

Operator growth in disordered spin chains: Indications for the absence of many-body localization

A. Weisse *Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany*R. Gerstner  and J. Sirker*Department of Physics and Astronomy and Manitoba Quantum Institute, University of Manitoba, Winnipeg, Canada R3T 2N2*

(Received 20 January 2024; revised 12 July 2024; accepted 11 June 2025; published 7 July 2025)

We consider the spreading of a local operator A in one-dimensional many-body systems with Hamiltonian H by calculating the k -fold commutator $[H, [H, [\dots, [H, A]]]]$. We derive bounds for the operator norm of this commutator in free and interacting systems with and without disorder thus directly connecting the *operator growth hypothesis* with questions of localization. We analytically show, in particular, that an almost-factorial growth of the operator norm—as recently proven for the random Ising model and strongly suggested here to also hold for the Heisenberg model with random fields—is inconsistent with an exponential localization of A . Assuming there exists a quasilocal unitary U which maps H onto an effective Hamiltonian $\tilde{H} = UHU^\dagger = \sum_n E_n \tau_n^z + \sum_{i,j} J_{ij} \tau_i^z \tau_j^z + \dots$, we show that $\tilde{A} = UAU^\dagger$ is a quasilocal operator which, in contrast to the Anderson case, indeed *does not remain exponentially localized* in the general many-body case, leading to an almost-factorial growth of the commutator norm. Therefore, either the unitary U in many-body systems with maximal norm growth does not exist, and such systems are always ergodic, or unusual nonergodic phases described by $\tilde{H}$ exist which violate the operator growth hypothesis and in which local operators spread over the entire lattice, implying that transport will eventually set in. To investigate this issue further, we concentrate on the XXZ chain with random magnetic fields. We analytically and symbolically verify our general results for the noninteracting Anderson and Aubry-André models. For the XXX case, the symbolic calculations are consistent with a maximal norm growth. Furthermore, we find no indication of a weakened exponential localization of A , expected for strong disorder and low commutator orders if the unitary U does exist. Finally, we study the differences between the interacting and noninteracting cases when trying to perturbatively construct U by consecutive Schrieffer-Wolff transformations. While it is straightforward to show that this construction converges in the Anderson case, we find no indications for a convergence in the interacting case, suggesting that U does not exist and that many-body localization is absent.

DOI: [10.1103/wgss-nt8t](https://doi.org/10.1103/wgss-nt8t)

I. INTRODUCTION

In a one-dimensional fermionic lattice model with short-range hoppings and without interactions, any amount of disorder leads to a localization of the eigenfunctions. This is known as Anderson localization [1–4]. An open question for the past 65 years has been “What happens if interactions are included?” One of the difficulties in investigating this question is that localization can no longer be defined in terms of single-particle eigenfunctions; new indicators of localization are needed.

Studying the possibility of localization in the many-body case has gained renewed interest in the last 17 years starting with the work by Basko, Aleiner, and Altshuler [5]. This was followed by a large number of numerical studies concentrating, in particular, on the Heisenberg (XXX) chain with random

magnetic fields [6–19]. By a Jordan-Wigner transform, this model is equivalent to a fermionic chain with nearest-neighbor hoppings, density-density interactions, and random on-site potentials. Based on an analysis of spectral properties such as the level spacing as well as of dynamical properties such as the growth of the entanglement entropy after a quantum quench from an unentangled initial state, an apparent consensus was reached that this model shows a phase transition in all eigenstates at a finite disorder strength, separating an ergodic from a many-body localized phase. This consensus was first called into question by studies showing that the transition point, defined for a finite system of size L , appears to slowly shift to infinite disorder strength for $L \rightarrow \infty$ [20,21]. These arguments were supported by studies of the number and Hartley entropies after a quantum quench [22–27] and of the fidelity susceptibility [28,29].

Here we want to propose and follow a new avenue by studying the properties of operators instead of wave functions which offer a unified perspective of the noninteracting and interacting cases. Our motivation is the following observation: In an Anderson localized model, the projectors onto the eigenstates $P_n = |E_n\rangle\langle E_n|$ are conserved *quasilocal operators*, i.e.,

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

they are centered at a site in the lattice and decay exponentially away from this site. These operators are also often called local integrals of motion. Without loss of generality, we can order the operators such that P_n is an operator localized around lattice site n . To be more precise, we can use the fermionic language in which the Anderson Hamiltonian and any unitary transformation of this Hamiltonian are bilinear. We can then write $P_n = \sum_{ij} a_{ij}^n c_i^\dagger c_j$ where $c_i^{(\dagger)}$ is a fermionic annihilation (creation) operator at site i and $|a_{ij}^n| \sim \exp(-|i - n|) \exp(-|j - n|)$ are amplitudes. In particular, the contribution of bilinear operators to P_n is exponentially suppressed with the distance they cover.

There is a unitary transformation U which diagonalizes the microscopic, noninteracting Anderson Hamiltonian H , bringing it into the form $\tilde{H} = U H U^\dagger = \sum_n E_n P_n$ where P_n is quasilocal and conserved, $[H, P_n] = 0$. Importantly, this unitary transformation itself is quasilocal and will always map local microscopic operators A_i onto quasilocal operators, $\tilde{A}_i = U A_i U^\dagger$. As we will show, this has important implications for the spreading of a local operator in Euclidean time. We can write

$$A_i(\tau) = e^{\tau H} A_i e^{-\tau H} = \sum_{k=0}^{\infty} \underbrace{[H, [H, [\dots, [H, A_i]]]]}_{k\text{-times}} \frac{\tau^k}{k!}. \quad (1.1)$$

That is, the spreading of the operator is encoded in the k -fold commutator $[H, A_i]^{(k)}$ with the Hamiltonian. It is important to note that the Frobenius norm of this commutator is not affected by the unitary transformation $\|[H, A_i]^{(k)}\|_2 = \|[H, \tilde{A}_i]^{(k)}\|_2$. We also note that the same k -fold commutator needs to be considered when calculating real-time two-point correlation functions such as a current-current correlation function which determines the transport properties of a system.

In a many-body localized phase, it has been argued that—similar to the Anderson case—a unitary transformation of the microscopic Hamiltonian H to a Hamiltonian $\tilde{H}$ of conserved quasilocal charges exists. The main difference is assumed to be that these quasilocal charges τ_n^z are interacting with each other [17,19]

$$\tilde{H} = \sum_n E_n \tau_n^z + \sum_{i,j} J_{ij} \tau_i^z \tau_j^z + \dots \quad (1.2)$$

Here, J_{ij} are exponentially decaying coupling constants which describe a slow dephasing between the conserved charges. For the Anderson case, $J_{ij} = 0$ and the τ_n^z can be chosen to be the projection operators P_n . We note that the choice of the quasilocal conserved operators τ_n^z is not unique. If we define a many-body localized phase by the property that a local operator in the microscopic model remains local—up to exponential tails—under Euclidean time evolution, as in the noninteracting case, then this implies, as we will show, that the total operator norm in both cases can grow at most exponentially with the commutator order k . We will also show that if local operators spread over the entire system then this strongly suggests that transport will eventually set in.

The localized case has to be contrasted with the general, ergodic case. In one dimension, there is no phase transition in a clean system at finite temperatures. Thus, the time evolution is always analytic and Eq. (1.1) implies that the norm

of the k -fold commutator $[H, A_i]^{(k)}$ has to grow slower than factorially. In Ref. [30], it has been shown that this is indeed the case; however, the derived bound for the total norm grows much faster than exponentially and almost factorially: $\|[H, A_i]^{(k)}\|_2 \sim (k/\ln k)^k$. The *operator growth hypothesis* states that any generic quantum chaotic system will asymptotically saturate this bound [30,31]. Thus, we have a sharp contrast between the growth of the operator norm of the k -fold commutator of a local operator with the Hamiltonian between a strictly exponentially localized, nonergodic system, where it can grow at most exponentially, and a nonlocalized system, where an almost-factorial growth is expected in general. This qualitative difference in the operator norm growth is what we will investigate here. To avoid misunderstandings, we want to stress already here that what we mean by a local operator and what we will investigate in the following is an operator A_i which resides on lattice site i in the microscopic model. In contrast, operators of the form $A = \sum_i A_i$ are also often called local conserved charges if $[H, A] = 0$ and are of particular importance in integrable models. Operators of the type A , which are sums of local densities but which extend over the entire lattice, are also the type of operators whose growth has been investigated in Refs. [31,32]. The latter type of operators are not of interest for a study of localization and we will not consider them here. From hereon we will therefore drop the lattice index with the understanding that A is always a local operator.

We note that in contrast to real-time evolution, the Euclidean time evolution of operators can be directly connected to questions of quantum chaos and the eigenstate thermalization hypothesis (ETH) [30,31]. First, the Frobenius norm of a local operator A under Euclidean time evolution in the energy eigenbasis for a finite system of size L is given by

$$\|A(\tau)\|_2^2 = \frac{\text{tr}(A^\dagger(\tau)A(\tau))}{\text{tr}\mathbb{1}} = \sum_{ij} e^{\tau(E_i - E_j)} \frac{|A_{ij}|^2}{\text{tr}\mathbb{1}}. \quad (1.3)$$

For a system satisfying ETH, most matrix elements A_{ij} are expected to be nonzero even for extensive energy differences $|E_i - E_j|$. This implies that $\|A(\tau)\|_2^2 \sim \exp(J\tau L)$ where J is the local energy scale. If a system is noninteracting or has local conservation laws, however, and thus does not satisfy ETH, matrix elements will be zero for extensive energy differences and we thus expect $\|A(\tau)\|_2^2 \sim \exp(J\tau)$. By deriving strict bounds directly for the Frobenius norm $\|A(\tau)\|_2$, one can use Eq. (1.3) to make precise statements about the scaling of off-diagonal matrix elements A_{ij} as a function of $|E_i - E_j|$. For the generic, chaotic case this scaling has been considered in Ref. [30] and is consistent with numerical investigations [33,34].

Another quantity which can be obtained from considering the Euclidean time evolution of local operators are Lieb-Robinson bounds. A clean, one-dimensional system with short-range interactions shows a linear spreading of information under real-time evolution independent of whether the system is chaotic or not [35,36]. In contrast, under Euclidean time evolution information spreads exponentially fast, $\ell \sim e^\tau$, over a distance ℓ in a 1D chaotic system [30] while it only spreads ballistically in a noninteracting system or a system with local conservation laws. It has also been speculated

that the time for the spreading of information in Euclidean time in a 1D chaotic system, $\tau \sim \ln(\ell)$, is indicative of the Thouless time in a system without conservation laws (for example a Floquet system), i.e., the time after which the system can be described by a random matrix. Furthermore, it is also possible to connect the characteristic time of Euclidean operator growth with the delocalization of this operator in Krylov space [31]. Overall, there is strong evidence that in a quantum chaotic system, an operator grows asymptotically at its maximally allowed rate and that a parametrically slower growth is related to ergodicity breaking.

Here we want to expand the study of operator growth to disordered systems. There is, in particular, a direct conflict between the operator growth hypothesis, which suggests that an almost-factorial growth is indicative of ergodicity, and systems such as the random transverse Ising model, which has been suggested to have a many-body localized phase but always shows almost-factorial operator growth independent of the disorder strength [37]. While we will consider Lieb-Robinson bounds and the decay of off-diagonal matrix elements in the Supplemental Material [38], it is important to note that these quantities are obtained by taking sums over k -fold commutators $[H, A]^{(k)}$ and thus some of the information, in particular about the spatial support of such commutators which is relevant for the question of localization, is lost. Here we follow a novel path by directly obtaining analytical results for the asymptotic scaling of k -fold commutators resolved by the extent of their spatial support. Our results are a step toward resolving the above mentioned conflict and suggest that the operator growth hypothesis is valid and that many-body systems, even for strong disorder, do not localize but rather stay ergodic.

In addition to these analytical results, we will also present exact results from symbolic calculations of commutators and norms. To be able to consider high orders k in the commutator, we have written code in Julia with Nemo [39] which is able to handle symbolic computations with up to ~ 200 million terms, limited only by the available memory. Nemo also provides arbitrary precision complex ball arithmetic which tracks error bounds rigorously. We typically use 200 or 400 digit precision for the amplitudes of the operators. While the absolute errors for large commutator orders can become large, we keep the relative errors always below 10^{-50} such that error bars are not visible in many of the plots presented in the following. We note that by using symbolic calculations we are able to consider system sizes for nearest-neighbor models which are about twice as large as the ones typically considered in exact diagonalizations.

II. OUTLINE AND MAIN RESULTS

Here, we give a summary of our main results, provide a guide to allow the reader to find these results, and point out parts of the paper which can be skipped in a first reading.

It has recently come to light that in the Ising model with random transverse and longitudinal fields [40,41],

$$\| [H, \sigma_0^z]^{(k)} \|_2 \sim \left(\frac{k}{\ln k} \right)^k \quad (2.1)$$

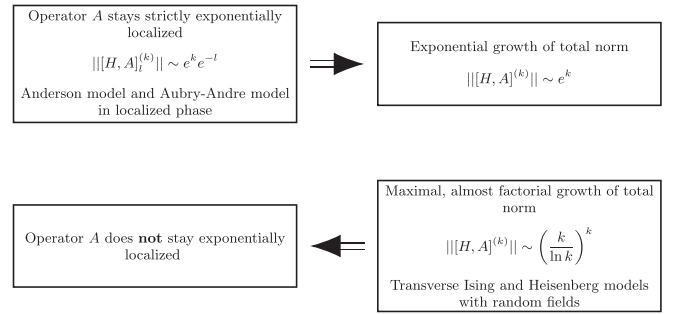


FIG. 1. If local operators remain strictly exponentially localized, then the total norm cannot grow faster than exponential. Conversely, in many-body systems such as the Ising and Heisenberg models with random fields, the almost-factorial growth of the total norm implies that local operators will not remain exponentially localized.

is *both* an upper and a lower bound [37]. This asymptotically maximal growth suggests, according to the operator growth hypothesis, that this model is ergodic for all disorder strengths. In Sec. V we show data from symbolic calculations which indicate that the same maximal growth also holds for the Heisenberg model with random fields. This means that either both models do not show an MBL phase or the operator growth hypothesis has to be modified such that a maximal operator growth is also possible in certain nonergodic models.

To investigate this issue further, we resolve the commutator by the number of sites l it covers. At order k , the commutator $[H, A]^{(k)}$ of a local operator A acting on a single site with a nearest-neighbor Hamiltonian contains operators which span at most $l = k + 1$ sites. We show in Sec. III A that the contribution of operators with length l to the total norm is always bounded by

$$\| [H, A]_l^{(k)} \|_2 \leq (2CJ)^k 2^l S(k, l), \quad (2.2)$$

where CJ is a constant and $S(k, l)$ is the Stirling number of the second kind. This bound, which is expected to be asymptotically tight for an ergodic system, has a maximum at some $l(k)$ that shifts to infinity for $k \rightarrow \infty$, that is, the operator spreads throughout the entire lattice. Summing over all operator lengths l leads to the almost-factorial bound (2.1). However, we might expect that in a strictly exponentially localized system, the operator A remains quasilocal, i.e., $\| [H, A]_l^{(k)} \|_2 \sim e^{-l}$. In this case, the total norm growth is dominated by terms with support on a few sites only. This can be understood in more detail by representing the commutator as graphs which we present in Sec. V A for the noninteracting case and in Sec. V B for the interacting case. In the strictly exponentially localized case, we then find that

$$\| [H, A]_l^{(k)} \|_2 \sim e^k e^{-l}. \quad (2.3)$$

This implies, in particular, that in the strictly exponentially localized case, the total norm cannot grow faster than exponentially with the commutator order k . Conversely, a faster, almost factorial growth of the total norm, as in the many-body random-field Ising and Heisenberg models, is *incompatible* with an exponential localization of operators. This important result is summarized in Fig. 1.

In the noninteracting case, we prove that local operators indeed remain strictly exponentially localized in a localized phase. In Sec. V A we demonstrate, in particular, that in an Anderson localized phase, Eq. (2.3) holds, while in a non-localized phase, $||[H, A]_l^{(k)}||_2 \sim e^k$ holds independently of l for $k \gg l$. That is, in the latter case, the operator spreads throughout the entire lattice. We show, furthermore, that this difference in scaling allows one to unambiguously identify the phase transition in the Aubry-André model. We find it important to rigorously establish this novel operator perspective on localization by first studying the noninteracting case in detail. However, the reader who accepts the noninteracting results and is mostly interested in what this approach has to say about MBL might skip Sec. V A in its entirety.

If one takes strict exponential localization of local operators under commutation with the Hamiltonian as the very definition of a localized phase, then one has to conclude that the Ising model with random transverse and longitudinal fields and—based on the analysis presented in this paper—the Heisenberg model with random fields do not possess a localized phase: In these two models, the total norm growth is faster than exponential and described by Eq. (2.1), independent of the disorder strength.

However, if one takes the existence of a unitary transformation U which maps these microscopic models onto an effective model (1.2) as the definition of a localized phase, then there appears to potentially be room for a faster-than-exponential growth of the total norm in the interacting case, as shown by the strict upper bounds derived in Sec. IV. Here, we summarize the main results. First, it is important to note that the Frobenius norm is invariant under unitary transformations, i.e., $||[H, A]_l^{(k)}||_2 = ||[\tilde{H}, \tilde{A}]_l^{(k)}||_2$ with $\tilde{H} = UHU^\dagger$ and $\tilde{A} = UAU^\dagger$. It is easy to show that for a local operator such as τ_0^x we have $||[\tilde{H}, \tau_0^x]_l^{(k)}||_2 \sim e^k e^{-l}$ with $\tilde{H}$ as given in Eq. (1.2). That is, the operator growth hypothesis does hold for $\tilde{H}$, clearly indicating that the model is not ergodic. However, it is crucial to note that a local operator A in the microscopic model gets mapped onto a *quasilocal* operator $\tilde{A}$ under the unitary transformation U . In the noninteracting case, this is still a one-body operator, and we can show that this operator remains strictly exponentially localized. Therefore, in the noninteracting case, the two aforementioned definitions of a localized phase—that A remains strictly exponentially localized and that a quasilocal unitary U exists which maps H to $\tilde{H}$ —are equivalent.

In the interacting case, however, $\tilde{A}$ is in general a generic quasilocal operator, and if no additional microscopic restrictions exist, then $||[\tilde{H}, \tilde{A}]_l^{(k)}||_2$ will show an almost-factorial growth. Furthermore, we will demonstrate in Sec. IV that we can obtain an upper bound for the spreading of a quasilocal operator of the form

$$||[\tilde{H}, \tilde{A}]_l^{(k)}||_2 \lesssim l^k e^{-\kappa l}, \quad (2.4)$$

where $\kappa = \kappa(D)$ increases with disorder strength D . We argue in Sec. IV that $||[H, A]_l^{(k)}||_2$ should show the same functional dependence as Eq. (2.4) for strong disorder. We note that this function has a maximum at $l = k/\kappa$, i.e., for large commutator order k , the local operator A spreads over the entire lattice. Furthermore, for very large k , the bound (2.2) will eventually

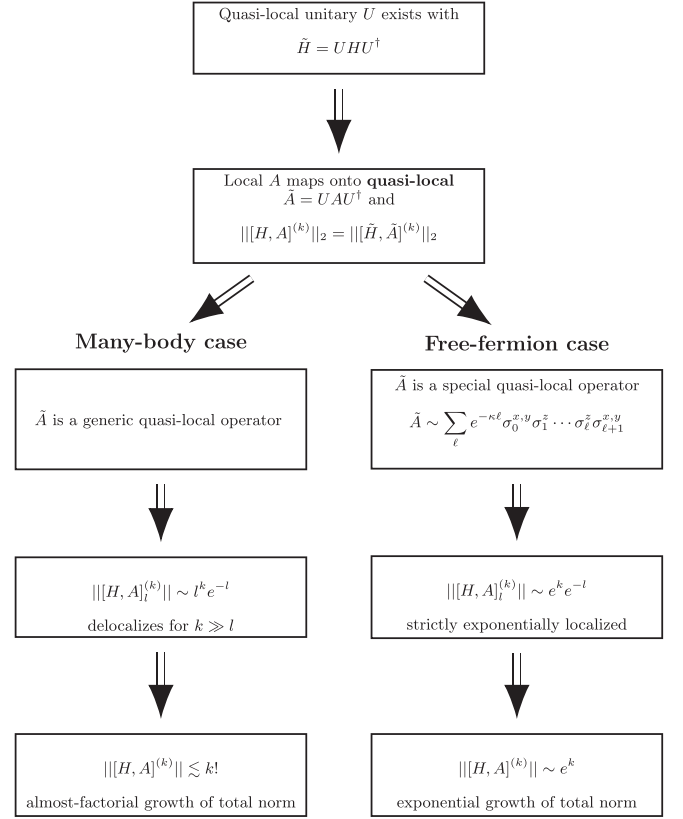


FIG. 2. Assuming that a quasilocal unitary U with $\tilde{H} = UHU^\dagger$ exists, local operators A are mapped onto quasilocal operators $\tilde{A} = UAU^\dagger$. In the free-fermion case, $\tilde{A}$ remains a one-body operator, implying that A is strictly exponentially localized and the total norm growth is exponential. In contrast, $\tilde{A}$ is a generic quasilocal operator in the many-body case in general, implying that A eventually delocalizes and that the total norm growth is almost factorial.

be tighter than Eq. (2.4), implying that the operator will spread through the lattice in the exact same way as in an ergodic system. This then leads to one of two possible scenarios: (1) The quasilocal unitary U does exist and the many-body Hamiltonian H has an unusual nonergodic phase in which it is equivalent to $\tilde{H}$ but shows maximal norm growth and delocalization of local operators A . We show in Sec. IV C that this strongly suggests that transport eventually sets in at long times. Therefore, even if such a phase exists, it should not be called a many-body localized phase. (2) The unitary U does not exist. These results are summarized in Fig. 2.

