

Unbounded-input explicit Bell inequalities for general quantum networks

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(Received 9 September 2025; accepted 13 January 2026; published 3 February 2026)

Quantum nonlocality in networks featuring multiple independent sources underpins large-scale quantum communication and poses fundamental challenges for its characterization. In this work, we construct a family of explicit nonlinear Bell inequalities to verify the nonlocality across the general multi-input quantum networks. The construction of these inequalities relies on the number of leaf nodes, a network parameter that can be identified by a linear-time algorithm. Our approach establishes a structural connection between bipartite full-correlation Bell inequalities and network Bell inequalities, enabling the analytical derivation of optimal quantum violations and the conditions under which they occur. We further quantify the upper bound on maximal violations achievable by arbitrary two-qubit mixed states in such networks, under separable measurements, and evaluate the noise robustness of the proposed inequalities via the visibilities of noisy states. Finally, we demonstrate that these inequalities can, in a device-independent manner, distinguish between network topologies of equal size that differ in the number of leaf nodes.

DOI: [10.1103/w2yy-mhp3](https://doi.org/10.1103/w2yy-mhp3)

I. INTRODUCTION

Spatially separated observers performing local measurements on a shared entangled quantum state can establish strong correlations that cannot be explained by any local hidden variable theory [1]. This phenomenon, referred as quantum nonlocality [2], lies at the heart of quantum theory and can be verified through the violation of a suitable Bell inequality. Beyond its fundamental significance, nonlocality serves as a vital resource for device-independent quantum tasks, such as quantum key distribution [3], random number generation [4], and reduction of communication complexity [5].

Quantum nonlocality arising in single-source scenarios has been extensively studied. In such scenarios, the achievable local correlations form a convex hull [2], and numerous linear Bell inequalities have been proposed to verify the nonlocality [2,6–8]. Network scenarios with multiple independent sources give rise to correlations that exhibit nonconvex features, making the construction of Bell inequalities for detecting network nonlocality more challenging [9]. Pioneering works

by Branciard *et al.* [10,11] established nonlinear Bell inequalities to verify nonbilocal correlations in the simplest quantum network. Subsequent works generalized this simplest scenario to several typical multisource network structures, like star-shaped [12–16], chain-shaped [17,18], and tree-shaped networks [19–21]. However, these works are specific to particular topologies, while deriving inequalities applicable to general network structures holds greater practical relevance.

Several methods have been developed to derive Bell inequalities for general networks. Refs. [22, 23] employed linear programming techniques and algebraic geometry methods, respectively, to construct Bell inequalities for small-scale general networks. An iterative approach was introduced in Ref. [24], where starting from a Bell inequality for a given network, one can construct a new one for an extended network obtained by adding a bipartite source. This iterative procedure was further shown to, in principle, allow the derivation of Bell inequalities for any acyclic networks with multipartite sources [25]. However, these construction methods do not provide explicit expressions for general networks and instead require case-by-case derivations, which limits their practical applicability. In 2018, Luo derived explicit Bell inequalities for general N -party networks with two inputs per party, based on identifying the largest set of mutually unconnected parties [26]. A computationally efficient algorithm with time complexity $O(N^{2.5})$ was also introduced in Ref. [26] to find this set, though it may yield suboptimal results in large-scale or structurally complex networks.

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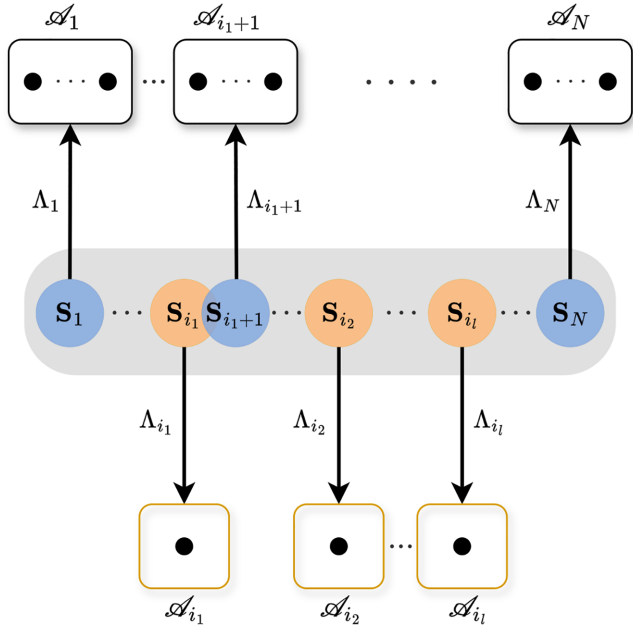


