

Inhomogeneity, fluctuations, and gap filling in disordered overdoped cuprates

Miguel Antonio Sulangi^{1,*}, Willem Farmilo², Andreas Kreisel³, Mainak Pal⁴, W. A. Atkinson², and P. J. Hirschfeld⁴

¹National Institute of Physics, *University of the Philippines Diliman, Quezon City 1101, Philippines*

²Department of Physics and Astronomy, *Trent University, Peterborough, Ontario K9L 0G2, Canada*

³Niels Bohr Institute, *University of Copenhagen, DK-2200 Copenhagen, Denmark*

⁴Department of Physics, *University of Florida, Gainesville, Florida 32611, USA*



(Received 31 March 2025; accepted 16 October 2025; published 21 November 2025)

Several recent experiments have challenged the premise that cuprate high-temperature superconductors approach conventional Landau-BCS behavior in the high-doping limit. We argue, based on an analysis of their superconducting spectra, that anomalous properties seen in the most-studied overdoped cuprates require a pairing interaction that is strongly inhomogeneous on nm length scales. This is consistent with recent proposals that the “strange-metal” phase above T_c in the same doping range arises from a spatially random interaction. We show, via mean-field Bogoliubov-de Gennes (BdG) calculations and time-dependent Ginzburg-Landau (TDGL) simulations, that key features of the observed tunneling spectra are reproduced when both inhomogeneity and thermal phase fluctuations are accounted for. In accord with experiments, BdG calculations find that low- T spectra are highly inhomogeneous and exhibit a low-energy spectral shoulder and broad coherence peaks. However, the spectral gap in this approach becomes homogeneous at high T , in contrast to experiments. This is resolved when thermal fluctuations are included within TDGL; in this case, global phase coherence is lost at the superconducting T_c via a broadened BKT transition, while robust phase-coherent superconducting islands persist well above T_c . The local spectrum remains inhomogeneous at T_c , and the gap is found to fill instead of close with increasing temperature.

DOI: [10.1103/x1m7-bs95](https://doi.org/10.1103/x1m7-bs95)

I. INTRODUCTION

Underdoped and optimally doped cuprates represent the epitome of doped Mott physics due to their proximity to the parent insulator, and exhibit not only superconductivity but other intertwined phases as well. Until relatively recently, however, it was generally accepted that the *overdoped* materials are uncomplicated by a pseudogap or other competing phases and should neatly approach the Landau-BCS paradigm with increasing doping.

Numerous features of the overdoped regime challenge this notion. First, the superfluid density at zero temperature in high-quality $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$ (LSCO) films was found to be proportional to the critical temperature T_c [1,2], in stark contrast to the BCS prediction that it should be equal to the electron density and independent of T_c . Second, scanning tunneling spectroscopy (STS) experiments in Pb- and La-doped $\text{Bi}_2\text{Sr}_2\text{CuO}_{6+\delta}$ (Bi-2201) repeatedly observe that the spectrum has a two-gap structure, with a homogeneous low-energy gap that is ascribed to superconductivity and an inhomogeneous high-energy gap that is reminiscent of the pseudogap found in underdoped cuprates [3–6]. A two-gap structure has also been observed in overdoped $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ (Bi-

2212) [7,8], though the two gaps are close in energy and hard to resolve. Third, overdoped Bi-2201 and Bi-2212 are highly inhomogeneous, with a spectral gap that varies by a factor of 2 or more over distances of a few nm [3,5,6,9–11]. Temperature-dependent STS [9,12,13] and angle-resolved photoemission spectroscopy (ARPES) experiments [14–16] have shown that in both materials these spectral gaps close well above T_c , raising questions about the nature of the superconducting state. Magnetoresistance measurements on LSCO found evidence for superconducting fluctuations extending a few tens of kelvin above T_c [17] throughout the overdoped regime; from this, it was inferred that T_c occurs at a Berezinskii-Kosterlitz-Thouless (BKT) transition. However, magnetic susceptibility measurements [18–20] in LSCO find diamagnetic signatures extending up to much higher temperatures. These invite a “granular” interpretation, in which superconducting T_c is determined by the strength of Josephson coupling between robustly superconducting islands [21].

Furthermore, it has been argued that granularity above T_c arises naturally if the overdoped cuprates contain a sufficient density of pair-breaking impurities [20] so that superconductivity persists only in rare regions. However, numerical simulations cast doubt that heavily disordered d -wave superconductors are truly granular [22]. Indeed, the disorder levels suggested in Ref. [20] would generate a much larger residual density of states [23] than is actually observed by ARPES measurements on very overdoped Bi-2212 [24], and suppress the average T_c to zero.

Here, we propose a rather different picture. We suggest that the observed spectroscopic properties of overdoped Bi-2201 and Bi-2212 depend on two ingredients: a localized

*Contact author: msulangi@nip.upd.edu.ph

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

pairing interaction, whose strength varies by an order of magnitude on a scale of nanometers, and *inhomogeneous* phase fluctuations. We further argue that pair-breaking disorder, which is essential to explain anomalous low-energy properties like the superfluid density [25–28], is less relevant on the energy scale of the gap. Our calculations show that a two-gap spectrum with a homogeneous subgap and rapidly varying large gap emerges naturally when the inhomogeneity in the pair potential is sufficiently strong, without the need to appeal to a pseudogap, and that both gaps are associated with superconductivity. Furthermore, thermal fluctuations have a dramatic effect on the subgap, which fills with increasing T and disappears when superconducting phase coherence is lost. We find that both Bi-2201 and Bi-2212 can be understood within our model, with the key difference between the two systems being the degree of inhomogeneity. Our calculations suggest that T_c in these cuprates is determined by a BKT-like transition that is broadened by disorder [29,30] and interlayer coupling [31] rather than a granular transition, and that a weak diamagnetic signal persists above T_c because of isolated superconducting islands with anomalously large pairing interactions.

The idea that phase-incoherent pairing can open a partial gap without superconductivity was originally explored in the context of underdoped cuprates [32,33]. It was deemed an unlikely model for the pseudogap because of the unreasonably large temperature window (~ 100 K) over which a vortex-antivortex plasma must persist [32]. Nonetheless, such models capture coarse features of ARPES and scanning tunneling microscopy (STM) data near T_c [34–42]. At low T , Nunner *et al.* [43] and Fang *et al.* [44] showed that pairing inhomogeneity is responsible for two key properties of spectra in underdoped Bi-2212 [45]—that they are largely particle-hole symmetric and homogeneous at low energies, and that there is an inverse correlation between coherence peak energy and peak height—that cannot be explained by conventional impurity models. The main feature of the current work is the focus on how large pairing and phase fluctuations modify local spectra in a strongly inhomogeneous superconductor near the transition.

II. RESULTS: MEAN-FIELD THEORY

We first solved the Bogoliubov-de Gennes (BdG) equations on a 40×40 lattice for the tight-binding Hamiltonian

$$H = \sum_{ij\sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + \sum_{ij} (\Delta_{ij}^* c_{j\downarrow} c_{i\uparrow} + \Delta_{ij} c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger), \quad (1)$$

with $t_{ij} = t$ ($t_{ij} = -0.35t$) for nearest-neighbor (next-nearest-neighbor) lattice sites i and j , and with $\Delta_{ij} = V_{ij} \langle c_{j\downarrow} c_{i\uparrow} \rangle$ determined self-consistently along each bond. V_{ij} is nonzero for nearest-neighbor sites only (see Appendix), and Δ_{ij} is constrained to be real so that there are no spontaneous supercurrents. We set $t = -1$ so that $|t|$ is the unit of energy, and the chemical potential is adjusted to maintain an electron density $n = 0.85$ per unit cell. Note that the value of n does not matter for this work so long as the Fermi surface is qualitatively reasonable. As in Ref. [43], the model assumes that the pairing interaction V_{ij} has a weak homogeneous component V_0 that is

modulated locally near randomly distributed impurities with a Yukawa spatial profile, $V_i e^{-r/\lambda}/r$, with r the distance to the impurity and λ the range of the modulation (see Appendix A 1 for details). Importantly, λ is comparable to the typical superconducting coherence length ξ at $T = 0$, so that there are large nanoscale spatial fluctuations of the order parameter.

Figure 1 compares the d -wave component of the order parameter, spectral gap, and bin-averaged local density of states (LDOS) for temperatures ranging from near zero to near the mean-field transition temperature, T_{mf} . The order parameter at site i is $\Delta_{\text{OP},i} = |\sum_{\delta} (-1)^{\delta_y} \Delta_{i,i+\delta}|$, and the spectral gap $\Delta_{\text{SG},i}$ is the energy of the largest positive-energy peak in the LDOS at site i . The homogeneous interaction ($V_0 = 0.8$) by itself gives $\Delta_{\text{OP}} = 0.19$ at $T = 0.001$ and has a critical temperature $T_c = 0.085$.