One way to investigate this question further is through symbolic calculations of $||[H, A]_l^{(k)}||_2$, which, according to Eq. (2.4), should be dominated by an exponential decay for strong disorder and finite commutator order k . Our results presented in Sec. V B are not inconsistent with Eq. (2.4), but even for disorder strengths much larger than the hopping amplitude, $D \gg J$, the exponential decay is never fully dominant. We therefore also study in Sec. VI, through symbolic calculations, whether a mapping $H \rightarrow \tilde{H}$ by successive Schrieffer-Wolff transformations is convergent. This has been suggested to be possible by Imbrie [40] for the Ising model with random fields and it is supposed to work similarly to the Anderson case. However, there are important differences:

(1) In the Anderson case, one-body operators remain one-body operators under Schrieffer-Wolff transformations. This means that two sites which are a distance ℓ apart only get coupled by $c_j^\dagger c_{j+\ell} + \text{H.c.}$ and that this coupling is suppressed by $(J/D)^{2\ell}$ if we ignore resonances. In the interacting case, however, there is no such restriction, and we can place any of the operators $\{\mathbb{I}, \sigma^x, \sigma^y, \sigma^z\}$ on the sites $j+1, \dots, j+\ell-1$ and Pauli operators on sites $j, j+\ell$ leading to $3^2 \cdot 4^{\ell-1}$ different operators coupling sites j and $j+\ell$. Even if each individual term is suppressed as $D^{-2\ell}$, the parameter for the total contribution $\sim 4^\ell D^{-2\ell}$ is no longer always small. This is, of course, not necessarily inconsistent with having a transition at some finite D .

(2) The smallest pairwise difference of random fields in a chain segment of length ℓ scales as ℓ^{-2} . This implies that the effective coupling scales as $J_{j,j+\ell} \sim D^{-2\ell} \ell^2$ in the noninteracting case and resonances therefore do not proliferate. However, in the interacting case, we have many-body resonances caused by some linear combination of *all* the random fields on the segment. We show in Fig. 18 results of symbolic Schrieffer-Wolff transformations which do not show any evidence that the amplitudes of the generated long-range terms converge.

(3) Finally, we note that for a finite system, we want to stop the Schrieffer-Wolff transformations once the norm of the off-diagonal parts of the effective Hamiltonian falls below some threshold ε . For the noninteracting case, this leads to the condition that $k \sim \ln(L/\varepsilon)$ steps are required for a chain of length L which is fully consistent with our symbolic calculations. In the interacting case, however, the number of generated off-diagonal terms scales exponentially with system size. If we assume that each individual contribution is still suppressed as D^{-2k} , i.e., resonances do not proliferate, then this leads to the condition $k \sim (L - \ln \varepsilon)/2 \ln D$. In particular, we now expect a linear scaling with system size, $k \sim L$, as compared to the logarithmic scaling in the noninteracting case. However, we find symbolically that the Schrieffer-Wolff transformations for typical disorder distributions do not converge in $k \sim L$ steps. Instead, our results appear to be consistent with $k \sim e^L$, i.e., all the terms generated will need to be eliminated. This suggests that many-body resonances prohibit the perturbative construction of an effective Hamiltonian via Schrieffer-Wolff transformations in the thermodynamic limit.

Our paper is organized as follows: In Sec. III, we discuss general bounds for the growth of the norm $||[H, A]^{(k)}||$ for a local operator A and a nearest-neighbor Hamiltonian H . We review the result in the general interacting case [30] before discussing the free-fermion case. In Sec. IV we then derive bounds for systems where local operators remain strictly exponentially localized and for systems which are unitarily equivalent to an effective Hamiltonian $\tilde{H}$ of local conserved charges. We also discuss the implications of our results for real-time dynamics and transport. In Sec. V, we then consider a specific model, the XXZ spin chain. We start with the free-fermion (XX) model without disorder before discussing the Anderson case of quenched disorder. We also consider the Aubry-André model which has a periodic potential incommensurate with the lattice and shows a phase transition at a finite disorder strength which, as we will demonstrate, is

clearly visible in the norm growth. We then turn to the interacting Heisenberg (XXX) model and show that results obtained from symbolic computations are inconsistent with a strict exponential localization for all studied disorder strengths. In Sec. VI, we consider Schrieffer-Wolff transformations which can be used to perturbatively transform the Hamiltonian into a basis of quasilocal conserved operators. We show that such a perturbative construction succeeds in the noninteracting case while it appears to fail once interactions are included. The final section contains a summary of our main results and a discussion of the remaining open issues.

III. GENERAL BOUNDS ON OPERATOR NORM GROWTH

We want to consider how the norm of the k th order commutator

$$[H, A]^{(k)} \equiv \underbrace{[H, [H, [\dots, [H, A]]]]}_{k\text{-times}} \quad (3.1)$$

between a one-dimensional lattice Hamiltonian

$$H = \sum_I h_{I,I+1} \equiv \sum_I h_I \quad (3.2)$$

which acts only locally at nearest-neighbor sites I and $I+1$ and a one-site operator A grows with the order of the commutator k . We consider the Hamiltonian as acting on L lattice sites which therefore can be described by an $M \times M$ matrix

$$H = \sum_I \mathbb{1}_1 \otimes \dots \otimes \mathbb{1}_{I-1} \otimes h_{I,I+1} \otimes \mathbb{1}_{I+2} \otimes \dots \otimes \mathbb{1}_L \quad (3.3)$$

with $M = n^L$, where n is the number of degrees of freedom per site. We place the operator A in the middle of the chain and choose chain lengths sufficiently large so that the support of $[H, A]^{(k)}$ is always less than L .

We consider two entry-wise matrix norms for a matrix A with elements A_{ij} :

$$\begin{aligned} ||A||_1 &= \sum_{i,j} |A_{ij}| / \sqrt{\text{tr} \mathbb{1}_{M \times M}} \\ ||A||_2 &= \sqrt{\frac{\text{tr}(A^\dagger A)}{\text{tr} \mathbb{1}_{M \times M}}} = \sqrt{\frac{\sum_{i,j} |A_{ij}|^2}{\text{tr} \mathbb{1}_{M \times M}}}. \end{aligned} \quad (3.4)$$

We note that the 2-norm is also called the Frobenius norm. For finite-dimensional spaces it is also equivalent to the Hilbert-Schmidt norm which is consistent with the standard scalar product in quantum mechanics at infinite temperatures. We divide here by the constant factor $\text{tr} \mathbb{1}_{M \times M}$ such that the identity operator acting on the entire system is normalized to one. We note that the 2-norm is invariant under unitary transformations. That is, the operator growth of a local operator A in the microscopic model with Hamiltonian H under Euclidean time evolution is exactly the same as that of the transformed operator $\tilde{A} = UAU^\dagger$ in the effective model $\tilde{H} = UHU^\dagger$ if U describes a unitary transformation. The 1-norm, however, is not invariant under basis changes. Let us therefore explain how we define the matrix A corresponding to an operator. If $\{\sigma_i^\alpha\}$ is a f -dimensional local operator basis, then we can

always write

$$A = \sum_{\{\alpha\}} a_{\alpha_1, \dots, \alpha_L} \sigma_1^{\alpha_1} \otimes \dots \otimes \sigma_L^{\alpha_L}. \quad (3.5)$$

There are f^L many coefficients $a_{\alpha_1, \dots, \alpha_L}$, and the 2-norm is given by $\|A\|_2 = \sqrt{\sum_{\{\alpha\}} |a_{\alpha_1, \dots, \alpha_L}|^2}$. We can therefore consider as entries of our matrix A the f^L many coefficients $\sqrt{\text{tr} \mathbb{1}_{M \times M} a_{\alpha_1, \dots, \alpha_L}}$. For the norms the order of these coefficients does not matter. The 1-norm is then given by $\|A\|_1 = \sum_{\{\alpha\}} |a_{\alpha_1, \dots, \alpha_L}|$. In symbolic calculations, where operators are obtained in the form (3.5), this means that we simply have to sum up the absolute values, or the absolute values squared, of the operator A written in the local basis $\{\sigma_i^\alpha\}$.

The two matrix norms which we have defined in Eq. (3.4) are subadditive and the 1-norm is an upper bound for the 2-norm

$$\|A + B\|_p \leq \|A\|_p + \|B\|_p, \quad \|A\|_2 \leq \|A\|_1. \quad (3.6)$$

However, due to the rescaling by the factor $(\text{tr} \mathbb{1}_{M \times M})^{-1/2}$ the two norms are no longer submultiplicative. We note, however, that if $A = J_A \sigma_1^{\alpha_1} \otimes \dots \otimes \sigma_L^{\alpha_L}$ and $B = J_B \sigma_1^{\beta_1} \otimes \dots \otimes \sigma_L^{\beta_L}$, i.e., both operators have only one coefficient in Eq. (3.5) which is nonzero, then

$$\|AB\|_p = \|A\|_p \|B\|_p = |J_A| |J_B|. \quad (3.7)$$

While the 2-norm (Hilbert-Schmidt norm) is physically more relevant because it is induced by the standard scalar product in quantum mechanics, the 1-norm is often easier to calculate and provides an upper bound for the 2-norm.

A. General case

The case of the Euclidean norm growth for a one-dimensional nearest-neighbor model without any further restrictions on the form of the nearest-neighbor Hamiltonian has been considered in Ref. [30]. Here we briefly recapitulate the derivation of the general bound to contrast it with the tighter bounds which we derive below in the special cases of free-fermion and localized models.

Let us define $H = \sum_I \sum_{a=1}^C h_I^a$, where h_I^a are local Hamiltonian densities of the form $h_I^a = J_I^a \sigma_1^{a_1} \otimes \sigma_{I+1}^{a_2}$. We note that the commutator (3.1) is zero if the local operator A and the local Hamiltonian densities h_I^a do not form a connected cluster. Let $\{I_1, \dots, I_k\}$ be a set of points which fulfills the adjacency condition, then

$$\begin{aligned} \|[H, A]^{(k)}\| &= \left\| \sum_{\{I_1, \dots, I_k\}} \sum_{\{a_I\}} [h_{I_1}^{a_{I_1}}, [h_{I_2}^{a_{I_2}}, \dots, [h_{I_k}^{a_{I_k}}, A]]] \right\| \\ &\leq \sum_{\{I_1, \dots, I_k\}} \sum_{\{a_I\}} \|[h_{I_1}^{a_{I_1}}, [h_{I_2}^{a_{I_2}}, \dots, [h_{I_k}^{a_{I_k}}, A]]]\|, \end{aligned} \quad (3.8)$$

where we have used the subadditivity property (3.6). The commutator now is determined by terms which are products of single operators in the $\{\sigma_i^\alpha\}$ basis and we can use Eq. (3.7) implying that

$$\|h_{I_1}^{a_{I_1}} h_{I_2}^{a_{I_2}} \dots h_{I_k}^{a_{I_k}} A\| = \|h_{I_1}^{a_{I_1}}\| \|h_{I_2}^{a_{I_2}}\| \dots \|h_{I_k}^{a_{I_k}}\| \|A\|. \quad (3.9)$$

There are 2^k terms of this kind and we therefore find

$$\begin{aligned} \|[H, A]^{(k)}\| &\leq 2^k \sum_{\{I_1, \dots, I_k\}} \sum_{\{a_I\}} \underbrace{\|h_{I_1}^{a_{I_1}}\| \dots \|h_{I_k}^{a_{I_k}}\|}_{k\text{-many}} \|A\| \\ &\leq (2CJ)^k \|A\| \sum_{\{I_1, \dots, I_k\}}, \end{aligned} \quad (3.10)$$

where we have used that $\|h_I^a\| = J_I^a \leq J = \max_{I,a} \{J_I^a\}$ and $\sum_{\{a_I\}} = C^k$.

Equation (3.10) is now a purely combinatorial problem. We need to figure out how many different arrangements of the lattice points $\{I_1, \dots, I_k\}$ exist such that a connected cluster is formed. Without repeated indices, j local adjacent Hamiltonians form a cluster with j bonds. At each step, one can either attach a local Hamiltonian to the right or to the left of the existing cluster. Thus, as for the random walk in one dimension, there are 2^j possibilities to form a cluster with j bonds. Last, consider the case that we have a cluster of length j and $k \geq j$ local Hamiltonians h_I . Then the question is how many ways there are to “distribute” those local Hamiltonians on the j bonds of the cluster. This is the same question as asking how many ways there are to partition k elements into j nonempty sets. The answer is given by the Stirling numbers of the second kind, $S(k, j)$. We therefore obtain

$$\sum_{\{I_1, \dots, I_k\}} = \sum_{j=1}^k 2^j S(k, j) = B_k(2), \quad (3.11)$$

where $B_k(x)$ is the Bell (Touchard) polynomial. The general bound is therefore given by

$$s^{(p)}(k) \equiv \|[H, A]^{(k)}\|_p \leq \|A\|_p (2CJ)^k B_k(2), \quad (3.12)$$

where $p = 1, 2$ is the 1-norm or the 2-norm defined earlier. We note that this bound is also valid for any submultiplicative norm.

We are, in particular, interested in the asymptotic growth of the operator norm for large k . In this limit, the Bell polynomial scales as

$$B_k(2) \sim \frac{\exp[-k(1 + \ln W(k/2) - W^{-1}(k/2))]}{\sqrt{W(k/2) + 1}} e^{-2} k^k, \quad (3.13)$$

where $W(x)$ is the Lambert W function [42]. Using the Stirling formula $k^k \sim k! \exp(k)/\sqrt{2\pi k}$ we can rewrite the asymptotics as

$$\begin{aligned} B_k(2) &\sim \frac{e^{-2} k!}{\sqrt{2\pi k(W(k/2) + 1)}} \\ &\times \exp \left[k \frac{1 - W(k/2) \ln W(k/2)}{W(k/2)} \right]. \end{aligned} \quad (3.14)$$

This means that the bound (3.12) grows faster than exponentially and almost factorially. We note that the term in the exponential in the second line of (3.14) only becomes negative for $k \gtrsim 20$ thus suppressing the initial almost-factorial growth. For large k , one can also use that $W(k) \sim \ln k - \ln \ln k$ to see from Eq. (3.13) that $B_k(2) \sim (k/\ln k)^k$.

We can also ask the question how many *distinct* terms are possible in the commutator $[H, A]^{(k)}$. Note that this question is different from the question of how many connected clusters exist which we addressed above because connected clusters

formed in different ways can result in the same operator. Let us assume that our system has a basis of $f - 1$ local operators plus the local identity $\mathbb{I}_j$. At order k , clusters have at most length $k + 1$ and, if the operator A was placed on site $j = 0$, extend from sites $-k, -k + 1, \dots, 0$ to the right. That is, there are $k + 1$ clusters of length $k + 1$ with each shifted by one lattice site to the right compared to the previous one. For the cluster starting at $-k$, we can put any of the $f - 1$ local operators or the identity on each of the $k + 1$ sites leading to f^{k+1} possible operators. For the cluster starting at site $-k + 1$, we have to put one of the $f - 1$ local operators at the right most site, $j = 1$, to obtain operators distinct from the ones already constructed. Thus, this cluster will add $(f - 1)f^k$ possible operators. The same construction also works for the remaining $k - 1$ clusters. We can therefore bound the total number of possible distinct operators occurring in the commutator $[H, A]^{(k)}$ by

$$n(k) \leq f^{k+1} + k(f - 1)f^k = (kf - k + f)f^k. \quad (3.15)$$

For a generic system, we thus expect that the number of distinct terms $n(k)$ in the commutator grows exponentially. As we will discuss in more detail below, the terms occurring in the commutator can be represented as nodes in a graph and $n(k)$ is directly related to the complexity of this graph.

To summarize, if a system does not have *local* operators which are conserved, then the hypothesis is that the bound on the total norm (3.12) is asymptotically tight [30]. Importantly, a norm growth $s(k) \sim (k/\ln k)^k$ is not possible in regular graphs or Cayley trees where the norm cannot grow faster than exponential. Many-body systems and the question of localization in such systems are therefore qualitatively different from studies of localization on regular graphs [43,44]. Next, we consider noninteracting systems which are one example of systems which do correspond to regular graphs and where stricter, exponential bounds on the norm can be derived.

B. Free fermions

The main difference between a general, interacting system and a free-fermion system is that in the latter case the commutator of a one-body operator with the Hamiltonian always remains a one-body operator. Therefore, many of the connected clusters leading to the bound (3.11) do not contribute. Here, a free-fermion model is a system with a bilinear Hamiltonian of the form $H = \sum_{j,k} h_{jk} c_j^\dagger c_k$, where $c_j^{(\dagger)}$ is a fermionic annihilation (creation) operator at site j .

Our starting point to prove a bound in the free-fermion case is the last line of Eq. (3.10). We now want to understand how many distinct connected clusters there are which give a nonzero contribution to the sum. The Hamiltonian in this case is, in general, a sum of C one-body terms, $H = \sum_l \sum_{a=1}^C h_l^a$ such as nearest-neighbor hoppings and local potentials. Now consider a connected cluster $\{I\} = \{I_1, \dots, I_k\}$ which gives a nonzero contribution to the sum in Eq. (3.10). Since the commutator $[H, A]^{(k)}$ is always a one-body operator if A is a one-body operator, there is one fermionic operator on a site i and another fermionic operator on a site j for each such cluster. Then, growing the cluster the only contributions which can potentially be nonzero will come from adding a one-body

term h^a on one of the bonds $I_{i-1}, I_i, I_j, I_{j+1}$. That is, there are at most four possibilities to build a new nonzero cluster with $k + 1$ elements. While this is, in general, overcounting the number of possibilities we can thus iteratively show that

$$\sum_{\{I_1, \dots, I_k\}} \leq 4^k, \quad (3.16)$$

resulting in the bound

$$|[H, A]^{(k)}| \leq \|A\| (8CJ)^k. \quad (3.17)$$

For a free-fermion system, the operator norm growth is thus at most exponential [45].

We can also again consider the question how many distinct terms can at most occur in the commutator $[H, A]^{(k)}$. As in the general case, the clusters have length of at most $k + 1$ and have $k + 1$ different starting points (leftmost lattice site in the cluster). The difference is that we now have to distribute only two operators, a fermionic creation operator $c_i^\dagger$ and a fermionic annihilation operator c_j , over the cluster. For the first cluster, we have $(k + 1)$ possibilities to place the creation and $(k + 1)$ possibilities to place the annihilation operator for a total of $(k + 1)^2$ possible distinct terms. For the other clusters, we have to place either $c^\dagger$ or c on the rightmost site to get new distinct terms. The other operator can then be placed on any of the $k + 1$ sites for a total of $2(k + 1) - 1 = 2k + 1$ (placing both operators on the rightmost site results in only one new term) distinct new terms for each of the k remaining clusters. A bound for the total number of distinct terms is thus given by

$$n(k) \leq (k + 1)^2 + k(2k + 1) = 3k(k + 1) + 1. \quad (3.18)$$

In a free-fermion system, the number of distinct operators in the commutator $[H, A]^{(k)}$ thus grows quadratically with the order of the commutator k . As we will show below when considering the example of the XX chain, the quadratic growth of the number of distinct operators is directly related to a regular structure of the corresponding graph with a constant number of nodes and edges at each level.

IV. LOCALIZATION

Studying the localization of operators when being commuted with the Hamiltonian has the advantage that it is applicable both to the noninteracting and the interacting case. This is in contrast to the localization of wave functions which is only a useful criterion in the noninteracting Anderson case. However, we then have to define precisely what we mean by a local operator staying localized. In the noninteracting Anderson case, there is a simple answer: As we will show below, the usual criterion of the single-particle wave function becoming exponentially localized is equivalent to a local operator remaining strictly exponentially localized when commuted with the Hamiltonian,

$$\frac{s_l(k)}{s_1(k)} \sim \exp(-l), \quad (4.1)$$

with $s_l(k) = |[H, A]_l^{(k)}|$. As we will also show below, however, strict exponential localization implies that the total

operator norm $s(k) \leq \sum_l s_l(k)$ cannot grow faster than exponentially. This does not lead to any contradictions in the Anderson case because we have already shown in Eq. (3.17) that an exponential norm bound always holds in the free-fermion case. It does, however, cause an issue in the many-body case because it is analytically known that the norm always grows faster than exponentially in the Ising model with random longitudinal and transverse fields, and we will provide strong evidence from symbolic calculations in Sec. VB2 that the same is also true in the Heisenberg chain with random fields. That is, we will demonstrate that a strict exponential localization is not possible in the many-body localized case. If one takes exponential localization of local operators as the defining feature of a localized system, then one has to conclude that many-body localization does not exist.