FIG. 1. Schematic diagram of a general network with l leaf nodes (orange squares). There are N parties (nodes) $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_N$, and M bipartite sources $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_M$. Each party $\mathcal{A}_i$ can receive particles from at most M sources $\mathbf{S}_i = \{\mathcal{S}_{j_1}, \mathcal{S}_{j_2}, \dots, \mathcal{S}_{j_{s_i}}\}$ that distribute hidden variables $\Lambda_i = \{\lambda_{j_1}, \lambda_{j_2}, \dots, \lambda_{j_{s_i}}\}$, where $\bigcup_{i=1}^N \mathbf{S}_i = \{\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_M\}$. Intermediate nodes (black squares) receive multiple particles, while each leaf node $\mathcal{A}_{i_s}$ only receives one particle from peripheral source $\mathcal{S}_{i_s}$ (orange circle), i.e., $\mathbf{S}_{i_s} = \{\mathcal{S}_{i_s}\}$, $s = 1, 2, \dots, l$.

In this work, we present efficient and simpler explicit nonlinear Bell inequalities for general N -party networks with M independent bipartite sources and multiple leaf nodes. The construction relies on identifying the number of leaf nodes, which can be efficiently computed in linear time $O(N + M)$. Our framework accommodates arbitrary inputs per party, under the assumption that all intermediate nodes receive the same number of inputs. We establish a direct link between the bipartite full-correlation Bell inequalities (FCBIs) associated with peripheral sources and the network inequality. To verify the nonlocality in noisy quantum networks using our inequalities, we derive the upper bound on maximal quantum violation for any ensemble of two-qubit mixed states produced by sources and measured with separable measurements, together with the conditions required to achieve it. Specifically, considering that all sources emit Werner states, we determine the constraints on visibilities under which network nonlocality can be revealed. Finally, we show that our inequalities provide an effective tool for distinguishing networks with same size but different leaf-node counts, including cases that cannot be discriminated by methods presented in Ref. [27].

II. EXPLICIT BELL INEQUALITIES

Consider a general network depicted in Fig. 1 consisting of N parties $\mathcal{A}_1, \dots, \mathcal{A}_N$, which are mediated by M independent bipartite sources $\mathcal{S}_1, \dots, \mathcal{S}_M$. Each source connects a distinct pair of parties and together forms the overall network.

Labeling the inputs for party $\mathcal{A}_i$ by $x_i \in \mathcal{X}_i$ with binary outputs $a_i \in \{0, 1\}$. The correlation generated in the network is local if the joint probability distribution admits the following form:

$$P(\vec{a}|\vec{x}) = \int \left(\prod_{j=1}^M d\lambda_j q_j(\lambda_j) \right) \prod_{i=1}^N p(a_i|x_i, \Lambda_i), \quad (1)$$

where $\vec{a} = (a_1, a_2, \dots, a_N)$, $\vec{x} = (x_1, x_2, \dots, x_N)$, λ_j denotes the hidden variable distributed by source $\mathcal{S}_j$ and follows the distribution $q_j(\lambda_j)$, satisfying the normalization condition $\int d\lambda_j q_j(\lambda_j) = 1$. $\Lambda_i = \{\lambda_{j_1}, \lambda_{j_2}, \dots, \lambda_{j_{s_i}}\}$ denotes the set of classical variables from sources $\mathbf{S}_i = \{\mathcal{S}_{j_1}, \mathcal{S}_{j_2}, \dots, \mathcal{S}_{j_{s_i}}\}$ associated with party $\mathcal{A}_i$. If the probability distribution cannot be decomposed into Eq. (1), it is said to be nonlocal. Any distribution that can be written in the above form must satisfy a suitable network Bell inequality, which can be violated by measuring quantum states distributed in network.

