

Space-time Floquet operator: Nonreciprocity and fractional topology of space-time crystals

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We introduce a *space-time Floquet operator* for crystals whose spatial and temporal translations are intertwined. Unlike the conventional Floquet operator, which describes evolution over a full time period, the space-time Floquet operator evolves the system over a fraction of the period paired with a spatial shift that generates the primitive space-time unit cell. Its eigenmode spectrum defines a space-time band structure that unfolds conventional Floquet bands to reveal the consequences of the intertwined crystal symmetry in reciprocal frequency-wavevector space. We relate the topology of these space-time bands to quantized transport phenomena, such as Bloch oscillations and adiabatic charge transport, and uncover an intrinsically fractional charge pump enabled by the primitive space-time cycle. Numerical simulations quantitatively support these predictions with small deviations controlled by the speed of the modulation. We also demonstrate how nonreciprocal parametric resonances are naturally anticipated and accurately predicted by our framework. Our approach applies broadly to both classical and quantum systems with space-time symmetry, including non-Hermitian crystals.

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Introduction. Periodic drive provides a powerful tool to design unconventional quantum [1,2] and classical [3] nonequilibrium phases, often with no equilibrium counterpart. In many cases, such driven systems exhibit a discrete symmetry under a combined translation in space and time that does not simply decompose into independent periodicities in the spatial and temporal dimensions. Such structures, called *space-time crystals* [4], are generically created when a static crystal has its local properties modulated by a traveling wave. Classically, this mixing of space and time underpins reciprocity breaking in linear response [5] and enables the transfer of energy from the driving field to electromagnetic or mechanical waves via traveling-wave parametric amplification [6–10]. In quantum condensed matter, space-time crystals are predicted to harbor novel topologically protected states [11,12] and nontrivial oscillatory dynamics under external electric fields [13]. However, these rich symmetries pose theoretical and computational challenges for current analysis methods.

For an ordinary lattice subject to a periodic drive, the primitive spatial and temporal translations are independent, and Floquet analysis [14,15] can be combined with the Bloch framework for spatially periodic systems. In space-time crystals, in contrast, an irreducible primitive symmetry advances space and time together, while the separate translations by a

larger spatial supercell and full modulation period are coarser symmetries that derive from the mixed one. If a space-time crystal is treated only as a conventional lattice plus periodic drive, the resulting Floquet band structure [16–18] provides a route to characterizing the linear response and topological properties of the system [19]. However, this Floquet-Bloch approach fails to preserve the primitive space-time symmetry because it involves integrating the dynamics over the full modulation period, missing the finer timescale set by the intertwined periodicity [20]. The resulting band structures exhibit spurious eigenvalue degeneracies (band crossings) that do not correspond to physical resonances [10,13,20] and provide little insight into reciprocity-breaking behavior [10,20]. Frequency-domain approaches [4,8,13] resolve these issues by using an oblique reciprocal lattice to expand the eigenmodes of the system while explicitly satisfying the space-time symmetry, but actual computations require truncation of an infinite-dimensional coupling matrix in Fourier space [8,18,19]. The number of harmonics that must be kept for an accurate evaluation of the spectrum can be very large [15] or even infinite [21–23] in strongly driven or strongly interacting systems, limiting the utility of the frequency-domain approach.

Here, we resolve these issues by defining a *space-time Floquet operator*. Unlike the Floquet operator, which captures the evolution of a state over a complete modulation period T , the operator we define generates time evolution over an interval T/β determined by the spatiotemporal symmetries of the crystal [20]. We define this operator by building on our finite-system framework introduced in Ref. [20] to construct its Bloch-resolved form for extended lattices and general spatiotemporal symmetry. Our results supersede both Floquet-Bloch and frequency-domain approaches by providing a framework based on direct time evolution that avoids *ad*

