

Extracting complete resonance characteristics from the phase of physical signals

Isam Ben Soltane^{✉*} and Nicolas Bonod^{✉†}

Aix Marseille Univ, CNRS, Centrale Med, Institut Fresnel, Marseille, France



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Resonances are ubiquitous in wave physics and shape the spectral response of every linear physical system. Yet their description remains fragmented, through partial and sometimes competing definitions. Here, we show that the resonance properties are fully encoded in the phase of the response functions of physical systems and can be decoded by deriving the spectral function of the phase. The frequency derivative of the phase is related to the Wigner-Smith time-delay operator, but is applied to individual elements of the scattering matrix. By deriving the meromorphic expansion of this scalar element-wise time delay in terms of poles and zeros in the complex frequency plane obtained from the knowledge of both phase and amplitude of the response function, we demonstrate that it yields a quality function whose peaks reveal both the strength and the position of each resonance. This expression proves that poles only are not sufficient to fully characterize resonances and that zeros bring the other part of the information. The numerical calculation of the quality function together with the analytical and balanced treatment of poles and zeros provide a rigorous framework for identifying and quantifying resonances without any prior knowledge of the physical system.

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I. INTRODUCTION

Resonances result from an efficient coupling between an excitation wave and a physical system. They strongly impact the response of the system in both harmonic and time domains and are at the core of many important research fields in wave physics. Resonances are at the basis of cavity-enhanced interactions for laser developments cavities [1–3], for cavity electrodynamics [4–6], cavity optomechanics [7,8], acoustics [9], but also for studying the stability of the response of systems [10–14], achieving strong coupling [15], detecting gravitational waves [16], enhancing light-matter interactions [15,17,18], or also for tuning the wavefront of light beams [19–23]. Recently, the combination of both has been achieved to develop lasers with arbitrarily shaped wavefronts [24]. Resonances have been at the core of tremendous efforts to design optical nanostructures featuring specific resonant spectra through the excitation of Fano anomalies [25–29], bound states in the continuum (BIC) [30–32], or also chiral BICs [33,34].

Classically, the strength of the resonance is quantified by the quality factor (Q factor). This factor is originally derived from the energy balance within a cavity, which is related to the spectral behavior of that cavity. As a consequence, the Q factor can be determined from this spectral response as the ratio between the frequency of the maximum of the response (in amplitude) and the width at midheight [35–38]. It can also

be expressed using the temporal-coupled mode theory, which describes the energy transfer between resonators, via the resonance frequency taken as the frequency of the maximum of amplitude divided by the sum of the inverse of the radiative and internal loss times [39,40]. The introduction of cavity modes is associated with complex frequencies called poles or singularities and results in an additional definition of the Q factor of a resonance associated with a pole p as $-\frac{\text{Re}[p]}{2\text{Im}[p]}$ [37,41].

These definitions of the Q factor are usually applicable in the specific case of Lorentzian-shaped amplitude curves induced by isolated poles of the response of the system. Nevertheless, the response functions of physical systems usually show more complex variations with no apparent symmetry, making it impossible to graphically evaluate the Q factor. A non-Lorentzian function can be described as a sum of Lorentzian functions, each function being associated with a pole and a residue in the complex ω space. As the Q factor does not take into account the complex residue, the Q factor of the non-Lorentzian function cannot be retrieved simply from the set of Q factors induced by the Lorentzian decomposition.

Alternatively, resonances have been associated with the Wigner-Smith time delay in the context of the S matrix scattering formalism. The Wigner-Smith time delay, τ_ω , describes the interaction time of the wave with the scatterer before leaking through a scattering channel in the far field. This concept was first developed in quantum physics to estimate the interaction time between a particle and a potential well [46,47], before being widely used in wave physics and, in particular, in electromagnetism in systems described by unitary S matrices through the so-called Wigner-Smith time-delay operator $Q = -iS^\dagger \partial_\omega[S]$ [48,49]. This concept has been associated with the characterization of group delays in multiport systems [50], or with the notion of density of states [51–53]. While

*Contact author: isam.ben-soltane@fresnel.fr

†Contact author: nicolas.bonod@fresnel.fr

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the Wigner-Smith operator is no longer defined in subunitary S matrices as its eigenvalues become complex, the imaginary part of the complex average time delay defined as $\tau_\omega = -\frac{i}{M} \frac{d}{d\omega} \text{Log}[\det(S)]$ can still be related to the loss in the system [54–56]. Important efforts have been carried out to find the expression of τ_ω in terms of poles and zeros of the scattering matrix [54,56]. The most recent advances show how the study of the real and imaginary parts of τ_ω can bring information on poles and zeros of the scattering matrix [57–59]. Within the Wigner-Smith time-delay formalism, resonances appear as changes of dynamics or regimes of physical systems, leading to varying interaction times. In agreement with this representation, it was pointed out that phase-line densities in the complex ω space of a single scattering coefficient are good indicators of the steepness of resonances [26]. This definition of resonances as changes in interaction times observed through the spectral derivative of the phase of scattering channels is appealing since it applies to any linear physical system described by the S matrix. In its general description, it comes at the expense of the calculation or measurement of the whole S matrix, which can often be fastidious or even impossible. It would be worth being able to characterize separately the resonances of every single scattering channel individually without requiring the knowledge of the whole scattering matrix. However, the link between the Wigner-Smith time delay and the other formalisms to characterize the resonance properties has not been established.

