

Erratum: Generalized Haldane map from the matrix product state path integral to the critical theory of the J_1 – J_2 chain [Phys. Rev. Research 7, L012037 (2025)]

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It has come to light that the derivation of the Wess-Zumino term from the MPS path integral that we gave in Ref. [1] is incomplete. The precise functional form we gave was incorrect because it failed to take into account the distinction between taking the continuum limit of differences on “strong” bonds vs “weak” ones. The central result of the work—that we construct the correct critical theory—is unchanged.

In Ref. [1], we showed how, by taking the path integral over a set of matrix product states, one may directly construct the theory for the critical antiferromagnetic phase of the $S = 1/2$ J_1 – J_2 chain—the $O(4)$ nonlinear sigma model (NLSM) with a WZ term,

$$\mathcal{Z} = \int \mathcal{D}u e^{-\mathcal{S}[u]}, \quad \mathcal{S} = i\Omega_{\text{WZ}} + \int d\tau dx \frac{1}{2\pi K} \left(\frac{1}{c} (\partial_\tau u)^2 + c (\partial_x u)^2 \right), \quad (1)$$

for some c and K that depend only on J_1 and J_2 , and where $u(\tau, x) = (\cos \alpha, \sin \alpha \hat{\mathbf{m}}) \in S^3$, for a Néel field $\hat{\mathbf{m}} \in S^2$ and dimerization field α . We have omitted the dimerization potential $V(\alpha) = -J_2 \cos^2 \alpha$ in the above since it is irrelevant at small J_2 [2]. The second term in the action above is the $O(4)$ NLSM; the first is the WZ term, which is, after explicitly integrating over the extension coordinate of its usual formulation [3],

$$i\Omega_{\text{WZ}} = iS \int d\tau dx f_{\text{WZ}}(\alpha) (\partial_\tau \phi \partial_x \cos \theta - \partial_x \phi \partial_\tau \cos \theta), \quad f_{\text{WZ}}(\alpha) = \frac{\sin 2\alpha - 2\alpha}{2\pi}, \quad (2)$$

where θ and ϕ are the coordinates of $\hat{\mathbf{m}} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. From our construction in Ref. [1], we obtained an action precisely of the form in Eq. (1), except that we obtained, in place of $i\Omega_{\text{WZ}}$,

$$i\Omega = iS \int d\tau dx f(\alpha) (\partial_\tau \phi \partial_x \cos \theta - \partial_x \phi \partial_\tau \cos \theta), \quad f(\alpha) = \begin{cases} -\frac{1}{2} \sin \alpha, & 0 \leq \alpha < \frac{\pi}{2}, \\ \frac{1}{2} \sin \alpha - 1, & \frac{\pi}{2} \leq \alpha \leq \pi. \end{cases} \quad (3)$$

At this point, we prematurely concluded that our argument was complete, reasoning that since f_{WZ} and f are monotonic, and have the same values at the poles $\alpha = 0, \pi$, we could recover Eq. (1) simply by making a homeomorphic deformation of the target space (i.e., a deformation of α). However, since $f(\alpha)$ is linear near the singlet covers ($\alpha = 0, \pi$), whereas $f_{\text{WZ}}(\alpha)$ is cubic, such a deformation of α would completely change the NLSM part of the action.

But in taking the continuum limit of Eq. (S42) [1] to arrive at Eq. (3), we did not account for the fact that the field $\hat{\mathbf{m}}$ is more heavily constrained on entangled bonds than nonentangled bonds (in the extreme case, if sites $i - 1$ and i are in a singlet, then $\hat{\mathbf{m}}_i = \hat{\mathbf{m}}_{i-1}$, but $\hat{\mathbf{m}}_{i+1}$ is completely independent of $\hat{\mathbf{m}}_i$). As discussed in Ref. [1] (and in detail in Sec. IV of the Supplemental Material), the ∂_x operators in the NLSM part of the action refer to the derivative of $\hat{\mathbf{m}}$ over the *weak* bonds, or, equivalently, over two sites (always encompassing one strong and one weak bond),

$$\partial_x \hat{\mathbf{m}} \sim \frac{\hat{\mathbf{m}}_{i+1} - \hat{\mathbf{m}}_{i-1}}{2}. \quad (4)$$

In Eq. (S42), however, only the differences of $\hat{\mathbf{m}}_i$ over the *strong* bonds appear, and in order to work out how to replace these by coarse-grained derivatives (4), we must estimate the relative strength of variation of $\hat{\mathbf{m}}$ over the weak bonds compared to the strong bonds, as a function of α .

