

Topology and edge modes surviving criticality in non-Hermitian Floquet systems

Longwen Zhou ^{*}

College of Physics and Optoelectronic Engineering, *Ocean University of China*, Qingdao 266100, China;
Qingdao Key Laboratory of Advanced Optoelectronics, Qingdao 266100, China;
and Engineering Research Center of Advanced Marine Physical Instruments and Equipment of MOE, Qingdao 266100, China



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The discovery of critical points that can host quantized nonlocal order parameters and degenerate edge modes relocates the study of symmetry-protected topological phases (SPTs) to gapless regions. In this Letter, we reveal gapless SPTs (gSPTs) in systems tuned out of equilibrium by periodic drivings and non-Hermitian couplings. Focusing on one-dimensional models with sublattice symmetry, we introduce winding numbers by applying the Cauchy's argument principle to generalized Brillouin zone, yielding unified topological characterizations and bulk-edge correspondence both in gapped phases and at gapless critical points. The theory is demonstrated in a broad class of Floquet bipartite lattices, unveiling unique topological criticality of non-Hermitian Floquet origin. Our findings identify gSPTs in driven open systems and uncover robust topological edge modes at phase transitions beyond equilibrium.

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Introduction. Featured by coexistent bulk fluctuations and symmetry-protected edge modes, gapless symmetry-protected topological phases (gSPTs) blend the physics of quantum criticality and topology into a single context [1]. The survival of topological edge modes at critical points further allows the storage of localized information across phase transitions [2], whose stability could be of great use in quantum technologies. Following theoretical progress [3–38], observations of gSPTs were made in superconducting processors and acoustic waveguides [39–41], attracting continued interest over the years.

Floquet drivings and non-Hermitian effects are two means to impel a system out of equilibrium. The former could induce symmetry-breaking and topological transitions [42–44], create long-range-coupled phases with large topological numbers [45–47], and generate anomalous edge modes without static counterparts [48–50]. The latter could enrich the symmetry classification of topological matter [51–54], yield exceptional points with enhanced sensitivity [55–57], and lead to non-Hermitian skin effects (NHSEs) that challenge the Bloch band theory [58–60]. Their cooperation further results in non-Hermitian Floquet phases with unique topology [61], whose characterizations were yet mostly focused on gapped cases.

This Letter establishes a framework to depict the gapless bulk topology, $(0, \pi)$ edge modes, and bulk-edge correspondence at non-Hermitian Floquet criticality. Our approach extends a recent theory of Floquet quantum criticality [62–64]

by incorporating the generalized Brillouin zone (GBZ) scheme [65–67], making it applicable to the gaplessness of both Bloch and non-Bloch Floquet bands [68–71]. The theory is demonstrated in a set of Floquet non-Hermitian Su-Schrieffer-Heeger (NHSSH) chains and developed more systematically in the Supplemental Material [72]. A unified view is offered by investigating the phase diagram, edge states, phase transitions along topological phase boundaries and scaling of entanglement entropy (EE). Notably, we identify nontrivial critical points and critical edge modes originated from non-Hermitian Floquet drivings.

Theory. We start by noting that any (non-)Hermitian Bloch Hamiltonian of a one-dimensional (1D), two-band system with sublattice symmetry can be formally expressed as $H(k) = f(k)\sigma_+ + g(k)\sigma_-$, where $\sigma_{\pm} \equiv (\sigma_x \pm i\sigma_y)/2$, $\sigma_{x,y,z}$ are Pauli matrices, and the sublattice symmetry $\Gamma = \sigma_z$ acts on $H(k)$ as $\Gamma H(k) \Gamma = -H(k)$. The 2π periodicity of $H(k)$ in quasimomentum k allows one to expand $f(k)$ and $g(k)$ as polynomials of e^{ik} . Continuing $e^{ik} \rightarrow z$ to the whole complex plane and tracing the zero/pole locations of polynomial $f(z)$, $g(z)$ yields a topological invariant w , which could classify all gapped phases of $H(z)$ and predict a bulk-edge correspondence consisting with the non-Bloch band theory [73–77]. However, this approach is not directly applicable at the gap-closing point of $H(z)$ [78], not to mention the Floquet case. The underlying reason is at least twofold. First, the non-Hermiticity of $H(z)$ implies $f(z) \neq g^*(z)$, so that the criticality in $f(z)$ and $g(z)$ must be treated simultaneously at phase transitions instead of being depicted by a single ratio. Second, the spectrum of a Floquet operator could be gapless at both 0 and π quasienergies [79–81]. Such a “gapless doubling” is of temporal origin, making it impossible to capture its associated topology and phase transitions by a single static Hamiltonian.