Leveraging geometric features of network topologies, we construct explicit nonlinear Bell inequalities for general quantum networks with at least two leaf nodes. In our framework, parties connected to a single source are referred to as leaf nodes, with the associated source termed a peripheral source. The set of indices for all peripheral sources is denoted by $\mathcal{I}$. We define the index of the leaf node connected to the peripheral source $\mathcal{S}_{i_s}$ as i_s . In a network with N nodes and l ($l \geq 2$) leaf nodes, let $\Gamma = \{i_1, i_2, \dots, i_l\}$ denote the set of indices of all leaf nodes, and $\bar{\Gamma} = [N] \setminus \Gamma$ (with $[N] := \{1, 2, \dots, N\}$) represent the set of indices of other nodes (i.e., intermediate nodes). By definition, we have $\Gamma = \mathcal{I}$. Assume that the number of inputs of the intermediate nodes is the same, i.e., $|\mathcal{X}_i| = k$, $i \in \bar{\Gamma}$. Let $A_{x_i}^i$ be the measurement of the party $\mathcal{A}_i$ with input x_i . Any correlation generated from a network (see Fig. 1) with classical variables is bounded by the following nonlinear Bell inequality:

$$\mathcal{S} = \sum_{j=1}^k |I_j(N, l)|^{\frac{1}{l}} \leq \left(\prod_{i \in \Gamma} \beta_i \right)^{\frac{1}{l}}, \quad (2)$$

where the correlator $I_j(N, l)$ is defined as

$$I_j(N, l) = \sum_{x_i, i \in \Gamma} \left(\prod_{i \in \Gamma} M_{x_i, j} \right) \langle A_{x_1}^1 A_{x_2}^2 \cdots A_{x_N}^N \rangle, \quad (3)$$

and $\langle A_{x_1}^1 A_{x_2}^2 \cdots A_{x_N}^N \rangle = \sum_{\vec{a}} (-1)^{\sum_{i=1}^N a_i} p(\vec{a}|\vec{x})$. In different correlators $\{I_j(N, l), j = 1, \dots, k\}$, the same intermediate party $\mathcal{A}_u$ ($u \in \bar{\Gamma}$) needs to use different measurement inputs x_u . Without loss of generality, we set the inputs of all intermediate parties in the correlator $I_j(N, l)$ to be j . Defining $\Delta_j^i = \sum_{x_i} M_{x_i, j} A_{x_i}^i$ for all $i \in \Gamma$, the expression in Eq. (3) is rewritten as $I_j(N, l) = \langle \prod_{u \in \bar{\Gamma}} A_j^u \prod_{i \in \Gamma} \Delta_j^i \rangle$. The quantity β_i represents the classical bound of bipartite FCBI $\mathcal{B}_i$ associated with peripheral source $\mathcal{S}_{i_s}$, which is fully characterized by a real coefficient matrix $\mathbf{M}^{(i)} = (M_{x_i, j})$ (see Appendix A 1 for details). Using Eq. (1) and the important inequality (4), the classical

bound $(\prod_{i \in \Gamma} \beta_i)^{\frac{1}{l}}$ in Eq. (2) is derived in Appendix A 2, where the connection between the network inequality and the bipartite FCBI is established.

For all $i \in \Gamma$, coefficients $\{M_{x_i, j}\}$ in inequality (3) are determined from bipartite FCBIs $\{\mathcal{B}_i\}$ related to the number of

measurement settings for the parties connected by peripheral sources $\{\mathcal{S}_i, i \in \Gamma\}$. For example, when both nodes connected via a peripheral source $\mathcal{S}_i$ have two inputs, Clauser–Horne–Shimony–Holt (CHSH) inequality is considered for $\mathcal{S}_i$ such that $M_{x_i,j} = \frac{1}{2}(-1)^{x_i \cdot j}$ ($x_i, j \in \{1, 2\}$). Further examples are provided in Sec. V. Another key quantity l corresponds to the number of degree-one vertices in the underlying network graph. It can be computed in linear time $O(N + M)$ using an adjacency list representation [28], as each bipartite source contributes one edge, making the number of edges equal to M .