Here, we show how a scalar time delay obtained through the frequency derivative of the phase of any response function, including each element of the S matrix, permits to rigorously characterize the resonances in the whole spectrum. By describing the response function with the singularity and zero factorization (SZF) [26,60,61], we write the scalar element-wise time delay as a meromorphic function and study the contribution of each complex singularity and zero to its variations, i.e., to the shape and the position of the resonances. We derive a quality function that quantifies the resonances and their effect at any frequency ω . This dimensionless function can be calculated numerically through the spectral derivative of the phase, but can also be calculated independently using the poles and zeros extracted from the response function. The accuracy of these two approaches is assessed in several optical cavities, and we show that they match very accurately. We analyze the different roles of poles and zeros for characterizing resonances.

II. CHARACTERIZING RESONANCES WITH POLES AND ZEROS

Any response function $H(\omega)$ is a complex-valued quantity at real and complex frequencies ω , thus reading as

$$H(\omega) = |H(\omega)|e^{i\Psi(\omega)} \\ \text{Log}[H(\omega)] = G(\omega) + i\Psi(\omega), \quad (1)$$

with $\omega = \frac{2\pi c}{\lambda}$, $|H(\omega)|$ the amplitude, Log refers to the principal determination of the complex natural logarithm, $G(\omega)$ the log-amplitude, and $\Psi(\omega)$ the phase. $H(\omega)$ possesses complex zeros $z^{(\ell_z)}$ and poles $p^{(\ell_p)}$ that can be used to expand $H(\omega)$ as a meromorphic function through the singularity expansion

method [26,60–63]. This method admits two equivalent analytical expressions taking the form of (1) a sum of rational functions depending on poles and associated residues and (2) of a rational function known as the SZF involving the poles $p^{(\ell_p)}$ and zeros $z^{(\ell_z)}$:

$$H(\omega) = H_0 \frac{\prod_{\ell_z} (\omega - z^{(\ell_z)})}{\prod_{\ell_p} (\omega - p^{(\ell_p)})}. \quad (2)$$

In this expression, H_0 is a known real-valued constant [61]. In practice, Eq. (2) is a truncated version of the SZF. In theory, a potentially infinite number of poles and zeros characterize the system. Outside of the spectral window of interest, the response function is approximated by a rational function, resulting in effective poles and zeros already accounted for by the sets of poles and zeros in Eq. (2).

Using Eqs. (1) and (2), the log-amplitude $G(\omega)$ and the phase $\Psi(\omega)$ can be obtained at real frequencies to draw Bode diagrams, or in the complex ω plane to get complex amplitude and phase maps (or complex Bode diagrams). These plots are drawn in Fig. 1 for two systems. The response function of both systems is their reflection coefficient: Figs. 1(a)–1(c) a two-port radio-frequency (RF) waveguide behaving as a Fabry-Perot due to the presence of thin silver slabs in between the waveguide ends and the coax-to-waveguide transitions. A perturbation is introduced in the center of the cavity by inserting two aluminium rods in holes drilled in the top face of the waveguide cavity. The perturbation leads to a shift of the complex zeros associated with the Fabry-Perot dips, yielding exceptional points at various insertion depths as shown in Fig. 1(b) of Ref. [42]. The experimental data used in this manuscript correspond to a penetration depth close to one such exceptional point, highlighted by the two almost overlapping dips at 7.2×10^{10} rad/s. The system and the experimental setup are described in Ref. [42]; Figs. 1(d)–1(f) show a grating designed to obtain a flat amplitude [64], associated with a Gires-Tournois effect [64–67]. The poles and zeros in the complex frequency plane can be retrieved through different means [59,68–72]. Here, they are obtained by fitting the simulated response function $H(\omega)$ with the SZF (red curves) introduced in Eq. (2), via autodifferentiation with an open access tool called sempo [45], and are shown in the complex amplitude map in Figs. 1(c) and 1(f). The distributions of log-singularities, i.e., poles and zeros, directly set the shape of the spectral responses. The amplitude of the reflection coefficient of the RF Fabry-Perot cavity in Fig. 1(a) exhibits several dips, related to the poles and zeros close to the real frequency axis displayed in Fig. 1(c). While the real parts of these poles and zeros seem evenly distributed, as expected in the case of a Fabry-Perot cavity, the presence of a perturbation disturbs that regularity and leads to different imaginary parts, i.e., dips of different depths. For the Gires-Tournois grating, the spectral range of quasiconstant amplitude is associated with a zero and a pole, which are complex conjugates of each other, and are both isolated from other log-singularities [65,66]. The Gires-Tournois grating behaves as a lossless one-port system, and in this case, zeros are mirror symmetric to poles across the real axis. This symmetry enforces the unit amplitude of the reflection response.