Approximately 40% of the lattice sites belong to the first or second bin, see legend in Fig. 1(o). At low T , these “small-gap” sites are characterized by a canonical d -wave density of states with well-defined coherence peaks. Spectra associated with “large-gap” sites, belonging to the third or higher bin, deviate from a simple d -wave form: They have a two-gap structure, with low-energy shoulders and high-energy coherence peaks. The shoulders are often interpreted as a subgap, and we adopt that terminology. The shoulders occur at the same energies for all bins, and coincide with the coherence peak energy of the second bin. We ascribe the shoulders to proximity coupling to small-gap sites; that is, they are the extension of neighboring coherence peaks onto large-gap sites [44]. Where they are distinct, the spectral gap (which coincides with the large gap) reflects the local order parameter, and the subgap reflects the typical order parameter.

The correspondence between Δ_{SG} and Δ_{OP} is not perfect [46]: Spatial variations of Δ_{SG} are a factor ~ 2 less than Δ_{OP} . Nonetheless, Δ_{SG} appears granular at high T , with robust superconducting islands embedded in a weakly superconducting, nearly uniform background. The granularity is not reflected in the subgap, which is spatially uniform. Indeed, it is striking that an order parameter with spatial variations exceeding an order of magnitude can generate an essentially homogeneous low-energy LDOS. Equally remarkable, the subgap persists to temperatures well above the subgap energy, where superconductivity would be suppressed in a homogeneous BCS superconductor. Its persistence at these elevated temperatures is the result of proximity coupling to nearby regions with large pairing interactions.

Because the typical coherence length is larger than the interimpurity spacing and grows near T_c , this proximity coupling produces a spectrum that is surprisingly homogeneous and d -wave-like near T_{mf} . This is evident in the binned LDOS shown in Fig. 1, and is summarized in Fig. 2, which shows the T -dependence of Δ_{OP} and Δ_{SG} , both sorted into ten equally sized bins according to the local values of Δ_{SG} . Both are inhomogeneous at low T ; however, by $T = 0.2 \approx T_{\text{mf}}$, Δ_{SG} is nearly the same for all bins, apart from the largest-gap bin, while Δ_{OP} varies by an order of magnitude or more between small- and large-gap bins. The homogeneity of Δ_{SG} near T_{mf} is incompatible with temperature-dependent STS experiments, primarily on Bi-2212 [9,10], which show that the spectral gap is strongly inhomogeneous near T_c , and that it fills rather than closes.

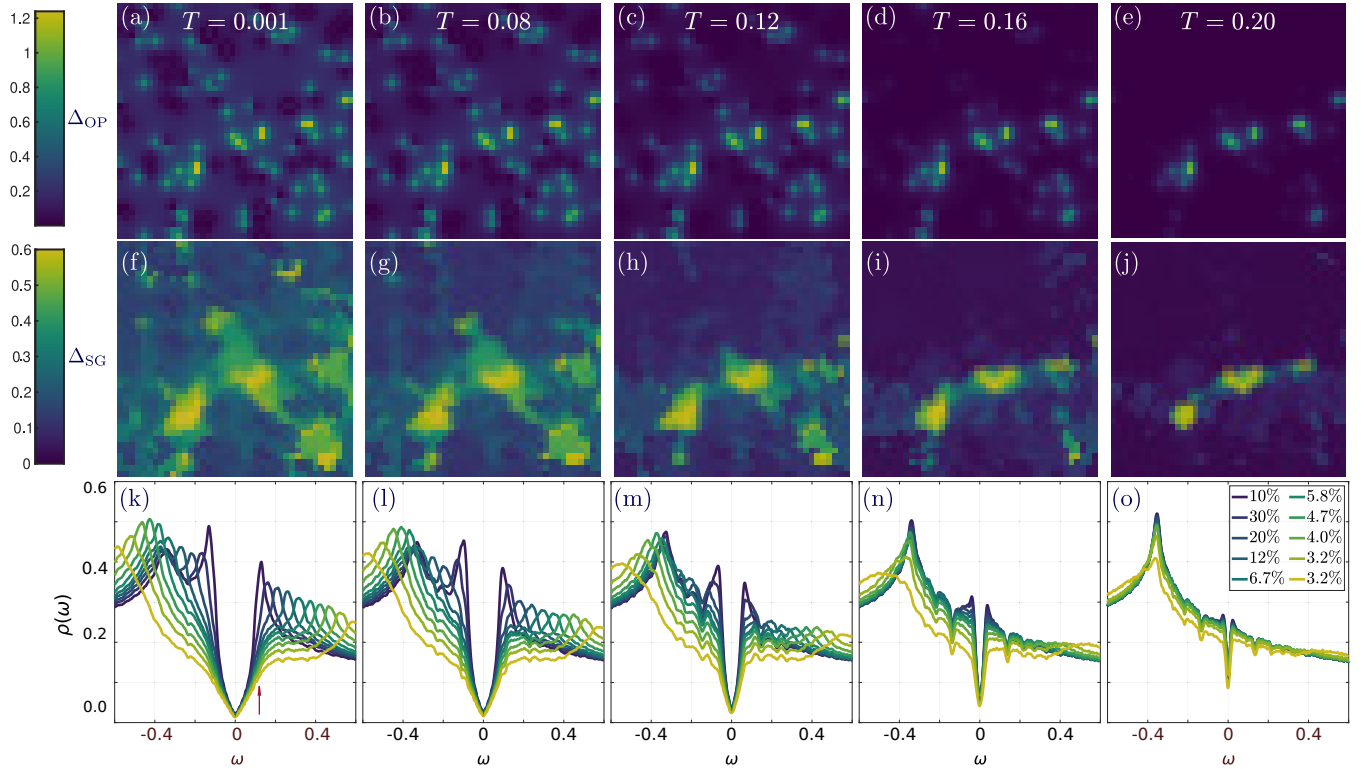


FIG. 1. Spatial maps of the d -wave order parameter Δ_{OP} (top row) and spectral gap Δ_{SG} (middle row) for a single realization of the pairing interaction. The LDOS (bottom row) is calculated for an ensemble of 50 random pairing realizations and for 15×15 supercells. Sites are binned by the value of Δ_{SG} at $T = 0.001$, and the average LDOS is shown for each bin (bottom row). The same sites are in each bin at all temperatures. Ten equal-width bins spanning $0.085 < \Delta_{SG} < 0.605$ are used; the legend shows the fraction of sites in each bin and darker colors correspond to lower-energy bins. Results are shown at five temperatures ranging from $T \ll T_{mf}$ to $T \approx T_{mf}$, for an impurity concentration of 7.5%. The red arrow in panel (k) indicates the subgap shoulder.

III. RESULTS: FLUCTUATIONS

To address these discrepancies, we consider order-parameter fluctuations in tandem with inhomogeneity. We first solved a set of time-dependent Ginzburg-Landau (TDGL) equations with thermal noise to obtain an ensemble of order parameter “snapshots.” These calculations naturally allow both amplitude and complex phase fluctuations. Inhomogeneity is built into the model via the lowest-order Ginzburg-Landau (GL) parameter, which has a similar

Yukawa spatial dependence as the BdG pairing interaction (see Appendix A 1). Once order-parameter ensembles were generated, we calculated the LDOS at selected sites using a microscopic non-self-consistent BdG Hamiltonian with the same tight-binding parameters as above and taking the TDGL snapshots as input. TDGL calculations were for a single disorder realization on a 300×300 lattice.

This approach follows a broad body of work in which an ansatz is made to generate an ensemble of order parameters that capture the essential ingredients of pair fluctuations across the phase transition. Most of the published work (see, e.g., Refs. [32–42]) takes their ensembles from classical Monte Carlo simulations of a GL free energy or two dimensional XY model. The TDGL calculations are closely related to classical Monte Carlo approaches in that they both generate pair fluctuations that are consistent with the equipartition theorem and they both generate vortex-antivortex excitations near T_c . The explicit time in the TDGL calculations is important for conductivity calculations since it determines the frequency-dependent response to an applied EM field, but is not relevant for the density of states since tunneling is an essentially instantaneous process. For our purposes, there is no difference between the real TDGL time and fictitious “Monte Carlo time.” The two approaches contain the same physics, and we chose the TDGL approach for computational convenience.