Suppose $\mathbf{X} = (x_{j,i}) \in \mathbb{R}^{p \times q}$ is a nonnegative matrix where $x_{j,i} \geq 0$. Ref. [29] introduced the following inequality:

$$\sum_{j=1}^p \left(\prod_{i=1}^q x_{j,i} \right)^{\frac{1}{q}} \leq \prod_{i=1}^q \left(\sum_{j=1}^p x_{j,i} \right)^{\frac{1}{q}}, \quad (4)$$

where equality holds if and only if either $\text{Rank}(\mathbf{X}) = 1$, or for some $i \in [q]$, $x_{j,i} = 0$ for all $j \in [p]$. This inequality is used to derive the classical bound of the network inequality (2) and is also crucial for deriving its quantum bound. For quantum networks with identical topology but replacing all classical variables $\lambda_1, \dots, \lambda_M$ with bipartite quantum states $\rho_1, \dots, \rho_M$, the quantum correlations are bounded by

$$\mathcal{S} = \sum_{j=1}^k |I_j(N, l)|^{\frac{1}{l}} \leq \left(\prod_{i \in \Gamma} (\mathcal{B}_i)_{\mathcal{Q}}^{\text{opt}} \right)^{\frac{1}{l}}, \quad (5)$$

where $(\mathcal{B}_i)_{\mathcal{Q}}^{\text{opt}}$ denotes the optimal quantum value of $\mathcal{B}_i$. The network correlators can be expressed as $\langle \otimes_{i=1}^N A_{x_i}^i \rangle_{\rho} = \text{Tr}[(\otimes_{i=1}^N A_{x_i}^i)(\otimes_{j=1}^M \rho_j)]$, where $A_{x_i}^i$ is the dichotomic measurement for party $\mathcal{A}_i$. The upper bound in Eq. (5) is derived using Eq. (4) and the elegant sum-of-squares (SOS) approach introduced in Ref. [15], with full proof given in Appendix A 3. We show that the upper bound can be achieved using separable measurements. Each intermediate party performs separable measurements on each of its subsystems. The measurement of an intermediate party $\mathcal{A}_i$ ($i \in \Gamma$) with input x_i factorizes as $A_{x_i}^i = \prod_{w \in \mathcal{I}_i} B_{x_i}^{(w)}$, where $\mathcal{I}_i$ is the set of indices of sources associated with $\mathcal{A}_i$, and $B_{x_i}^{(w)}$ denotes the local observable performed by intermediate party $\mathcal{A}_i$ on the subsystem from ρ_w . The upper bound is achieved under the following conditions, with the detailed proof provided in Appendix A 4.

(1) For each peripheral source $\mathcal{S}_i$ ($i \in \mathcal{I}$), the local observables $\{A_{x_i}^i, x_i \in \mathcal{X}_i\}$ and $\{B_j^{(i)}, j \in [k]\}$ acting on the subsystems of ρ_i achieve $(\mathcal{B}_i)_{\mathcal{Q}}^{\text{opt}}$.

(2) The equality condition of Eq. (4) should be satisfied. Here, the matrix in Eq. (4) is given by $\mathbf{X} = (x_{j,i}) \in \mathbb{R}^{k \times |\mathcal{I}|}$ with $x_{j,i} = \sqrt{\text{Tr}[(\Delta_j^i)^{\dagger} \Delta_j^i \rho_i]} \geq 0$, where $i \in \mathcal{I}$, $j \in [k]$, and $\Delta_j^i = \sum_{x_i} M_{x_i,j} A_{x_i}^i$.

(3) For each intermediate source $\mathcal{S}_i$ ($i \in [M] \setminus \mathcal{I}$), the state ρ_i is measured by local observables $\{B_{x_{\square=j}}^{(i)}, j \in [k]\}$ and $\{B_{x_{\Delta=j}}^{(i)}, j \in [k]\}$ such that $\langle B_{x_{\square=j}}^{(i)} B_{x_{\Delta=j}}^{(i)} \rangle_{\rho_i} = 1$ for all inputs $j \in [k]$. The symbols $\square, \Delta \in \Gamma$ denote the indices of intermediate parties connected via the intermediate source ρ_i .