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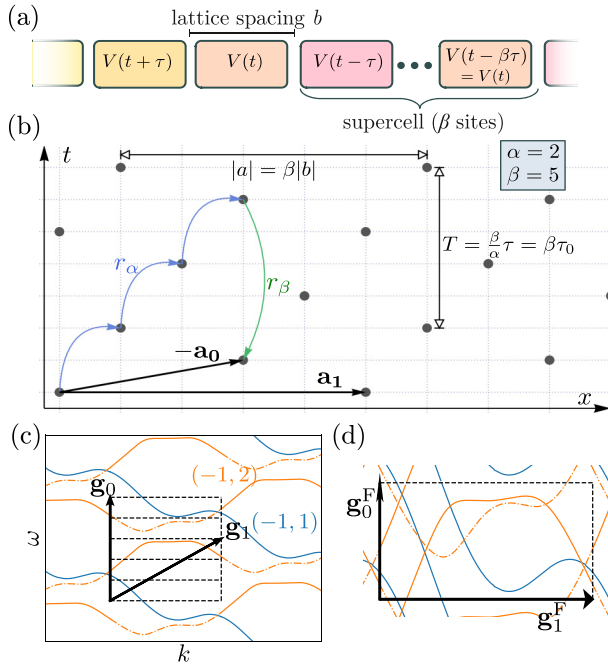


FIG. 1. (a) Schematic of a 1D tight-binding system with space-time periodic on-site potential $V(t)$. (b) Space-time periodic lattice with $\beta = 5$ and $\alpha = 2$ showing the primitive vectors ($\mathbf{a}_0$, $\mathbf{a}_1$) and the geometric relationships between the quantities r_β and r_α (here, -1 and 3 , respectively), the time period T , and the smallest time step τ_0 . (c) Space-time band structure on the reciprocal lattice generated by primitive vectors ($\mathbf{g}_0$, $\mathbf{g}_1$). Two bands are shown with their associated windings; the orange band combines two Bloch states with distinct indices (solid and dashed). (d) The Floquet band structure collapses $\beta = 5$ plaquettes [dashed boxes from panel (c)] onto a smaller Floquet-Bloch Brillouin zone spanned by ($\mathbf{g}_0^F$, $\mathbf{g}_1^F$).

hoc truncation schemes while respecting the oblique reciprocal lattice structure.

Space-time Floquet operator. While our framework holds for systems in any spatial dimension as long as they satisfy periodic boundary conditions, we frame our discussion using a one-dimensional (1D) system with a Hamiltonian of the form $\mathcal{H}(t) = \frac{p^2}{2m} + V(x, t)$ such as in Fig. 1(a). The system has space-time symmetry [4, 12, 13] if it satisfies $V(x + b, t + \tau) = V(x, t)$. That is, a traveling wave with velocity $v = \frac{b}{\tau}$ modulates the system. The time step τ is taken to be positive, while the sign of b is set by the direction of the traveling wave [24]. The displacement $|b|$ is the shortest length for which such an equation may be written.

We denote the system size by $L := n\beta|b|$, where n is the number of supercells in the periodic system and $|a| := \beta|b|$ is the size of each supercell (β is a positive integer). If $n > 1$, we have a lattice, and when $n = 1$ we have a space-time symmetric ring [20] without translational symmetry. Since $V(x, t) = V(x + \beta b, t) = V(x, t - \beta\tau)$, the potential $V(x, t)$ (and, by consequence, the Hamiltonian) is required to be time periodic with some time period T whose general form is $T = \frac{\beta}{\alpha}\tau$, with α a positive integer coprime with β [25].

While the space-time Bravais lattice of the crystal is spanned by any pair of independent lattice vectors, we make a

special choice of primitive vectors involving the *shortest time step* needed to capture the long-time evolution of the system, $\tau_0 := \frac{T}{\beta} = \frac{\tau}{\alpha}$ [graphically represented in Fig. 1(b)]. We use Bézout's identity to find the integers r_β and r_α satisfying $r_\beta\beta + r_\alpha\alpha = 1$ [26] such that

$$V(x, t) = V(x + r_\alpha b, t + r_\alpha \tau + r_\beta T) = V(x + r_\alpha b, t + \tau_0).$$

The requisite primitive vectors are then $\mathbf{a}_0 = -(r_\alpha b, \tau_0)$ and $\mathbf{a}_1 = (\beta b, 0)$ [Fig. 1(b)]. These quantities, together with the translation operator $S(r) := e^{irp}$ (we set $\hbar = 1$ throughout), allow us to express the periodicity conditions on the Hamiltonian as

$$\mathcal{H}(t) = S(a)\mathcal{H}(t)S^{-1}(a), \quad (1a)$$

$$\mathcal{H}(t) = \mathcal{H}(t + T), \quad (1b)$$

$$\mathcal{H}(t) = S(r_\alpha b)\mathcal{H}(t + \tau_0)S^{-1}(r_\alpha b), \quad (1c)$$