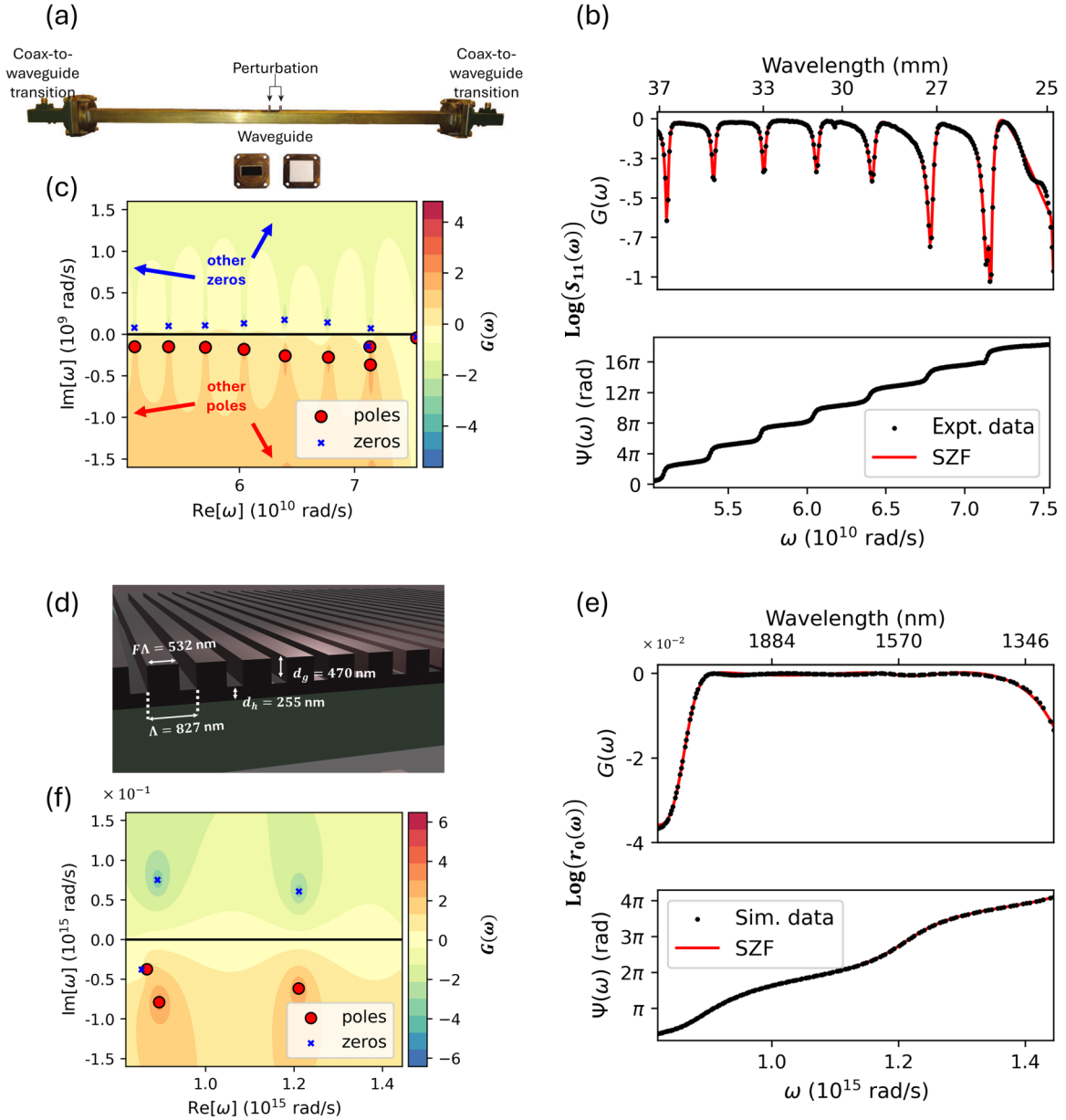


FIG. 1. Reflection spectra in amplitude and phase of optical and radio-frequency cavities and their associated distributions of zeros and poles in the complex ω space. (a) Radio-frequency Fabry-Perot cavity composed of a waveguide connected to coax-to-waveguides via thin silver slabs. The full system is described in Ref. [42]. (d) Grating of Si on a thin layer of Si with thickness $d_h = 255$ nm, over a substrate of glass with a period $\Lambda = 827$ nm, a duty cycle $F = 64\%$, and a height $d_g = 470$ nm. (b), (e) Bode diagrams of the reflection parameter S_{11} [panel (a)] and the specular reflection coefficient $r_0(\omega)$ for an illumination at a linearly polarized normal incidence, in the transverse-magnetic polarization [panel (d)]. The phase is numerically obtained by unwrapping the values with the unwrap function of the Python package numpy [43]. See the Supplemental Material for more details [44]. Top scale displays the wavelength λ and the bottom scale displays the associated angular frequency $\omega = 2\pi c/\lambda$. (c), (f) Log-amplitudes of the reflection coefficients in the complex ω plane. The associated poles and zeros are indicated by the red and blue dots. They are used to reconstruct the amplitude and phase (SZF, red curves) of the two reflection coefficients in panels (c) and (d) through Eq. (2). The SZF reconstruction (red curves) is performed via the SEMPO package [45]. Consequently, the quality of this reconstruction is subject to the ability of the package to accurately retrieve the poles and zeros, and depends on the signal-to-noise ratio and available data.

In both cases, the amplitude spectra does not feature true Lorentzian shapes. In the case of the Gires-Tournois grating, the presence of resonances cannot even be inferred from the amplitude curve. In both systems, the classical means to characterize resonances using the amplitude cannot be used, and we will now show that even the poles do not suffice to

characterize the resonance through the ratio between their real and imaginary parts.

Traces of the resonances must be searched for in the phase plots of the Bode diagrams shown in Figs. 1(b) and 1(e), where considerable phase variations are observed. Let us stress that the phase of the reflection coefficient of the grating

shows a phase gain exceeding 2π over the constant amplitude range [64–66,73].