Equations (2) and (5) establish a direct link between bipartite FCBIs and the constructed network Bell inequality. Both the classical bound and optimal quantum violation of

the network inequality can be determined using the results of well-developed bipartite FCBIs.

III. QUANTUM VIOLATIONS FOR ARBITRARY TWO-QUBIT MIXED STATES

To investigate nonlocality in noisy networks, we model each source as emitting an arbitrary two-qubit mixed state $\rho_i = \frac{1}{4}(\mathbb{I} \otimes \mathbb{I} + \sum_{u=1}^3 a_u^{(i)} \sigma_u \otimes \mathbb{I} + \sum_{v=1}^3 b_v^{(i)} \mathbb{I} \otimes \sigma_v + \sum_{u,v=1}^3 t_{uv}^{(i)} \sigma_u \otimes \sigma_v)$, with σ_u ($u=1, 2, 3$) the Pauli operators and $t_{uv}^{(i)} = \text{Tr}[\rho_i(\sigma_u \otimes \sigma_v)]$ the entries of the correlation matrix T_{ρ_i} of ρ_i . Denote $1 \geq t_{i,0} \geq t_{i,1} \geq t_{i,2} \geq 0$ the three singular values of T_{ρ_i} . Each party performs dichotomic measurements that are separable across the qubit systems.

For any ensemble of two-qubit mixed states $\rho = \otimes_{i=1}^M \rho_i$, the maximal quantum value of Eq. (2) obtained using local qubit observables satisfies

$$\mathcal{S}^{\max}(\rho) \leq \prod_{i \in \mathcal{I}} [\mathcal{B}_i^{\max}(\rho_i)]^{\frac{1}{l}} \prod_{u \in [M] \setminus \mathcal{I}} t_{u,0}^{\frac{1}{l}}, \quad (6)$$

where $\mathcal{B}_i^{\max}(\rho_i)$ denotes the maximal quantum value of $\mathcal{B}_i$ that can be achieved for the shared two-qubit state ρ_i , and $t_{i,0}$ is the largest singular value of T_{ρ_i} .

The upper bound in Eq. (6) is achieved when the local operators acting on each peripheral source ρ_i ($i \in \mathcal{I}$) both maximize $\mathcal{B}_i(\rho_i)$ and satisfy the equality condition of Eq. (4). Here, the matrix in Eq. (4) is $\mathbf{X} = (x_{j,i}) \in \mathbb{R}^{k \times |\mathcal{I}|}$ with $x_{j,i} = \langle B_j^{(i)} \Delta_j^i \rangle_{\rho_i} \geq 0$, where $j \in [k]$ and $i \in \mathcal{I}$. Meanwhile, the operators acting on the intermediate source ρ_i ($i \in [M] \setminus \mathcal{I}$) should enable $\langle B_{x_{\square=j}}^{(i)} B_{x_{\Delta=j}}^{(i)} \rangle_{\rho_i} = t_{i,0}$. The detailed proof is provided in Appendix B. As mentioned in Ref. [29], the correlation matrix can be diagonalized and ordered as $\text{diag}\{t_{i,1}, t_{i,2}, t_{i,0}\}$. This implies that it is sufficient to measure the systems connected by intermediate sources in the σ_3 basis.

In deriving the upper bound in Eq. (6), we considered only separable measurements, which are experimentally feasible and offer convenient theoretical simplifications. This approach is commonly used in studies of the maximal violation achievable by a given state, $\rho = \otimes_{i=1}^M \rho_i$, in noisy networks [29–33]. However, it is important to explore the maximal violations achievable using general joint measurements in future research, as they may lead to larger violations in such networks.