corresponding to translational symmetry over the supercell, Floquet symmetry, and space-time symmetry, respectively. The first two conditions are the symmetries of an ordinary periodically driven lattice and constrain the regular Floquet band structure. The third condition is the additional space-time constraint: Evolution by the fractional time step τ_0 returns the Hamiltonian to itself only after a spatial translation, so it cannot be captured by treating the drive and lattice periodicity as independent. The time-propagation operator generated by the Hamiltonian evolution, $U(t) := \mathcal{T}e^{-i\int_0^t dt' \mathcal{H}(t')}$, inherits these symmetries and satisfies

$$U(t) = S(a)U(t)S^{-1}(a), \quad (2a)$$

$$U(t + T) = U(t)U(T), \quad (2b)$$

$$U(t + \tau_0) = S^{-1}(r_\alpha b)U(t)S(r_\alpha b)U(\tau_0). \quad (2c)$$

Equation (2c), which we prove in the Supplemental Material [27], shows explicitly that evolution over the shortest time step is sufficient to reproduce the long-time behavior of the system. Specifically, it implies that the evolution over m integral time steps is

$$U(t + m\tau_0) = S^{-m}(r_\alpha b)U(t)[S(r_\alpha b)U(\tau_0)]^m. \quad (3)$$

This form is reminiscent of the Floquet form for evolution in multiples of the time period of the potential, $U(t + mT) = U(t)[U(T)]^m$, where $U(T)$ is the Floquet operator.

To proceed, we transform to a Bloch basis to account for the spatial periodicity. We change basis to the eigenstates of $S(a)$, which simultaneously block diagonalizes the operators $\mathcal{H}(t)$, $U(t)$, and $S(r_\alpha b)$ into N -dimensional blocks, where N is the number of system degrees of freedom in a supercell of size a . Each block of $S(a)$ is given by $S_k(a) = e^{ika}\mathbb{I}_N$, where $k \in (-\frac{\pi}{a}, \frac{\pi}{a}]$ is the wave vector in reciprocal space. Using this basis, Eq. (2c) is represented as (see the Supplemental Material [27] for details)

$$U_k(T) = U_k(\beta\tau_0) = e^{-ir_\alpha ka}[S_k(r_\alpha b)U_k(\tau_0)]^\beta, \quad (4)$$

where $U_k(T)$ is the Floquet operator in wave vector space (whose spectrum generates the conventional Floquet-Bloch band structure) and we used $S_k^{-\beta}(r_\alpha b) = S_k^{-1}(r_\alpha a) = e^{-ir_\alpha ka}\mathbb{I}_N$. Taking the principal β th root of this relation, we

define

$$X_k := e^{-ir_a kb} S_k(r_a b) U_k(\tau_0) \quad (5)$$

as the k th block of the *space-time Floquet operator*, which factorizes the Bloch-transformed Floquet operator as

$$U_k(T) = X_k^\beta. \quad (6)$$

Following the approach used to generate Floquet band structures, we define the N -dimensional effective Hamiltonian $\mathcal{H}_k^{\text{eff}}$ of the system by $e^{-i\mathcal{H}_k^{\text{eff}}\tau_0} := X_k$ and denote its j th eigenvalue by $\omega_j(k)$. Henceforth, we will refer to the quantum number j as the state index of the eigenstate. The N functions $\omega_j(k)$ define the *space-time band structure* for space-time crystals. Each space-time frequency eigenvalue $\omega_j(k)$ is defined modulo $\frac{2\pi}{\tau_0} = \frac{2\pi\beta}{T}$. The corresponding eigenvectors $\Psi_{k,j}(x)$ are Bloch states satisfying $\Psi_{k,j}(x+a) = e^{ika}\Psi_{k,j}(x)$. A state initialized in an eigenvector $\Psi_{k,j}(x)$ at $t = 0$ evolves over an integer multiple of the shortest time step as

$$U_k(m\tau_0)\Psi_{k,j}(x) = e^{im(r_a kb - \omega_j\tau_0)} S_k^{-m}(r_a b)\Psi_{k,j}(x); \quad (7)$$

i.e., the stroboscopic dynamics of the eigenstate at intervals of duration τ_0 are generated simply by appropriate translations in space. This feature translates to the stroboscopic dynamics of arbitrary initial states when $\Psi_{k,j}(x)$ form a complete basis for the respective Bloch subspace, such as when $\mathcal{H}_k(t)$ is Hermitian.