To proceed, we express the Floquet operator $U(k) = \mathcal{T} e^{-\frac{i}{\hbar} \int_0^T H(k,t) dt}$ of the system in symmetric time frames

^{*}Contact author: zhoulw13@u.nus.edu

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[79] by selecting the initial time of evolution t' , so that the symmetrized Floquet operator $U_s(k)$ unfolds the sublattice symmetry as $\Gamma U_s(k) \Gamma = U_s^{-1}(k)$ ($s = 1, 2$). Here, the Hamiltonian $H(k, t) = H(k, t + T)$ has the driving period T , and $\hat{T}$ performs time ordering. In time frame s , the effective Hamiltonian of $U_s(k)$ reads $[U_s^{-1}(k) - U_s(k)]/(2i)$, which must have the form $H_s(k) = p_s(k)\sigma_+ + q_s(k)\sigma_-$ due to its sublattice symmetry. Taking the continuation $k \rightarrow -i \ln z$, we find a characteristic function of $H_s(z)$ as $f_s(z) = p_s(z) (= q_s(z))$ if $m > m'$ ($m' > m$), where m and m' are the highest positive powers of $p_s(z)$ and $q_s(z)$ in z . Finding the numbers of zeros N_s^z and poles N_s^p of $f_s(z)$ inside the GBZ of $H_s(z)$ then yields a winding number w_s following Cauchy's argument principle, i.e.,

$$w_s \equiv \begin{cases} N_s^z - N_s^p, & \Delta_\pi \neq 0, \\ N_s^z - N_s^p/2, & \Delta_\pi = 0, \end{cases} \quad (1)$$

where $\Delta_\pi \equiv \min_{z \in \text{GBZ}} |E(z) - \pi|$ infers whether $U_s(z)$ has a gap at quasienergy $E = \pi$ along the Floquet GBZ (FGBZ). Combining the (w_1, w_2) defined in two time frames yields the topological invariants

$$(w_0, w_\pi) = \frac{1}{2}(w_1 + w_2, w_1 - w_2) \in \mathbb{Z} \times \mathbb{Z}. \quad (2)$$

They offer a unified characterization for all gapped and gapless non-Hermitian Floquet phases in 1D, two-band, sublattice-symmetric systems. The numbers of edge modes at 0 and π quasienergies are related to these invariants according to the bulk-edge correspondence

$$(N_0, N_\pi) = 2(|w_0|, |w_\pi|). \quad (3)$$

We emphasize that Eqs. (1)–(3) are generic, in the sense that they do not concern whether there exists a (point or line) gap in the spectrum and whether the underlying band theory is in Bloch or non-Bloch forms. The only relevant factors are spatial dimensions, number of bands, and symmetries of the system. In Hermitian limits, where FGBZ reduces to the usual Brillouin zone, we end with the approach in Ref. [62]. Figures 1(a)–1(c) illustrate our theory applied to a case with gapless spectrum [72]. Below, we work out an example to show the unique topology and edge modes at criticality in non-Hermitian Floquet systems.

Model. We focus on a class of periodically quenched NHSSH (PQNHSSH) models, which may be viewed as a stacking of SSH chains with two distinct hopping ranges in time domain. A sketch of the model is given in Fig. 1(d), where the Floquet operator reads $\hat{U} = e^{-i\hat{T}_\alpha} e^{-i\hat{T}_{\alpha'}}$, $\hat{H}_\alpha \equiv \sum_n (J_\alpha^L \hat{b}_n^\dagger \hat{a}_{n+\alpha} + J_\alpha^R \hat{a}_{n+\alpha}^\dagger \hat{b}_n)$, the hopping amplitude $J_\alpha^{L,R} \equiv t_\alpha^{L,R} T / (2\hbar)$ for $\alpha = \alpha'$, and $\hat{a}_n^\dagger$ ($\hat{b}_n^\dagger$) creates a particle in sublattice A (B) of unit cell n . To evaluate the winding numbers, we need to transform $\hat{U}$ to symmetric time frames. Taking the initial time of evolution as $t' = T/4$ and $3T/4$ yields Floquet operators

$$\hat{U}_{1(2)} = e^{-\frac{i}{2}\hat{H}_{\alpha'(\alpha)}} e^{-i\hat{H}_{\alpha(\alpha')}} e^{-\frac{i}{2}\hat{H}_{\alpha'(\alpha)}}, \quad (4)$$