We can do this semiclassically. Let us first consider the case that the bonds $(2i - 1, 2i)$ are stronger, such that $0 \leq \alpha \leq \pi/2$, and, further, assume that α takes a constant value (or, rather, varies sufficiently slowly). Then, the expectation value of the Hamiltonian over the MPS ansatz is

$$\langle H \rangle = \sum_{2i} -J_1 (1 + \cos \alpha) \hat{\mathbf{m}}_{2i-1} \cdot \hat{\mathbf{m}}_{2i} - J_1 \sin^2 \alpha \hat{\mathbf{m}}_{2i} \cdot \hat{\mathbf{m}}_{2i+1} + J_2 \sin^2 \alpha (\hat{\mathbf{m}}_{2i-1} \cdot \hat{\mathbf{m}}_{2i+1} + \hat{\mathbf{m}}_{2i} \cdot \hat{\mathbf{m}}_{2i+2}). \quad (5)$$

Now consider some reference configuration $\hat{\mathbf{m}}_i = \hat{\mathbf{z}}$; of course, there is no long-range order, but we need to only consider a few sites. We now perturb $\hat{\mathbf{m}}_0$ and ask how $\hat{\mathbf{m}}_1$ (which is weakly bonded to $\hat{\mathbf{m}}_0$ and strongly bonded to $\hat{\mathbf{m}}_2$) responds. To lowest order, we assume that $\hat{\mathbf{m}}_{-1}$ and $\hat{\mathbf{m}}_3$ are completely determined by their strong bond,

$$\hat{\mathbf{m}}_{-1} \sim \hat{\mathbf{m}}_0, \quad \hat{\mathbf{m}}_3 \sim \hat{\mathbf{m}}_4 = \hat{\mathbf{z}} = \hat{\mathbf{m}}_2. \quad (6)$$

Over the three sites in question, then,

$$\langle H \rangle \approx (-J_1 \sin^2 \alpha + J_2 \sin^2 \alpha) \hat{\mathbf{m}}_0 \cdot \hat{\mathbf{m}}_1 + (-J_1(1 + \cos \alpha) + J_2 \sin^2 \alpha) \hat{\mathbf{m}}_1 \cdot \hat{\mathbf{m}}_2, \quad (7)$$

where, recall, $\hat{\mathbf{m}}_0$ and $\hat{\mathbf{m}}_2$ are held fixed by assumption. $\hat{\mathbf{m}}_1$ that minimizes this lies between, and in the same plane as, $\hat{\mathbf{m}}_0$ and $\hat{\mathbf{m}}_2$. Denoting the angles between these vectors by φ_{01} and φ_{12} , and assuming the perturbation to $\hat{\mathbf{m}}_0$ was small such that $\sin \varphi \sim \varphi$, we find

$$\varphi_{12} \sim \frac{(J_1 - J_2) \sin^2 \alpha}{J_1(1 + \cos \alpha) - J_2 \sin^2 \alpha} \varphi_{01} \Rightarrow \partial_x \mapsto \frac{(J_1 - J_2) \sin^2 \alpha}{J_1(1 + \cos \alpha) - J_2 \sin^2 \alpha} \partial_x, \quad (8)$$

which tells us how we should modify Eq. (3). Finally, we note that if the other set of bonds were strong, i.e., $\pi \geq \alpha \geq \pi/2$, we simply map $\alpha \mapsto \pi - \alpha$ in the above factor. Absorbing this into the weighting function $f(\alpha)$ in the Wess-Zumino term, we thus have

$$f(\alpha) = \begin{cases} -\frac{1}{2} \frac{(J_1 - J_2) \sin^3 \alpha}{J_1(1 + \cos \alpha) - J_2 \sin^2 \alpha}, & 0 \leq \alpha < \frac{\pi}{2}, \\ \frac{1}{2} \frac{(J_1 - J_2) \sin^3 \alpha}{J_1(1 + \cos(\pi - \alpha)) - J_2 \sin^2 \alpha} - 1, & \frac{\pi}{2} \leq \alpha \leq \pi. \end{cases} \quad (9)$$

If we *now* define a deformed $\tilde{\alpha}$ such that $f(\alpha) = f_{\text{WZ}}(\tilde{\alpha})$, both of these functions are cubic near the singlet covers, so the only effect on the NLSM action is to change the overall constant in Eq. (1),

$$K \mapsto \left(\frac{3\pi(J_1 - J_2)}{8J_1} \right)^{2/3} K \quad (10)$$

($K \rightsquigarrow 1.115K$ at $J_2 = 0$). The original conclusion of Ref. [1], that the MPS path integral constructs the critical theory, is thus now properly supported.

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[1] F. Azad, A. J. McRoberts, C. Hooley, and A. G. Green, Generalized Haldane map from the matrix product state path integral to the critical theory of the J_1 - J_2 chain, [Phys. Rev. Res. 7, L012037 \(2025\)](#).

[2] I. Affleck and F. Haldane, Critical theory of quantum spin chains, [Phys. Rev. B 36, 5291 \(1987\)](#).

[3] T. Senthil and M. P. Fisher, Competing orders, nonlinear sigma models, and topological terms in quantum magnets, [Phys. Rev. B 74, 064405 \(2006\)](#).