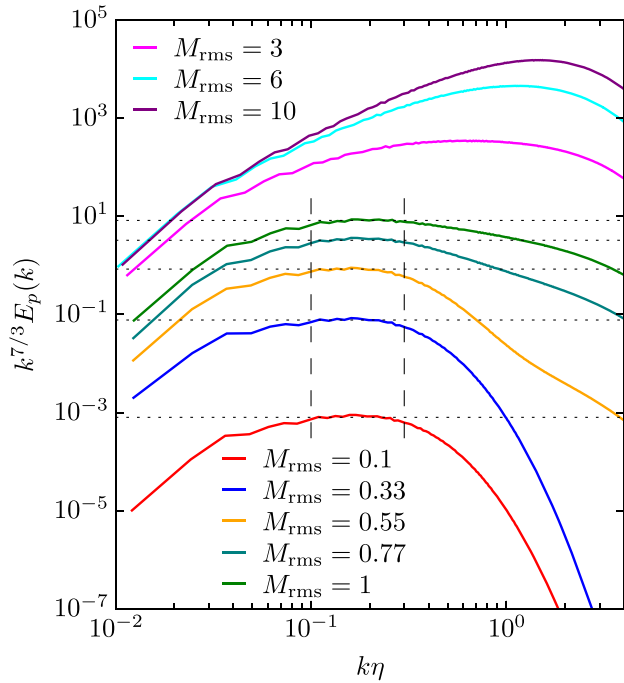


FIG. 9. Compensated pressure spectrum  $k^{7/3}E_p(k)$  vs the normalized wave number  $k\eta$  for simulations of  $2048^3$  grid points ( $\text{Re}_\lambda \approx 100$ ). For  $M_{\text{rms}} \leq 1$ , the compensated spectra exhibit a plateau over the interval  $0.1 \lesssim k\eta \lesssim 0.3$ , indicated by the vertical dashed lines. The horizontal dotted lines show the corresponding plateau amplitudes. For  $M_{\text{rms}} > 1$ , the pressure spectrum is shallower than  $k^{-7/3}$ .

numbers will determine the extent of the predicted  $k^{-7/3}$  scaling range. We leave this as a future task. For  $M_{\text{rms}} > 1$ , the compensated spectra become progressively shallower than  $k^{-7/3}$ ; this scaling no longer exists as seen in the figure.

### APPENDIX C: SCALING EXPONENTS OF ENSTROPY MOMENTS

For incompressible turbulence, Elsinga *et al.* [53] investigated the anomalous scaling of enstrophy moments with

TABLE II. Scaling exponents of enstrophy moments in incompressible homogeneous isotropic turbulence, as reported in a DNS study [53].

| $n$ | $\tilde{\beta}_{\text{ens},n}$ | $\beta_{\text{ens},n}$ |
|-----|--------------------------------|------------------------|
| 2   | 0.21                           | 0.23                   |
| 3   | 0.42                           | 0.70                   |
| 4   | 0.57                           | 1.27                   |

respect to the Taylor microscale Reynolds number  $\text{Re}_\lambda$ . In their analysis, the scaling was characterized using the  $n$ th-order moment raised to the power  $1/n$ , defined as

$$\tilde{M}_{\text{ens},n} = \frac{\langle \mathcal{E}^n \rangle^{1/n}}{\langle \mathcal{E} \rangle} \sim \text{Re}_\lambda^{\tilde{\beta}_{\text{ens},n}}. \quad (\text{C1})$$

This definition can be expressed with respect to the large-scale Reynolds number  $\text{Re}$  and written directly in terms of the normalized  $n$ th-order moment, without taking the  $1/n$ , i.e.,

$$M_{\text{ens},n} = \frac{\langle \mathcal{E}^n \rangle}{\langle \mathcal{E} \rangle^n} \sim \text{Re}^{\beta_{\text{ens},n}}, \quad \text{where} \quad \beta_{\text{ens},n} = \frac{n \tilde{\beta}_{\text{ens},n}}{C}. \quad (\text{C2})$$

Here, we have used the relation  $\text{Re} \sim \text{Re}_\lambda^C$  reported by Elsinga *et al.* [53], based on a fit to the DNS data of Schumacher *et al.* [27] with the fit coefficient  $C = 1.8$ . The resulting scaling exponents are listed in Table II. In the incompressible limit ( $M_{\text{rms}} \ll 1$ ), the dynamic viscosity  $\mu = \rho\nu$  is constant, and the solenoidal dissipation component  $\epsilon_s$  is therefore directly proportional to the local enstrophy. Consequently, the scaling exponents of the moments of  $\epsilon_s$  coincide with the enstrophy exponents, i.e.,  $\beta_{s,n} \approx \beta_{\text{ens},n}$  [see again Fig. 5(b)].

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