Another possible definition of many-body localization is—as we have alluded to earlier—to demand that a unitary transformation exists which maps the microscopic Hamiltonian onto the effective Hamiltonian (1.2). If one could show that the existence of such a transformation implies that local operators in the microscopic model remain exponentially localized when commuted with the Hamiltonian, then one would have proven that many-body localization is also excluded if defined in this way. We are, however, not able to prove a strict exponential bound but rather only the weaker bound (2.4) which does not exclude an almost-factorial growth of the total norm. However, even the bound (2.4) predicts an almost-exponential decay of the norm as a function of the length of the operator support l if $l \gg k/\kappa$ at fixed order k . We will show in Sec. VB2 through symbolic computations that there is no evidence for such a weaker type of localization even at very strong disorder.

A. Strict exponential localization

In a localized phase, one might expect that a local operator A remains localized to a finite region of space of length ξ_{loc} (up to exponentially small tails) when commuted k times with the Hamiltonian. We call this strict exponential localization and prove in Sec. VA2 that it does hold in the Anderson case.

In any system which is strictly exponentially localized, we expect that connected clusters of length $j \gg \xi_{\text{loc}}$ give almost no contribution to the norm. Starting from the general bound (3.10), which is valid both for noninteracting and interacting systems, and neglecting such clusters leads to

$$\sum_{\{l_1, \dots, l_k\}}' = \sum_{j=1}^{\min(k, \xi_{\text{loc}})} 2^j S(k, j). \quad (4.2)$$

Asymptotically, for large k , we thus obtain

$$\sum_{j=1}^{\xi_{\text{loc}}} 2^j S(k, j) \sim \sum_{j=1}^{\xi_{\text{loc}}} \frac{2^j j^k}{k!} < \frac{2^{\xi_{\text{loc}}}}{(\lceil \xi_{\text{loc}} \rceil - 1)!} \xi_{\text{loc}}^k, \quad (4.3)$$

where we have used the asymptotic scaling $S(k, j) \sim j^k/j!$ of the Stirling numbers of the second kind for $k \gg j$. The operator growth in a strictly localized phase is therefore at

most exponential:

$$\|[H, A]^{(k)}\| \leq \|A\| \frac{2^{\xi_{\text{loc}}}}{(\lceil \xi_{\text{loc}} \rceil - 1)!} (2^{\xi_{\text{loc}}} C J)^k. \quad (4.4)$$

In deriving this bound, we have made the simplifying assumption that the contributions of all clusters with lengths $j \gg \xi_{\text{loc}}$ can be neglected. Given that with increasing length, more possibilities exist to distribute the local Hamiltonians h_i^a , one might worry that this assumption is too naive. We will show in the Supplemental Material [38] that the argument remains valid if clusters with $j > \xi_{\text{loc}}$ are included but are assumed to be exponentially suppressed with their length l .

Another way to think about strict exponential localization is to take the norm growth of terms in the commutator of order k which have support on l lattice sites

$$s_l^{(p)}(k) \equiv \|[H, A]_l^{(k)}\|_p \sim e^k e^{-l} \quad (4.5)$$

directly as the definition of such a phase. Here, the exponential decay with l is the strict exponential localization of the operator, while the exponential growth with the order k is a consequence of having a finite number of possible clusters with lengths less than l that can be visited before returning to a cluster of length l . Each of the k steps in the commutator path then introduces a multiplicative weight, leading to an e^k growth. This is best understood in a graph-theoretical language which is discussed in Sec. V. From Eq. (4.5) it immediately follows that the total norm satisfies $s^{(p)}(k) \leq \sum_l s_l^{(p)}(k) \sim e^k$ because the sum over l converges.

We note, furthermore, that for free fermions both the non-localized and the localized case show an exponential growth of the operator norm. Thus, the total operator norm is insufficient to distinguish between them. What does distinguish them is that the local norm $s_l^{(p)}(k)$ will follow Eq. (4.5) in the localized case while in a nonlocalized phase the initially local operator A will spread over the entire lattice, implying that terms in the commutator with $l \ll k$ will contribute approximately equally to the total norm.

B. Localization defined by local conservation laws

Another way localization might possibly be defined is to demand that a unitary transformation U exists which maps the microscopic model H onto an effective model $\tilde{H}$ given in Eq. (1.2) that has an infinite set of conserved charges τ_i^z . These conserved charges are quasilocal when expressed in the basis of the original Pauli operators σ_i^α of the microscopic model. The effective model $\tilde{H}$ is strictly exponentially localized when considering operators which are fully local in the τ basis. This is a consequence of the exponentially decaying couplings J_{ij} in $\tilde{H}$ and the fact that one of the operators in the Hamiltonian has to overlap with the local operator. If we consider, for example, the commutator of the nearest-neighbor interaction with the local operator τ_n^x , then we find that $[\sum_i J_{i,i+1} \tau_i^z \tau_{i+1}^z, \tau_n^x]^{(k)}$ only produces terms of the form τ_n^x , $\tau_n^y \tau_{n+1}^z$, $\tau_n^z \tau_{n+1}^y$, and $\tau_{n-1}^z \tau_n^x \tau_{n+1}^z$. That is, the length of the operators produced by the commutator cannot grow beyond three sites. Considering also longer-range interactions, terms with additional τ^z 's which are farther apart are produced as well, but the corresponding couplings J_{ij} are exponentially suppressed.

However, it is crucial to recognize that local operators in the microscopic model are generally mapped by the unitary transformation U onto quasilocal operators. We therefore must consider the operator norm growth of a generic quasilocal operator $\tilde{A} = UAU^\dagger$ when commuted with the effective Hamiltonian $\tilde{H}$. We can assume w.l.o.g. that for infinite disorder the conserved charges are given by $\tau_i^z \equiv \sigma_i^z$ and are fully local. For finite disorder, the conserved charges then become quasilocal, i.e., $\tau_i^z = \sigma_i^z + \text{exp.-decaying terms}$. For the derivation of a norm bound, it suffices to consider just the leading term and to write $\tilde{H} = \sum_n \varepsilon_n \sigma_n^z$ where we have also dropped the interaction terms between the conserved charges. We consider a local operator A which is situated at site $i = 0$. This operator then gets mapped onto a quasilocal operator

which we can express as

$$\tilde{A} = UAU^\dagger = \sum_{j,\ell} \sum_{\{\alpha\}} e^{-\kappa_1|j|} e^{-\kappa_2\ell} \sigma_j^{\alpha_0} \otimes \cdots \otimes \sigma_{j+\ell}^{\alpha_\ell}, \quad (4.6)$$

with $\alpha_0, \alpha_\ell \neq 0$. Here, j is the leftmost site of the operator and ℓ is the operator's length. The factors κ_1, κ_2 encode the strength of the exponential decay of the quasilocal operator. We have used the fact that we can expand every operator in the local basis $\sigma_i^\alpha = \{\mathbb{1}_i, \sigma_i^x, \sigma_i^y, \sigma_i^z\}$. We remind the reader that the Frobenius norm is invariant under unitary transformations, i.e., $\|[\tilde{H}, A]^{(k)}\|_2 = \|[\tilde{H}, \tilde{A}]^{(k)}\|_2$, and so a bound for the latter will also be a bound for the former. Using the properties of the norm, we find

$$\begin{aligned} \|[\tilde{H}, \tilde{A}]^{(k)}\|_2 &= \left\| \sum_{j,\ell} \sum_{i_1=j}^{j+\ell} \cdots \sum_{i_k=j}^{j+\ell} \sum_{\{\alpha\}} \varepsilon_{i_1} \cdots \varepsilon_{i_k} e^{-\kappa_1|j|} e^{-\kappa_2\ell} [\sigma_{i_1}^z, [\sigma_{i_2}^z, \dots, [\sigma_{i_k}^z, \sigma_j^{\alpha_0} \otimes \cdots \otimes \sigma_{j+\ell}^{\alpha_\ell}]]] \right\|_2 \\ &\leq \sum_{j,\ell} \sum_{i_1=j}^{j+\ell} \cdots \sum_{i_k=j}^{j+\ell} \sum_{\{\alpha\}} |\varepsilon_{i_1}| \cdots |\varepsilon_{i_k}| e^{-\kappa_1|j|} e^{-\kappa_2\ell} \|[\sigma_{i_1}^z, [\sigma_{i_2}^z, \dots, [\sigma_{i_k}^z, \sigma_j^{\alpha_0} \otimes \cdots \otimes \sigma_{j+\ell}^{\alpha_\ell}]]]\|_2 \\ &\leq 2^k \sum_{j,\ell} \sum_{i=j}^{j+\ell} \cdots \sum_{i_k=j}^{j+\ell} \sum_{\{\alpha\}} |\varepsilon_{i_1}| \cdots |\varepsilon_{i_k}| e^{-\kappa_1|j|} e^{-\kappa_2\ell} \leq (2J)^k \sum_j e^{-\kappa_1|j|} \sum_\ell (\ell+1)^k 4^{\ell+1} e^{-\kappa_2\ell} \\ &= \frac{4(2J)^k}{1-e^{-\kappa_1}} \sum_\ell (\ell+1)^k 4^\ell e^{-\kappa_2\ell} = \frac{4(2J)^k}{1-e^{-\kappa_1}} \sum_\ell (\ell+1)^k e^{-\tilde{\kappa}_2\ell}. \end{aligned} \quad (4.7)$$

Here, we have used the fact that there are 2^k terms in the commutator, and we have bounded $|\varepsilon_i| \leq J$. Furthermore, in the chosen normalization of the norm, we have $\|\sigma_{i_1}^z \sigma_{i_2}^z \cdots \sigma_{i_k}^z \sigma_j^{\alpha_0} \otimes \cdots \otimes \sigma_{j+\ell}^{\alpha_\ell}\|_2 = \|\sigma_{i_1}^z\|_2 \|\sigma_{i_2}^z\|_2 \cdots \|\sigma_{i_k}^z\|_2 \|\sigma_j^{\alpha_0}\|_2 \cdots \|\sigma_{j+\ell}^{\alpha_\ell}\|_2 = 1$. Finally, we have defined $\tilde{\kappa}_2 = \kappa_2 - \ln 4$, and we assume that the disorder D is strong enough such that $\kappa_2 = \kappa_2(D) > \ln 4$, ensuring that we still have an exponential decay. We note that this is consistent with the fact that in the general, interacting case, a finite disorder strength is required for a potentially localized phase, whereas any amount of disorder leads to localization in the noninteracting case.

If the bound (4.7) is tight, then it implies that quasilocal operators, in contrast to fully local operators, do not remain strictly exponentially localized but rather evolve as

$$s_l(k) = \|[\tilde{H}, \tilde{A}]_l^{(k)}\|_2 \sim (Jl)^k e^{-\tilde{\kappa}_2 l}. \quad (4.8)$$

This function is first increasing up to a maximum at $l = k/\tilde{\kappa}_2$ before an almost-exponential decay with l sets in. That is, for any finite k and sufficiently strong disorder, the maximum will be at some small cluster length l and we should observe a decay for larger length that is almost indistinguishable from being exponential. However, no matter how strong the disorder is, and consequently, how large $\tilde{\kappa}_2$ is, a generic quasilocal operator $\tilde{A}$ in the effective Hamiltonian $\tilde{H}$ will eventually fully delocalize for large k . We also note that we have made no assumptions about the structure of the initial microscopic Hamiltonian. If this Hamiltonian only has nearest-neighbor

terms, as we have assumed in Sec. III, then the bound (2.2) will ultimately be the tighter bound for very large k . This is shown in Fig. 3 where the two bounds are compared. We conclude that generic quasilocal operators in a system described by $\tilde{H}$ will delocalize for very large k in exactly the same way as they do in an ergodic system.

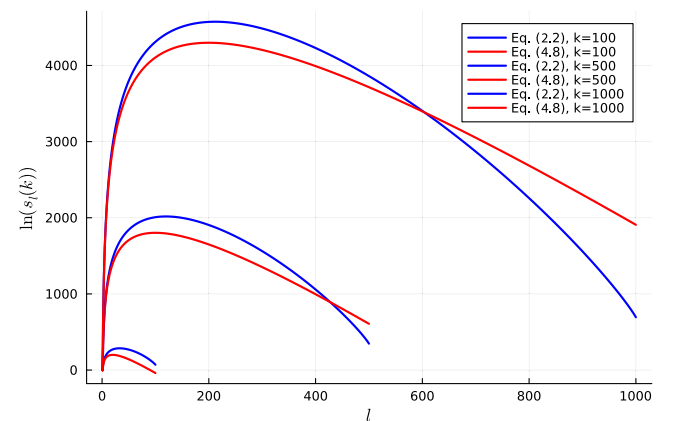


FIG. 3. Comparison between the generic local bound for nearest-neighbor models, Eq. (2.2), and the bound for quasilocal operators in a model described by $\tilde{H}$; see Eq. (4.8) for $J = 1$ and $\tilde{\kappa}_2 = 5$. Note that for very large k , the bound (4.8) will ultimately converge to the bound (2.2) for $l \lesssim k/\tilde{\kappa}_2$ and Eq. (2.2) will then be a tighter bound for $l \gtrsim k/\tilde{\kappa}_2$.

Before we go any further, we have to understand if the bound (4.8) can be tight, and how the Anderson case fits into this picture as a model which is indeed unitarily equivalent to $\tilde{H}$ and in which quasilocal operators $\tilde{A} = UAU^\dagger$, with A fully local, are expected to remain localized. First, we note that the factor of $(\ell + 1)^k$ comes from bounding the commutator in the second line of (4.7) by assuming the “worst case scenario” that every $\sigma_{j+n}^{\alpha_n}$ in the expansion of the quasilocal operator in the microscopic Pauli basis is either a σ_{j+n}^x or a σ_{j+n}^y operator so that the commutator never vanishes no matter where the $\sigma_{l_m}^z$ operators are placed. Can operators that are local in the microscopic model indeed unitarily map onto quasilocal operators of this form? We can first consider the noninteracting case. W.l.o.g., we choose $A = \sigma^z = c^\dagger c - 1/2$. A unitary U will always map one-body to one-body operators, i.e., the operator $\tilde{\sigma}^z = U\sigma U^\dagger$ will also only contain one fermionic creation and one annihilation operator. In the spin language, this means that the quasilocal operator $\tilde{\sigma}^z$ is a sum of terms of the form $\sigma^{x,y} \sigma^z \dots \sigma^z \sigma^{x,y}$, i.e., each term has either a σ^x or a σ^y operator at the end, an arbitrary number of σ^z operators in between, and an amplitude which decays exponentially with the length of the string. In this case, the commutator in the second line of Eq. (4.7) is only nonzero if $i_n = j$ or $i_n = j + \ell$. Therefore, the factor $(\ell + 1)^k$ is replaced by 2^k , and we find

$$\begin{aligned} ||[H, A]^{(k)}||_2 &= ||[\tilde{H}, \tilde{A}]^{(k)}||_2 \leq \sum_{\ell} ||[\tilde{H}, \tilde{A}]_{\ell}^{(k)}||_2 \\ &\leq \frac{4(4J)^k}{1 - e^{-\kappa_1}} \sum_{\ell} e^{-\tilde{\kappa}_2 \ell} = \frac{4(4J)^k}{(1 - e^{-\kappa_1})(1 - e^{-\tilde{\kappa}_2})}. \end{aligned} \quad (4.9)$$

This is consistent with our previous findings (3.17) that the total norm in the free-fermion case can never grow faster than exponentially and that local operators remain strictly exponentially localized; see Eq. (4.5).

In the interacting case, however, no such restrictions exist, and one-body operators can be mapped onto many-body operators. Consider, for example, again the case where $A = \sigma^z$ and therefore $A^2 = (\sigma^z)^2 = \mathbb{1}$. Because the transformation is unitary, this implies that $(\tilde{\sigma}^z)^2 = U\sigma^z U^\dagger U\sigma^z U^\dagger = \mathbb{1}$ as well. This imposes some restrictions on the quasilocal operator $\tilde{\sigma}^z$, and there might be other, less trivial conditions which can be imposed. We note that if such conditions can be found for a microscopic model, and if they lead to a tighter bound, then there may be a direct contradiction between an almost-factorial norm growth and the existence of the unitary U . Without such additional constraints, we can explicitly construct an operator which is quasilocal, fulfills the condition $(\tilde{\sigma}^z)^2 = \mathbb{1}$, and for which the bound (4.7) is asymptotically tight. Such an operator is, for example, given by

$$\tilde{\sigma}^z = \sqrt{1 - \sum_l e^{-2\kappa} \sigma_0^z + \sum_l e^{-\kappa l} \sigma_0^y \otimes \dots \otimes \sigma_{l-1}^y \otimes \sigma_l^x}, \quad (4.10)$$

where we define to have just a single σ^x in the sum for $l = 0$. This example thus demonstrates that in the many-body case, local operators may get unitarily mapped onto quasilocal operators which do not remain exponentially localized when commuted with the effective Hamiltonian.

Given that we therefore have to take the bound (4.7) seriously, there are two important remaining questions to address: (1) How does the total norm scale? (2) What does this bound imply for the spreading of the fully local operator A in the microscopic model with Hamiltonian H ? To answer the first question, we can perform the sum over the length ℓ of the terms of the quasilocal operator in Eq. (4.7) to find

$$\begin{aligned} ||[H, A]^{(k)}||_2 &= ||[\tilde{H}, \tilde{A}]^{(k)}||_2 \leq \frac{4(2J)^k}{1 - e^{-\kappa_1}} \sum_{\ell=0}^{\infty} (\ell + 1)^k e^{-\tilde{\kappa}_2 \ell} \\ &= \frac{4(2J)^k e^{\tilde{\kappa}_2}}{1 - e^{-\kappa_1}} \sum_{\ell=1}^{\infty} \ell^k e^{-\tilde{\kappa}_2 \ell} = \frac{4(2J)^k e^{\tilde{\kappa}_2}}{1 - e^{-\kappa_1}} \text{Li}_{-k}(e^{-\tilde{\kappa}_2}) \\ &\stackrel{k \gg 1}{\sim} \frac{4e^{\tilde{\kappa}_2}}{\tilde{\kappa}_2(1 - e^{-\kappa_1})} \left(\frac{2J}{\tilde{\kappa}_2}\right)^k k!, \end{aligned} \quad (4.11)$$

where in the last step we used the asymptotics of the polylogarithm $\text{Li}_{-k}(e^{-\tilde{\kappa}_2})$ for large commutator order k . For a quasilocal operator $\tilde{A}$ in Eq. (4.6) where the contribution of terms decays quickly with their length $\tilde{\kappa}_2 = \kappa_2 - \ln 4 \gg 1$, i.e., in the limit of strong disorder, we thus find again a bound which grows slightly slower than factorially, similar to the bound (3.12) we derived in Sec. III A. Note that in deriving Eq. (4.11), we have not made any assumptions about the structure of the Hamiltonian H . Therefore, this bound is not quite as tight as Eq. (3.12), which was derived for a Hamiltonian with only nearest-neighbor terms.

To answer the second question, we can use the fact that a unitary transformation leaves the Frobenius norm invariant, and that the inverse transformation makes operators more local, to show

$$\begin{aligned} ||[H, A]_m^{(k)}|| &= ||(U^\dagger [\tilde{H}, \tilde{A}]^{(k)} U)_m|| \leq \sum_{l=m}^{m+\xi} ||[\tilde{H}, \tilde{A}]_l^{(k)}|| \\ &\leq \frac{4(2J)^k}{1 - e^{-\kappa_1}} \sum_{l=m}^{m+\xi} (l + 1)^k e^{-\tilde{\kappa}_2 l} \\ &= \frac{4(2J)^k}{1 - e^{-\kappa_1}} (m + 1)^k e^{-\tilde{\kappa}_2 m} \\ &\quad \times \left[1 + \left(\frac{m+2}{m+1}\right)^k e^{-\tilde{\kappa}_2} + \dots \right] \\ &\approx \frac{1 + e^{-\tilde{\kappa}_2}}{1 - e^{-\kappa_1}} 4(2J)^k (m + 1)^k e^{-\tilde{\kappa}_2 m}. \end{aligned} \quad (4.12)$$

That is, mostly terms in the commutator $[\tilde{H}, \tilde{A}]^{(k)}$ with length $l \in [m, m + \xi]$ will unitarily map to terms in $[H, A]^{(k)}$ with length m . Furthermore, we assume that we are in the strong disorder regime in which $\tilde{\kappa}_2 \gg 1$ so that the function in the square brackets in the third line of Eq. (4.12) has weak m and k dependence and can therefore be approximated by $1 + e^{-\tilde{\kappa}_2}$. In other words, the norm of terms with length m in the commutator $[H, A]^{(k)}$ is given by a “running average” of the norms of terms with lengths $l \in [m, m + \xi]$ in the unitarily transformed commutator $[\tilde{H}, \tilde{A}]^{(k)}$. If the quasilocal operator $\tilde{A}$ decays quickly, as is expected for strong disorder, then this will only renormalize the prefactor and will not change the functional form of the norm of such terms. We thus conclude

that in a model where the unitary transformation U maps the local operator A onto a generic quasiloc operator $\tilde{A}$, the local operator will *not remain exponentially localized*, but will rather spread throughout the entire lattice as $||[H, A]_l^{(k)}|| \sim (2Jl)^k e^{-\tilde{k}_2 l}$, which means that for large commutator order k , the operator spreads exactly as in an ergodic system.