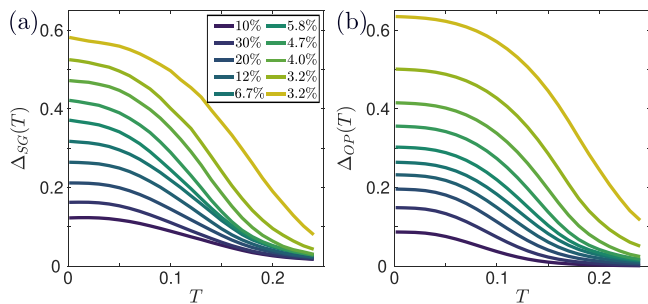


FIG. 2. The binned spectral gap (a) and order parameter (b) as functions of temperature. At the lowest measured temperature, each site is sorted its local value of Δ_{SG} and put into one of ten bins; the values of Δ_{SG} and Δ_{OP} are then averaged over each bin at each temperature.

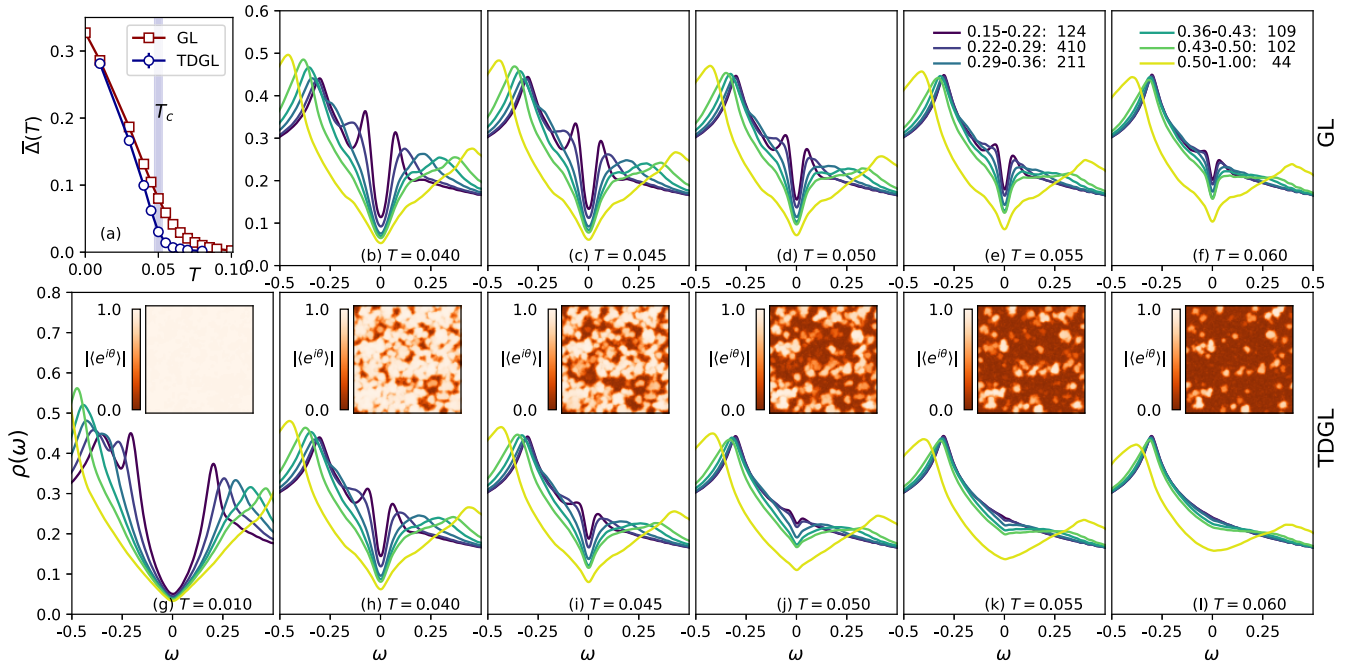


FIG. 3. Effect of thermal fluctuations on the density of states. Temperature dependence of (a) the root-mean-square correlation function, (b)–(f) the GL local density of states, and (g)–(l) the TDGL local density of states. Insets show the time-averaged phase. In panel (a), T_c is determined from the scaling of the spatial correlation function (see Appendix C 2). The legend shows the spectral-gap ranges for each bin at $T = 0.010$ and the number of sites within each bin.

Superconducting fluctuations contain a longitudinal component that does not satisfy the time-independent equation $\nabla \cdot \mathbf{J} = 0$, but should satisfy the full continuity equation $\partial_t \rho + \nabla \cdot \mathbf{J} = 0$. In all of the phenomenological approaches mentioned above, the microscopic BdG Hamiltonian violates the continuity equation for any given order-parameter snapshot, but satisfies it in the ensemble average. It is widely believed that this phenomenological approach accurately captures the essential qualitative physics of the superconducting spectrum. Indeed, the physics is intuitive: Currents generated by phase fluctuations shift the quasiparticle excitation spectrum up or down in energy (so-called Doppler shifts), which broadens features in the density of states and produces gap filling. (See Ref. [32] for a discussion of this physics.) To characterize the effects of fluctuations, we show in Fig. 3(a) the equal-time pair correlation function,

$$\overline{\Delta}(T) = \frac{1}{N} \left[\sum_{i,j} \langle \Delta_{OP,i}(t) \Delta_{OP,j}(t)^* \rangle_t \right]^{1/2}, \quad (2)$$

where $\langle \cdots \rangle_t$ indicates a time average. For the (static) GL calculations, $\overline{\Delta}(T)$ is the spatially averaged order parameter. In Fig. 3(a), an extrapolation of the low- T slope would suggest a GL transition near $T = 0.06t$; however, $\overline{\Delta}(T)$ develops upwards curvature near this temperature (a result of the pairing inhomogeneity; similar behavior is seen in BdG calculations [23]), and only vanishes at $T_{GL} \approx 0.14t$. For the TDGL calculations, $\overline{\Delta}(T)$ is expected to scale as $L^{-\eta(T)}$ [with $\eta(T)$ a T -dependent power and L the linear size of the system] below T_c and vanish exponentially above; this provides a simple way to estimate the superconducting transition in finite systems.

In Fig. 3(a), we estimate that fluctuations push the transition temperature down by a factor of nearly 3 to $T_c \approx 0.05t$. This value is supported by analysis of the spatial phase correlations (see Fig. 4).

The remaining panels in Fig. 3 show the binned LDOS taken from 1000 randomly selected sites, following the same binning scheme as before. At low T , the GL spectra are indistinguishable from the TDGL spectra in Fig. 3(g); $\sim 50\%$ of the sites belong to the first or second bin and have a typical d -wave spectrum; these are characterized as “small-gap.” The spectra of the remaining “large-gap” sites have a two-gap structure, with a spatially uniform subgap and an inhomogeneous spectral gap. The subgap and spectral gap are less distinct here than in the BdG calculations because the GL parameters correspond to a less inhomogeneous system. As T is increased [Figs. 3(b)–3(f)], the subgap closes, approximately following the temperature dependence of $\overline{\Delta}(T)$.

Spectra from the TDGL calculations are shown in Figs. 3(g)–3(l). The magnitude of the time-averaged phase, $|\langle e^{i\theta_i(t)} \rangle_t|$, is shown in the insets: It is one (zero) for sites whose correlation time is much longer (shorter) than the simulation time. Thermal fluctuations are increasingly prominent as T increases, and the phase correlations vanish in progressively larger regions of space with increasing T . Slightly below T_c , much of the system is phase coherent but with patches of phase-incoherent pairing, while slightly above T_c , the pairing is incoherent everywhere except for small islands of phase-coherent superconductivity. The effect of these fluctuations is to cause a T -dependent filling of the subgap; interestingly, while the binned spectra for the large-gap sites are smoothed by fluctuations, the size of the spectral gap changes very little with increasing temperature. At the highest temperature