where the sublattice symmetry $\hat{\Gamma} = \sum_n (\hat{a}_n^\dagger \hat{a}_n - \hat{b}_n^\dagger \hat{b}_n)$ is retained as $\hat{\Gamma} \hat{U}_s \hat{\Gamma} = \hat{U}_s^{-1}$ for $s = 1, 2$. This symmetry enforces the eigenstates of $\hat{U}$ to come in pairs of opposite quasienergies $\pm E$. The Floquet bands of $\hat{U}$ could thus meet at either $E = 0$ or π , yielding phase transitions (criticality), with the

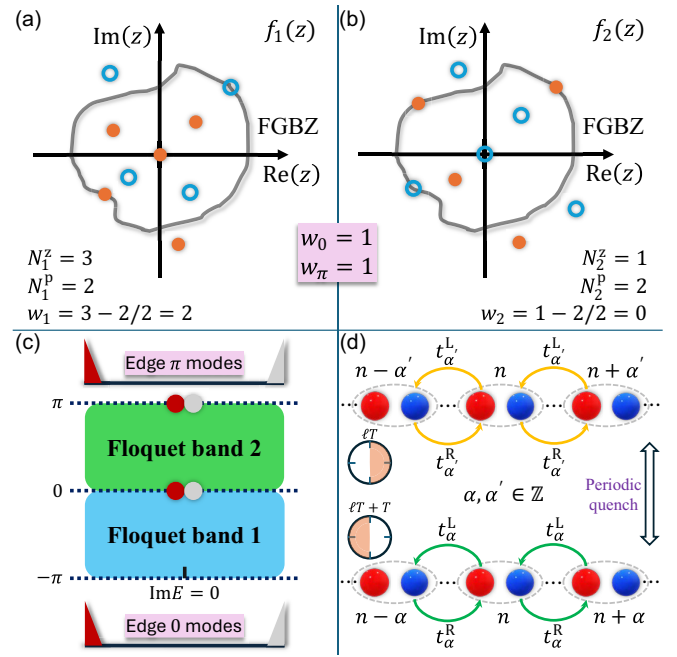


FIG. 1. Schematic diagram. Panels (a) and (b) show the zeros/poles of $f_{1,2}(z)$ on complex plane, with the wiggled contour showing the FGBZ. Orange dots (cyan circles) denote the zeros (poles) of $f_{1,2}(z)$. The spectrum related to panels (a) and (b) is sketched in panel (c). The two bands are touched at $E = 0, \pi$. The critical 0 (π) edge modes are depicted by solid balls at $E = 0$ (π), with their profiles illustrated at the bottom (top) of panel (c). Panel (d) shows the periodically quenched NHSSH model with $\alpha, \alpha' \in \mathbb{Z}$. Each encircled set of red and blue balls denotes a unit cell (indexed by n) with two sublattices A and B.

latter being of Floquet origin. Below, we take $0 \leq \alpha < \alpha'$ and let $J_\alpha^L = J_\alpha^R = J_\alpha \in \mathbb{R}$ without losing the essence. Non-Hermitian effects then arise from nonreciprocal couplings $J_{\alpha'}^L \neq (J_{\alpha'}^R)^*$. The model we introduced here will be called the PQNHSSH α chain (see Sec. II of the Supplemental Material [72] for details).

Topology and bulk-edge correspondence. Under periodic boundary condition, the Floquet operators in Eq. (4) can be expressed in momentum space, and the FGBZ is obtained as (see Sec. II of the Supplemental Material [72] for derivations)

$$\beta = (J_{\alpha'}^R/J_{\alpha'}^L)^{1/(2\alpha' - 2\alpha)} e^{i\theta}, \quad \theta \in [-\pi, \pi), \quad (5)$$

which is a circle of radius $r = |\beta|$. It deviates from the unit circle once $|J_{\alpha'}^L| \neq |J_{\alpha'}^R|$, yielding physics unique to non-Hermitian systems. The gap-closing condition along FGBZ generates phase boundaries. Working it out explicitly with $(J_{\alpha'}^L, J_{\alpha'}^R) \equiv (J - \gamma, J + \gamma)$, we find the critical lines $J_\alpha \pm \sqrt{J^2 - \gamma^2} = v\pi$ for $|\gamma| \leq |J|$ and $\cos(J_\alpha) \cosh(\sqrt{\gamma^2 - J^2}) = (-1)^v$ for $|\gamma| > |J|$ ($v \in \mathbb{Z}$). They yield the curves in Fig. 2. They are distinct from those found in Hermitian [62] or non-driven [78] limits, and can be topologically allocated into four groups.