To further investigate the content encoded within the phase, we derive an analytical expression of $\Psi(\omega)$. From Eq. (1), we have $\Psi(\omega)$ as the imaginary part of the principal value complex logarithm function Log of $H(\omega)$:

$$\Psi(\omega) \equiv \text{Im}[\text{Log}[H(\omega)]] \quad (2\pi) \quad (3)$$

which yields, with Eq. (2), the following expression:

$$\begin{aligned} \Psi(\omega) \equiv & +\frac{i}{2} \sum_{\ell_p} \text{Log} \left[\frac{\omega - p^{(\ell_p)}}{\omega - \overline{p^{(\ell_p)}}} \right] \\ & - \frac{i}{2} \sum_{\ell_z} \text{Log} \left[\frac{\omega - z^{(\ell_z)}}{\omega - \overline{z^{(\ell_z)}}} \right] \quad (2\pi) \end{aligned} \quad (4)$$

where $\bar{p}$ and $\bar{z}$ stand for complex conjugates. The details of the calculations are provided in Eqs. (S1)–(S7) of the Supplemental Material [44]. Let us point out that the SZF in Eq. (2) is obtained from the singularity expansion of $H(\omega)$, which requires the poles and residues [61]. While zeros and residues are related to each other, the retrieval of the zeros from the knowledge of the residues is not straightforward. The use of poles and zeros, rather than poles and residues, is usually preferred for studying the phase.

In Eq. (4), $\Psi(\omega)$ appears as a meromorphic function. However, the presence of the log function hints at a non-unique definition of the phase, associated with branch cuts in the complex ω plane [74]. This is naturally solved for by differentiating Eq. (4) with respect to the frequency ω :

$$\begin{aligned} \partial_\omega[\Psi](\omega) = & +\frac{i}{2} \sum_{\ell_p} \left[\frac{1}{\omega - p^{(\ell_p)}} - \frac{1}{\omega - \overline{p^{(\ell_p)}}} \right] \\ & - \frac{i}{2} \sum_{\ell_z} \left[\frac{1}{\omega - z^{(\ell_z)}} - \frac{1}{\omega - \overline{z^{(\ell_z)}}} \right]. \end{aligned} \quad (5)$$

Equation (5) highlights the crucial role of the zeros $z^{(\ell_z)}$, symmetric to that of the poles $p^{(\ell_p)}$, in the calculation of $\partial_\omega[\Psi](\omega)$ [and the phase $\Psi(\omega)$ by extension]. Equation (5) is a singularity expansion [61,62,75,76] of $\partial_\omega[\Psi](\omega)$, which implies that poles and zeros fully characterize $\partial_\omega[\Psi](\omega)$ [54,56,77].

Let us now investigate each term $D_s(\omega)$ in the expression of $\partial[\Psi](\omega)$ in Eq. (5):

$$D_s(\omega) = \frac{i}{2} \left[\frac{1}{\omega - s} - \frac{1}{\omega - \bar{s}} \right] = -\frac{\text{Im}[s]}{|\omega - s|^2}, \quad (6)$$

where $s = p^{(\ell_p)}$ for the terms involving a pole and $s = \overline{z^{(\ell_z)}}$ for those involving a zero. $D_s(\omega)$ is a real-valued function with a perfect Lorentzian shape when evaluated at real frequencies, as shown in Fig. 2, where we calculate this function for one pole $s = p$ and two zeros $s = z_1$ or $s = z_2$. It reaches its extremum value $-\frac{1}{2|\text{Im}[s]|}$ at $\omega = \text{Re}[s]$. That maximum (or minimum) value is divided by two as ω is at a distance $|\text{Im}[s]|$ of the frequency $\text{Re}[s]$.

By integrating $D_s(\omega)$ over the full real ω axis, from $-\infty$ to $+\infty$, a $\pm\pi$ phase shift is obtained, for any log-singularity, regardless of its type (zero or pole) and its position [20,26,60]. We show that a phase variation $\Delta\Psi_B = \pm\pi/2$ is