The space-time band structure inherits the intertwined space-time translational symmetries in the reciprocal wave vector–frequency space. The space-time lattice depicted in Fig. 1(b) is associated with a reciprocal lattice with primitive vectors $\mathbf{g}_i$ defined via $\mathbf{g}_i \cdot \mathbf{a}_j = 2\pi\delta_{ij}$ [4,13]. For our choice of direct primitive vectors, these evaluate to $\mathbf{g}_0 = (0, \frac{2\pi}{\tau_0})$ and $\mathbf{g}_1 = (\frac{2\pi}{\beta b}, \frac{2\pi r_a}{\beta\tau_0})$, signifying an oblique reciprocal lattice that mixes frequency and wave vector directions. Space-time symmetry ensures that the space-time band structure $\omega_j(k)$ generates a periodic tiling of the frequency-momentum space with these primitive vectors, as shown schematically in Fig. 1(c) for a system with two unique noncrossing bands.

Equation (6) shows that the space-time bands (the spectrum of X_k) are closely related to Floquet bands (the spectrum of $U_k(T)$) but the latter do not capture the finer space-time symmetric structure. Conventional Floquet-Bloch analysis treats the temporal and spatial periodicities of the space-time crystal, with periods $T = \beta\tau_0$ and $a = \beta b$, respectively, as independent [2,17,18,28]. This construction is appropriate for a generic periodically driven lattice, but for a space-time crystal it discards the primitive mixed translation. The primitive vectors $\mathbf{a}_0^F = -(0, \beta\tau_0)$ and $\mathbf{a}_1^F = (\beta b, 0)$ generate a rectangular reciprocal lattice spanned by $\mathbf{g}_0^F = (0, \frac{2\pi}{\beta\tau_0})$ and $\mathbf{g}_1^F = (\frac{2\pi}{\beta b}, 0)$. The Floquet bands $\omega_j^F(k)$ are the eigenvalues of $\frac{i}{T} \ln U_k(T)$; by Eq. (6), they “fold” the space-time bands onto a narrower frequency range via $\omega_j^F(k) = \omega_j(k) \bmod \frac{2\pi}{T}$ [Fig. 1(d)], giving rise to spurious band crossings that do not reflect actual eigenvalue couplings as they are not present in the space-time band structure. The corresponding eigenvectors $\Psi_{k,j}^F(x)$ are also Bloch states with period a , but do not necessarily satisfy the finer structure of Eq. (7).

Topological phenomena. For any point (k, ω) on a band in the space-time band structure, we can move along the band to its first lattice copy, $(k, \omega) + w(\mathbf{g}_0)\mathbf{g}_0 + w(\mathbf{g}_1)\mathbf{g}_1$. The integers $w(\mathbf{g}_0)$ and $w(\mathbf{g}_1)$ define the winding numbers of the band [29]. A key advance of our operator construction is a unified formulation in which the two windings $w(\mathbf{g}_0)$ and $w(\mathbf{g}_1)$ determine both quantized transport and field-driven wave-packet dynamics. This formulation builds on earlier studies of oblique space-time bands [4,13] and generalizes established winding-based characterizations of Floquet bands [2,17,18] to the space-time symmetric context.

In Hermitian systems, Floquet band topology relates to quantized adiabatic charge transport or “Thouless pumping” under time-modulated potentials [30,31]. In a conventional Floquet analysis, the natural closed cycle is the full temporal period T , and the current in a filled Floquet band integrated over this period is quantized to an integer given by the winding number of the band in the periodic frequency direction [17]. Space-time symmetry supplies an alternate closed cycle—evolution over the shortest time step $\tau_0 = T/\beta$ followed by the spatial translation appearing in X_k —which correctly captures the mixed primitive cell. Upon integrating the spatial average of the instantaneous current [17] of a filled band over this time step, we obtain (see the Supplemental Material [27])

$$\begin{aligned} Q_{\tau_0} &:= \int_0^{\tau_0} dt J(t) = \int_{\text{BZ}} \frac{dk}{2\pi} i \text{Tr} [P_k X_k^{-1} \partial_k X_k] \\ &= w(\mathbf{g}_0) + \frac{r_a}{\beta} w(\mathbf{g}_1), \end{aligned} \quad (8)$$

where the trace is over the full Bloch subspace, with the projector P_k selecting the occupied eigenstates. Thus, the charge pumped across the system over a time interval $\tau_0 = T/\beta$ is topologically quantized to a rational fraction, rather than an integer.