To unveil the bulk topology, we compute the (w_0, w_π) in Eq. (2) analytically and report them for each phase in Fig. 2 (see Sec. II of the Supplemental Material [72] for details). Although the gapped phases with $(w_0, w_\pi) = (\alpha, 0)$

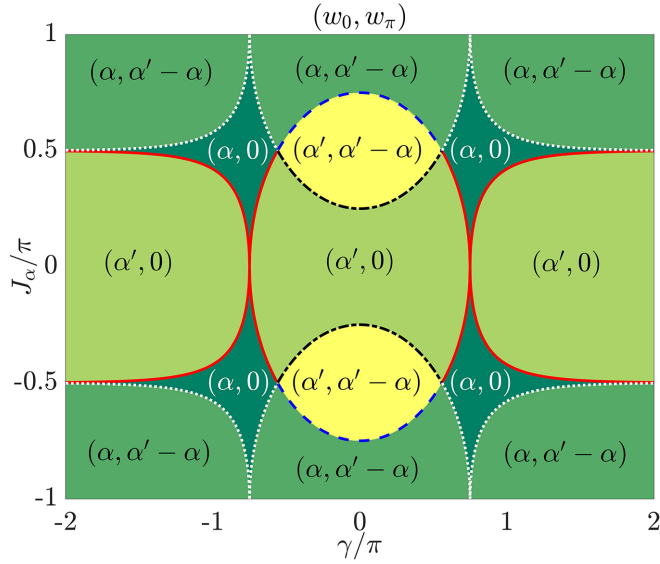


FIG. 2. Phase diagram of the PQNHSSH α chain. Hopping amplitudes over α' unit cells are $J_{\alpha'}^{L(R)} = J - (+)\gamma$. Regions of distinct colors are gapped phases, with their topological indices (w_0, w_π) shown explicitly. The solid (dotted) lines have $(w_0, w_\pi) = (\alpha, 0)$, with bulk gap closing at $E = 0$ (π). The dash-dotted lines have $(w_0, w_\pi) = (\alpha', 0)$, with bulk gap closing at $E = \pi$. The dashed lines have $(w_0, w_\pi) = (\alpha, \alpha' - \alpha)$, with bulk gap closing at $E = 0$.

and $(\alpha', 0)$ are realizable in equilibrium [78], the phases with $w_\pi = \alpha' - \alpha$ are unique to non-Hermitian Floquet setups, with each of them having $2(\alpha' - \alpha)$ topological edge modes at $E = \pi$ [Eq. (3)]. The topology of these Floquet phases is thus captured by our theory, even though they both subject to NHSEs and the violation of standard bulk-edge correspondence if $\gamma \neq 0$.

Strikingly, the invariants in Eq. (2) are also well defined and quantized at phase transitions. Along the solid critical lines where $|\tan(\sqrt{J^2 - \gamma^2}/2)| = |\tan(J_\alpha/2)|$ in Fig. 2, the gap closes at $E = 0$ and $(w_0, w_\pi) = (\alpha, 0)$. They are thus topologically nontrivial, having $N_0 = 2\alpha$ edge 0 modes with a gapless bulk at $E = 0$ for $\alpha > 0$. Along the dash-dotted critical lines where $|\tan(\sqrt{J^2 - \gamma^2}/2)\tan(J_\alpha/2)| = 1$ in Fig. 2,

the gap closes at $E = \pi$ and $(w_0, w_\pi) = (\alpha', 0)$. They are always topologically nontrivial, having $N_0 = 2\alpha'$ edge 0 modes with a gapless bulk at $E = \pi$. The dotted critical lines in Fig. 2 also satisfy $|\tan(\sqrt{J^2 - \gamma^2}/2)\tan(J_\alpha/2)| = 1$, where the gap closes at $E = \pi$ and $(w_0, w_\pi) = (\alpha, 0)$. They are topologically nontrivial, with 2α edge 0 modes coexisting with a gapless bulk at $E = \pi$ if $\alpha > 0$. Notably, these two groups of transitions are absent when the driving is switched off [78], which implies their Floquet origins. Finally, the dashed critical lines in Fig. 2 satisfy $|\tan(\sqrt{J^2 - \gamma^2}/2)| = |\tan(J_\alpha/2)|$ with the gap closing at $E = 0$ and $(w_0, w_\pi) = (\alpha, \alpha' - \alpha)$. They are topologically nontrivial with 2α and $2(\alpha' - \alpha)$ edge modes at $E = 0$ and π . Non-Hermitian Floquet gSPTs arise along these critical lines for any $0 \leq \alpha < \alpha'$. A summary of the topology and edge modes at four distinct phase boundaries in Fig. 2 is given in Table I. It deserves to mention that all critical lines in Fig. 2 refer to gap closings between Floquet non-Bloch bands over FGBZ. A naive application of Bloch band theory cannot yield consistent phase boundaries and bulk-edge correspondence at criticality.