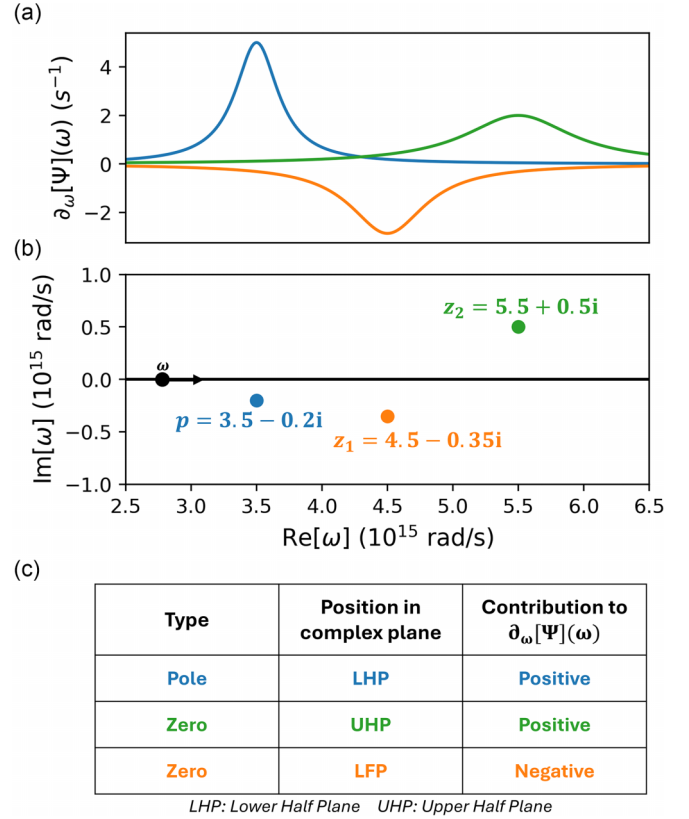


FIG. 2. Contribution of poles and zeros to the spectral derivative of the phase: Poles located in the lower half ω space always have a positive contribution, while zeros can have positive or negative contribution if located in the upper or lower half ω space, respectively. (a) Differentiated-phase term $D_s(\omega)$ for a pole $s = p$ (blue), and two zeros $s = z_1$ (orange) and $s = z_2$ (green). (b) The positions of the log-singularities are shown in the complex ω plane. The sharpness and peak value of each resonant term are directly related to the imaginary part of the log-singularity, i.e., to their distance to the real frequency axis (black horizontal line). (c) Table summarizing the contribution of the poles and zeros to the spectral derivative of $\Psi(\omega)$. The poles are always in the lower half plane (LHP) and have a positive contribution. The zeros can be either in the LHP where they have a negative contribution, or in the upper half plane, where they contribute positively.

obtained by integrating over the smaller $[-|\text{Im}[s]|, +|\text{Im}[s]|]$ spectral range, centered around the real part of the log-singularity $\text{Re}[s]$, which we refer to as the bandwidth of the log-singularity:

$$\begin{aligned} \Delta\Psi_B &= \int_{\text{Im}[s]}^{-\text{Im}[s]} D_s(\text{Re}[s] + \omega) d\omega \\ &= \pm \frac{i}{2} \text{Log} \left(\frac{\text{Im}[s](1-i)(-1-i)}{\text{Im}[s](1+i)(-1+i)} \right) = \pm \frac{\pi}{2}. \end{aligned} \quad (7)$$

This phase variation is a phase gain if $\text{Im}[s] < 0$ and a phase loss if $\text{Im}[s] > 0$. Poles are always assumed to be stable and have thus a negative imaginary part. As a consequence, a phase gain is observed. Zeros can have either a positive or negative imaginary part, i.e., $\text{Im}[s] = -\text{Im}[z^{(\ell_z)}]$ is positive if the zero is in the upper part of the complex frequency plane,