C. Consequences for transport

One might then ask what the consequences of the spreading of local operators for the transport properties of a system are. Let us consider a globally conserved charge $Q = \sum_i q_i$ where the local densities are, for example, $q_i = \sigma_i^z$ for a spin model or $q_i = c_i^\dagger c_i$ for a fermionic chain. Let us denote the corresponding current by $\mathcal{J} = \sum_i j_i$ where j_i is the current density. We are interested in the transport properties in the infinite temperature limit. Assuming that the model has no Drude weight in this limit, which will always be the case once a possible integrability is broken by disorder, we can obtain the diffusion constant $\mathcal{D}$ from

$$T\mathcal{D} = \lim_{\omega \rightarrow 0} \lim_{L \rightarrow \infty} \frac{1}{2\chi L} \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \mathcal{J}(t) \mathcal{J}(0) \rangle, \quad (4.13)$$

where χ is the corresponding susceptibility [56]. We note that in the high-temperature limit, $T\mathcal{D}$ is the proper nontrivial quantity to consider. We can write the correlator as

$$\langle \mathcal{J}(t) \mathcal{J}(0) \rangle = \sum_{n,m} \langle j_n(t) j_m(0) \rangle = \sum_{n,m,k} \langle [H, j_n]^{(k)} j_m \rangle \frac{(it)^k}{k!}. \quad (4.14)$$

That is, the behavior of the correlator is determined by the same k -fold correlator we have been considering earlier.

For a translationally invariant system, we can write $\sum_{n,m} \langle j_n(t) j_m(0) \rangle = L \sum_m \langle j_0(t) j_m(0) \rangle \equiv LC(t)$, implying that $T\mathcal{D} = \frac{1}{2\chi} C(\omega \rightarrow 0)$, which is in general nonzero in the thermodynamic limit. For an exponentially localized system, however, the amplitudes of terms in $[H, j_n]^{(k)}$ will decay exponentially with the length of their support on some length scale ξ , implying that $\sum_{n,m} \langle j_n(t) j_m(0) \rangle \approx \sum_n \sum_{m=n-\xi}^{n+\xi} \langle j_n(t) j_m(0) \rangle \equiv \sum_n C_n(t)$. For $L \gg \xi$, and rounding ξ to the nearest integer, we then obtain

$$\begin{aligned} \frac{1}{L} \sum_n C_n(t) &= \frac{1}{L} [C_0(t) + C_\xi(t) + \dots] \\ &\quad + (C_1(t) + C_{1+\xi}(t) + \dots) + \dots \\ &\xrightarrow{L \rightarrow \infty} 0, \end{aligned} \quad (4.15)$$

because correlators that are farther than ξ apart are essentially uncorrelated, and so for each time t , the terms $\frac{1}{L}(C_i(t) + C_{i+\xi}(t) + \dots)$ are averages over random numbers and thus tend to zero. We therefore find that, as expected, the diffusion constant $T\mathcal{D}$ vanishes in a phase in which local operators remain exponentially localized.

For the case in which the current operator at large commutator orders eventually spreads throughout the entire lattice, i.e., $[H, j_n]_l^{(k)} \sim l^k e^{-l}$, no similar argument for the vanishing of the diffusion constant can be made because all local current

densities will become correlated with each other. We have also seen that for very large k , the asymptotics of $[H, j_n]^{(k)}$ will be exactly the same as those for an ergodic system. In this case, one would thus generically expect that $\frac{1}{L} \sum_n C_n(t)$ is nonzero in the thermodynamic limit, indicating transport of the globally conserved charge. This finding is consistent with the results in Refs. [23–25] where the transport of charges between subsystems of the disordered Heisenberg chain was studied numerically. Of course, in contrast to Euclidean time evolution, the real-time evolution (4.14) cannot be related to a norm because the i^k factor allows for interference effects. While one can imagine some fine-tuning where such interference effects lead to a vanishing of the diffusion constant, it is difficult to see how this can be a generic scenario which is stable against small changes in the microscopic parameters.

We also want to point out that it is a common misconception that the operator growth considered here is related to short-time dynamics. As one can immediately see from Eq. (4.14), the short-time dynamics are determined only by $[H, j_n]^{(2)}$, whereas we have been focusing on the asymptotic behavior of the k -fold commutator for large k . It is correct though that the asymptotic behavior of the commutator is insufficient to fully determine the long-time behavior of real-time correlation functions. To do so, we would need additional information about the behavior at small k and possibly also about subleading terms to the large k asymptotics. This point is discussed in more detail in Ref. [31]. For the discussion here, this means that while the spreading of local current densities over the entire lattice implies transport of the globally conserved charge and diffusion seems to be the most natural scenario, other more exotic scenarios of anomalous diffusion are not ruled out.

V. XXZ MODEL WITH (QUASI)RANDOM FIELDS

As a specific example, we want to consider the XXZ chain with random or quasirandom magnetic fields,

$$H = \sum_j (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z + 2h_j \sigma_j^z). \quad (5.1)$$

Here, $\sigma^{x,y,z}$ are Pauli spin-1/2 matrices, Δ characterizes the spin exchange anisotropy, and h_j are magnetic fields. As the operator which we will commute with the Hamiltonian H we consider a single local σ_j^z . The model (5.1) is equivalent to a spinless fermion model using the Jordan-Wigner transformation

$$\begin{aligned} \sigma_j^+ &= S_j c_j^\dagger, \quad \sigma_j^- = S_j^\dagger c_j, \quad \sigma_j^z = 2c_j^\dagger c_j - 1, \\ S_j &= \exp \left[-i\pi \sum_{k=1}^{j-1} c_k^\dagger c_k \right] = \prod_{k=1}^{j-1} [1 - 2c_k^\dagger c_k], \end{aligned} \quad (5.2)$$

with $\sigma_j^\pm = (\sigma_j^x \pm i\sigma_j^y)/2$ leading to

$$\begin{aligned} \frac{H}{4} &= \frac{1}{2} \sum_j \{c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j\} \\ &\quad + \sum_j \{\Delta(n_j - 1/2)(n_{j+1} - 1/2) + h_j(n_j - 1/2)\}. \end{aligned} \quad (5.3)$$

Here $c_j^{(\dagger)}$ are fermionic annihilation (creation) operators and $n_j = c_j^\dagger c_j$. In the following, we will investigate the noninteracting case, $\Delta = 0$, and the interacting isotropic Heisenberg case, $\Delta = 1$.

A. Noninteracting case

We will first concentrate on the noninteracting free-fermion case, $\Delta = 0$, where localization is well understood. Here we want to show that the general bounds for the total norm of the k th order commutator derived earlier do apply and that we can distinguish localized from nonlocalized phases by resolving the contributions to the norm by the length of the spatial support of each term in the commutator.

1. No disorder

The first specific case we will consider is the free-fermion case, $\Delta = 0$, without any magnetic fields, $h_j = 0$. In this case, the entire structure of the commutator $[H, \sigma_0^z]^{(k)}$ can be worked out analytically. We find these results instructive and will discuss them in detail here to motivate a description of the commutator by graphs where the vertices are the terms

generated and the edges denote their contribution to the 1-norm.

The full analytical solution is best understood in the fermionic language: the commutator $[H, c_0^\dagger c_0]^{(k)}$ will only contain terms which are one-particle operators, i.e., terms which contain exactly one annihilation and one creation operator. More specifically, we find for the case that the order of the commutator k is odd that

$$\begin{aligned} [H, \sigma_0^z]^{(k)} &= \sum_{l=0}^{\frac{k-1}{2}} \sum_{s=-\frac{k+1}{2}-l}^{\frac{k-1}{2}-l} a_{kls} \left\{ \sigma_s^x \bigotimes_{j=s+1}^{s+2l} \sigma_j^z \sigma_{s+2l+1}^y \right. \\ &\quad \left. - \sigma_s^y \bigotimes_{j=s+1}^{s+2l} \sigma_j^z \sigma_{s+2l+1}^x \right\} \\ &= -\frac{2}{i} \sum_{l=0}^{\frac{k-1}{2}} \sum_{s=-\frac{k+1}{2}-l}^{\frac{k-1}{2}-l} a_{kls} \{c_s^\dagger c_{s+2l+1} - \text{H.c.}\}, \end{aligned} \quad (5.4)$$

with coefficients a_{kls} which are discussed below. For k even, we find instead

$$\begin{aligned} [H, \sigma_0^z]^{(k)} &= \sum_{s=-\frac{k}{2}}^{\frac{k}{2}} a_{k0s} \sigma_s^z + \sum_{l=1}^{\frac{k}{2}} \sum_{s=-\frac{k}{2}-l}^{\frac{k}{2}-l} a_{kls} \left\{ \sigma_s^x \bigotimes_{j=s+1}^{s+2l-1} \sigma_j^z \sigma_{s+2l}^x + \sigma_s^y \bigotimes_{j=s+1}^{s+2l-1} \sigma_j^z \sigma_{s+2l}^y \right\} \\ &= 2 \sum_{s=-\frac{k}{2}}^{\frac{k}{2}} a_{k0s} (n_s - 1/2) - 2 \sum_{l=1}^{\frac{k}{2}} \sum_{s=-\frac{k}{2}-l}^{\frac{k}{2}-l} a_{kls} \{c_s^\dagger c_{s+2l} + \text{H.c.}\}. \end{aligned} \quad (5.5)$$

Note that the commutator has the form of a current density for k odd and that of a charge density for k even. From the formulas (5.4) and (5.5) we can immediately read off that there are $n(k) = \frac{k+1}{2}(k+1) \cdot 2 = k+1 + \frac{k}{2}(k+1) \cdot 2 = (k+1)^2$ different terms generated at order k , both in the odd and in the even case. We confirm this result by calculating the commutator using symbolic manipulations up to high order; see Fig. 4. We want to stress that the quadratic growth of the number of terms is a special property of a free fermionic model where a one-body operator, when commuted with the Hamiltonian, always remains a one-body operator. What changes is only the separation between the creation and annihilation operator (support of the one-body operator) and the number of different sites where the first of the fermionic operators is located. Both the allowed separations as well as the possible sites for the first fermionic operator grow linearly in k , leading to the overall quadratic growth in the number of distinct terms in the commutator. We already want to note here that this behavior is expected to change dramatically in the interacting case.

In the simple, nondisordered XX case considered here, we can even take it one step further and calculate the coefficients a_{kls} exactly. These coefficients are completely determined by the coefficient matrices C_k which are defined by

$$(C_k)_{ij} = 2^k k \binom{k}{i-1} \binom{k}{j-1}, \quad (5.6)$$

with $1 \leq i, j \leq k+1$. In the following, we concentrate on the case of k odd which has a slightly simpler structure; see Eq. (5.4). In this case we find, in particular, that

$$a_{kls} = i(-1)^s (C_k)_{s+\frac{k+3}{2}, l, \frac{k+1}{2}-l}. \quad (5.7)$$

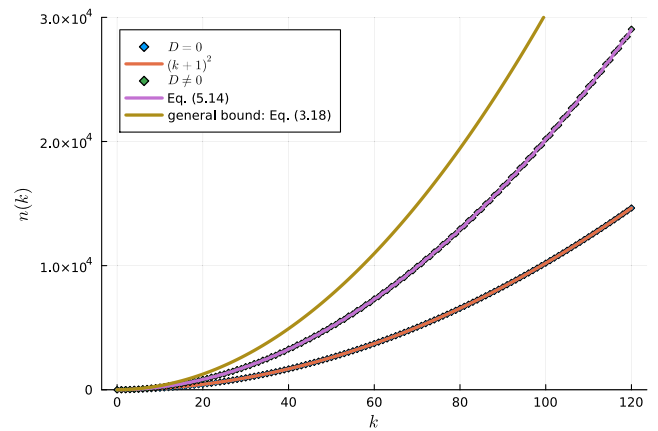


FIG. 4. XX model: Without disorder, the number of distinct terms $n(k)$ in $[H, \sigma_0^z]^{(k)}$ grows as $(k+1)^2$ while it grows according to Eq. (5.14) with disorder. The general bound (3.18), valid for any free-fermion model, is also shown.

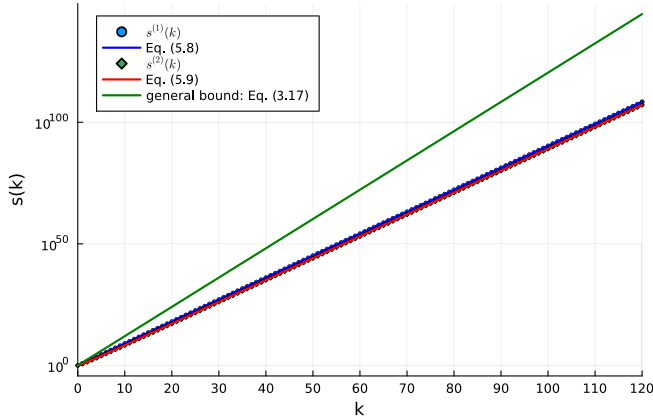


FIG. 5. In the XX model without disorder the norms grow as $s^{(1)}(k) = 8^k$ and $s^{(2)}(k) = 2^k \binom{2k}{k}$. Note that on this scale both results are very close to each other. The 2-norm is only slightly smaller than the 1-norm. Also shown is the general bound, Eq. (3.17), with $C = 2$.

From the result (5.7) we can now determine the total norms of the commutator and, furthermore, we can even determine which operators contribute how much to the total norm. From Eqs. (5.4) and (5.7) we find that the set of the coefficients $\{|a_{kl}|^2\}$ is the same as $\{(C_k)_{ij}\}^2$, i.e., each entry in C_k appears two times. We therefore find

$$s^{(1)}(k) = \|[H, \sigma_0^z]^{(k)}\|_1 = 2 \sum_{i=1}^{k+1} \sum_{j=1}^{\frac{k+1}{2}} (C_k)_{ij} = 8^k, \quad (5.8)$$

where we have used the explicit form of the coefficient matrix (5.6). Similarly, we find

$$(s^{(2)}(k))^2 = \|[H, \sigma_0^z]^{(k)}\|_2^2 = 2 \sum_{i=1}^{k+1} \sum_{j=1}^{\frac{k+1}{2}} (C_k)_{ij}^2 = 2^{2k} \binom{2k}{k}^2. \quad (5.9)$$

Asymptotically, the 2-norm therefore grows as

$$s^{(2)}(k) = 2^k \binom{2k}{k} \approx \frac{8^k}{\sqrt{\pi k}}, \quad (5.10)$$

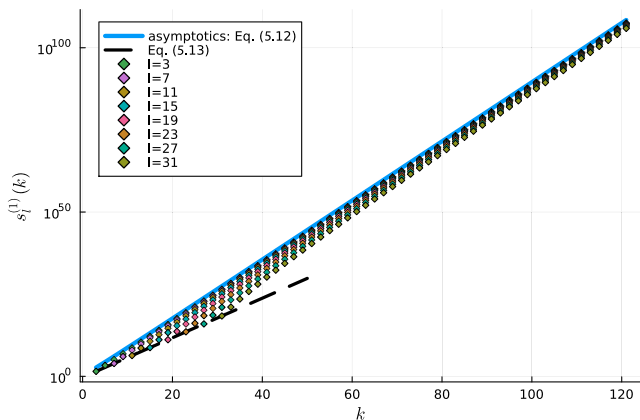


FIG. 6. For $k \gg l$ the norm of operators with support on l sites is asymptotically described by Eq. (5.12) and is independent of l . The exact value (5.13) when $s_l^{(1)}(k)$ first becomes nonzero is also shown.

where we have used Stirling's formula. This shows that the 2-norm grows also exponentially in k but with a correction which makes it always smaller than the 1-norm as expected on general grounds; see Eq. (3.6). We can again check these results for finite k by symbolic manipulations; see Fig. 5.

Finally, we can study how much terms which extend over a distance l contribute to the total norm. We consider the case k odd where the amplitude a_{kls} for terms which have support on $2l + 2$ sites is given by Eq. (5.7). We find again that the same entries appear multiple times and that all prefactors for a given l are represented by the column $j = \frac{k+1}{2} - l$ of the coefficient matrix $(C_k)_{ij}$. Summing over all initial sites we thus obtain

$$\begin{aligned} s_{2l+2}^{(1)}(k) &= \|[H, \sigma_0^z]_{2l+2}^{(k)}\|_1 = \sum_i |a_{kli}| \\ &= 2 \sum_{i=1}^{k+1} (C_k)_{i, \frac{k+1}{2}-l} = 2^{2k+1} \binom{k}{\frac{k-1}{2}-l}, \\ &\rightarrow s_l^{(1)}(k) = 2^{2k+1} \binom{k}{\frac{k-1}{2}}, \end{aligned} \quad (5.11)$$

where in the last line we transformed the equation to show the support on l sites rather than on $2l + 2$ sites. In the limit $k \gg l$ we can apply Stirling's formula and find

$$s_l^{(1)}(k) \approx \sqrt{\frac{8}{k\pi}} 8^k. \quad (5.12)$$

This means that all the terms *contribute equally independent of the length of their support l* which clearly indicates delocalization. We also note that the largest possible support at commutator order $k - 1$ are k sites with

$$s_k^{(1)}(k - 1) = 2^{2k-1}. \quad (5.13)$$

These analytical results can be checked by symbolic manipulations; see Fig. 6.

While we have been able to exactly calculate the k th order commutator in the noninteracting, nondisordered case, see Eqs. (5.4), (5.5), and (5.7), this is clearly no longer possible, even in principle, if random fields are included. On a positive note, this is also more than what we need. To distinguish between free-fermionic localized and nonlocalized phases and interacting localized and nonlocalized phases it is sufficient to consider the total commutator norm and the norm of the terms in the commutator which are supported on l sites. This can be achieved more efficiently by developing graphs where the vertices represent the possible terms at order k and the edges represent the contribution to the 1-norm. Figure 7 shows such a graph for the noninteracting model. The case without random magnetic fields discussed so far corresponds to the case $D = 0$ where only the terms in the left column are present. In this case, the graph is a 2-regular graph with one node per level and two edges per node. The constant number of nodes per level is directly reflected in the quadratic growth of the number of distinct terms $n(k)$.

To obtain $s^{(1)}(k)$, we have to sum over all weighted walks through the graph which start at the top and have length k by multiplying the values of all traversed edges, irrespective of the vertex the walk ends on. If, however, we want to calculate

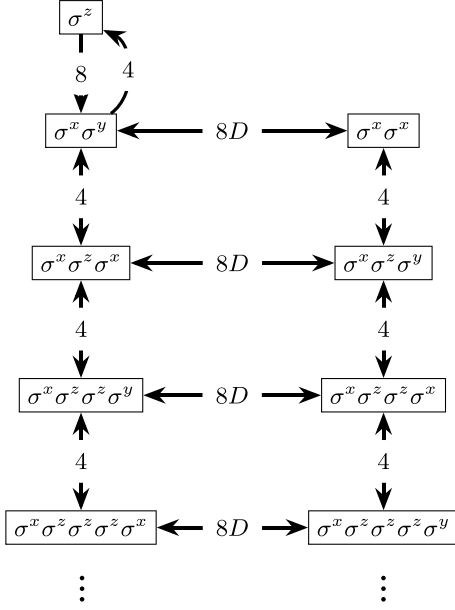


FIG. 7. Graph for the commutator $[H, \sigma_0^z]^{(k)}$ in the noninteracting case, $\Delta = 0$. The vertices represent the possible terms with support on l sites ($l = 1, 2, \dots$ from top to bottom) while the edges represent the contribution to the 1-norm. The number of traversed edges equals the order of the commutator k . Terms obtained by $\sigma^x \leftrightarrow \sigma^y$ are grouped together into a single vertex. Without disorder, $D = 0$, only the terms in the left column will be present.

only the contributions of terms which have support on l sites, $s_l^{(1)}(k)$, then we have to sum only over those weighted walks with k steps which end at a vertex with terms on l sites. We discuss this approach to calculating or bounding the norm of the commutator in more detail in the Supplemental Material [38]. Figure 7 also already shows how the graph has to be modified in the Anderson case which we discuss next.

2. Anderson localization

Next, we consider the case of quenched disorder where the magnetic fields h_j in the Hamiltonian (5.1) are drawn randomly from a box distribution, $h_j \in [-D, D]$. We note that in the one-dimensional case considered here any amount of disorder is a relevant perturbation. The model is localized for all $D > 0$. Since h_j is random, we cannot expect exact results for the norm. Instead, we will derive strict upper bounds.