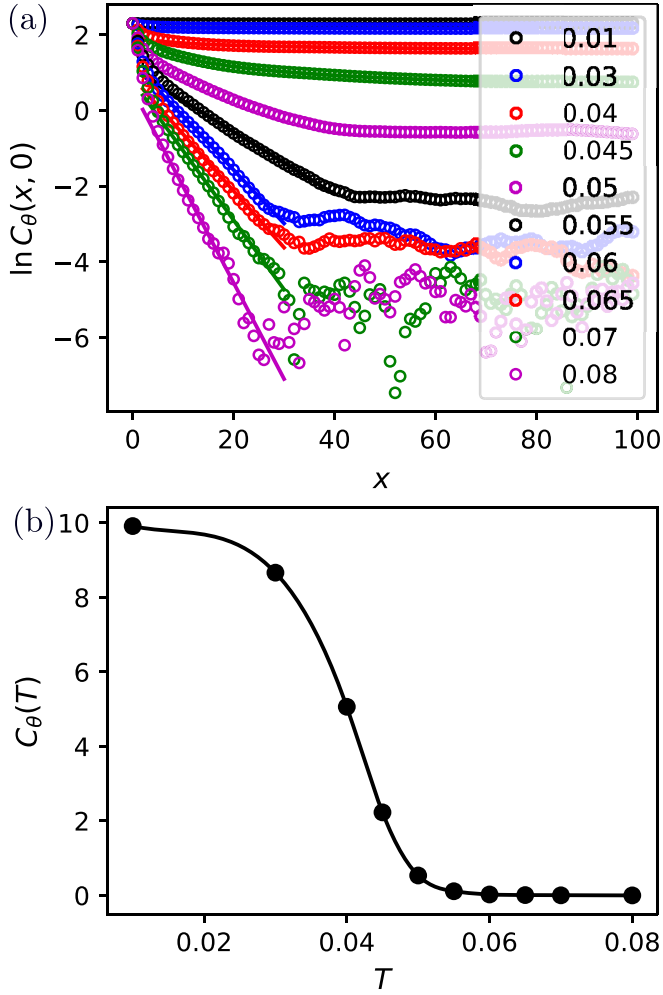


FIG. 4. (a) Pair-phase correlation function $C_\theta(\mathbf{r})$ along $\mathbf{r} = (x, 0)$ at different temperatures. Symbols show numerical results for Eq. (B12) taken from TDGL data for cuts along the x axis, while solid curves are fits to the asymptotic forms in Eq. (B13). Fits are to the power law form for $T \leq 0.045$ and the exponential form for $T \geq 0.055$. The data at $T = 0.050$ cannot be fitted by either form. (b) Temperature dependence of the spatially averaged phase correlation function. The average is performed over sites separated by $50 < |\mathbf{r}| < 150$ lattice constants.

shown [Fig. 3(i)], there is no trace of superconductivity on $\sim 75\%$ of the sites in the system. Conversely, the sites on which spectra have large gaps, which comprise $\sim 25\%$ of our system, have only partially filled gaps: The proximity-induced subgaps fill quickly with increasing temperature while the high-energy spectral gap persists as a pseudogap [Figs. 3(j)–3(l)].

IV. DISCUSSION

As mentioned above, we attribute the subgap to proximity coupling between the large-gap region where the spectrum is measured and neighboring regions with smaller gaps (see also Ref. [47]). Two-gap structures like those obtained here have been observed repeatedly at low temperatures in Pb- and La-doped Bi-2201 [3–6] and Bi-2212 [7,8,48]. In those

experiments, the subgap was interpreted as a superconducting gap, and the larger gap was attributed to a nonsuperconducting pseudogap. However, the STM data show that there are small-gap regions whose coherence peaks coincide with the shoulders observed in large-gap regions [6], which is a key requirement for proximity-induced subgaps. Our calculations predict that these subgaps will fill in when thermal fluctuations wipe out the superconductivity in small-gap regions; that is, the same proximity physics that induces subgaps at low T also wipes them out at high T . Conversely, the high-energy coherence peaks, which have been taken for “pseudogaps,” should remain above T_c , reflecting a robust and phase-coherent local superconducting patch. Unfortunately, detailed measurements of the T -dependent STM spectra binned according to gap size have not been reported for the overdoped materials.

We remark that two-gap structures are also ubiquitous in underdoped Bi-2212 [8,49] and Bi-2201 [4,11]. Superficially, the measured STS spectra look very much like the results presented here. However, these are complicated by the presence of a pseudogap, charge order, and possibly pair density waves [50]. In particular, it is likely that short-range nematic [51], charge- [52], or pair-density [53,54] order contribute to the LDOS. However, the mechanism is not understood and simple models of intertwined order generally do not reproduce the measured two-gap spectrum [51] (see, however, Refs. [53,54]); we suggest this points to the importance of pairing inhomogeneity in underdoped cuprates. Nonsuperconducting order disappears rapidly with overdoping Bi-2201 [5] and Bi-2212 [55,56], where our model applies without modification.

We have compared in detail with Bi-2212 and Bi-2201, since high-quality STM and ARPES data are readily available for these systems, and because the distribution of local pairing interactions needed to reproduce gap maps was known from previous theoretical work [43,57]. However, we expect our conclusions continue to hold qualitatively for other disordered cuprates, such as $\text{HgBa}_2\text{CuO}_{4+\delta}$, LSCO, and $\text{Tl}_2\text{Ba}_2\text{CuO}_{6+\delta}$, since the main spectral features identified here are robust against changes in model parameters (see Fig. 5). On the other hand, stoichiometric $\text{YBa}_2\text{Cu}_3\text{O}_7$, which is slightly overdoped, is both homogeneous and more three-dimensional, and is not described by our model.

One of the remarkable implications of this work is that the pairing interaction is both local and varies by an order of magnitude over nm length scales. This can occur, for example, if the pairing bosons are spin fluctuations and the system is near a spin-density wave transition; in that case, inhomogeneity can create large local differences in the dynamical spin susceptibility [58].

It was recently shown that a spatially random interaction can produce a “strange-metal” linear-in- T resistivity [59–61] like that measured in optimally and overdoped cuprates. Our work suggests that this same strange-metal physics is responsible for anomalous features in the superconducting spectrum.

To the extent that our model describes them, overdoped cuprates should exhibit aspects of both BKT and granular superconductivity. The superconducting transition in our model is essentially BKT-like, but with an inhomogeneous distribution of local phase-disordering temperatures [29] and gaps [62] that broadens the transition. Our use of the Nunner model [43] for pair inhomogeneity leads

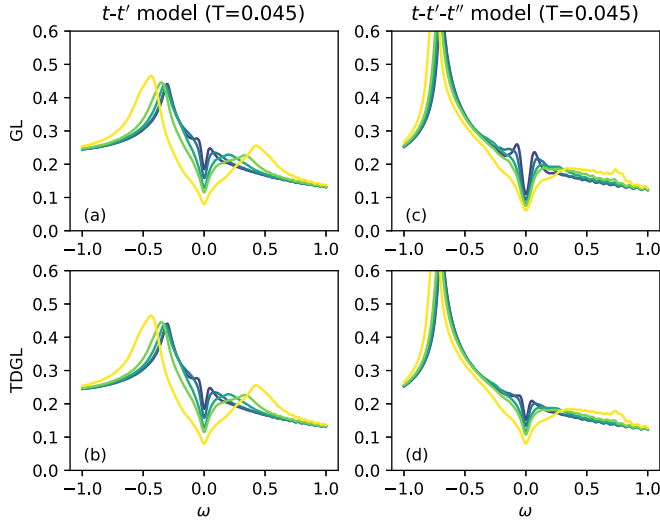


FIG. 5. Density of states at $T = 0.045$ for (a), (b) the t - t' model shown in the main text, and (c), (d) a t - t' - t'' model. Results are shown for (a), (c) static GL calculations and (b), (d) thermally fluctuating TDGL calculations. The calculations use the same inhomogeneous GL parameters, $T_{i\alpha}^*$, as in the main text but different band structures for the calculated LDOS. The LDOS is binned such that the same sites are allocated to each bin for the two models.

to a dilute concentration of nanoscale regions with high mean-field transition temperatures. (The importance of pair-breaking disorder for the transition in overdoped cuprates is debated [1,2,23,25,62,63].) In layered materials, the BKT transition is further broadened by interlayer coupling [31].

Thus, while superconductivity disappears via a smeared BKT transition, a weak diamagnetic signal will persist well above T_c due to these nanoscale regions. This provides a natural resolution to an apparent dichotomy between magnetoresistance [17] and magnetic susceptibility [18,20] measurements. The magnetoresistance measurements, which probe the entire sample, will depend on diamagnetic fluctuations in *typical* superconducting regions, while the onset temperature for a diamagnetic response in susceptibility measurements will depend on the GL temperature of the *large-gap* regions. We therefore believe that the anomalous spectral properties of the overdoped phase, as measured on inhomogeneous families of cuprates, can be explained by our analysis.

ACKNOWLEDGMENTS

M.A.S. is grateful to the late J. Zaanen for numerous inspiring discussions and collaborations on related projects, and to UFIT Research Computing for providing computational resources and support that have contributed to the research results reported in this publication. W.A.A. acknowledges the support of the Natural Sciences and Engineering Research Council of Canada (NSERC). This work was made possible by the facilities of the Shared Hierarchical Academic Research Computing Network [64] and the Digital Research Alliance of Canada. P.J.H. was supported by NSF-DMR-2231821. A.K. acknowledges support by the Danish National Committee for Research Infrastructure (NUFI) through the ESS-Lighthouse Q-MAT.