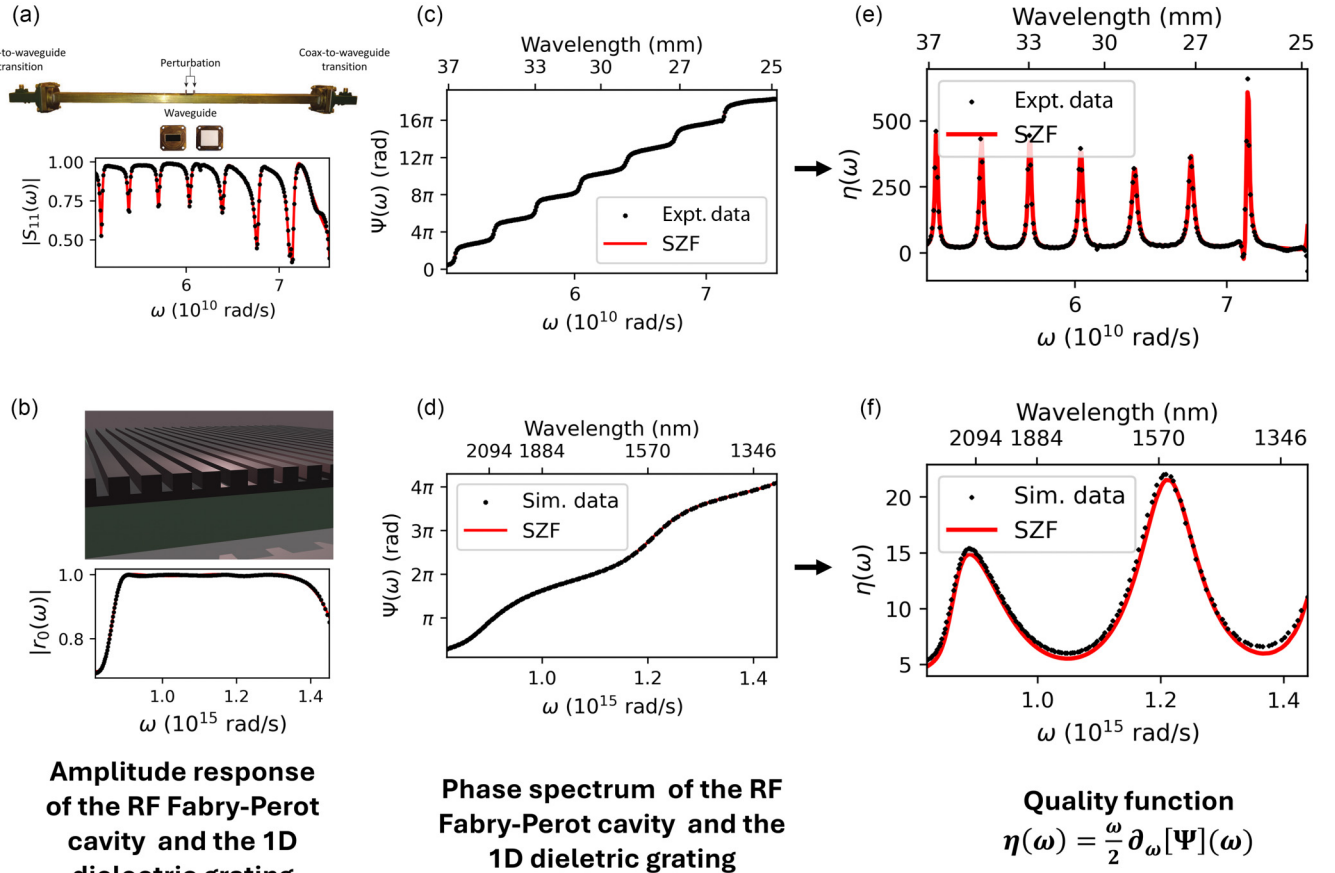


FIG. 3. Extraction of the quality function $\eta(\omega)$ from the phase spectrum of the systems from the numerical calculation of the spectral derivative of the phase (black dots) or with the poles and zeros (red line). (a), (b) Amplitude response of the two systems introduced in Fig. 1, i.e., the RF Fabry-Perot cavity in panel (a), and the Gires-Tournois grating in panel (b). (c), (d) Phase spectrum associated with the two optical systems. Simulated data in black dots are compared with the reconstruction through poles and zeros in the SZF expression from Eq. (2). (e), (f) Quality function $\eta(\omega)$ calculated for these two systems. The quality function $\eta(\omega)$ is obtained directly from the experimental data with Eq. (7) using a first-order divided difference. It is compared to the meromorphic singularity expansion in Eq. (8) using the poles and zeros in the SZF expression (2). (c)–(f) Top scale displays the wavelength λ and the bottom scale displays the associated angular frequency $\omega = 2\pi c/\lambda$. The SZF reconstruction (red curves) is again performed via the SEMPO package [45].

yielding a $\pi/2$ phase gain over the aforementioned range. Otherwise, a phase loss is observed. The resonances are thus identified via the peaks in the phase-derivative curve, while the amplitude and width of these peaks are directly related to the imaginary part of the pole or zero. Let us point out that a phase loss is always brought about by the presence of zeros in the lower part of the complex ω plane. It is, however, impossible to discriminate between a pole and a nonreversible zero, i.e., a zero with a positive imaginary part, when a phase gain is observed, in which case the amplitude is required to identify a pole or zero, as described in Sec. I of the Supplemental Material [44].

III. QUALITY FUNCTION OF AN ARBITRARY RESPONSE FUNCTION

At high frequencies, fields oscillate faster within the systems, allowing them to interact over smaller time periods. Therefore, their amplitude varies faster than at low frequencies. Consequently, the poles and zeros tend to move away from the real ω axis as we move into higher spectral domains.

Each term $D_s(\omega)$ quantifies the proximity of the frequency of interest ω to the log-singularity s . It is thus often lower valued at high frequencies than at low frequencies. As a response to that effect, we introduce the quality function of the response function, defined as the nondimensional phase variation function:

$$\eta(\omega) := \frac{\omega}{2} \partial_{\omega} [\Psi](\omega). \quad (8)$$

The quality function $\eta(\omega)$ can be calculated from experimental or numerical data by numerically approximating the frequency derivative of $\Psi(\omega)$ in Eq. (7) via a first-order divided difference. We discuss in the Supplemental Material the influence of the noise on the numerical treatment of the frequency derivative [44]. More advanced techniques can be used to get the numerical derivatives of experimental data [59]. An alternative method to retrieve the spectral derivative of a function response is to calculate the poles and zeros from which the analytic expression of the spectral derivative Eq. (5) can be obtained (see red lines in Figs. 1 and 3). A thorough study of the effect of noise on the retrieval of complex

parameters (poles, zeros, and residues) has been carried out in Ref. [45]. The method sempo was proved to be very robust to the noise when extracting the complex parameters (see Fig. 10 in Ref. [45]). The results obtained with a first-order frequency derivative are displayed in Figs. 3(e) and 3(f) for the RF cavity and the grating introduced in Fig. 1. Each extremum in the curve of $\eta(\omega)$ corresponds to a resonance. In the case of the metasurface, this approach provides a highly accurate localization of the resonances, which cannot be done classically by considering the amplitude or phase curves in the Bode diagram. Resonances are clearly highlighted and quantified by the quality function $\eta(\omega)$, while they are invisible in the Bode diagram. The quality function unveils the resonances of any arbitrary spectral response function. Let us stress that $\eta(\omega)$ reveals the information encoded in the phase of the spectral response without any *a priori* knowledge on the physical system itself.