What we can do exactly, however, is to count the number of distinct terms $n(k)$ in the commutator which occur at order k and which is independent of the specific disorder realization. This is easiest done by making use of the graph, see Fig. 7, and by taking also into account the different possibilities where in the lattice these terms can be located. Doing so we find

$$n(k) = 2k(k+1) + \begin{cases} 1 & \text{even,} \\ 1-k & \text{odd.} \end{cases} \quad (5.14)$$

This result is compared to symbolic calculations in Fig. 4. We stress once more that Eq. (5.14) is independent of the type of disorder or the specific disorder configuration and is thus also valid in the Aubry-André case discussed in Sec. V A 3. Note that while the precise number of terms has changed as

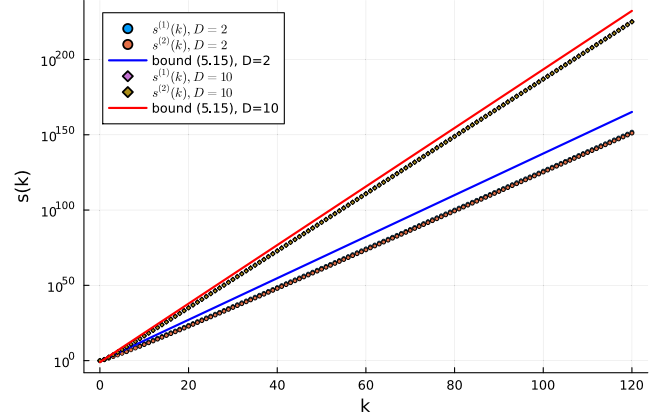


FIG. 8. $s^{(1)}(k)$ and $s^{(2)}(k)$ for the Anderson model with $D = 2$ and $D = 10$. The results are averaged over 100 samples. Note that the 2-norm is only slightly smaller than the 1-norm and the two norms are almost indistinguishable on this scale. The bound (5.15) is shown as well.

compared to the nondisordered case, importantly the number of terms still only increases quadratically with k . To bound the 1-norm, we use the graph shown in Fig. 7. With random fields, the additional terms in the right column and the corresponding additional paths are present. In the first step, we collect a factor of 8 in the 1-norm. In every other step, we can collect at most a factor $8 + 8D$ where we have bounded the disorder by $|h_j| \leq D$. We therefore obtain the following bound:

$$s^{(1)}(k) \leq 8(8 + 8D)^{k-1} = 8^k(D+1)^{k-1}. \quad (5.15)$$

As expected, we find that the norm has an exponential bound. Note that this bound increases with increasing disorder D ; see Fig. 8. Also note that for $D = 0$ we obtain the exact result for the norm in the nondisordered case, Eq. (5.8). Apart from the initial node and the node on the top right, the graph with disorder is a 3-regular graph. That is, each node is connected to three edges. We note that in every n -regular graph the norm $s^{(1)}(k)$ can not grow faster than exponential. This remains true even if the number of nodes increases with k as, for example, in a Cayley tree as long as the number of edges per node is constant. This follows immediately from the fact that $s^{(1)}(k)$ is the sum of the product of edge values along all possible paths of length k . That $n(k)$ grows only quadratically means that the graph in the free-fermion case is particularly simple and has a constant number of nodes at each level.

To see the difference between a free-fermionic model, where an initially localized operator spreads over the entire chain, and a system where the operator remains localized it is thus not sufficient to study the total operator norm. Instead, we have to investigate the spatial structure of the terms in the commutator at order k . Based on the graph in Fig. 7, we can derive a bound for the contribution of terms with support on l sites to the norm at order k . We use again that $|h_j| \leq D$. The derivation of this bound is discussed in detail in the Supplemental Material [38]. Here, we will consider the simplest possible approach where we only take into account those terms with support on l sites which have the largest possible number of factors h_j . This approach is justified for $D \gg 1$

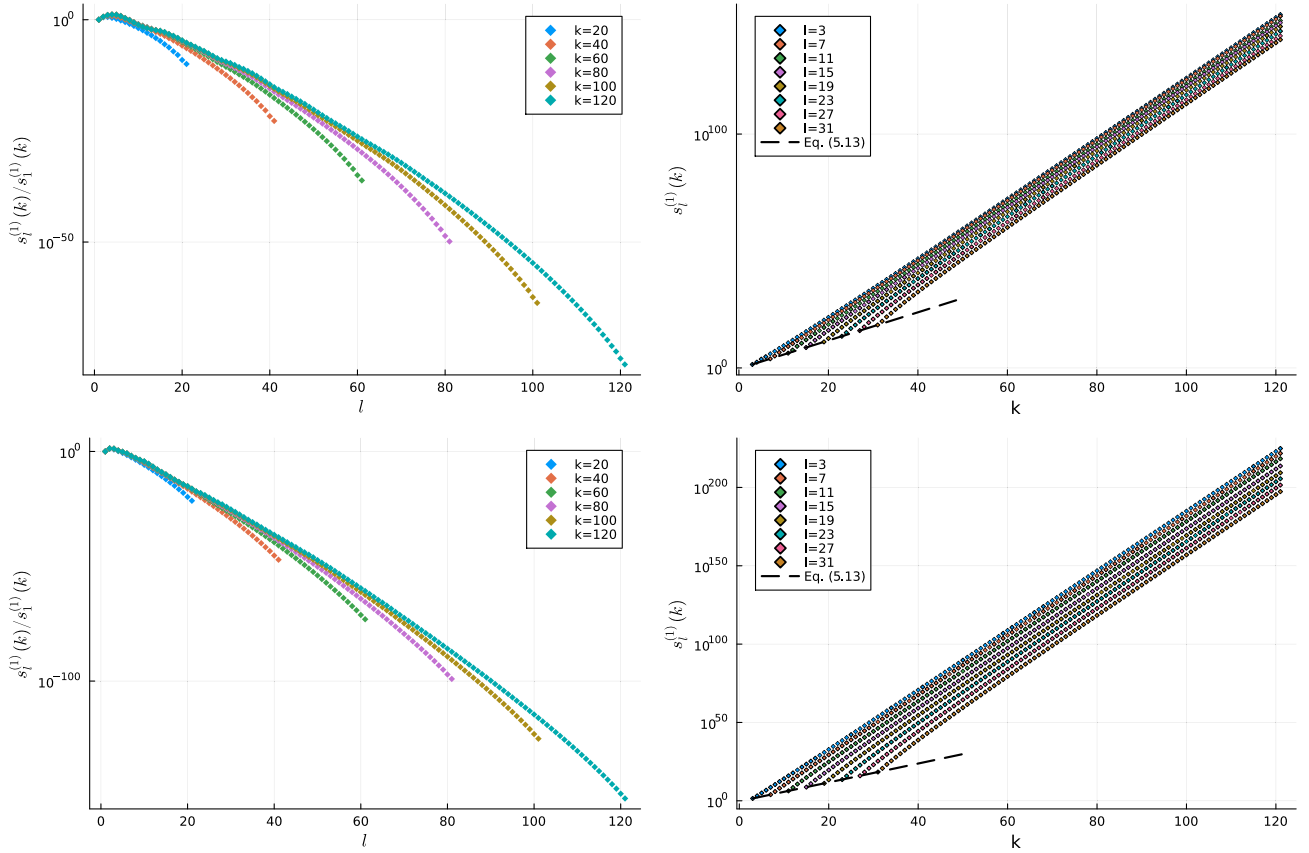


FIG. 9. Left column: Norm ratio $s_l^{(1)}(k)/s_1^{(1)}(k)$ for a fixed k as a function of l for the Anderson model with $D = 2$ (top row) and $D = 10$ (bottom row). Right column: $s_l^{(1)}(k)$ for a fixed l as a function of k . The data are averaged over 100 samples. Note that the value of $s_l^{(1)}(k)$ when it first becomes nonzero at $k = l - 1$ is not affected by disorder; see Eq. (5.13).

and, as we will argue, gives the same asymptotics in the limit $k \gg l \gg 1$ as the more sophisticated approach discussed in the Supplemental Material [38]. Let us first consider terms which have support on a single site, i.e., which are of the form σ_j^z . As is obvious from the graph in Fig. 7, such terms are only generated at even orders of k . We collect the largest number of factors h_j if we go down one level, then travel $k - 2$ times along the vertical direction, and then go back up to the top of the graph. The 1-norm of such terms is thus bounded by

$$s_1^{(1)}(k) \leq 8 \times (8D)^{k-2} \times 4 = 2^{2k+1}(2D)^{k-2}. \quad (5.16)$$

Next, we consider the norm of terms with support on $l \geq 2$ sites with the largest number of possible h_j factors. In the first step, we have to go one level down in the graph in Fig. 7 which contributes a factor of 8. To get to terms with support on $l \geq 2$ sites we have to take another $l - 2$ steps down, contributing a factor 4^{l-2} . We can distribute these steps among a total of $k - 1$ remaining steps, i.e., there are $\binom{k-1}{l-2}$ possibilities. Finally, we get a factor $(8D)^{k-1-(l-2)}$ from the remaining steps along the vertical lines in the graph. Thus, the one norm of such terms is bounded by

$$s_l^{(1)}(k) \leq \binom{k-1}{l-2} 2^{2k+1} (2D)^{k-l+1}. \quad (5.17)$$

The relative contribution to the norm of terms with support on l sites when normalized to the contribution coming from single-site operators is thus given by

$$\frac{s_l(k)}{s_1(k)} \lesssim \binom{k-1}{l-2} (2D)^{3-l} \approx \frac{1}{\sqrt{2\pi l}} \left(\frac{ke}{l}\right)^l e^{-\frac{l^2}{2k}} (2D)^{3-l}. \quad (5.18)$$

We note that while for a fixed k we indeed find that the ratio (5.18) goes exponentially to zero for large l , there is a maximum in the ratio which shifts to larger l with increasing k . This is not quite the behavior we expect. Instead, we expect that with increasing k the ratio (5.18) converges to a function which is exponentially decreasing in l . Here it is important to note that (5.18) is only an upper bound. In particular, we have only considered those paths which have a maximum number of h_j factors. There are, however, many additional paths and we expect that the prefactor in (5.18) will eventually converge with k for a given l . This is consistent with what we observe from symbolic manipulations to finite orders of k ; see Fig. 9. We note that in contrast to the nondisordered case where $s_l(k)$ grows exponentially and independent of l for $k \gg l$, see Eq. (5.12) and Fig. 6, we find that with quenched disorder

$$s_l(k) \sim e^k D^{-l}, \quad (5.19)$$

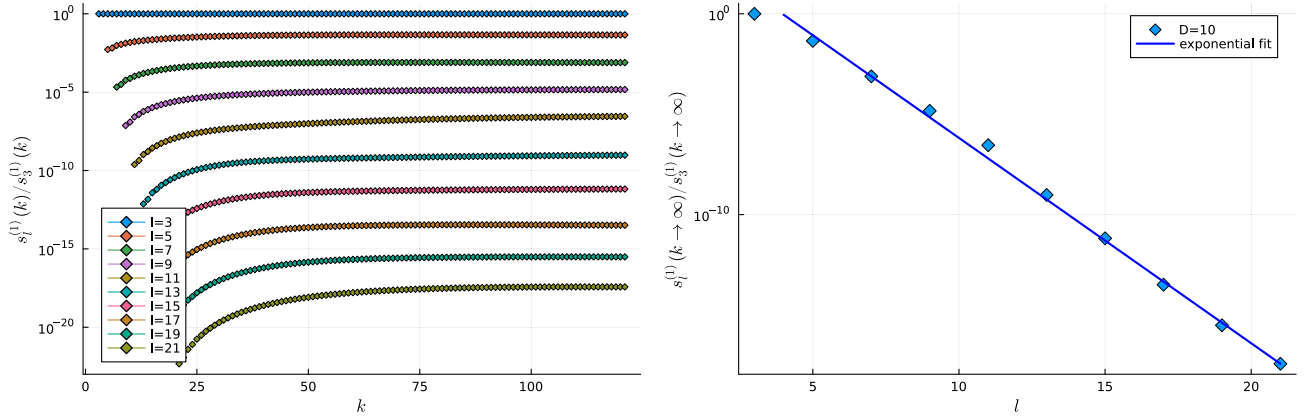


FIG. 10. Anderson model with $D = 10$. Left: Norm ratio $s_l^{(1)}(k)/s_3^{(1)}(k)$ for a fixed l as a function of k . Right: The norm ratio $s_l^{(1)}(k \rightarrow \infty)/s_3^{(1)}(k \rightarrow \infty)$ decays exponentially with the support l . The data are averaged over 100 samples.

see Fig. 10. The spatial structure of the norm contributions $s_l(k)$ thus provide a clear distinction between the nonlocalized and localized phases.

3. Aubry-André model

To further check that the exponential decay of $s_l(k) = ||[H, \sigma_0^z]_l^{(k)}||$ with l is indeed the signature of a localized system, we consider the Aubry-André model next. This is a model which is typically formulated as a fermionic model (5.3) where instead of quenched disorder we have a periodic potential

$$h_j = \frac{\lambda}{2} \cos(2\pi\beta j + \phi). \quad (5.20)$$

Then, for β Diophantine and almost any ϕ —excluding a set of measure zero of resonant phases—the eigenstates of the Aubry-André model are localized for $\lambda > 2$ [46,47]. A common choice is to use the golden ratio $\beta = (1 + \sqrt{5})/2$. The model thus provides an important test case where both a nonlocalized and a localized phase exist. Below the transition in the nonlocalized phase we expect that $s_l(k)$ is qualitatively described by Eq. (5.12), i.e., grows with k independent of l for $k \gg l$. In the localized phase, however, we expect that $s_l(k)$ is exponentially localized; see Eq. (5.19). Note that the graph for the Aubry-André model is the same as the one in the Anderson case shown in Fig. 7 with $D \rightarrow \lambda/2$ on the edges. With this replacement, also the bound (5.15) holds. We note that the considerations which let us to Eq. (5.19) in the Anderson case were valid only in the case of strong disorder. In this limit, they remain valid for the Aubry-André model. That the exponential decaying structure for the spatial support of the norm remains unchanged for all $D \neq 0$ in the Anderson case while there is a transition to the structure (5.12), which is independent of l for $k \gg l$, for the Aubry-André model is beyond the strong disorder arguments we used in the previous section. All these results and expectations outlined above are fully consistent with the symbolic computations shown in Fig. 11. Note that the data in the left panels for the nonlocalized phase are shown on a linear scale while the results in the right panels for the localized phase are shown on a logarithmic scale.

While other methods might be better suited to determine the exact phase transition point, there is a clear *qualitative* difference in the scaling of $s_l(k)$ between the extended and the localized phase. The operator growth $s_l(k)$ therefore allows for an unambiguous distinction between these two phases.

B. Interacting case

After having checked the analytical results derived in Secs. III and IV for the noninteracting nonlocalized and localized cases, we can now look at the interacting isotropic Heisenberg (XXX) model, i.e., the Hamiltonian (5.1) with $\Delta = 1$. A fundamental general difference in interacting as compared to noninteracting models is that one-body operators do not remain one-body operators when commuted with the Hamiltonian. This leads to a qualitatively different phenomenology in the operator growth with exponentially many more terms and paths.

We also note that the Heisenberg model without disorder is a so-called integrable model which has, what is commonly called, an infinite set of local conserved charges O with $[H, O] = 0$ in the thermodynamic limit. It is crucial to note though that “local” is meant here in an entirely different way. These charges are in no way localized but rather can be written as $O = \sum_j o_j$ where o_j is a local density acting on a finite number of lattice sites. That is, the charge densities are local but the entire conserved operator O is not and rather acts on the entire lattice. The trivial examples are the Hamiltonian itself, $H = \sum_l h_l$ and the magnetization $M = \sum_l \sigma_l^z$. What is special about integrable models is that an infinite set of operators of this type exist which allows to uniquely characterize all eigenstates by the corresponding quantum numbers. The eigenspectrum of the model can thus be calculated exactly by using the Bethe ansatz [48,49]. For the question of *local operator growth*, integrability is not relevant and we do not expect to see any difference to a nonintegrable model. Even if we break integrability, the magnetization M is still conserved as long as $U(1)$ symmetry is preserved. That is, we are in any case studying the operator growth of a local operator σ_j^z which belongs to a globally conserved charge. Differences between integrable and nonintegrable models are, of course, expected if we study the Euclidean time dynamics of operators

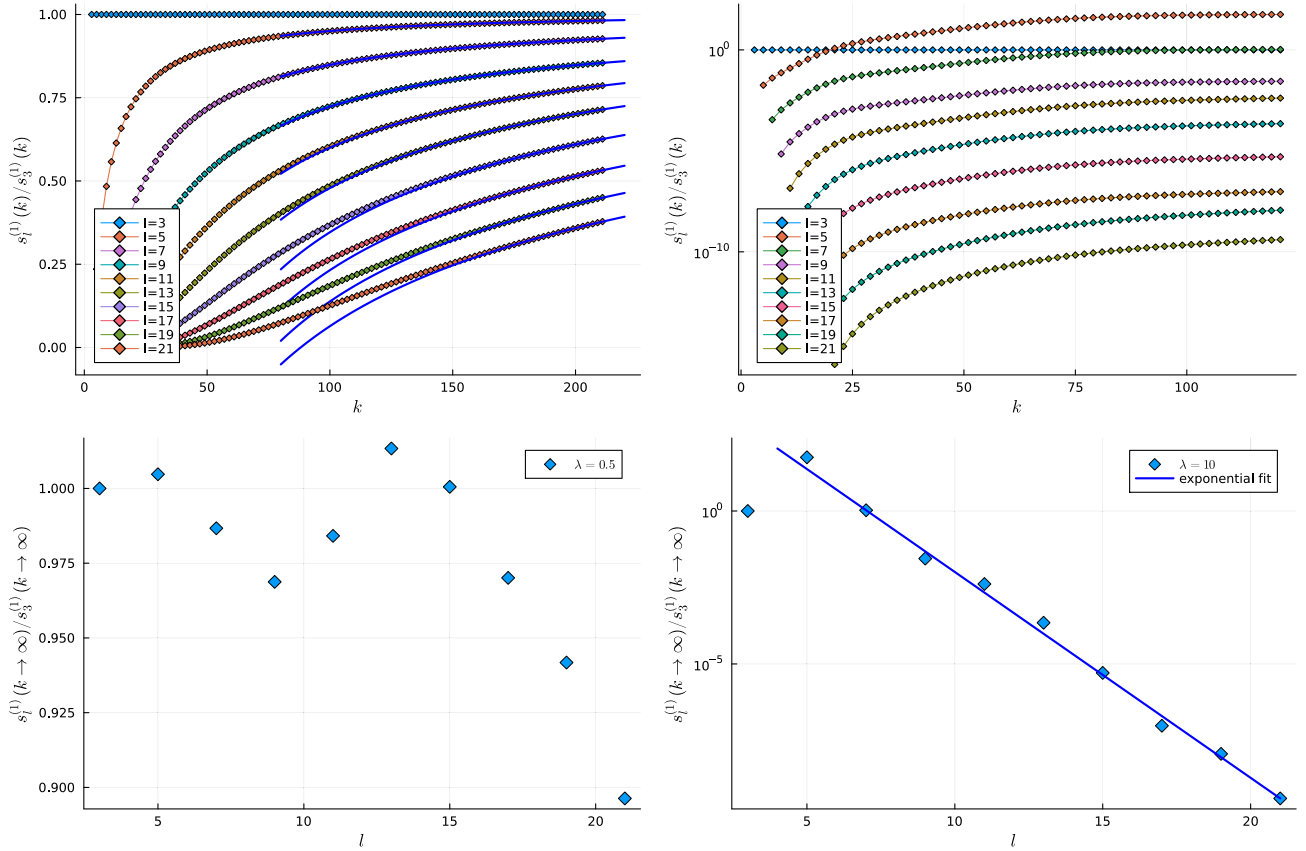


FIG. 11. Aubry-André model for $\beta = (1 + \sqrt{5})/2$, $\phi = 0$ where $\lambda = 0.5$ (left column, linear scale) and $\lambda = 10$ (right column, logarithmic scale). For $\lambda = 0.5$, $s_l^{(1)}(k)$ is extrapolated (blue lines) and the results are consistent with $s_l^{(1)}(k) \sim Ce^k$ with a constant C independent of l . This leads to $s_l^{(1)}(k \rightarrow \infty)/s_3^{(1)}(k \rightarrow \infty) \approx 1$ (bottom left panel) clearly showing that the model is not in the localized phase. In contrast, for $\lambda = 10$ the norm scales as $s_l^{(1)}(k) \sim e^k e^{-l}$, i.e., is exponentially suppressed with increasing l for k fixed, demonstrating that the model is now in the localized phase (bottom right panel).

of the type $O = \sum_j o_j$ [31]. However, for the Euclidean time dynamics of *local* operators in a generic system, the only thing which matters is whether or not this system has local conserved charges, i.e., whether it is localized or not. We also note that integrability is immediately destroyed if we introduce even an infinitesimal amount of disorder.

We will first briefly consider the case without disorder before turning to the most interesting case, the XXX model with random magnetic fields $h_j \in [-D, D]$ which has been suggested to have a localized phase for sufficiently strong disorder.