DATA AVAILABILITY

The data supporting this study's findings are available within the article.

APPENDIX A: BOGOLIUBOV-DE GENNES CALCULATIONS

1. Pairing inhomogeneity

The pairing interaction is assumed to be inhomogeneous because of local modulations due to out-of-plane dopant atoms [43]. The pairing interaction V_{ij} along each nearest-neighbor bond is obtained from the ansatz $V_{ij} = \frac{1}{2}(V_i + V_j)$, where

$$V_i = V_0 + \sum_{j=1}^{N_i} \pm V_I \frac{e^{-\frac{\sqrt{|\mathbf{r}_i - \mathbf{r}_j|^2 + r_z^2}}{\lambda}}}{\sqrt{|\mathbf{r}_i - \mathbf{r}_j|^2 + r_z^2}} \quad (\text{A1})$$

is defined for each lattice site, N_i is the number of impurities, the sign ($\pm$) is selected randomly for each impurity, $V_0 = 0.8$, $V_I = 3.125$, $r_z = 0.7$, and $\lambda = 0.7$. For these calculations, we take $N_i = 0.075L^2$. The created impurity realizations for these parameters are available in the Supplemental Material [66] and are used for impurity averaging.

2. Density of states

The LDOS at site i is obtained from eigenvalues E_n and eigenvectors U_{in} of the BdG Hamiltonian

$$\rho_i(E) = \frac{1}{L^2 N_k} \sum_n \sum_{\mathbf{k}} |U_{i,n}(\mathbf{k})|^2 \delta(\omega - E_{n\mathbf{k}}), \quad (\text{A2})$$

where we sum over $N_k = 15 \times 15$ supercells to minimize finite-size effects [23]. In Eq. (A2), the δ function is a Lorentzian of width $\eta = 0.005$.

APPENDIX B: TIME-DEPENDENT LANDAU-GINZBURG CALCULATIONS

1. Order-parameter simulations

We define singlet bond order parameters $\eta_{i\alpha}$, where i is a site label and $\alpha = x, y$ indicates bonds connected to i in the positive x or y directions. These are related to the usual nearest-neighbour bond order parameters by

$$\Delta_{i+\alpha,i} = \Delta_{i,i+\alpha} = V \langle c_{i+\alpha\downarrow} c_{i\uparrow} \rangle = \eta_{i\alpha}, \quad (\text{B1})$$

where $i + \alpha$ indicates the nearest-neighbour site to i along the direction α and V is a mean-field pairing interaction. Rather than solving the mean-field equations directly, which is computationally prohibitive for large system sizes, we obtain the superconducting order parameter from solutions of the TDGL equations with thermal noise. That is, we write

$$\frac{\partial \eta_{i\alpha}}{\partial t} = -\Gamma \frac{\delta \mathcal{F}}{\delta \eta_{i\alpha}^*} + \sqrt{\frac{k_B T \Gamma}{dt}} \zeta_i, \quad (\text{B2})$$

where $\mathcal{F}$ is the GL free energy, Γ is a relaxation rate, and ζ_i is a site-dependent Gaussian-distributed complex random variable, taken from the distribution

$$P(\zeta_i) = \frac{1}{2\pi} e^{-|\zeta_i|^2/2}, \quad (\text{B3})$$

and satisfying $\langle \zeta_i(t)^* \zeta_j(t') \rangle = 2\delta_{i,j}\delta_{t,t'}$. Here, t and t' are discrete time variables with timestep dt , and the time derivatives are understood as finite differences. The specific value of Γ does not matter for the current calculations so long as the fractional changes in $\eta_{i\alpha}$ at each time step are small. In the

absence of thermal noise ($\zeta_i = 0$), and at large times, Eq. (B2) asymptotically approaches the usual GL equation,

$$\frac{\delta \mathcal{F}}{\delta \eta_{i\alpha}^*} = 0. \quad (\text{B4})$$

The free energy is

$$\begin{aligned} \mathcal{F} = a_0 \sum_i \sum_{\alpha=x,y} & \left[\left(\frac{T - T_{i\alpha}^*}{T_0} \right) |\eta_{i\alpha}|^2 + \frac{1}{2\Delta_0^2} |\eta_{i\alpha}|^4 + \frac{\xi_d^2}{2} |\nabla \eta_{i\alpha}|^2 \right] \\ & + \frac{d}{4} \sum_i [4(|\eta_{ix}|^2 + |\eta_{iy}|^2) + (\eta_{ix} + \eta_{i-x,x})^* (\eta_{iy} + \eta_{i-y,y}) + (\eta_{ix} + \eta_{i-x,x})(\eta_{iy} + \eta_{i-y,y})^*]. \end{aligned} \quad (\text{B5})$$

In this expression, a_0 , d , Δ_0 , and ξ_d are GL parameters, T_0 is a characteristic temperature scale for pairing, and $T_{i\alpha}^*$ is the local GL temperature below which the singlet pairing interaction along the bond $(i, i + \alpha)$ becomes attractive. Note, however, that $T_{i\alpha}^*$ is not the local critical temperature, even at the mean-field level, because the order parameter on the bond $(i, i + \alpha)$ depends on the pairing interaction in the neighborhood of the bond. The parameter d determines whether the dominant pairing channel has d -wave ($d > 0$) or extended s -wave ($d < 0$) symmetry. In the former case, the lowest-energy GL solution for a homogeneous pairing interaction ($T_{i\alpha}^* = T_0$) has d -wave symmetry with $\eta_x = -\eta_y = \Delta_0$ at $T = 0$. An extended s -wave solution is also possible, with $\eta_x = \eta_y = \Delta_0 \sqrt{(a_0 - 2d)/a_0}$; however, this has higher energy. Values for the GL parameters are shown in Table I. Both the temperature scale T_0 and the pairing scale Δ_0 are somewhat large for a quantitative description of overdoped cuprates; however, these choices help mitigate finite-size effects in our simulations.

The first step in the TDGL simulations is to find the solutions to the GL equations, Eq. (B4), by iterating Eq. (B2) to self-consistency with the thermal noise term set to zero. The thermal noise is then turned on, and a typical TDGL simulation runs for 10^5 time steps. Thermal equilibrium is typically achieved within a few hundred time steps. The density of states can be obtained for both the GL and TDGL simulations; in the latter case, the density of states is averaged over 100 snapshots taken from the simulation.

TABLE I. TDGL parameters. All parameters are written in units of the nearest-neighbour hopping t , which in cuprates is ~ 150 meV. These parameters correspond to a filling $n = 0.85$ and a low- T correlation length ξ of approximately 3 unit cells.

Parameter	Value
a_0	3
T_0	0.01
Δ_0	0.04
d	3
Γdt	10^{-3}

2. Density of states

To obtain the local density of states for a given order-parameter snapshot, we write a mean-field Hamiltonian,

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} & \cdots \\ \mathbf{H}_{21} & \mathbf{H}_{22} & \\ \vdots & & \ddots & \\ & & & \mathbf{H}_{NN} \end{bmatrix}, \quad (\text{B6})$$

where $\mathbf{H}_{ij}$ is a 2×2 block corresponding to a particular pair of sites i and j , with

$$\mathbf{H}_{ij} = \begin{bmatrix} t_{ij} & \Delta_{ij} \\ \Delta_{ji}^* & -t_{ji} \end{bmatrix}. \quad (\text{B7})$$

In Eq. (B7), the hopping matrix elements t_{ij} are the same as for the BdG calculations. The superconducting order parameters Δ_{ij} are only nonzero for nearest-neighbor sites and are taken either from solutions of the GL equations or from snapshots of the TDGL equations, using Eq. (B1). To mitigate finite-size effects, results in this work are shown for a 300×300 lattice. This precludes standard matrix diagonalization approaches to obtain the density of states. Rather, we use Haydock's recursion algorithm to find the density of states at selected individual sites [65]. For a single configuration Δ_{ij} , the recursion algorithm evaluates the local density of states at site i ,

$$\rho_i(\omega) = -\frac{1}{\pi} \text{Im} [\omega + i\delta - \mathbf{H}]_{ii}^{-1} \quad (\text{B8})$$

where $[\cdots]^{-1}$ denotes a matrix inverse and $\delta = 0.02$ is a Lorentzian broadening parameter.