Let us now expand the quality function in terms of the poles and zeros of $H(\omega)$. Since $\partial\Psi(\omega)$ is a meromorphic function, $\eta(\omega)$ is also a meromorphic function, which means that it is also fully characterized by the poles and zeros of $H(\omega)$:

$$\eta(\omega) = +\frac{i}{4} \sum_{\ell_p} \left[\frac{\omega}{\omega - p^{(\ell_p)}} - \frac{\omega}{\omega - \overline{p^{(\ell_p)}}} \right] - \frac{i}{4} \sum_{\ell_z} \left[\frac{\omega}{\omega - z^{(\ell_z)}} - \frac{\omega}{\omega - \overline{z^{(\ell_z)}}} \right]. \quad (9)$$

This expression is central to this work since it shows that poles and zeros must be both considered to characterize resonances of a spectral response, i.e., that even a full knowledge of the poles does not bring the full information about resonances of the transfer function. Once the poles and zeros associated with a given spectrum of the transfer function have been identified [45], their number and values are plugged into Eq. (9) to obtain the quality function $\eta(\omega)$. We show in Figs. 3(e) and 3(f) that this description via the poles and zeros matches the function obtained directly by numerically solving Eq. (8).

Equation (9) can be further simplified by using the notion of log-singularities $s^{(\ell)}$:

$$\eta(\omega) = +\frac{i}{4} \sum_{\ell} \left[\frac{\omega}{\omega - s^{(\ell)}} - \frac{\omega}{\omega - \overline{s^{(\ell)}}} \right]. \quad (10)$$

This simplified formulation of the quality function $\eta(\omega)$ in Eq. (10) shows that it can be fully expanded on the set of log-singularities $s^{(\ell)}$ without the need to discriminate between a pole and the complex conjugate of a zero. Consequently, the resonance characteristics can be extracted from the phase of the signal without requiring an explicit discrimination of poles and zeros.

Similarly to the previous study of the function $\partial_\omega[\Psi](\omega)$, we consider the role of each log-singularity s to the function $\eta(\omega)$, via the terms $\eta_s(\omega) := \frac{\omega}{2} D_s(\omega)$:

$$\eta_s(\omega) = -\frac{\omega \text{Im}[s]}{(\sqrt{2}|\omega - s|)^2}. \quad (11)$$

Each term $\eta_s(\omega)$ exhibits a single extremum at the positive frequency $\omega = |s|$, which is commonly referred to as the

resonance frequency in the case of pure, isolated Lorentzian functions:

$$\partial_\omega[\eta](\omega) = -\text{Im}[s] \frac{-\omega^2 + |s|^2}{2|\omega - s|^4}, \quad (12)$$

$$\partial_\omega[\eta](\pm|s|) = 0. \quad (13)$$

As discussed in the analysis of $D_s(\omega)$ in the previous section, the bandwidth of a logarithmic singularity is twice the modulus of its imaginary part. Consequently, all log-singularities whose imaginary part is of the same order of magnitude as their real part correspond to very broad resonances, which can be approximated as nearly constant offsets in both amplitude and phase. Therefore, the study of $\eta_s(\omega)$ is only meaningful for log-singularities whose imaginary part is small compared to their real part, i.e., $|s| \approx \text{Re}[s]$. Under this condition, the extremum of $\eta_s(\omega)$ is reached close to $\text{Re}[s]$, where the function takes the following form:

$$\eta_s(\text{Re}[s]) = -\frac{\text{Re}[s]}{2\text{Im}[s]}. \quad (14)$$

When the log-singularity s is a pole, Eq. (14) identifies with the commonly used Q factor $Q_{p^{(\ell_p)}}$:

$$\eta_{p^{(\ell_p)}}(\text{Re}[p^{(\ell_p)}]) = Q_{p^{(\ell_p)}} > 0. \quad (15)$$

In the case of a zero, we obtain a definition of the classical Q factor $Q_{z^{(\ell_z)}}$ of the zero $z^{(\ell_z)}$, which can be negative if the zero is located in the lower part of the complex ω plane. The classical Q factor Q_s of an isolated resonance therefore corresponds to a particular case of $\eta_s(\omega)$, for $\omega = \text{Re}[s]$.

IV. PHASE VARIATION ANALYSIS WITH POLES AND ZEROS

Equation (11) shows $\eta(\omega)$ as a nondimensional function, corresponding to the ratio of two quantities with the same unit, i.e., $-\omega \text{Im}[s]$ and $(\sqrt{2}|\omega - s|)^2$. These quantities are the areas of two surfaces in the complex frequency plane, as shown in Fig. S1 of the Supplemental Material [44], each associated with a different effect.

(1) The numerator $-\omega \text{Im}[s]$ is associated with the distance from the log-singularity s to the real-frequency axis. It quantifies the information or phase variations that can be retrieved over the real-frequency axis. To dissociate its effect from that of the denominator $(\sqrt{2}|\omega - s|)^2$, we set ω far enough from the log-singularity s for the term $|\omega - s|$ to be almost constant. The effect of the numerator is quantified by the phase variations $\Delta\Psi_1$ and $\Delta\Psi_2$ in Fig. 4(b), associated with the log-singularities s_1 and s_2 in the complex ω plane [see Fig. 4(b)], each corresponding to a resonance, with s_1 being closer to the real frequency axis than s_2 . These phase shifts are induced by a fixed frequency shift $\Delta\omega$ at an arbitrary frequency ω . We see in Fig. 4(b) that $\Delta\Psi_1 \leq \Delta\Psi_2$, which shows that the farther the log-singularity is from the real ω axis, the bigger the associated phase shift $\Delta\Psi$. This explains why the quantity in the numerator grows as $|\text{Im}[s]|$ increases.