Critical edge modes. To further reveal edge modes associated with non-Hermitian Floquet topology at phase transitions, we obtain the quasienergies under open boundary condition (OBC) with $(\alpha, \alpha') = (1, 2)$. The spectra in Figs. 3(a), 3(b), 3(d), and 3(e) are obtained at exemplary points along the dashed-dotted, solid, dashed, and dotted critical lines of Fig. 2 in its $\mathcal{PT}$ -unbroken region ($|\gamma| < |J|$). The probability distributions of their edge modes are shown in panels (a), (b), (c), and (d) of Fig. 4, with the bands touching at $E = \pi$ ($F = -1$), 0 ($F = 1$), 0, and π . Our bulk theory [Eq. (3)] predicts the numbers of edge modes $(N_0, N_\pi) = (4, 0), (2, 0), (2, 2),$ and $(2, 0)$ at these critical points, which are in perfect agreement with those observed in Fig. 4, verifying the bulk-edge correspondence at phase transitions. In the $\mathcal{PT}$ -broken region ($|\gamma| > |J|$), the range of exponential e^{-iE} goes beyond the unit circle $|F| = 1$, as shown in Figs. 3(c) and 3(f). Nevertheless, we find two edge 0 modes at both these critical points, regardless of whether the bulk gap closes at $E = 0$ [$F = 1$ in Fig. 3(c)] or $E = \pi$ [$F = -1$ in Fig. 3(f)], and their distributions are identical to those in panels (b) and (d) of Fig. 4. Lying along the solid and dotted lines in

TABLE I. Summary of the topology and edge modes surviving criticality in PQNHSSH α chain along phase boundaries in Fig. 2. Here, $J_{\alpha'} \equiv \sqrt{J_{\alpha'}^L J_{\alpha'}^R}$ and $J_{\alpha'}^L J_{\alpha'}^R = J^2 - \gamma^2$. Two Floquet bands are touched at the gapless quasienergy. The case of having π edge modes when the bulk spectrum is gapless at $E = \pi$ is discussed in Sec. II H of the Supplemental Material [72].

Critical line	Condition	Gapless quasienergy	Topological invariants (w_0, w_π)	Number of edge modes (N_0, N_π)
Solid	$ \tan(J_{\alpha'}/2) = \tan(J_\alpha/2) $ $ \tan(J_{\alpha'}/2)\tan(J_\alpha/2) < 1$	$E = 0$	$(\alpha, 0)$	$(2\alpha, 0)$
Dashed	$ \tan(J_{\alpha'}/2) = \tan(J_\alpha/2) $ $ \tan(J_{\alpha'}/2)\tan(J_\alpha/2) > 1$	$E = 0$	$(\alpha, \alpha' - \alpha)$	$(2\alpha, 2\alpha' - 2\alpha)$
Dotted	$ \tan(J_{\alpha'}/2)\tan(J_\alpha/2) = 1$ $ \tan(J_{\alpha'}/2) < \tan(J_\alpha/2) $	$E = \pm\pi$	$(\alpha, 0)$	$(2\alpha, 0)$
Dash-dotted	$ \tan(J_{\alpha'}/2)\tan(J_\alpha/2) = 1$ $ \tan(J_{\alpha'}/2) > \tan(J_\alpha/2) $	$E = \pm\pi$	$(\alpha', 0)$	$(2\alpha', 0)$
Multicritical	$ \tan(J_{\alpha'}/2) = \tan(J_\alpha/2) $ $ \tan(J_{\alpha'}/2)\tan(J_\alpha/2) = 1$	$E = 0, \pm\pi$	$(\alpha, 0)$	$(2\alpha, 0)$