The fractional quantization of pumped charge revealed by the space-time Floquet operator is not merely the result of slicing an ordinary Floquet pump into β time windows: After each interval τ_0 , the state returns to an equivalent point in the space-time crystal only after the accompanying spatial translation. Fractional charge pumping was previously proposed for systems with special symmetries by partitioning the integral of the Berry curvature of instantaneous eigenstates over periodic momentum and time into β equal parts [32]. In contrast, Eq. (8) derives the fractional response directly from the two windings of the space-time bands defined by X_k on the oblique reciprocal lattice. It recovers Thouless’s integer result over a full period [30] (see the Supplemental Material [27]), while identifying the topological contribution of a single primitive space-time cycle that will also control the Floquet-Bloch dynamics below.

To test the prediction of fractional charge quantization, we simulated wave-packet dynamics in a one-dimensional tight-binding chain with a discrete space-time symmetric Hamiltonian

$$\begin{aligned} \mathbf{H} = \sum_j &\left[-J|j\rangle\langle j+1| - J|j\rangle\langle j-1| \right. \\ &\left. + V_0 \cos\left(\frac{2\pi\alpha}{\beta}j - \frac{2\pi}{T}t\right)|j\rangle\langle j| \right]. \end{aligned} \quad (9)$$

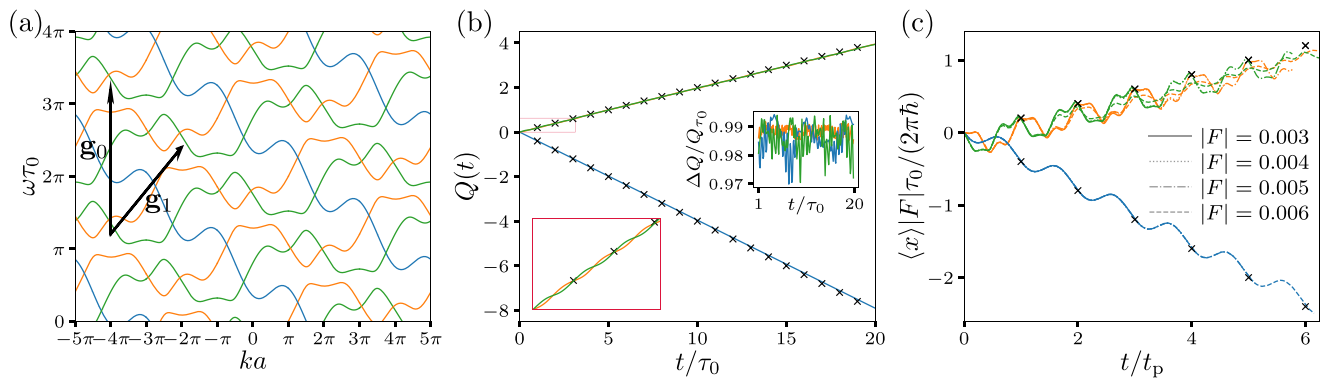


FIG. 2. (a) Space-time band structure of Hamiltonian in Eq. (9) with $\beta = 5$, $\alpha = 2$, $V_0 = 1.5$, $J = 1$, and $T = 29$. Blue, orange, and green bands have windings $(w(\mathbf{g}_0), w(\mathbf{g}_1)) = (-1, 1)$, $(-1, 2)$, and $(-1, 2)$, respectively. Band assignments and quantized transport predictions neglect the minuscule gaps at the crossing points, the largest of which is $\Delta\omega \approx 0.0119/\tau_0$, and treat these points as true crossings (see the Supplemental Material [27]). (b) Net pumped charge $Q(t)$ for filled bands, obtained by numerically integrating Schrödinger's equation on a chain with $n = 400$. Colors indicate the bands from panel (a). Crosses show the quantized expectation $Q(n\tau_0) = nQ_{\tau_0}$ at integer multiples of τ_0 . Left inset: Close-up of boxed region to show detail. Right inset: Pumped charge $\Delta Q(t) = Q(t) - Q(t - \tau_0)$ normalized by the winding prediction. (c) Shift in center of mass during Floquet-Bloch oscillations for wave packets initialized using eigenstates of the distinct bands from panel (a) (colors) at three different values of the external field (dashing) in a chain with $n = 200$. Crosses show the accumulation of the predicted shift over one oscillation [Eq. (12)].