Inhomogeneity is achieved through spatial variations of the coefficient $T_{i\alpha}^*$ of the quadratic term in the GL energy, Eq. (B5), with

$$T_{i\alpha}^* = (T_i^* + T_{i+\alpha}^*)/2, \quad (\text{B9})$$

$$T_i^* = T_0 + \delta T^* \sum_s f_{is}, \quad (\text{B10})$$

where the sum is over impurities, $f_{is} = \exp(-r_{is}/\kappa)/r_{is}$ for impurity s , r_{is} is the distance between the impurity and site i , κ is a characteristic length scale over which pairing is enhanced, and δT^* sets the scale for the local enhancement. We set $\delta T^* = 10T_0$ and $\kappa = 2.8$ lattice constants, take the dopant

atoms to sit a distance 0.8 lattice constants above the plane, and set the dopant concentration to be 2.5%. The resulting distribution of $T_{i\alpha}^*$ used in the simulations is available in the Supplemental Material [66].

3. BKT scaling

In homogeneous systems, the BKT transition is characterized by a power-law pair correlation function below T_{BKT} and an exponential correlation function above T_{BKT} . We define the phase of the d -wave component of the order parameter via

$$e^{i\theta_j(t)} = \frac{\Delta_{d,j}(t)}{|\Delta_{d,j}(t)|}. \quad (\text{B11})$$

The phase correlation function is then

$$C_\theta(\mathbf{r}) = \frac{1}{N} \sum_j \langle e^{i[\theta_j(t) - \theta_{j+\mathbf{r}}(t)]} \rangle, \quad (\text{B12})$$

where $\langle \dots \rangle$ denotes a time average. Figure 4 shows $C_\theta(\mathbf{r})$ for $\mathbf{r} = (x, 0)$ (i.e., for a cut along the x axis). The correlation function is fitted to expressions for the homogeneous system, namely,

$$C_\theta(\mathbf{r}) = \begin{cases} a[x^{-\eta(T)} + (L-x)^{-\eta(T)}], & T < T_{\text{BKT}}, \\ a'e^{-x/\xi(T)}, & T > T_{\text{BKT}}, \end{cases} \quad (\text{B13})$$

with a , η , a' , and ξ taken as fitting parameters. These fits are shown as solid lines in Fig. 4(a). Convincing fits are obtained for the power law when $T \leq 0.045$ and for the exponential when $T \geq 0.055$; however, the data at $T = 0.050$ are not well reproduced by either asymptotic form, indicating that the superconducting transition temperature is $T_c \sim 0.050$.

As further confirmation, we plot the temperature dependence of $C_\theta(\mathbf{r})$, averaged over separations $50 < |\mathbf{r}| < 150$ in Fig. 4(b). This figure shows that phase stiffness is lost near $T = 0.055$.

APPENDIX C: SENSITIVITY TO PARAMETERS

1. Band-structure parameters

Figure 5 compares the binned LDOS for the band structure used in the main text (the t - t' model) and another common tight-binding model for the cuprates (the t - t' - t'' model). For the latter model, we take $t = -1$, $t' = 0.4$, and $t'' = -0.05$. The main difference between the two models is that the van Hove singularity lies further from the Fermi level in the t - t' - t'' model, although their electron densities are the same ($n = 0.85$ electrons per unit cell). Since the primary effect of doping in our model is to move the van Hove singularity toward or away from the Fermi level, these two models also provide insight into the effects of doping.

Although there are quantitative differences between the two GL calculations, Figs. 5(a) and 5(c), the important qualitative structures are the same: both exhibit a two-gap structure with a homogeneous subgap and an inhomogeneous spectral gap. Similarly, the TDGL calculations show identical trends, namely, that the subgap fills relative to the GL gap but that the high-energy spectral gap is largely unaffected by thermal fluctuations.

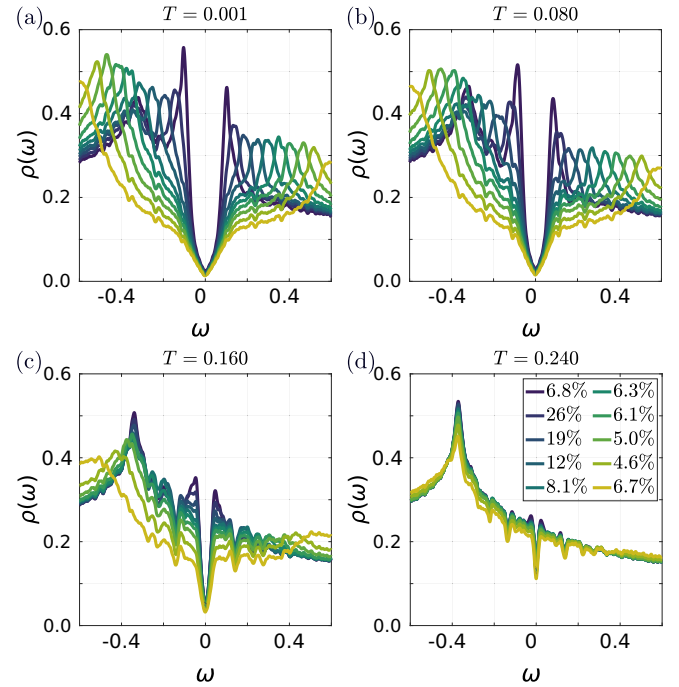


FIG. 6. Bin-averaged LDOS (bottom row) with inhomogeneity parameters $V_0 = 0.8$, $V_I = 3.125$, $r_z = 1.4$, and $\lambda = 1.4$. 50 random pairing realizations and 15×15 supercells are used. As with the inhomogeneity model shown in the main text, the sites are binned by the value of Δ_{SG} at $T = 0.001$, and the LDOS within each bin is averaged over. The same sites are in each bin at all temperatures. Ten equal-width bins spanning $0.054 < \Delta_{SG} < 0.605$ are used; the legend shows the fraction of sites in each bin and darker colors correspond to lower-energy bins.

2. Pairing interaction length scale

Figure 6 shows the binned LDOS for a modified version of the inhomogeneity model used in the main text with a larger characteristic length scale. Here, we set $0.075L^2$, $V_0 = 0.8$, $V_I = 3.125$, $r_z = 1.4$, and $\lambda = 1.4$ in Eq. (A1). Notice that only r_z and λ are changed from the values used in the main text (both are doubled in this case); all other parameters are kept the same. It can be seen that much of the same

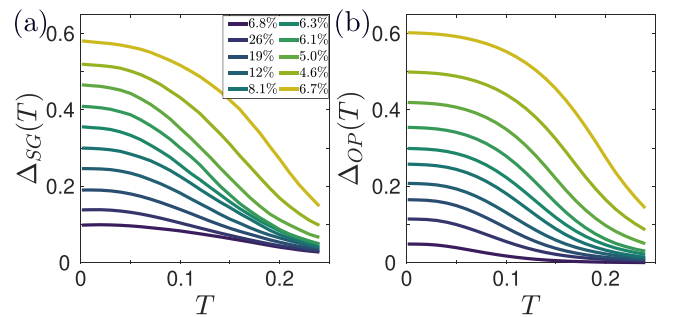


FIG. 7. The binned spectral gap (a) and order parameter (b) as functions of temperature for the inhomogeneity model with $N_i = 0.075L^2$, $V_0 = 0.8$, $V_I = 3.125$, $r_z = 1.4$, and $\lambda = 1.4$. At the lowest measured temperature, each site is sorted according to its local value of Δ_{SG} and put into one of ten bins; the values of Δ_{SG} and Δ_{OP} are then averaged over each bin at each temperature.

striking behavior seen in the main text (i.e., the two-gap structure consisting of low-energy homogeneity and shoulders and an inhomogeneous spectral gap, and the overall anticorrelation between the spectral gap and the coherence peak) remains present even when the inhomogeneity length scale is doubled.

We show in Fig. 7 the T dependence of Δ_{SG} and Δ_{OP} , both of which have been binned using the same procedure as in the main text. Much of what is observed in the main text is also seen here: At high temperatures, Δ_{SG} is homogeneous, whereas Δ_{OP} continues to show large variations.

Some differences between this model and the one used in the main text are as follows: (1) the spectral-gap-coherence-peak height anticorrelation is less strong in the long-ranged inhomogeneity model compared to the short-ranged one and (2) the gap persists to a larger temperature in the long-ranged model compared to the short-ranged one. However, the important qualitative features we have highlighted in the main text remain robust to changes in the inhomogeneity length scale. We defer an in-depth discussion of the length-scale dependence of the spectra and the superfluid density to a future manuscript.