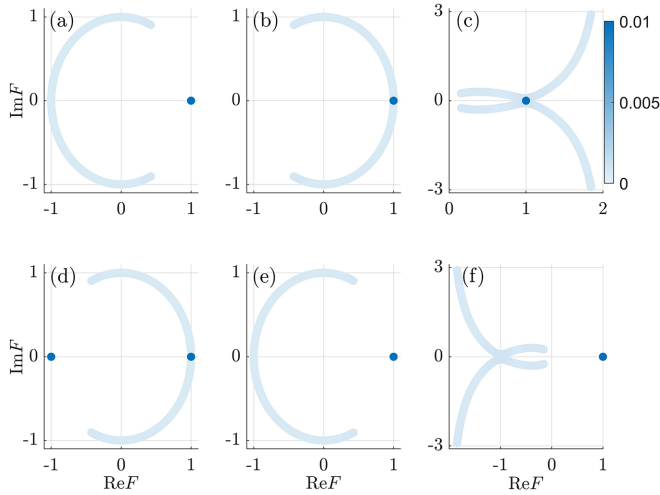


FIG. 3. Spectra of PQNHSSH α chain at criticality with $(\alpha, \alpha') = (1, 2)$, shown in terms of the Floquet operator's eigenvalue $F \equiv e^{-iE}$. The color of each point records the inverse participation ratio (IPR) [82] of related state in a lattice with 1000 unit cells. System parameters are $J_1^L = J_1^R = J_1$, $(J_2^L, J_2^R) = (J - \gamma, J + \gamma)$, $J = 3\pi/4$ for all panels, and $J_1 = 0.32\pi$ (0.68π) for panels (a)–(c) [panels (d)–(f)]. $\gamma = \sqrt{J^2 - (\pi - J_1)^2}$ for panels (a) and (e), $\sqrt{J^2 - J_1^2}$ for panels (b) and (d), $\sqrt{J^2 + \text{acosh}^2(1/\cos J_1)}$ for panel (c), and $\sqrt{J^2 + \text{acosh}^2(-1/\cos J_1)}$ for panel (f).

Fig. 2, the numbers of edge modes at these critical points are $(N_0, N_\pi) = (2, 0)$, verifying again Eq. (3). Overall, we confirmed the nontrivial topology and edge modes surviving criticality in our setting, regardless of whether the underlying quasienergy spectrum is in $\mathcal{PT}$ -invariant or $\mathcal{PT}$ -broken regime.

For the case with $(\alpha, \alpha') = (1, 2)$, we find two Floquet eigenstates at quasienergy $E = 0$ and one at $E = \pi$ when considering a half-infinite chain with the OBC taken at its left

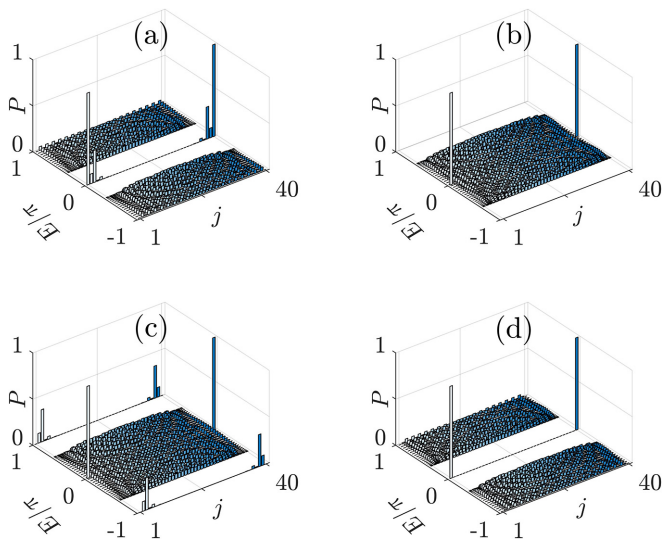


FIG. 4. Quasienergy-resolved probability distributions P of bulk and edge modes in the PQNHSSH α chain under OBC with 20 unit cells. j is the lattice index. Other parameters of panels (a)–(d) are the same as those of panels (a), (b), (d), and (e) in Fig. 3.