In the resulting space-time band structure, shown in Fig. 2(a), the five degrees of freedom per Bloch subspace appear to organize into three contiguous bands that cross each other as they wind around the Brillouin zone. Upon closer inspection (Fig. S1 of the Supplemental Material [27]), the crossings are revealed to be weakly avoided crossings with gaps $\Delta\omega$ of order $0.01/\tau_0$ that disrupt the windings of the space-time bands. However, these gaps are expected to fall exponentially with the modulation period T [33], producing small nonadiabatic corrections to the idealized quantized pumping [34,35]. Accordingly, we ignore the minuscule gaps at avoided crossings to organize the bands into effective diabatic branches with nontrivial windings [Fig. 2(a)]. For this diabatic band assignment, Eq. (8) is no longer exact, but rather is accurate only up to nonadiabatic corrections of order $\tau_0\Delta\omega \sim 0.01$ (see the Supplemental Material [27]).

We computed the net pumped charge associated with wave packets assembled from all Bloch states of each diabatic band (see the Supplemental Material [27] for simulation details). The pumped charge $Q(t)$ at intervals $t = n\tau_0$ closely tracks integer multiples of the fractional quantized charge predicted by Eq. (8), which has values $Q_{\tau_0} = 1/5$ for the green and orange bands, and $Q_{\tau_0} = -2/5$ for the blue band [Fig. 2(b)]. The deviation from the exact quantization over each cycle is 3% or less (inset), consistent with the expected correction due to the weakly avoided crossings. These simulations show that the fractional quantization predicted by Eq. (8), exact for isolated space-time bands, can be observed with small quantifiable corrections for realistic systems with weakly avoided crossings.

The periodicity of the space-time band structure in momentum and frequency directions causes an electron to move back and forth in response to a unidirectional electric field, in a phenomenon termed Floquet-Bloch oscillations [13], which we also relate to winding topology. In the presence of a constant external force F , weak enough that the resulting dynamics are slow compared to the timescales in the band

structure, the adiabatic evolution of the quasimomentum of each state is well described by $k(t) = k(0) + \frac{F}{\hbar}t$ [13,36]. As the momentum-frequency pair traverses the band and returns to successive lattice images of itself, the group velocity of the eigenstate varies periodically with time period

$$t_p = \frac{\hbar}{|F|} \frac{2\pi w(\mathbf{g}_1)}{a} = \frac{w(\mathbf{g}_1)}{\beta} \frac{h}{|bF|} \quad (10)$$

corresponding to the time taken for the momentum to span the Brillouin zone in the mixed direction $\mathbf{g}_1$.

Nontrivial windings of space-time bands can also generate an overall drift of particle positions during each period of the Floquet-Bloch oscillation. Consider a wave packet representing a localized particle, which traverses a space-time band over the time interval t_p . The shift in its position over that period is given by the integrated group velocity [13],

$$\Delta x = \int_0^{t_p} \frac{\partial \omega}{\partial k} dt = \int_k^{k+\Delta k} \frac{1}{|k|} \frac{\partial \omega}{\partial k} dk = \frac{\hbar}{|F|} \Delta\omega, \quad (11)$$

where Δk and $\Delta\omega$ are, respectively, the shift in quasimomentum and quasifrequency of the band when the momentum-frequency pair reaches its lattice image. This shift can be expressed using winding numbers as

$$\Delta x = \frac{\hbar}{|F|} \left[\frac{2\pi}{\tau_0} w(\mathbf{g}_0) + \frac{2\pi r_\alpha}{\beta \tau_0} w(\mathbf{g}_1) \right] = \frac{2\pi \hbar}{|F| \tau_0} Q_{\tau_0}, \quad (12)$$

revealing a connection between the fractional charge pumped during the shortest time period τ_0 for a completely filled band and the drift of a localized particle comprised only of a *subset* of the eigenstates of that band under a constant electric field. A similar connection was made for Floquet band structures in Ref. [37]; Eq. (12) generalizes the result to space-time crystals.