1. No disorder

From the exact solution of the model, we know for sure that the eigenstates of the XXX model are not localized. We can calculate the number of distinct terms $n(k)$ in the commutator $[H, \sigma_0^z]^{(k)}$ at order k by symbolic calculations up to order $k = 18$ exactly. We have not been able to identify the series but asymptotically the number of terms $n(k)$ does grow exponentially with a multiplicative algebraic correction which is consistent with the general bound (3.15) which we derived earlier. The data and a corresponding fit are shown in Fig. 12. The fit works extremely well with a least square deviation

of $\chi^2 \sim 10^{-2}$. The local operator basis for the Heisenberg model has dimension $f = 4$ and consists of the operators

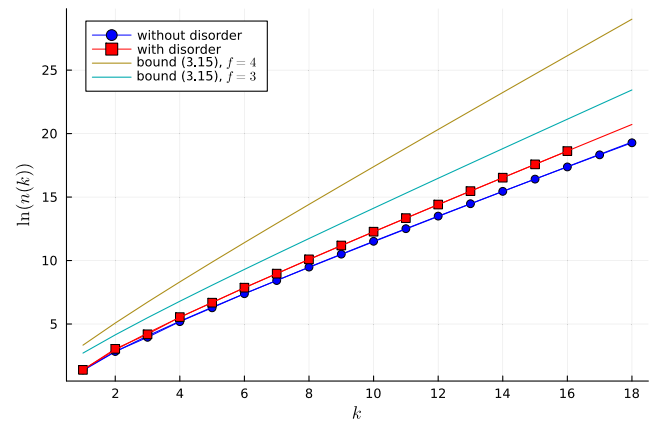


FIG. 12. In the interacting model with $\Delta = 1$, the number of different terms $n(k)$ in the commutator $[H, \sigma_0^z]^{(k)}$ grows exponentially (symbols) and is very well fitted by $n(k) \sim 1.514 k^{0.803} \exp(0.922k)$ in the nondisordered case and $n(k) \sim 1.492 k^{0.815} \exp(0.998k)$ in the case with disorder (solid lines). The general bounds (3.15) with $f = 3$ and $f = 4$ are shown as well.

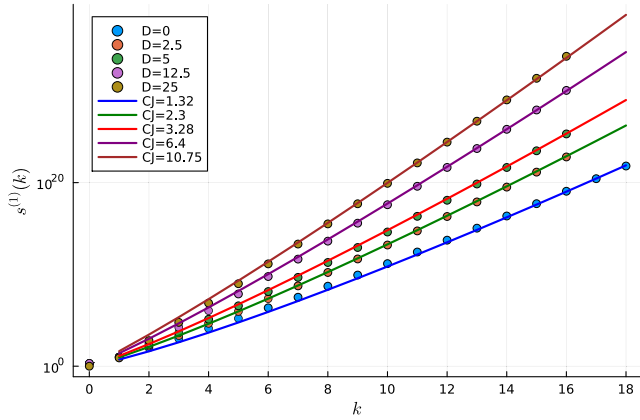


FIG. 13. The 1-norm in the interacting case with $\Delta = 1$ grows faster than exponential for all disorder strengths shown, consistent with the rescaled general bound, Eq. (3.12). The data are averaged over 30 disorder realizations for $D = 2.5$, $D = 5$ and $D = 12.5$ and over 50 realizations for $D = 25$.

$\mathbb{1}, \sigma^x, \sigma^y, \sigma^z$. We note, however, that only a small subset of terms in $[H, \sigma_0^z]^{(k)}$ contains an identity in between Pauli matrices. Ignoring such terms, we can set $f = 3$ in Eq. (3.15) which gives a much tighter bound; see Fig. 12. The exponential growth of $n(k)$, instead of the quadratic growth in the

free-fermion case, indicates that in the corresponding commutator graph the number of nodes and edges per level are no longer constant.

We note that the number of distinct terms is different and always smaller than the number of connected clusters which we have calculated in Eq. (3.11). The reason is that different connected clusters can result in the same term. For the calculation of the norm this does not matter though. We can bound the contribution of each connected cluster as in Eq. (3.12) even if some of the resulting terms are the same. In Fig. 13 we show that this general bound indeed describes the norm growth well if we use the bound on the local Hamiltonian, $||h_l^a|| \leq J$, as a fitting parameter. In particular, the norm clearly grows faster than exponential for the orders of the commutator which we can handle symbolically. We also note that we obtain qualitatively similar results if we use the 2-norm instead of the 1-norm; see Fig. 14. This difference between the noninteracting and the interacting case becomes even more evident if we consider the norm $s_l^{(1)}(k)$ of terms supported on l sites; see Fig. 15. Whereas in the noninteracting, nondisordered case the norm of the terms supported on l sites grows asymptotically in the same way independent of l if the order of the commutator k is much larger than l , see Eq. (5.12), $s_l^{(1)}(k)/s_1^{(1)}(k)$ has a maximum which shifts to larger and larger l with increasing k in the interacting case. For large k , the total norm is thus completely dominated by terms which have support on a large

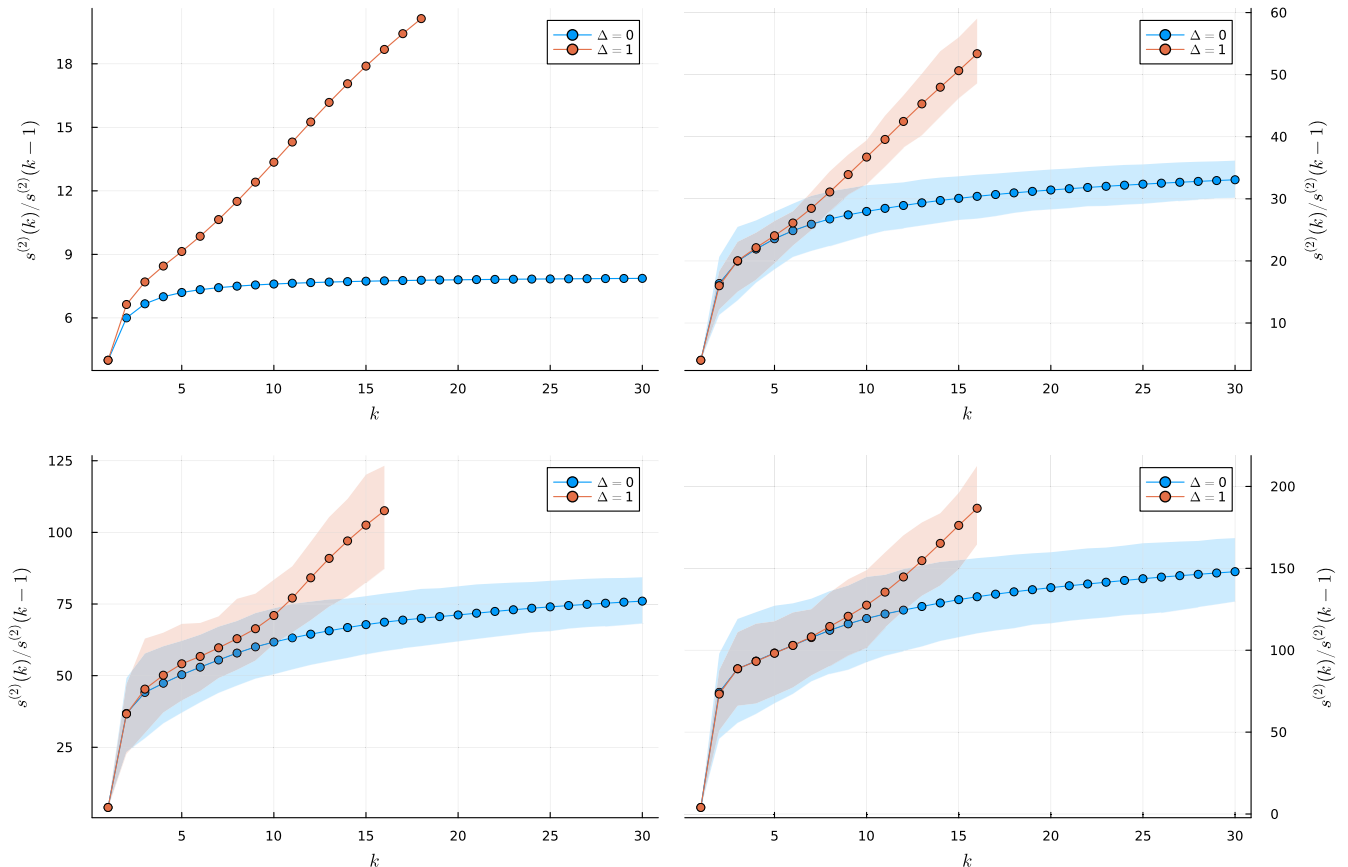


FIG. 14. Average of the 2-norm ratio $s^{(2)}(k)/s^{(2)}(k-1)$ for the XXZ chain with $\Delta = 0$ and $\Delta = 1$ for disorder strengths $D = 0, 5, 12.5, 25$. The data for $D = 5$ and $D = 12.5$ are averaged over 50, and the data for $D = 25$ over 70 realizations. The shaded regions indicate the intervals into which 50% of all realizations fall. The norm ratio has to become a constant for large k if the norm grows exponentially.

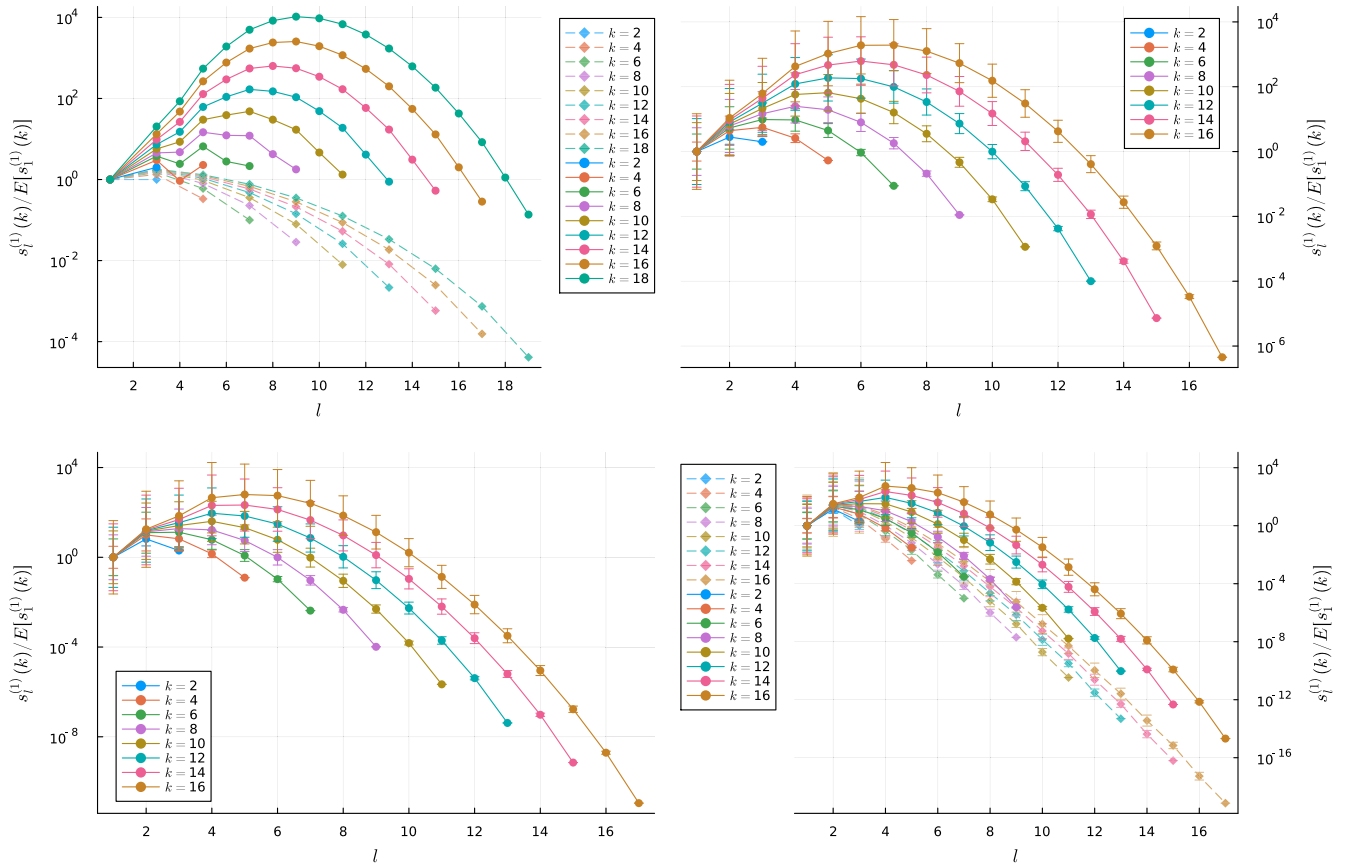


FIG. 15. The norm of operators $s_l^{(1)}(k)/E[s_1^{(1)}(k)]$ with support on l sites for the Heisenberg model with disorder $D = 0, 5, 12.5, 25$. The data for $D = 5$ and $D = 12.5$ are averaged over 50, the data for $D = 25$ over 70 disorder realizations. There is no indication for a convergence with increasing order k to an exponential or almost-exponential decay with l even for very strong disorder. For $D = 0$ and $D = 25$ the results for the noninteracting case are shown for comparison (symbols with dotted lines).

number of lattice sites. This is to be expected, given that the number of distinct terms in the commutator is increasing exponentially which also leads to a rapidly growing number of possible paths between those terms; see also the graph in Fig. 16. Note that the graph is no longer regular; the number of edges per node is not constant and is increasing with the level the node is situated on. Note that this increase of the

number of edges is a prerequisite for a faster than exponential growth of the norm.

2. Many-body localization

Based on this graph representation, it is obvious that a strict exponential localization of operators in a many-body system is thus an extremely challenging requirement. In a

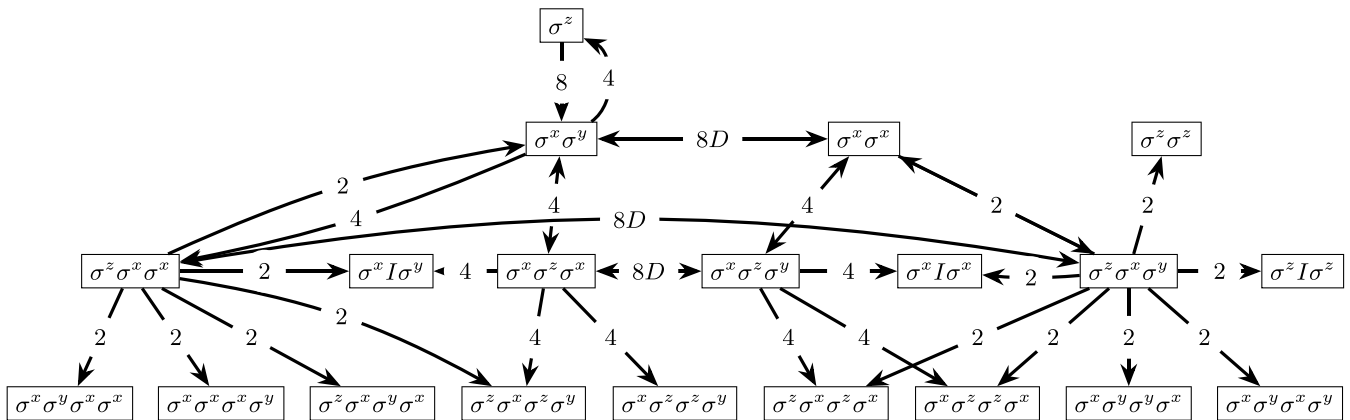


FIG. 16. Graph for the 1-norm $s^{(1)}(k)$ up to order $k = 3$ for the Heisenberg model with random fields $h_j \in [-D, D]$. The number of nodes grows exponentially with the length of the support.

graph with an equal number of edges at each level as realized in a noninteracting model, see Fig. 7, it is easy to see—at least in the limit of large disorder—how the norm of terms with support on only a few sites can dominate the total norm. The exponential growth of edges in the interacting case (see Fig. 16), however, suggests that the total norm for large k will be entirely dominated by terms with support on many lattice sites. That is, a local operator is generically expected to delocalize. This aspect also makes it difficult to see how a perturbative construction of local integrals of motion can be arranged. This is a point which seems to have not been sufficiently appreciated in Refs. [40,41] where such a construction was suggested. We will get back to this point in Sec. VI. The only way a local operator can avoid to spread through the entire system in the many-body case thus seems to be intricate interference effects between the exponentially many different paths in the graph; see Fig. 16. While such fine-tuned destructive interference seems implausible as a generic mechanism for many-body localization we cannot, in general, show that such effects do not exist. We note, however, that for the transverse Ising chain with random fields—the model considered in the proof of MBL in Refs. [40,41]—a gauge can be defined such that the commutator has a stoquastic-type property [37]. That is, all terms in the commutator in this gauge have a positive amplitude. For this model, interference effects as a mechanism for localization are thus strictly excluded and we have shown in Sec. IV that the almost-factorial norm growth is indeed incompatible with exponential localization. For the Heisenberg model considered here, such a gauge does not exist. We will instead consider data from symbolic computations up to order $k = 16$ and show that there is no evidence for destructive interference effects and a corresponding localization. The data are consistent with a stronger than exponential norm growth. We note that at order k , we are considering operators which are spread over $2k + 1$ sites in the lattice. That is, at order $k = 16$ we are dealing effectively with a chain of 33 lattice sites which is well beyond what is possible in exact diagonalizations. We will also consider disorder strengths up to $D = 25$ which is much larger than the critical values which have been suggested in the past [8,10,11,50].

Let us start by considering the number of distinct terms in the commutator at order k which is shown in Fig. 12. We find that the number of terms grows faster in the disordered than in the case without disorder. This is easy to understand: additional terms appear in the commutator due to the random magnetic field terms while all the terms present before remain. The different classes of terms which appear up to order $k = 3$ are shown in the graph in Fig. 16. Next, we consider the growth of the total norm which should be at most exponential in a strictly exponentially localized phase as we have proven in Eq. (4.4). The data, however, are overall much better described by the general bound (3.12) than by an exponential, with the norm bound of the local Hamiltonian, $\|h_l^q\| \leq J$, used as a fitting parameter. While this does not exclude the possibility that for larger k the scaling eventually does become exponential, we can say with confidence that no indications for a strict exponential localization are observed in spin chains up to 33 sites. These system sizes are much larger than those used in exact diagonalizations, the method which claims of a many-body localized phase have been mostly based on,

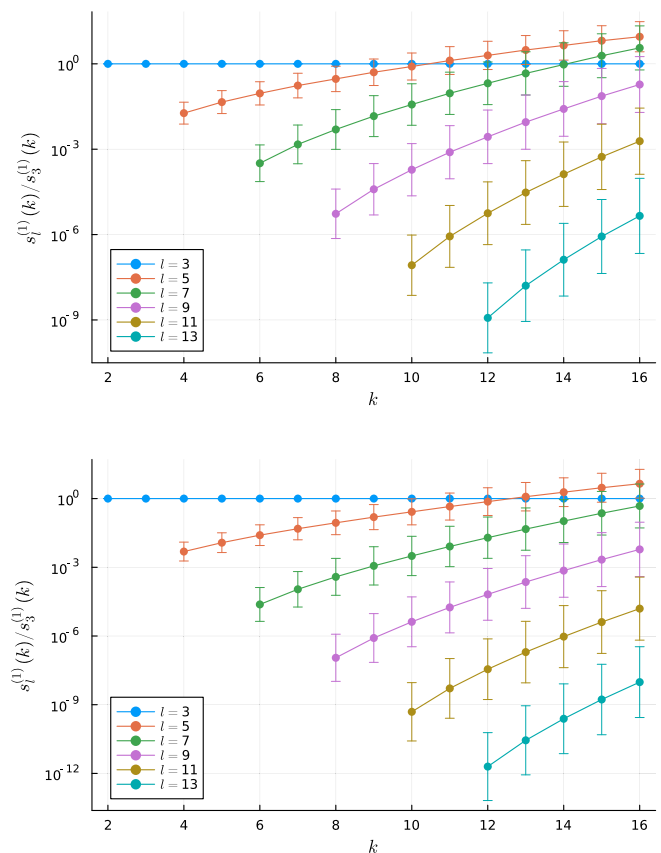


FIG. 17. Norm ratio $s_l^{(1)}(k)/s_3^{(1)}(k)$ for a fixed l as a function of k for the Heisenberg model with $D = 12.5$ (top) and $D = 25$ (bottom). These results are to be compared with the noninteracting case shown in Fig. 10.

see Refs. [8,10,11,19]. The stronger than exponential growth of the norm is further supported by considering the ratio of norms $s^{(2)}(k)/s^{(2)}(k-1)$ which is shown in Fig. 14. For the orders of the commutator which we can handle symbolically, there is no indication that this norm ratio saturates, even for extremely strong disorder, as it has to in the strictly localized case where $s(k) \sim \exp(k)$. As the figure shows, the behavior in the interacting case is in stark contrast to the localized noninteracting case.

Finally, let us also consider the norm ratio $s_l^{(1)}(k)/E[s_1^{(1)}(k)]$ of terms with support on l sites which is shown in Fig. 15 for various disorder strengths. Here $E[s_1^{(1)}(k)]$ denotes the mean of terms with support on one site. Even for very strong disorder $D = 25$ there is no indication of a convergence to an exponential decay with l as is expected in the strictly exponentially localized case. To the contrary, there is always a maximum which shifts to larger l with increasing order k just as in the case without disorder. To further highlight this point, we show in Fig. 17 the norm ratio $s_l^{(1)}(k)/s_3^{(1)}(k)$ as a function of the order k for various fixed l . In contrast to the noninteracting case shown in Fig. 10, the curves do not saturate for large k but rather are consistent with the weight in the total norm shifting to larger and larger l with increasing k . While we cannot exclude that this trend changes for even larger k than the ones we can handle in

symbolic calculations, we want to stress once more that the terms in the commutator are spread over 33 sites for $k = 16$. This is already much larger than the system sizes typically considered in exact diagonalizations. At the very least, we can say that for chains of lengths up to 33 sites, there is no indication that a phase with strict exponential localization is realized even at disorder strengths up to $D = 25$.