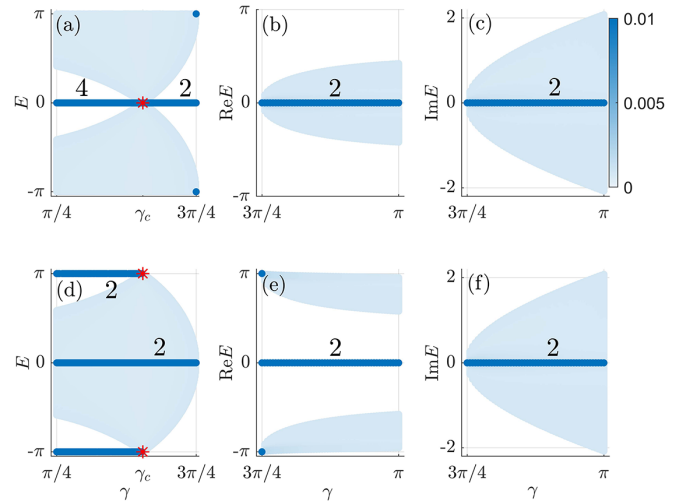


FIG. 5. Floquet spectra along critical lines. The color of each point records the IPR [82] of related state in a lattice with 1000 unit cells. Numbers of 0 and π edge modes are given in each panel. In panel (a) [panel (d)], parameters are varied along the dash-dotted to dotted (dashed to solid) lines in Fig. 2 for $\gamma = \pi/4 \rightarrow 3\pi/4$. In panels (b) and (c) [panels (e) and (f)], parameters are varied along the dotted (solid) line in Fig. 2 for $\gamma = 3\pi/4 \rightarrow \pi$.

edge. Analytically, these eigenmodes are given by (see Sec. II of the Supplemental Material [72] for details) $|\varphi_{0,1}\rangle = \hat{a}_1^\dagger |\emptyset\rangle$, $|\varphi_{0,2}\rangle = \sum_{n=1}^{\infty} [-\frac{\tan(J_1/2)}{\tan(J_2/2)}]^n [\cos(J_1/2)\hat{a}_{n+1}^\dagger - i\sin(J_1/2)\hat{b}_n^\dagger] |\emptyset\rangle$, and $|\varphi_\pi\rangle = \sum_{n=1}^{\infty} [\frac{1}{\tan(J_1/2)\tan(J_2/2)}]^n [\sin(J_1/2)\hat{a}_{n+1}^\dagger + i\cos(J_1/2)\hat{b}_n^\dagger] |\emptyset\rangle$, where $J_2 \equiv \sqrt{J^2 - \gamma^2}$. Each of the states, when exists, has a degenerate partner in thermodynamic limit due to the sublattice symmetry. We see that the state $|\varphi_{0,1}\rangle$ represents an edge 0 mode throughout the phase diagram, ensuring the nontrivial topology of all critical lines. The mode $|\varphi_{0,2}\rangle$ is localized, delocalized, and eliminated when $|\tan(J_1/2)/\tan(J_2/2)| < 1$, $= 1$, and > 1 , respectively. It thus depicts an edge 0 mode when $|\tan(J_2/2)| > |\tan(J_1/2)|$, which survives along the critical lines where $|\tan(J_1/2)\tan(J_2/2)| = 1$ with the bulk gap closing at quasienergy π . The mode $|\varphi_\pi\rangle$ is localized, delocalized, and eliminated when $|\tan(J_1/2)\tan(J_2/2)| > 1$, $= 1$, and < 1 , respectively. It then describes an edge π mode when $|\tan(J_1/2)\tan(J_2/2)| > 1$, which survives along the critical lines where $|\tan(J_1/2)| = |\tan(J_2/2)|$ and the bulk gap closes at quasienergy 0. These results are consistent with the predictions collected in Table I.

Phase transitions between g SPTs. With four critical lines of different topology, we expect to see phase transitions along topological phase boundaries via changing parameters across the multicritical point. In Fig. 5, we present the spectrum of our model with $(\alpha, \alpha') = (1, 2)$ along phase boundaries, where the star denotes the multicritical point $(J_1, \gamma_c) = (\pi/2, \sqrt{5}\pi/4)$. In panels (a) and (d) of Fig. 5, we find a change of 0 and π edge modes by 2 when going from one side to the other side of multicritical point, during which the spectrum remains gapless at $E = \pi$ and 0, respectively. The (w_0, w_π) in Eq. (2) changes from $(2, 0)$ [(1, 1)] to $(1, 0)$ across the multicritical point in Fig. 5(a) [Fig. 5(d)]. These