-
- [1] I. Božović, X. He, J. Wu, and A. Bollinger, Dependence of the critical temperature in overdoped copper oxides on superfluid density, *Nature (London)* **536**, 309 (2016).
 - [2] F. Mahmood, X. He, I. Božović, and N. Armitage, Locating the missing superconducting electrons in the overdoped cuprates $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$, *Phys. Rev. Lett.* **122**, 027003 (2019).
 - [3] M. C. Boyer, W. D. Wise, K. Chatterjee, M. Yi, T. Kondo, T. Takeuchi, H. Ikuta, and E. W. Hudson, Imaging the two gaps of the high-temperature superconductor $\text{Bi}_2\text{Sr}_2\text{CuO}_{6+x}$, *Nat. Phys.* **3**, 802 (2007).
 - [4] Y. He, Y. Yin, M. Zech, A. Soumyanarayanan, M. M. Yee, T. Williams, M. Boyer, K. Chatterjee, W. Wise, I. Zeljkovic *et al.*, Fermi surface and pseudogap evolution in a cuprate superconductor, *Science* **344**, 608 (2014).
 - [5] X. Li, C. Zou, Y. Ding, H. Yan, S. Ye, H. Li, Z. Hao, L. Zhao, X. Zhou, and Y. Wang, Evolution of charge and pair density modulations in overdoped $\text{Bi}_2\text{Sr}_2\text{CuO}_{6+\delta}$, *Phys. Rev. X* **11**, 011007 (2021).
 - [6] S. Ye, M. Xu, H. Yan, Z.-X. Li, C. Zou, X. Li, Z. Hao, C. Yin, Y. Chen, X. Zhou, D.-H. Lee, and Y. Wang, Emergent normal fluid in the superconducting ground state of overdoped cuprates, *Nat. Commun.* **15**, 4939 (2024).
 - [7] J. W. Alldredge, J. Lee, K. McElroy, M. Wang, K. Fujita, Y. Kohsaka, C. Taylor, H. Eisaki, S. Uchida, P. J. Hirschfeld, and J. C. Davis, Evolution of the electronic excitation spectrum with strongly diminishing hole density in superconducting $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$, *Nat. Phys.* **4**, 319 (2008).
 - [8] A. Pushp, C. V. Parker, A. N. Pasupathy, K. K. Gomes, S. Ono, J. Wen, Z. Xu, G. Gu, and A. Yazdani, Extending universal nodal excitations optimizes superconductivity in $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$, *Science* **324**, 1689 (2009).
 - [9] A. N. Pasupathy, A. Pushp, K. K. Gomes, C. V. Parker, J. Wen, Z. Xu, G. Gu, S. Ono, Y. Ando, and A. Yazdani, Electronic origin of the inhomogeneous pairing interaction in the high- T_c superconductor $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$, *Science* **320**, 196 (2008).
 - [10] C. V. Parker, A. Pushp, A. N. Pasupathy, K. K. Gomes, J. Wen, Z. Xu, S. Ono, G. Gu, and A. Yazdani, Nanoscale proximity effect in the high-temperature superconductor $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ using a scanning tunneling microscope, *Phys. Rev. Lett.* **104**, 117001 (2010).
 - [11] W. O. Tromp, T. Benschop, J.-F. Ge, I. Battisti, K. M. Bastiaans, D. Chatzopoulos, A. H. M. Vervloet, S. Smit, E. Van Heumen, M. S. Golden, Y. Huang, T. Kondo, T. Takeuchi, Y. Yin, J. E. Hoffman, M. A. Sulangi, J. Zaanen, and M. P. Allan, Puddle formation and persistent gaps across the non-mean-field breakdown of superconductivity in overdoped (Pb, Bi) $_2\text{Sr}_2\text{CuO}_{6+\delta}$, *Nat. Mater.* **22**, 703 (2023).
 - [12] K. K. Gomes, A. N. Pasupathy, A. Pushp, S. Ono, Y. Ando, and A. Yazdani, Visualizing pair formation on the atomic scale in the high- T_c superconductor $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$, *Nature (London)* **447**, 569 (2007).
 - [13] Y. Li, A. Sapkota, P. M. Lozano, Z. Du, H. Li, Z. Wu, A. K. Kundu, R. J. Koch, L. Wu, B. L. Winn *et al.*, Strongly overdoped $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$: Evidence for Josephson-coupled grains of strongly correlated superconductor, *Phys. Rev. B* **106**, 224515 (2022).
 - [14] M. Hashimoto, I. M. Vishik, R.-H. He, T. P. Devereaux, and Z.-X. Shen, Energy gaps in high-transition-temperature cuprate superconductors, *Nat. Phys.* **10**, 483 (2014).
 - [15] Y. He, S.-D. Chen, Z.-X. Li, D. Zhao, D. Song, Y. Yoshida, H. Eisaki, T. Wu, X.-H. Chen, D.-H. Lu *et al.*, Superconducting fluctuations in overdoped $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$, *Phys. Rev. X* **11**, 031068 (2021).
 - [16] S.-D. Chen, M. Hashimoto, Y. He, D. Song, J.-F. He, Y.-F. Li, S. Ishida, H. Eisaki, J. Zaanen, T. P. Devereaux, D.-H. Lee, D.-H. Lu, and Z.-X. Shen, Unconventional spectral signature of T_c in a pure d -wave superconductor, *Nature (London)* **601**, 562 (2022).
 - [17] P. M. Rourke, I. Mouzopoulou, X. Xu, C. Panagopoulos, Y. Wang, B. Vignolle, C. Proust, E. V. Kurganova, U. Zeitler, Y. Tanabe *et al.*, Phase-fluctuating superconductivity in overdoped $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$, *Nat. Phys.* **7**, 455 (2011).
 - [18] J. E. Sonier, M. Ilton, V. Pacradouni, C. V. Kaiser, S. A. Sabok-Sayr, Y. Ando, S. Komiya, W. N. Hardy, D. A. Bonn, R. Liang, and W. A. Atkinson, Inhomogeneous magnetic-field response of $\text{YBa}_2\text{Cu}_3\text{O}_y$ and $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$ persisting above the bulk superconducting transition temperature, *Phys. Rev. Lett.* **101**, 117001 (2008).
 - [19] Y. Koike, T. Adachi, Y. Tanabe, K. Omori, T. Noji, and H. Sato, Inhomogeneous superconductivity in both underdoped and overdoped regimes of high- T_c cuprates, *J. Phys.: Conf. Ser.* **108**, 012003 (2008).
 - [20] Z.-X. Li, S. A. Kivelson, and D.-H. Lee, Superconductor-to-metal transition in overdoped cuprates, *npj Quantum Mater.* **6**, 36 (2021).
 - [21] B. Spivak, P. Oreto, and S. Kivelson, d -wave to s -wave to normal metal transitions in disordered superconductors, *Phys. B: Condens. Matter* **404**, 462 (2009).
 - [22] C. N. Breið, P. J. Hirschfeld, and B. M. Andersen, Supercurrents and spontaneous time-reversal symmetry breaking by nonmagnetic disorder in unconventional superconductors, *Phys. Rev. B* **105**, 014504 (2022).