(2) The denominator is related to the distance $|\omega - s|$. A log-singularity s introduces a $\pm\pi$ phase shift as ω goes from $-\infty$ to $+\infty$. In order to quantify how easily that

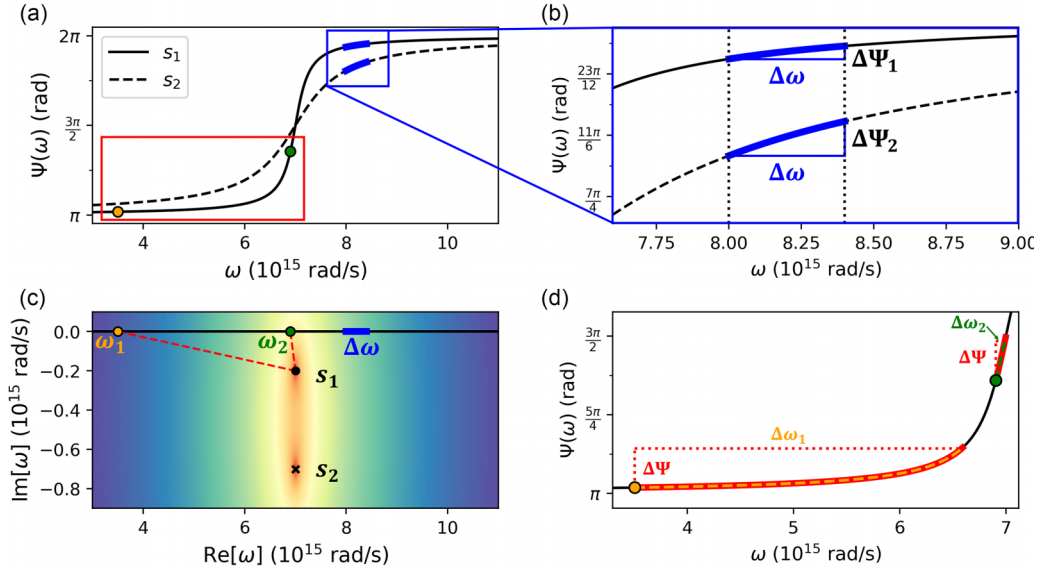


FIG. 4. Graphical interpretation. (a) Phase $\Psi(\omega)$ of the function $H(\omega) = \frac{1}{\omega - s}$ for two log-singularities s_1 and s_2 . The yellow and green dots on the curve of s_1 are associated with the real frequencies ω_1 and ω_2 . The blue lines correspond to a spectral range of width $\Delta\omega$. (b) Phase functions of both s_1 (full black curve) and s_2 (dashed black curve). For a fixed frequency variation $\Delta\omega$, a phase shift $\Delta\Psi_1$ is obtained for the log-singularity s_1 , and $\Delta\Psi_2$ for s_2 , with $\Delta\Psi_2 \geq \Delta\Psi_1$. (c) Position of the log-singularities (black dot and black cross), of ω_1 and ω_2 , and of $\Delta\omega$ in the complex frequency plane. The distance from ω_2 to s_1 is smaller than that from ω_1 to s_1 (dashed red lines). (d) Phase function associated with s_1 only. To obtain a fixed phase variation $\Delta\Psi = \frac{\pi}{8}$, a frequency shift $\Delta\omega_1$ is required as we start from ω_1 (yellow dot), greater than the frequency shift $\Delta\omega_2$ required when we start from ω_2 (green dot).

potential phase shift can be exploited at a given frequency ω , we look, in Fig. 4(d) at the frequency variations $\Delta\omega_1$ and $\Delta\omega_2$ required to obtain a fixed phase shift $\Delta\Psi$ (red curves), starting from the two real frequencies ω_1 (yellow dot) and ω_2 (green dot). The distance from ω_2 to the log-singularity s_1 is greater than that from ω_2 to s_2 , as shown in Fig. 4(c), and $\Delta\omega_1 \geq \Delta\omega_2$ in Fig. 4(c). It turns out that the farther ω is from the log-singularity s , the bigger $\Delta\omega$ is. The quantity in the denominator thus becomes smaller when ω is close to $\text{Re}[s]$.

The real-valued quality function $\eta(\omega)$ thus quantifies how much of the phase variation resulting from the presence of each log-singularity can be exploited at any frequency.

V. CONCLUSION

To conclude, resonances can be characterized from arbitrary continuous spectral functions by the quality function simply given by $\eta(\omega) := \frac{\omega}{2} \partial_\omega [\Psi](\omega)$. The quality function $\eta(\omega)$ is a meromorphic function that can be expanded in terms of poles and zeros of the response function $H(\omega)$, providing an analytical continuation into the complex ω plane. Poles and zeros are derived from the knowledge of amplitude and phase spectra of the response function. This expansion demonstrates that poles alone are not enough to describe resonances and

that zeros are also required. Zeros can have either positive or negative contributions to the quality function depending on their position in the lower (negative contribution) or upper (positive contribution) part of the complex frequency plane; poles being always located in the lower half plane, they systematically bring about a positive contribution. This shows that the commonly admitted π phase shift induced by a resonance or singularity, both being classically considered as equivalent, should be replaced by a more general analysis of the phase variation brought about by both poles and zeros. It shows that the resonance characteristics can be extracted from the phase spectrum of a signal without any *a priori* knowledge about the physical system. Far from being restricted to optical devices, this approach can be applied to any linear system in wave physics.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [78].

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