This leaves open the second scenario where the Hamiltonian is unitarily equivalent to the effective Hamiltonian $\tilde{H}$ and where, instead of a strict exponential localization, local operators spread across the lattice according to Eq. (2.4). First, we note that the decay rate $\tilde{\kappa}_2 = \tilde{\kappa}_2(D)$ is increasing with increasing D so that for very strong disorder there should be an almost-exponential decay with l . There is no indication for such a behavior in Fig. 15 even for $D = 25$. However, it is of course possible that D is still not large enough. Second, we note that Eq. (4.12) predicts a maximum at $l = k/\tilde{\kappa}_2$. This is not inconsistent with Fig. 15 where we indeed observe an approximately linear shift of the maximum with increasing commutator order k . For the norm ratios shown in Fig. 17, the result (4.12) predicts that $s_l(k)/s_3(k) \sim l^k$, which is also consistent with the data from the symbolic calculations (note the logarithmic scale).

To summarize, the symbolic data are consistent with a maximal, almost-factorial total norm growth in the disordered Heisenberg chain for all disorder strengths, as in the random field Ising model. This and the data for the norms with support on l sites seem to exclude the existence of a phase in which operators remain strictly exponentially localized. However, the data are not inconsistent with the second scenario where H is unitarily equivalent to the effective Hamiltonian $\tilde{H}$ describing a nonergodic phase where local operators eventually spread over the entire lattice and do not remain localized. We therefore investigate next if the unitary transform U can be constructed perturbatively.

VI. SCHRIEFER-WOLFF TRANSFORMATIONS

We have so far considered the commutator of a microscopic lattice Hamiltonian H with a local operator. If this Hamiltonian is in a strictly exponentially localized phase, then the operator will remain localized—up to exponential tails—under Euclidean time evolution. However, we have also seen that if we only demand that the Hamiltonian is unitarily equivalent to an effective Hamiltonian of (interacting) conserved charges, then this might open up the possibility for a nonergodic phase in which local operators, however, do not remain localized. Our symbolic calculations of the commutator were unable to completely rule out this scenario.

We therefore next consider if we can explicitly construct the unitarily equivalent Hamiltonian $\tilde{H}$ written in terms of local conserved charges τ_j^z [29,40,41,51–53]. It is important to note that these charges are not uniquely defined: if, for example, τ_j^z and τ_{j+1}^z are conserved charges localized near lattice site j with $[H, \tau_j^z] = [H, \tau_{j+1}^z] = 0$, then $\tau_j^z \tau_{j+1}^z$ is also local and conserved, $[H, \tau_j^z \tau_{j+1}^z] = 0$.

One way to obtain these charges is to try to perturbatively construct them in the limit of strong disorder [40,41,51]. In this limit, one can think of these charges as the local operators σ_i^z smeared out by the hopping and dressed by the interactions.

As in our previous considerations, there is then again an important difference between the noninteracting and the interacting case: While in the former the one-particle operators $\sigma_i^z = c_i^\dagger c_i - 1$ remain one-particle operators, they become many-particle operators in the latter case. As we will see, the perturbative transformation of the Hamiltonian will lead us again to multiple commutators which are closely related to the ones studied in the previous sections.

The Hamiltonian of a quantum system can be diagonalized by a unitary transformation,

$$H \rightarrow \exp(T)H \exp(-T) = \sum_{k=0}^{\infty} \frac{[T, H]^{(k)}}{k!}, \quad (6.1)$$

with $T^\dagger = -T$ being an antiunitary operator. For a Hamiltonian $H = H_0 + V$ where H_0 is diagonal in the operator basis chosen, we can instead of a full diagonalization demand that the new Hamiltonian $H^{(1)}$ is diagonal only to first order in the perturbation V . This is the Schrieffer-Wolff transformation [54,55]. Going one step further, we can consider successive Schrieffer-Wolff transformations where in each step the perturbation is eliminated to leading order. This is closely related to the approach in Refs. [40,41] which tries to generalize a proof of Anderson localization to the many-body case. In these works, the author is mostly concerned with the impact of resonances and splits the off-diagonal part of the Hamiltonian into a resonant and a nonresonant part. We will discuss the impact of resonances as well but will point out an even more fundamental issue which might call into question the entire perturbative construction of local conserved charges in the interacting case even for disorder configurations where no resonances are present.

To be specific, we consider again the XXZ Hamiltonian (5.1). We note that in Refs. [40,41] the transverse field Ising model with random couplings and random fields was considered instead. However, the arguments made in the following are quite general and will apply to the latter model as well [37]. We want to investigate the case of large disorder where the magnetic field and the interaction terms constitute H_0 with $[H_0, \sigma_j^z] = 0$ and the hopping terms are the perturbation. The condition the operator T has to fulfill to lowest order is then given by

$$V + [T, H_0] = 0 \quad (6.2)$$

and the new, transformed Hamiltonian is obtained as

$$H^{(1)} = H_0 + \frac{1}{2}[T, V] + \frac{1}{3}[T, [T, V]] + \dots \quad (6.3)$$

In the Schrieffer-Wolff transformation, one typically stops at the first order correction to H_0 . To justify this perturbative approach, we can rescale the Hamiltonian (5.1) by $1/D$ which changes the energy scales but leaves the physical properties unchanged. Then, the hopping terms have an amplitude $\tilde{J}_{j,j+1}^{(0)} = 1/D$ and the interaction terms an amplitude $\tilde{\Delta} = \Delta/D$ which are both small parameters for large D . After this rescaling, the random magnetic fields are always given by $h_j \in [-1, 1]$.

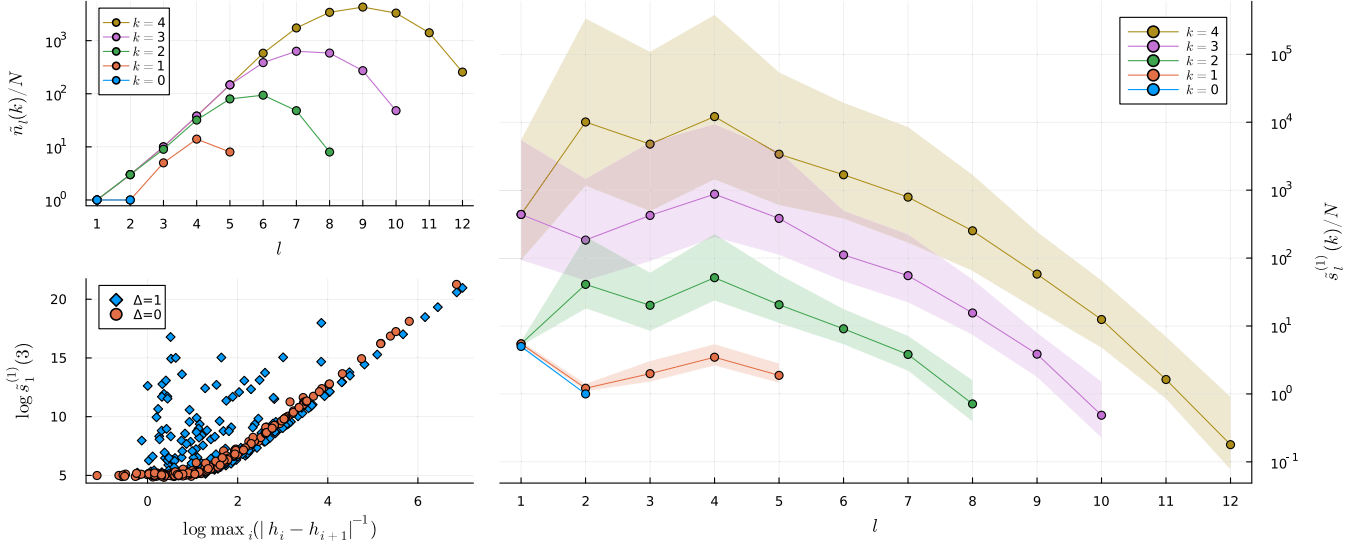


FIG. 18. Top left: Number of terms in the Hamiltonian with support on l sites after performing the Schrieffer-Wolff transformation (6.9) up to order k for the XXX chain with disorder and $N = 100$ lattice sites. For large k , this number increases approximately exponentially with l . Note that $\tilde{n}_l(k)$ does not depend on the disorder realization; the results are exact. Bottom left: Total norm of terms with support on one site after $k = 3$ transformations for 200 samples versus $\max_i |h_i - h_{i+1}|^{-1}$. Right: Total norm of terms with support on l sites after k transformations averaged over 200 realizations for $N = 100$ and $D = 5$. While each term is expected to have an amplitude $\sim D^{-2l}$ if there are no resonances, the exponentially increasing number of terms means that the total coupling of distant sites is no longer controlled by a small parameter. For example, we have $\tilde{s}_1^{(1)}(k) \approx \tilde{s}_8^{(1)}(k)$ for $k = 4$.

A. Noninteracting case

First, we consider again the noninteracting case. An ansatz for the operator T can be obtained by commuting H_0 and V . This leads to $T = \sum_j A_j (\sigma_j^+ \sigma_{j+1}^- - \sigma_{j+1}^+ \sigma_j^-)$. The amplitudes A_j can then be determined from Eq. (6.2) leading to

$$T = \frac{1}{2} \sum_j \frac{\bar{J}_{j,j+1}^{(0)}}{h_j - h_{j+1}} (\sigma_j^+ \sigma_{j+1}^- - \sigma_{j+1}^+ \sigma_j^-). \quad (6.4)$$

We note that the Schrieffer-Wolff operator T is thus proportional to the spin current operator $\mathcal{J} = \sum_i j_i$ which can be obtained from the lattice continuity equation [56]

$$\partial_t \sigma_i^z = -i[\sigma_i^z, H] = -(j_i - j_{i-1}). \quad (6.5)$$

We note, furthermore, that $|\overline{h_{j+1} - h_j}| = 2/3$, i.e., the factor $\bar{J}_{j,j+1}^{(0)}/|h_j - h_{j+1}|$ is small on average if the disorder strength D is large. However, if two neighboring sites have roughly the same potential, then this factor can become large or might even diverge. Rare resonances thus might destroy the perturbative construction if they proliferate. An example how such resonances can affect the transformed Hamiltonian is shown in Fig. 18, bottom left panel. We will come back to this point below.

After the Schrieffer-Wolff transformation, the new Hamiltonian is given by

$$\begin{aligned} H^{(1)} &= H_0^{(0)} + \frac{1}{2}[T^{(0)}, V^{(0)}] + \frac{1}{3}[T^{(0)}, [T^{(0)}, V^{(0)}]] \\ &= \sum_j \{ \bar{J}_{j,j+2}^{(1)} (\sigma_j^+ \sigma_{j+1}^z \sigma_{j+2}^- + \text{H.c.}) + 2h_j^{(1)} \sigma_j^z \} \\ &\quad + \sum_j \bar{J}_{j,j+1}^{(1)} (\sigma_j^+ \sigma_{j+1}^- + \text{H.c.}), \end{aligned} \quad (6.6)$$

with $H_0^{(0)} = H_0$, $V^{(0)} = V$, and $T^{(0)} = T$. The coupling constants and renormalized fields are given by

$$\begin{aligned} \bar{J}_{j,j+2}^{(1)} &= \frac{(\bar{J}_{j,j+1}^{(0)})^2}{2(h_{j+1} - h_j)} + \frac{(\bar{J}_{j,j+1}^{(0)})^2}{2(h_{j+1} - h_{j+2})}, \\ h_j^{(1)} &= h_j^{(0)} + \frac{(\bar{J}_{j,j+1}^{(0)})^2}{4(h_j - h_{j-k})} + \frac{(\bar{J}_{j,j+1}^{(0)})^2}{4(h_j - h_{j+1})}, \end{aligned} \quad (6.7)$$

and $\bar{J}_{j,j+1}^{(1)} \sim 1/D^3$ (the exact expression is lengthy and not needed in the following). The first two terms correspond to those generated by $[T^{(0)}, V^{(0)}]$ while the renormalized nearest-neighbor hopping $\bar{J}_{j,j+1}^{(1)}$ stems from the next higher commutator $[T^{(0)}, [T^{(0)}, V^{(0)}]]$. That is, the transformation to lowest order eliminates the nearest-neighbor hopping processes but generates a next-nearest-neighbor hopping $\bar{J}_{j,j+2}^{(1)} \sim 1/D^2$ (note that $\sigma_j^+ \sigma_{j+1}^z \sigma_{j+2}^- + \text{H.c.} = c_j^\dagger c_{j+2} + \text{H.c.}$ in the fermionic language) and renormalizes the local fields $h_j^{(0)} \rightarrow h_j^{(1)}$.

If we stop at the leading order of the transformation, then we obtain a Hamiltonian $H^{(1)}$ which looks like the original Hamiltonian $H^{(0)}$ but with the nearest-neighbor hopping replaced by a weaker next-nearest-neighbor hopping. We can then again define a Schrieffer-Wolff transformation to eliminate these next-nearest hopping terms which has the form

$$T^{(k)} = \frac{1}{2} \sum_j \frac{\bar{J}_{j,j+k+1}^{(k)}}{h_{j+k+1}^{(k)} - h_j^{(k)}} (\sigma_j^+ \sigma_{j+1}^z \dots \sigma_{j+k}^z \sigma_{j+k+1}^- - \text{H.c.}), \quad (6.8)$$

with $k = 1$. This will again, to leading order, renormalize the local fields and generate a third-nearest-neighbor hopping which again can be eliminated to leading order by

the Schrieffer-Wolff transformation (6.8) with $k = 2$. That is, we can define a series of successive Schrieffer-Wolff transformations which lead to longer and longer range interactions which are, however, suppressed as D^{-2k} . At the same time, they define an exponentially converging series for the renormalized random fields $h_j^{(k)}$. If there are no resonances, then we can drop the hopping term when all the amplitudes $J^{(k)}$ have fallen below some threshold, giving us an effective Hamiltonian with renormalized magnetic fields. In this approximation, which can be justified at very strong disorder and which corresponds to summing up exactly certain classes of diagrams, the conserved local charges are just the σ_j^z themselves but with renormalized amplitudes. This can be seen as a consistency check that at very large disorder such an effective description exists because the hopping is an irrelevant perturbation.

In the noninteracting case, it is also well understood that distant resonances have a vanishingly small probability to proliferate [1,3,4]. In our construction, this scaling argument arises as follows: Two lattice sites j and $j + \ell$ a distance ℓ apart are coupled by an effective long-range hopping $\bar{J}_{j,j+\ell}^{(\ell-1)}(c_j^\dagger c_{j+\ell} + \text{H.c.})$ with an effective amplitude obtained by $\ell - 1$ successive Schrieffer-Wolff transformations. The amplitude of this long-range effective hopping term is given by $\bar{J}^{\ell-1} \sim D^{-2\ell+2} \times f[\{(h_j^{\ell-1} - h_{j+\ell}^{\ell-1})^{-1}\}]$ where f is some function of the inverse differences of the random fields within the cluster of ℓ sites. Now the smallest pairwise difference on this cluster on average scales as ℓ^{-2} . Thus the effective long-range hopping amplitude scales as $\bar{J}_{j,j+\ell}^{(\ell-1)} \sim D^{-2\ell} \ell^2$ and is thus typically exponentially suppressed. Therefore, resonances have a vanishingly small probability to proliferate and to destroy the perturbative construction above. This is also supported by our symbolic calculations on finite chains of length N . First, we note that the size of the norm of terms with support on one site is entirely determined by how close to a nearest-neighbor resonance a given sample is. Distant resonances play no role; all the data fall onto a “hockey stick” curve; see Fig. 18, bottom left panel. Second, we consider successive lowest order Schrieffer-Wolff transformations and stop the transformations if the total 1-norm of the off-diagonal part is below some threshold value ε . We have $\sim N$ possible off-diagonal terms of a given range and if the amplitudes of all those terms, on average, are suppressed as D^{-2k} at order k of the transformation then our condition to stop is $ND^{-2k} < \varepsilon$. We thus expect that we need $k \sim \ln(N/\varepsilon)$ many transformation steps. This is consistent with our observations.

Quite generally we may ask why we expect a perturbative construction of the local conserved charges to work in the non-interacting case? In the lowest-order construction discussed above, the Hamiltonian $H^{(k)}$ always has $3N$ terms for a chain of length N for every k . That is, the number of terms does not increase during successive Schrieffer-Wolff transformations. If we take higher order commutators in the transformation into account, then the number of terms will increase, however, the Hamiltonian $H^{(k)}$ and the transformation $T^{(k)}$ will always consist of a sum of one-body operators and, for a chain of length N , there are only N^2 distinct one-body operators. Therefore, the problem of finding local conserved charges is

at most quadratic, and, in any finite-order approximation, one in fact always deals with a constant, nonincreasing number of terms in the transformed Hamiltonians.

B. Interacting case

The interacting case, however, is completely different and none of the above arguments hold. We are, however, able to generalize Eq. (6.4) and to obtain the exact Schrieffer-Wolff transformation for the XXZ chain and arbitrary interaction strength Δ . We find

$$T = \sum_j \frac{\bar{J}_{j,j+1}^{(0)}(\sigma_j^+ \sigma_{j+1}^- - \text{H.c.})}{2(h_j - h_{j+1})(2\bar{\Delta}^2 - 2(h_j - h_{j+1})^2)} \times [\bar{\Delta}^2 - 2(h_j - h_{j+1})^2 + \bar{\Delta}^2 \sigma_{j-1}^z \sigma_{j+2}^z + \bar{\Delta}(h_j - h_{j+1})(\sigma_{j-1}^z - \sigma_{j+2}^z)], \quad (6.9)$$

with $\bar{J}_{j,j+1}^{(0)} = 1/D$ and $\bar{\Delta} = \Delta/D$. We note that for $\Delta \neq 0$, the transformation T consists of $8N$ distinct operators for a chain of length N , including operators acting on two, three, and four neighboring sites.

If we stop the transformation (6.3) at first order, the transformed Hamiltonian $H^{(1)}$ already has $29N$ distinct operators if the chain length N is sufficiently large. This includes terms acting on five neighboring sites but also terms such as $\sigma_j^z \sigma_{j+1}^z$ and $\sigma_j^z \sigma_{j+2}^z \sigma_{j+3}^z$. Even for very strong disorder, the transformation thus does not just lead to a renormalization of the random fields but necessarily leads to a broadening of the diagonal basis. Symbolically, we can also construct $T^{(1)}$ to eliminate the off-diagonal terms in $H^{(1)}$. We find that $T^{(1)}$ already consists of $786N$ distinct operators. The transformed Hamiltonian $H^{(2)}$ then has $30618N$ many terms. That is, even if we stop the Schrieffer-Wolff transformation at lowest order, the number of terms in $T^{(k)}$ and $H^{(k)}$ is growing exponentially. Furthermore, there is no distinct set of diagrams which we can simply sum up to all orders. Exponentially many new diagrams are created at each order.

The related and most fundamental problem, however, is that lattice sites j and $j + \ell$ a distance ℓ apart will be coupled by $\sim 3^2 4^{\ell-1}$ distinct operators because on each site we can put in principle one of the operators $\{\sigma^x, \sigma^y, \sigma^z, \mathbb{I}\}$ but the string of operators of length $\ell + 1$ has to start and end with one of the three Pauli matrices. While not all of these combinations of operators will be realized in a microscopic model such as the XXX chain, there will be exponentially many distinct operators which couple these two sites; see Fig. 18. Thus, there is, in general, no reason to expect this perturbative construction to work because the overall effective amplitude of terms coupling these two sites will be $\bar{J}_{j,j+\ell} \sim D^{-\ell} 4^\ell \sim \mathcal{O}(1)$. That is, there is not necessarily a small parameter anymore. This reasoning is supported by the results of symbolic calculations shown in Fig. 18. Here we have calculated the sum $\tilde{s}_\ell^{(1)}(k)$ of the amplitudes of all terms obtained in the Schrieffer-Wolff transformation (6.9) to order k which have support on ℓ sites, i.e., which connect sites a distance $\ell - 1$ apart. For a periodic chain with even length N , there are $\sim N$ distinct pairs of sites which are a distance $1, \dots, N/2$ apart.

That is, for a specific pair the overall coupling is on average $\sim \tilde{s}_\ell^{(1)}(k)/N$ and, as can be seen in Fig. 18, is in general not exponentially decreasing with ℓ . That the number of terms generated in such a transformation with support on ℓ sites is exponentially increasing with ℓ makes it very questionable that a construction of local conserved charges in this perturbative manner is a controlled approximation. This is a major issue even before addressing the issue of resonances and thermal inclusions (regions for a given disorder configuration which, by chance, are only weakly disordered). It seems to us that the exponential increase of the number of terms in such transformations is a point which has not been properly addressed in the proof of MBL in Refs. [40,41] where the author seems to be only concerned about the effect of resonances. While it is not impossible that for disorder configurations without resonances and thermal inclusions the perturbative construction of local charges is working in the thermodynamic limit for very strong disorder, any proof needs to provide rigorous arguments that the exponentially many terms do not overwhelm the exponential small parameter in the expansion. Such arguments seem to be missing so far. We note that for a finite system the construction will always eventually converge because the finite length provides a cutoff and $\tilde{s}_\ell^{(1)}(k)$ will eventually drop off exponentially with ℓ .