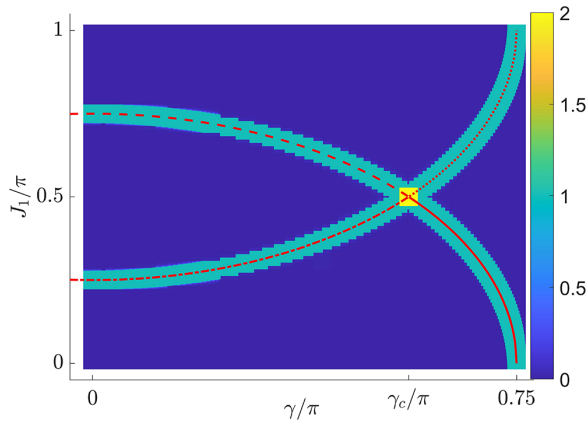


FIG. 6. Central charge of PQNHSSH α chain with $(\alpha, \alpha') = (1, 2)$. System parameters are $(J_2^L, J_2^R) = (J - \gamma, J + \gamma)$ with $J = 3\pi/4$. The length of lattice is $L = 1000$.

cases represent two examples of phase transitions between distinct non-Hermitian Floquet gSPTs, which are identified by the survival of edge zero modes at criticality. In Figs. 5(b), 5(c), 5(e), and 5(f), we show the spectra along two critical lines in the $\mathcal{PT}$ -broken region, with bulk bands touched at $E = 0$ and π . There are no multicritical points on these phase boundaries, and we find two edge 0 modes in both cases. In all examples, the degeneracy of 0 and π edge modes are protected by the sublattice symmetry.

The necessity of discriminating phase transitions via topology can also be seen from entanglement viewpoints. In Fig. 6, we extract the central charge c [83–88] of bipartite EE $S(L, l)$ at half-filling in a $\mathcal{PT}$ -unbroken region ($|\gamma| < |J|, J_\alpha > 0$). A finite-size scaling is performed versus the fitting curve $S(L, l) \sim c \ln[\sin(\pi l/L)]/3$ for EE under a fixed system size L and varied subsystem sizes l to extract c (see Sec. III of Supplemental Material [72] for details). Despite the multicritical point where $c \simeq 2$, we find $c \simeq 1$ along critical lines. The former comes about due to the simultaneous gap closing of Floquet non-Bloch bands at $E = 0$ and π , which is thus of nonequilibrium origin. The latter implies that the critical lines in Fig. 6 may belong to the same universality class, which can only be distinguished by their topology and edge modes. For other hopping ranges (α, α') , we expect $c = \alpha' - \alpha$ along critical lines (with $J_1 \mapsto J_\alpha$ in Fig. 6) and $c = 2\alpha' - 2\alpha$ at multicritical points.

Discussion. Experimentally, the Hermitian version of our model is realizable in acoustics [41], where the quasienergies

are accessible by a pump-probe method, and the driving is emulated by modulations along the propagation direction of sound. By recording both the amplitude and phase of pressure fields, the Floquet bands can be constructed from phase-resolved measurements, allowing the identification of whether the system is away from or at a gapless phase transition [41]. The critical 0 and π modes can be further detected by tracking the dynamics of edge excitations [89], in which the former shows a periodic revival while the latter shows a periodic doubling (see Sec. III of the Supplemental Material [72] for more details). Finally, the non-Hermiticity, as introduced by asymmetric couplings in our case, can be realized with directional amplifiers in acoustic crystals [90]. Acoustic metamaterials thus offer a flexible means to realize our model and detect its topology and edge modes surviving criticality. Other setups where our model is within reach include photonics [91–93] and electrical circuits [94–96].

Overall, we introduced a unique class of driven non-Hermitian models that are exactly solvable and unveiled a distinctive type of nonequilibrium gapless topology. A theoretical framework, based on a unified description of bulk Floquet non-Bloch bands, edge states, bulk-edge correspondence, and entanglement, was developed to characterize the topology and edge modes surviving criticality. Moreover, we identified π edge modes sandwiching between gapless Floquet non-Bloch bands and critical topological transitions via closing Floquet non-Bloch band gaps at $E = \pi$, with both being unavailable in static or Hermitian setups and are thus of non-Hermitian Floquet origins. These discoveries refined the organization of phase transitions in driven open systems, in which topology enriches criticality beyond the standard paradigms of continuous and topological phase transitions. In future work, it would be interesting to generalize our scheme to systems with multiple bands [97–99], in other symmetry classes [100], and beyond 1D cases [101]. Exceptional topology [102] of non-Hermitian Floquet origin and its impact on critical edge modes are awaited to be further clarified. The roles of disorder and interactions [103–106] in non-Hermitian Floquet gSPTs also deserve more thorough explorations.

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Data availability. The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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