- [23] M. Pal, A. Kreisel, W. A. Atkinson, and P. J. Hirschfeld, Simulating superconducting properties of overdoped cuprates: The role of inhomogeneity, *Phys. Rev. B* **107**, 144501 (2023).
- [24] C. Yin, Q. Wang, Y. Xie, Y. Chen, J. Liu, J. Yang, J. Jia, X. Zhang, W. Lv, H. Yan *et al.*, Negligible normal fluid in superconducting state of heavily overdoped $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ detected by ultra-low temperature angle-resolved photoemission spectroscopy, *Chin. Phys. B* **33**, 077405 (2024).
- [25] N. Lee-Hone, J. Dodge, and D. Broun, Disorder and superfluid density in overdoped cuprate superconductors, *Phys. Rev. B* **96**, 024501 (2017).
- [26] N. R. Lee-Hone, V. Mishra, D. M. Broun, and P. J. Hirschfeld, Optical conductivity of overdoped cuprate superconductors: Application to $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$, *Phys. Rev. B* **98**, 054506 (2018).
- [27] N. R. Lee-Hone, H. U. Özdemir, V. Mishra, D. M. Broun, and P. J. Hirschfeld, Low energy phenomenology of the overdoped cuprates: Viability of the Landau-BCS paradigm, *Phys. Rev. Res.* **2**, 013228 (2020).
- [28] H. U. Özdemir, V. Mishra, N. R. Lee-Hone, X. Kong, T. Berlijn, D. M. Broun, and P. J. Hirschfeld, Effect of realistic out-of-plane dopant potentials on the superfluid density of overdoped cuprates, *Phys. Rev. B* **106**, 184510 (2022).
- [29] L. Benfatto, C. Castellani, and T. Giamarchi, Broadening of the Berezinskii-Kosterlitz-Thouless superconducting transition by inhomogeneity and finite-size effects, *Phys. Rev. B* **80**, 214506 (2009).
- [30] F. Mahmood, D. Ingram, X. He, J. A. Clayhold, I. Božović, and N. P. Armitage, Effect of radiation-induced defects on the superfluid density and optical conductivity of overdoped $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$, *Phys. Rev. B* **105**, 174501 (2022).
- [31] L. Benfatto, C. Castellani, and T. Giamarchi, Kosterlitz-Thouless behavior in layered superconductors: The role of the vortex core energy, *Phys. Rev. Lett.* **98**, 117008 (2007).
- [32] M. Franz and A. Millis, Phase fluctuations and spectral properties of underdoped cuprates, *Phys. Rev. B* **58**, 14572 (1998).
- [33] H.-J. Kwon and A. T. Dorsey, Effect of phase fluctuations on the single-particle properties of underdoped cuprates, *Phys. Rev. B* **59**, 6438 (1999).
- [34] T. Eckl, D. Scalapino, E. Arrigoni, and W. Hanke, Pair phase fluctuations and the pseudogap, *Phys. Rev. B* **66**, 140510 (2002).
- [35] M. Mayr, G. Alvarez, C. Şen, and E. Dagotto, Phase fluctuations in strongly coupled d -wave superconductors, *Phys. Rev. Lett.* **94**, 217001 (2005).
- [36] D. Valdez-Balderas and D. Stroud, Single-particle density of states of a superconductor with a spatially varying gap and phase fluctuations, *Phys. Rev. B* **74**, 174506 (2006).
- [37] A. V. Chubukov, M. R. Norman, A. J. Millis, and E. Abrahams, Gapless pairing and the Fermi arc in the cuprates, *Phys. Rev. B* **76**, 180501 (2007).
- [38] E. Berg and E. Altman, Evolution of the Fermi surface of d -wave superconductors in the presence of thermal phase fluctuations, *Phys. Rev. Lett.* **99**, 247001 (2007).
- [39] D. Valdez-Balderas and D. Stroud, Effects of inhomogeneities and thermal fluctuations on the spectral function of a model d -wave superconductor, *Phys. Rev. B* **77**, 014515 (2008).
- [40] G. Alvarez and E. Dagotto, Fermi arcs in the superconducting clustered state for underdoped cuprate superconductors, *Phys. Rev. Lett.* **101**, 177001 (2008).
- [41] W. A. Atkinson, J. D. Bazak, and B. M. Andersen, Robust nodal d -wave spectrum in simulations of a strongly fluctuating competing order in underdoped cuprate superconductors, *Phys. Rev. Lett.* **109**, 267004 (2012).
- [42] L. E. Hayward, D. G. Hawthorn, R. G. Melko, and S. Sachdev, Angular fluctuations of a multicomponent order describe the pseudogap of $\text{YBa}_2\text{Cu}_3\text{O}_{6+x}$, *Science* **343**, 1336 (2014).
- [43] T. S. Nunner, B. M. Andersen, A. Melikyan, and P. J. Hirschfeld, Dopant-modulated pair interaction in cuprate superconductors, *Phys. Rev. Lett.* **95**, 177003 (2005).
- [44] A. C. Fang, L. Capriotti, D. J. Scalapino, S. A. Kivelson, N. Kaneko, M. Greven, and A. Kapitulnik, Gap-inhomogeneity-induced electronic states in superconducting $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$, *Phys. Rev. Lett.* **96**, 017007 (2006).
- [45] K. McElroy, J. Lee, J. A. Slezak, D. H. Lee, H. Eisaki, S. Uchida, and J. C. Davis, Atomic-Scale sources and mechanism of nanoscale electronic disorder in $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$, *Science* **309**, 1048 (2005).
- [46] M. A. Sulangi, W. A. Atkinson, and P. J. Hirschfeld, Correlations among STM observables in disordered unconventional superconductors, *Phys. Rev. B* **104**, 144501 (2021).
- [47] M. A. Sulangi, The low-temperature phenomenology of gap inhomogeneity in the cuprate superconductors: High-energy granularity, low-energy homogeneity, and spectral kinks (unpublished).
- [48] J. Lee, K. Fujita, A. R. Schmidt, C. K. Kim, H. Eisaki, S. Uchida, and J. C. Davis, Spectroscopic fingerprint of phase-incoherent superconductivity in the underdoped $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$, *Science* **325**, 1099 (2009).
- [49] A. Schmidt, K. Fujita, E. Kim, M. Lawler, H. Eisaki, S. Uchida, D. Lee, and J. Davis, Electronic structure of the cuprate superconducting and pseudogap phases from spectroscopic imaging STM, *New J. Phys.* **13**, 065014 (2011).
- [50] M. H. Hamidian, S. D. Edkins, S. H. Joo, A. Kostin, H. Eisaki, S. Uchida, M. J. Lawler, E.-A. Kim, A. P. Mackenzie, K. Fujita, J. Lee, and J. C. S. Davis, Detection of a Cooper-pair density wave in $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+x}$, *Nature (London)* **532**, 343 (2016).
- [51] K. Lee, S. A. Kivelson, and E.-A. Kim, Cold-spots and glassy nematicity in underdoped cuprates, *Phys. Rev. B* **94**, 014204 (2016).
- [52] W. A. Atkinson, S. Ufkes, and A. P. Kampf, Structure of the charge density wave in cuprate superconductors: Lessons from NMR, *Phys. Rev. B* **97**, 125147 (2018).
- [53] P. Choubey, S. H. Joo, K. Fujita, Z. Du, S. Edkins, M. Hamidian, H. Eisaki, S. Uchida, A. Mackenzie, J. Lee *et al.*, Atomic-scale electronic structure of the cuprate pair density wave state coexisting with superconductivity, *Proc. Natl. Acad. Sci. USA* **117**, 14805 (2020).
- [54] S. Wang, P. Choubey, Y. X. Chong, W. Chen, W. Ren, H. Eisaki, S.-i. Uchida, P. J. Hirschfeld, and J. S. Davis, Scattering interference signature of a pair density wave state in the cuprate pseudogap phase, *Nat. Commun.* **12**, 6087 (2021).
- [55] K. Fujita, C. K. Kim, I. Lee, J. Lee, M. Hamidian, I. Firmo, S. Mukhopadhyay, H. Eisaki, S. Uchida, M. Lawler *et al.*, Simultaneous transitions in cuprate momentum-space topology and electronic symmetry breaking, *Science* **344**, 612 (2014).
- [56] S. Mukhopadhyay, R. Sharma, C. K. Kim, S. D. Edkins, M. H. Hamidian, H. Eisaki, S.-i. Uchida, E.-A. Kim, M. J. Lawler, A. P. Mackenzie *et al.*, Evidence for a vestigial nematic state in

- the cuprate pseudogap phase, *Proc. Natl. Acad. Sci. USA* **116**, 13249 (2019).
- [57] A. Melikyan, B. M. Andersen, T. S. Nunner, and P. Hirschfeld, Gap distributions in cuprate superconductors, *AIP Conf. Proc.* **850**, 571 (2006).
- [58] A. T. Rømer, S. Graser, T. S. Nunner, P. J. Hirschfeld, and B. M. Andersen, Local modulations of the spin-fluctuation-mediated pairing interaction by impurities in d -wave superconductors, *Phys. Rev. B* **86**, 054507 (2012).
- [59] N. Bashan, E. Tulipman, J. Schmalian, and E. Berg, Tunable non-Fermi liquid phase from coupling to two-level systems, *Phys. Rev. Lett.* **132**, 236501 (2024).
- [60] A. A. Patel, P. Lunts, and S. Sachdev, Localization of overdamped bosonic modes and transport in strange metals, *Proc. Natl. Acad. Sci. USA* **121**, e2402052121 (2024).
- [61] N. Bashan, E. Tulipman, S. A. Kivelson, J. Schmalian, and E. Berg, Extended strange metal regime from superconducting puddles, [arXiv:2502.08699](https://arxiv.org/abs/2502.08699).
- [62] B. M. Andersen, A. Melikyan, T. S. Nunner, and P. J. Hirschfeld, Thermodynamic transitions in inhomogeneous d -wave superconductors, *Phys. Rev. B* **74**, 060501 (2006).
- [63] D. Juskus, J. Ayres, R. Nicholls, and N. Hussey, Insensitivity of T_c to the residual resistivity in high- T_c cuprates and the tale of two domes, *Front. Phys.* **12**, 1396463 (2024).
- [64] www.sharcnet.ca.
- [65] R. Haydock and R. L. Te, Accuracy of the recursion method, *Phys. Rev. B* **49**, 10845 (1994).
- [66] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/x1m7-bs95> for the data of impurity configurations used in the study.