The destabilizing effects of resonances and thermal inclusions have received considerable attention in the literature already [45,57,58]. From the perspective of perturbative Schrieffer-Wolff transformations, they lead to the following additional problems: The renormalized couplings for an operator acting on ℓ lattice sites will in its denominator typically contain a linear combination of *all* the random fields on these ℓ sites. Given that there are exponentially many such operators, the simple scaling argument for the nonproliferation of resonances in the Anderson case is clearly no longer applicable in the interacting case. This is confirmed by the symbolic calculations shown in Fig. 18, bottom left panel, where the sum of the amplitudes of terms in the Hamiltonian with support on one site, $\tilde{s}_1(k)$, in the interacting case is large even for samples which do not have nearest-neighbor resonances. Thermal inclusions—regions of the lattice which have similar potentials—lead, furthermore, to an additional destabilization of the perturbative transformation. This can be seen, for example, in symbolic calculations of the unitary transformation (6.1) with the operator (6.9) to higher orders. In configurations where such regions exist, the transformation generates long-range terms with large amplitudes even if the disorder D is extremely strong. If, however, we exclude configurations with such regions and with any pairwise resonances by hand, then the amplitudes of any individual operator typically decay with the length of its support. Another manifestation of this issue can be seen when performing k successive lowest order Schrieffer-Wolff transformations for a finite chain of length N . We want to stop the transformations again once the 1-norm of the off-diagonal part is below some threshold ε . In contrast to the noninteracting case, we now have exponentially many terms connecting sites a given distance apart. If we assume that individual amplitudes in the off-diagonal part are suppressed as D^{-2k} , then the condition for stopping the transformations becomes $e^N D^{-2k} < \varepsilon$

and we would thus expect that we need $k \sim N$ steps for N sufficiently large. This is, however, not what we observe for typical disorder configurations. The results of our symbolic calculations point instead to an increase of the number of transformations $k \sim e^N$. This implies that some amplitudes in the off-diagonal part are $\sim \mathcal{O}(1)$ and that we have to eliminate almost all of the off-diagonal terms to achieve convergence.

In summary, the iterative transformation of the Hamiltonian cannot simply be transferred from the noninteracting to the interacting case. There are two main issues: Even if one considers only a lowest order transformation, one is not dealing with a constant number of operators but rather with an exponentially increasing number of operators, both in the Schrieffer-Wolff transformation $T^{(k)}$ as well as in the transformed Hamiltonian $H^{(k)}$. Even if one tries to only perform a single unitary transformation (6.1) with the operator T , the number of distinct operators in the transformed Hamiltonian is increasing exponentially with the support of the operator which thus can overwhelm the exponential decay of the amplitude of individual operators with the length of their support. We note that different but related arguments for the failure of a perturbative construction of conserved charges in the interacting case have recently been made in Ref. [59]. Furthermore, regions with similar lattice potentials result in individual operators with large support which, nevertheless, have large amplitudes even for very strong disorder. The latter issue is related to the well-known avalanche instability [45,57,58,60].

VII. CONCLUSIONS

We have considered operator growth in one-dimensional lattice models and its connection to localization from two different perspectives. First, we studied the multiple commutator $[H, A]^{(k)}$, which appears both in real-time and Euclidean time evolution, for a local operator A and a Hamiltonian H which is a sum of nearest-neighbor terms. For a generic, ergodic system, we have shown that the number of distinct terms generated is exponentially increasing with the order k of the commutator. This leads to a strict bound for the commutator norm $s(k)$ in nearest-neighbor models which grows almost factorially with k , a result which was first demonstrated in Ref. [30]. Since this result is based on a simple counting of connected clusters generated by the commutator, it is expected that this bound is asymptotically tight for an ergodic system. This *operator growth hypothesis* is challenged by recent results for systems such as the random field Ising chain which are supposed to have a localized phase but, nevertheless, always show almost-factorial operator norm growth independently of the strength of the disorder. Here we have systematically addressed this issue. We started by analytically obtaining strict upper bounds for the operator norm for free fermion and for localized systems. For a free-fermion system, all the terms generated by the commutator of the Hamiltonian H with a one-body operator A will also always be one-body operators. For a short-range Hamiltonian, the spatial range of operators generated at order k will be proportional to k . The number of distinct terms in the commutator can therefore only grow $\sim k^2$ as compared to the exponential growth in the

general, interacting case. This leads to a rigorous exponential upper bound for the commutator norm in this case. Second, we considered the case of strict exponential localization where we demanded that a local operator A remains exponentially localized when commuted with H up to exponential tails. We proved that in this case, a strict exponential upper bound for the commutator norm does exist as well. The almost-factorial norm growth proven for the Ising model with random fields for all disorder strengths—and strongly suggested for the random field Heisenberg model by symbolic calculations in this work—is thus *inconsistent with a strict exponential localization of operators*. We then considered the alternative definition of localization based on the unitary transformation U of a microscopic Hamiltonian H to an effective Hamiltonian $\tilde{H}$ which has an infinite set of local conserved charges. Importantly, local operators in H are mapped by U onto quasilocal operators. For the noninteracting case, we proved that this definition is equivalent to the strict exponential localization of local operators. The contribution of terms in the commutator with support on l sites to the norm does scale as $s_l(k) \sim \exp(k) \exp(-l)$. For the interacting case, however, we were only able to prove a weaker bound $s_l(k) \sim l^k \exp(-l)$ which is consistent with an almost-factorial growth of the total norm. Furthermore, this bound means that for very large k , $s_l(k)$ grows in exactly the same way as in a generic ergodic system. The question then becomes whether the transformation U exists, and thus whether models such as the random field Ising and Heisenberg models have a nonergodic phase in which local operators eventually completely delocalize, or if the transformation U simply does not exist. We have also shown that even if the quasilocal unitary U does exist, the spreading of local operators through the entire lattice is generically expected to lead to transport of globally conserved charges, so such phases, if they do exist, should not be called many-body localized.

To further study the existence of the transformation U , we considered unitary Schrieffer-Wolff transformations of the XXZ chain to perturbatively eliminate the hopping terms and construct U . We showed that such an approach will succeed in the Anderson case because the Hamiltonian in this case consists of one-body operators and this property is not changed by unitary transformations. Consequently, there are only two terms possible in the effective Hamiltonian which can couple two sites j and k : $c_j^\dagger c_k$ and $c_j c_k^\dagger$. This allows to define consecutive lowest order Schrieffer-Wolff transformations and to sum up the entire series. In this approximation, valid at very strong disorder, the local conserved charges are simply the σ_j^z operators with renormalized fields. The longer-range hoppings generated by the transformations have amplitudes which decay exponentially with the range of the hopping process. The minimum gap between random fields on a cluster of length ℓ , however—which enters the effective hopping in the denominator—scales as ℓ^{-2} and thus cannot overcome the general suppression of the hopping process which scales as $D^{-2\ell}$. This means, in particular, that resonances cannot proliferate.

All of this changes drastically once interactions are included. In particular, there are then $\sim 4^\ell$ different terms which can connect sites a distance ℓ apart. We confirm this exponential growth with the length of the support by symbolic

calculations. A priori, there is thus no reason to believe that a perturbative construction of the local conserved charges by unitary transformations is still possible in the interacting case. Even if we assume that there are no resonances or thermal inclusions (regions with random fields of similar strengths), there is potentially a fundamental problem with this construction: while the amplitude of an individual term connecting sites a distance ℓ apart will still be $\sim D^{-2\ell}$, there are exponentially many such terms so that the total effective amplitude is $\sim 4^\ell D^{-2\ell} \sim \mathcal{O}(1)$. The small parameter in the perturbative construction is in general lost even before considering the issue of resonances. While this simple scaling argument does not preclude that the perturbative construction in the absence of resonances can succeed for very strong disorder such that $4^\ell D^{-2\ell} \ll 1$, our symbolic calculations show that the total effective amplitude is not decaying with the support of the operator in contrast to the noninteracting case. As far as we can tell, this issue is not addressed in the proof of MBL by Imbrie [40,41] which uses the same kind of unitary transformations but seems to be only concerned about resonances. The latter is also a much more severe problem in the interacting case. Even if we assume that the perturbative construction is convergent without resonances and thermal inclusions, many-body resonances will destroy the approach when present. A long-range term typically involves operators on all sites between the left most and the right most site. Consequently, the renormalization will involve linear combinations of all the random fields on this cluster and there are exponentially many of such terms. Clearly, the argument for the nonproliferation of resonances made in the noninteracting case is then no longer valid. At the very least, we believe that we have shown that there are certain aspects of the perturbative proof in Refs. [40,41] which would need to be addressed to make it more convincing. In particular, the discussion of the construction seems to ignore that, in general, the number of terms connecting distant sites grows exponentially with the distance between the sites. It is not obvious why the construction remains controlled by a small parameter despite the exponential number of terms, or why resonances do not proliferate. Our symbolic calculations for finite chains and finite orders of the transformation suggest that such an argument, in fact, cannot be made and that a perturbative construction of local conserved charges in this way is impossible except for $D = \infty$ where the model is trivially localized. We therefore conclude that we have found strong evidence that the unitary U does not exist and that the almost-factorial growth of the operator norm for the random field Ising and Heisenberg models indicates that these models are ergodic in the thermodynamic limit for all disorder strengths, consistent with the operator growth hypothesis. In a generic scenario, this implies that these systems in the thermodynamic limit and at very long times will ultimately exhibit finite transport coefficients.

As an outlook, we note that under the assumption that the unitary transformation U exists, we have only been able to prove the bound $s_l(k) \sim l^k \exp(-l)$, which is consistent with an almost-factorial growth $s(k) \sim (k/\ln k)^k$ of the total norm. However, we have not used any specific properties of a microscopic model such as the random Ising or Heisenberg model, so it may be possible that a stricter bound for $s_l(k)$ can

be derived for special cases. This could lead to a bound with parametrically slower growth than $s(k) \sim (k/\ln k)^k$, and thus a direct contradiction with the rigorous result for the norm growth from Ref. [37]. This seems to be a promising direction for future research.

ACKNOWLEDGMENTS

J.S. acknowledges discussions with D. Sels and M. Kiefer-Emmanouilidis and funding by the NSERC Discovery program and by the DFG via the research unit FOR 2316.

-
- [1] P. W. Anderson, Absence of diffusion in certain random lattices, *Phys. Rev.* **109**, 1492 (1958).
 - [2] J. T. Edwards and D. J. Thouless, Numerical studies of localization in disordered systems, *J. Phys. C* **5**, 807 (1972).
 - [3] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Scaling theory of localization: Absence of quantum diffusion in two dimensions, *Phys. Rev. Lett.* **42**, 673 (1979).
 - [4] E. Abrahams (ed.), *50 Years of Anderson Localization* (World Scientific, Singapore, 2010).
 - [5] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states, *Ann. Phys.* **321**, 1126 (2006).
 - [6] M. Žnidarič, T. Prosen, and P. Prelovšek, Many-body localization in the Heisenberg XXZ magnet in a random field, *Phys. Rev. B* **77**, 064426 (2008).
 - [7] V. Oganesyan and D. A. Huse, Localization of interacting fermions at high temperature, *Phys. Rev. B* **75**, 155111 (2007).
 - [8] A. Pal and D. A. Huse, Many-body localization phase transition, *Phys. Rev. B* **82**, 174411 (2010).
 - [9] J. H. Bardarson, F. Pollmann, and J. E. Moore, Unbounded growth of entanglement in models of many-body localization, *Phys. Rev. Lett.* **109**, 017202 (2012).
 - [10] D. J. Luitz, N. Laflorencie, and F. Alet, Many-body localization edge in the random-field Heisenberg chain, *Phys. Rev. B* **91**, 081103 (2015).
 - [11] D. J. Luitz, N. Laflorencie, and F. Alet, Extended slow dynamical regime close to the many-body localization transition, *Phys. Rev. B* **93**, 060201 (2016).
 - [12] M. Serbyn, Z. Papić, and D. A. Abanin, Local conservation laws and the structure of the many-body localized states, *Phys. Rev. Lett.* **111**, 127201 (2013).
 - [13] R. Vosk, D. A. Huse, and E. Altman, Theory of the many-body localization transition in one-dimensional systems, *Phys. Rev. X* **5**, 031032 (2015).
 - [14] E. Altman and R. Vosk, Universal dynamics and renormalization in many-body-localized systems, *Annu. Rev. Condens. Matter Phys.* **6**, 383 (2015).
 - [15] E. V. H. Doggen, F. Schindler, K. S. Tikhonov, A. D. Mirlin, T. Neupert, D. G. Polyakov, and I. V. Gornyi, Many-body localization and delocalization in large quantum chains, *Phys. Rev. B* **98**, 174202 (2018).
 - [16] P. Sierant, D. Delande, and J. Zakrzewski, Thouless time analysis of anderson and many-body localization transitions, *Phys. Rev. Lett.* **124**, 186601 (2020).
 - [17] R. Nandkishore and D. A. Huse, Many-body localization and thermalization in quantum statistical mechanics, *Annu. Rev. Condens. Matter Phys.* **6**, 15 (2015).
 - [18] A. C. Potter, R. Vasseur, and S. A. Parameswaran, Universal properties of many-body delocalization transitions, *Phys. Rev. X* **5**, 031033 (2015).
 - [19] D. A. Huse, R. Nandkishore, and V. Oganesyan, Phenomenology of fully many-body-localized systems, *Phys. Rev. B* **90**, 174202 (2014).
 - [20] J. Šuntajs, J. Bonca, T. Prosen, and L. Vidmar, Quantum chaos challenges many-body localization, *Phys. Rev. E* **102**, 062144 (2020).
 - [21] J. Šuntajs, J. Bonca, T. Prosen, and L. Vidmar, Ergodicity breaking transition in finite disordered spin chains, *Phys. Rev. B* **102**, 064207 (2020).
 - [22] M. Kiefer-Emmanouilidis, R. Unanyan, J. Sirker, and M. Fleischhauer, Bounds on the entanglement entropy by the number entropy in non-interacting fermionic systems, *SciPost Phys.* **8**, 083 (2020).
 - [23] M. Kiefer-Emmanouilidis, R. Unanyan, M. Fleischhauer, and J. Sirker, Evidence for unbounded growth of the number entropy in many-body localized phases, *Phys. Rev. Lett.* **124**, 243601 (2020).
 - [24] M. Kiefer-Emmanouilidis, R. Unanyan, M. Fleischhauer, and J. Sirker, Slow delocalization of particles in many-body localized phases, *Phys. Rev. B* **103**, 024203 (2021).
 - [25] M. Kiefer-Emmanouilidis, R. Unanyan, M. Fleischhauer, and J. Sirker, Unlimited growth of particle fluctuations in many-body localized phases, *Ann. Phys.* **435**, 168481 (2021).
 - [26] M. Kiefer-Emmanouilidis, R. Unanyan, M. Fleischhauer, and J. Sirker, Particle fluctuations and the failure of simple effective models for many-body localized phases, *SciPost Phys.* **12**, 034 (2022).
 - [27] M. Kiefer-Emmanouilidis, R. Unanyan, M. Fleischhauer, and J. Sirker, Comment on “Resonance-induced growth of number entropy in strongly disordered systems,” *arXiv:2203.06689*.
 - [28] D. Sels and A. Polkovnikov, Dynamical obstruction to localization in a disordered spin chain, *Phys. Rev. E* **104**, 054105 (2021).
 - [29] T. LeBlond, D. Sels, A. Polkovnikov, and M. Rigol, Universality in the onset of quantum chaos in many-body systems, *Phys. Rev. B* **104**, L201117 (2021).
 - [30] A. Avdoshkin and A. Dymarsky, Euclidean operator growth and quantum chaos, *Phys. Rev. Res.* **2**, 043234 (2020).
 - [31] D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, and E. Altman, A universal operator growth hypothesis, *Phys. Rev. X* **9**, 041017 (2019).
 - [32] R. Heveling, J. Wang, and J. Gemmer, Numerically probing the universal operator growth hypothesis, *Phys. Rev. E* **106**, 014152 (2022).
 - [33] L. D’Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics, *Adv. Phys.* **65**, 239 (2016).
 - [34] T. LeBlond, K. Malayya, L. Vidmar, and M. Rigol, Entanglement and matrix elements of observables in interacting integrable systems, *Phys. Rev. E* **100**, 062134 (2019).

- [35] E. Lieb, and D.W. Robinson, The finite group velocity of quantum spin systems, *Commun. Math. Phys.* **28**, 251 (1972).
- [36] S. Bravyi, M.B. Hastings, and F. Verstraete, Lieb-Robinson bounds and the generation of correlations and topological quantum order, *Phys. Rev. Lett.* **97**, 050401 (2006).
- [37] X. Cao, A statistical mechanism for operator growth, *J. Phys. A: Math. Theor.* **54**, 144001 (2021).
- [38] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/wgss-nt8t> for more details about the graphs and analytic norm bounds and a discussion of Lieb-Robinson bounds and off-diagonal matrix elements of an operator in the energy eigenbasis.
- [39] C. Fieker, W. Hart, T. Hofmann, and F. Johansson, Nemo/Hecke: Computer algebra and number theory packages for the Julia programming language, in *Proceedings of the ACM on International Symposium on Symbolic and Algebraic Computation (ISSAC'17)* (Association for Computing Machinery, New York, NY, 2017), pp. 157–164.
- [40] J. Z. Imbrie, Diagonalization and many-body localization for a disordered quantum spin chain, *Phys. Rev. Lett.* **117**, 027201 (2016).
- [41] J. Imbrie, On many-body localization for quantum spin chains, *J. Stat. Phys.* **163**, 998 (2016).
- [42] D. Dominici, Asymptotic analysis of the Bell polynomials by the ray method, *J. Comput. Appl. Math.* **233**, 708 (2009), 9th OPSFA Conference.
- [43] V. E. Kravtsov, I. M. Khaymovich, E. Cuevas, and M. Amini, A random matrix model with localization and ergodic transitions, *New J. Phys.* **17**, 122002 (2015).
- [44] V. E. Kravtsov, B. L. Altshuler, and L. B. Ioffe, Nonergodic delocalized phase in Anderson model on Bethe lattice and regular graph, *Ann. Phys.* **389**, 148 (2018).
- [45] D. Sels, Bath-induced delocalization in interacting disordered spin chains, *Phys. Rev. B* **106**, L020202 (2022).
- [46] S. Aubry and G. André, Analyticity breaking and Anderson localization in incommensurate lattices, *Ann. Israel Phys. Soc.* **3**, 133 (1980).
- [47] S. Y. Jitomirskaya, Metal-insulator transition for the almost Mathieu operator, *Ann. Math.* **150**, 1159 (1999).
- [48] H. Bethe, Zur Theorie der Metalle, *Z. Phys.* **71**, 205 (1931).
- [49] F. H. L. Essler, H. Frahm, F. Göhmann, A. Klümper, and V. E. Korepin, *The One-Dimensional Hubbard Model* (Cambridge University Press, Cambridge, UK, 2005).
- [50] A. Morningstar, D. A. Huse, and J. Z. Imbrie, Many-body localization near the critical point, *Phys. Rev. B* **102**, 125134 (2020).
- [51] J. Z. Imbrie, V. Ros, and A. Scardicchio, Local integrals of motion in many-body localized systems, *Ann. Phys.* **529**, 1600278 (2017).
- [52] B. Lu, C. Bertoni, S. J. Thomson, and J. Eisert, Measuring out quasi-local integrals of motion from entanglement, *Commun. Phys.* **7**, 17 (2024).
- [53] S. J. Thomson and M. Schirò, Local integrals of motion in quasiperiodic many-body localized systems, *SciPost Phys.* **14**, 125 (2023).
- [54] J. R. Schrieffer and P. A. Wolff, Relation between the Anderson and Kondo Hamiltonians, *Phys. Rev.* **149**, 491 (1966).
- [55] S. Bravyi, D. P. DiVincenzo, and D. Loss, Schrieffer-Wolff transformation for quantum many-body systems, *Ann. Phys.* **326**, 2793 (2011).
- [56] J. Sirker, Transport in one-dimensional integrable quantum systems, *SciPost Phys. Lect. Notes* **17** (2020).
- [57] A. Morningstar, L. Colmenarez, V. Khemani, D. J. Luitz, and D. A. Huse, Avalanches and many-body resonances in many-body localized systems, *Phys. Rev. B* **105**, 174205 (2022).
- [58] H. Ha, A. Morningstar, and D. A. Huse, Many-body resonances in the avalanche instability of many-body localization, *Phys. Rev. Lett.* **130**, 250405 (2023).
- [59] D. Sels and A. Polkovnikov, Thermalization of dilute impurities in one-dimensional spin chains, *Phys. Rev. X* **13**, 011041 (2023).
- [60] J. C. Peacock and D. Sels, Many-body delocalization from embedded thermal inclusion, *Phys. Rev. B* **108**, L020201 (2023).