We tested these predictions in simulations of Gaussian wave packets in the tight-binding chain described in Fig. 2. The external force field was implemented by adding a linear

on-site potential $\mathbf{H}_F = -\sum_j F|j\rangle\langle j|$ to the Hamiltonian. For strong fields, the Floquet-Bloch states tunnel seamlessly across the weakly avoided crossings due to Landau-Zener tunneling [38], making the winding topology implied by the diabatic band assignment in Fig. 2(a) accessible (see the Supplemental Material [27] for a detailed justification). Accordingly, the period and drift of the simulated wave packets match Eqs. (10) and (12) evaluated for the diabatic bands [Fig. 2(c)], except at the largest field value where the adiabatic evolution assumption breaks down.

Nonreciprocal wave transport. Traveling-wave modulation breaks spatial inversion symmetry, enabling nonreciprocal response [5,6] and directional amplification [3,5,39] in wave systems. These phenomena are typically studied using frequency-domain approaches [6,21–23,40–43], with eigenmodes expressed using a discrete but infinite set of Fourier components that are coupled by the parametric modulation. Practical calculations keep a limited subset of components (similar to techniques for quantum systems [12,13,19]), but such perturbative approaches can break down under strong drive or interactions. Generalizing our previous results on directional amplification in a ring geometry [20], we show that the space-time band structure provides a formally exact description of nonreciprocal wave transport based on time evolution over τ_0 that, unlike its Floquet counterpart, explicitly anticipates nonreciprocity and avoids spurious band crossings.

We illustrate these features in a 1D spring-mass chain with a traveling-wave modulation that generates broadband parametric resonance in its vibrational spectrum [10]. The chain is formed by connecting unit masses with springs of rest length b and base stiffness κ_0 . Upon modulating the j th spring stiffness as

$$\kappa_j(t) = \kappa_0 \left[1 + \delta \cos \left(\frac{2\pi\alpha}{\beta b} j - \frac{2\pi}{T} t \right) \right], \quad (13)$$

a stiffness wave with speed $v_m = \frac{\beta b}{\alpha T} = \frac{b}{\alpha \tau_0}$ is set up in the chain [Fig. 3(a)] that singles out a particular direction and thus breaks reciprocity. The ensuing equations of motion can be cast into the form of a time-dependent Schrödinger equation with a non-Hermitian space-time symmetric Hamiltonian [10,20].

The static chain ($\delta \rightarrow 0$) has a dispersion relation that is linear as $q \rightarrow 0$, $\Omega_{\pm}(q) \approx \pm v_s q$, where $v_s := \sqrt{\kappa_0}$ is the speed of sound in the chain. To anticipate the effect of modulation, we fold the static band structure along the reciprocal lattice vectors of the effective Hamiltonian; coincidences in folded eigenvalues signify resonances that generate parametrically amplified modes when the modulation is turned on [20,44]. In the Floquet framework (as in Ref. [10]), the folding is dictated by the rectangular reciprocal lattice vectors $\mathbf{g}_0^F$ and $\mathbf{g}_1^F$ [Fig. 3(c)]. However, since these vectors are symmetric with respect to $k \mapsto -k$ symmetry, the folding also respects this symmetry. Consequently, the nonreciprocal response of the modulated spectrum is not anticipated in the Floquet approach [notice the $k \mapsto -k$ inversion symmetry in Fig. 3(c), which is actually broken upon evaluating the Floquet spectrum in Fig. 3(d)]. The Floquet band structure

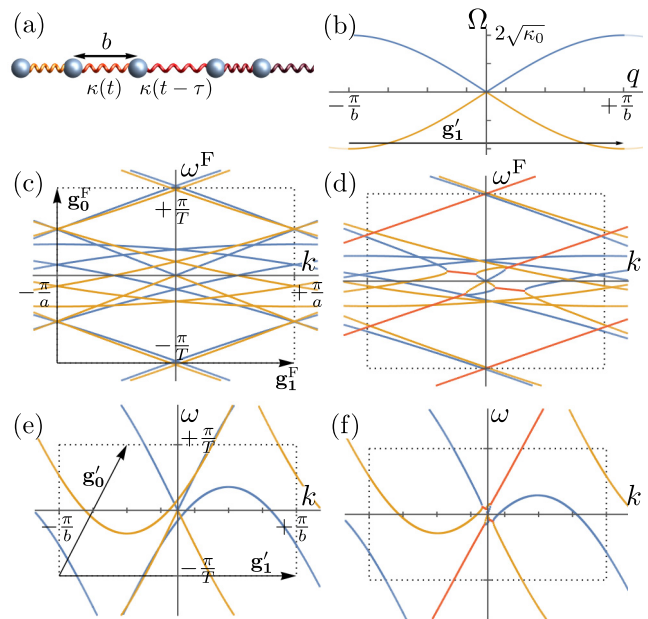


FIG. 3. (a) Spring-mass chain with coupling springs modulated according to Eq. (13) (here $\alpha = 2$, $\beta = 7$). (b) Static band structure is invariant on translation by reciprocal vector $\mathbf{g}_1' = (\frac{2\pi}{b}, 0)$. (c) Floquet band structure in $\delta \rightarrow 0$ limit, obtained by folding the bands in panel (b) along $\mathbf{g}_0^F = (0, \frac{2\pi}{T})$, $\mathbf{g}_1^F = (\frac{2\pi}{b}, 0)$. (d) Floquet bands for $\delta = 0.5$. Amplified modes ($\text{Im}(\omega) \neq 0$) in red. (e), (f) Space-time band structures at infinitesimal and finite modulation.

exhibits extraneous band crossings that do not translate into resonances.

In contrast, the reciprocal lattice vectors of the space-time band structure lie along mixed frequency and momentum directions, and explicitly incorporate the breaking of inversion symmetry. For convenience, we choose reciprocal lattice primitive vectors $\mathbf{g}_0' = (\frac{2\pi\alpha}{\beta b}, \frac{2\pi}{\beta\tau_0}) = \alpha\mathbf{g}_1 + r_\beta\mathbf{g}_0$ and $\mathbf{g}_1' = (\frac{2\pi}{b}, 0) = \beta\mathbf{g}_1 - r_\alpha\mathbf{g}_0$. Folding the static band structure along these vectors [Fig. 3(e)] gives an inversion-asymmetric band structure that anticipates amplification only for rightward-moving waves (dispersion with positive slope) [10,20] while avoiding any extraneous crossings. The overlap in the linear part of the spectrum is maximized when the slope of $\mathbf{g}_0'$, i.e., the modulation wave speed v_m , equals the sound speed v_s (the so-called sonic limit [9,10,21]). The anticipated resonance is confirmed by the exact evaluation of the space-time band structure for $\delta > 0$ [Fig. 3(f)]: Nearly the entire rightward-propagating band is amplified. Frequency-domain approaches can become inaccurate near such broadband instabilities [21]; in contrast, the space-time bands preserve the accuracy of the direct Floquet approach while correctly anticipating the nonreciprocal response.

Conclusion. The space-time Floquet operator we have introduced enables analysis of space-time crystals through direct time evolution, avoiding truncation ambiguities inherent to frequency-domain methods. This approach applies across platforms including natural [45] and artificial [46,47] quantum systems, photonic crystals [48] and active metamaterials.

materials [3]. The two windings of its oblique bands provide a unified description of fractional charge transport and the period and drift of Floquet-Bloch oscillations, while the band representation exposes nonreciprocal amplification without frequency-domain truncation ambiguities. The relationship between windings and transport phenomena, which is exact for isolated bands, persists with small quantifiable corrections for band structures with weakly avoided crossings. Promising future directions include extending our analysis to systems with synthetic dimensions [49–51], connecting Berry phases of space-time band eigenfunctions to physical observables [52], analyzing the effect of gauge fields [53], and investigating the non-Hermitian skin effect and non-Bloch theory

[54–56] for the space-time band structure. The algebraic relation in Eq. (6) suggests connections to n th root topological insulators [57,58] and sublattice [59–61] and time-glide symmetry [11,62].

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Data availability. All graphs were generated by numerical evaluation of the mathematical operations described in the text and can be reproduced using standard scientific computing software. The specific code used to generate the figures in the manuscript is available from the authors upon reasonable request.

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- [25] The integer α can always be chosen to be smaller than β , which we will assume is the case. If α is larger than β , replace α with $\alpha \bmod \beta$.
- [26] All such solutions are of the form $(r_\beta, r_\alpha) \in (r_\beta^0, r_\alpha^0) + \mathbb{Z}(\alpha, -\beta)$, with (r_β^0, r_α^0) a particular solution. We choose the one such that $0 < r_\alpha < \beta$.
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