IV. RESISTANCE TO NOISE

To analyze the noise robustness of the inequalities derived in this work, we assume that each source emits a noisy state of the form $\rho_i = v_i |\phi_i\rangle\langle\phi_i| + \frac{1-v_i}{4} \mathbb{I}$, where $v_i \in [0, 1]$ and $|\phi_i\rangle = a_i |00\rangle + b_i |11\rangle$, with real coefficients a_i, b_i satisfying $a_i^2 + b_i^2 = 1$. In the two-input case with $x_i, j \in \{1, 2\}$, the CHSH inequality is considered for each peripheral source such that $M_{x_i,j} = \frac{1}{2}(-1)^{x_i \cdot j}$ and $(\prod_{i \in \Gamma} \mathcal{B}_i)^{\frac{1}{l}} = 1$. From Eq. (6), the Bell inequality (2) is violated whenever the visibilities satisfy

$$\prod_{i=1}^M v_i > \frac{1}{\prod_{i \in \mathcal{I}} \sqrt{1 + 4a_i^2 b_i^2}}. \quad (7)$$

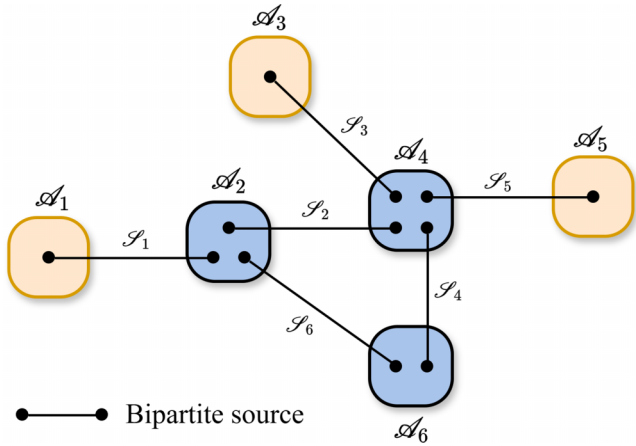


FIG. 2. The network consisting of six sources $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_6$ and six parties $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_6$, where $\mathcal{A}_1, \mathcal{A}_3, \mathcal{A}_5$ are the leaf nodes ($M = 6, l = 3$).

We further extend the analysis to the case of k inputs per party with $x_i, j \in [k]$. In this setting, the chained Bell inequality is considered for each peripheral source, which leads to $M_{x_i, j} = \frac{1}{2}[\delta_{x_i, j} + (1 - 2\delta_{j, k})\delta_{x_i, j+1(\text{mod } k)}]$ and $(\prod_{i \in \Gamma} \beta_i)^{\frac{1}{l}} = k - 1$ [34]. Using Eq. (6), we obtain $S^{\max}(\rho) \leq k \cos \frac{\pi}{2k} \prod_{i \in [M]} v_i^{\frac{1}{l}}$, where the bound is tight whenever the largest singular value of $T_{\rho_i} (i \in \mathcal{I})$ is degenerate. For instance, equality holds when all sources emit the Werner states that mixed the maximally entangled state $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ with white noise $\mathbb{I}$. In this case, the nonlocality in the quantum network can be revealed when the visibilities satisfy

$$\prod_{i \in [M]} v_i > \left(\frac{k-1}{k \cos \frac{\pi}{2k}} \right)^l. \quad (8)$$

When all sources produce the same Werner state with visibility v and $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, the critical visibility per source simplifies to $\left(\frac{k-1}{k \cos \frac{\pi}{2k}} \right)^{\frac{l}{M}}$.

V. EXAMPLES

In what follows, we randomly select a six-party, six-source network ($N = 6, M = 6$) with three leaf nodes (see Fig. 2) to illustrate our construction method. The leaf nodes are $\mathcal{A}_1, \mathcal{A}_3, \mathcal{A}_5$, while the peripheral sources are $\mathcal{S}_1, \mathcal{S}_3, \mathcal{S}_5$, implying $\Gamma = \mathcal{I} = \{1, 3, 5\}$. For different measurement configurations, we provide the corresponding network Bell inequalities, their maximal quantum violations, and the measurement schemes required to achieve these violations.

Example 1. (Two-input case) Each party has two inputs and outputs. In this setting, the CHSH inequality is considered for each peripheral source such that $M_{x_i, j} = \frac{1}{2}(-1)^{x_i \cdot j}$ and $(\prod_{i \in \Gamma} \beta_i)^{\frac{1}{l}} = 1$. Thus, after concretizing the inequalities (2) and (3), we obtained that the quantum nonlocal correlations from the network are demonstrated by violating the following inequality:

$$\mathcal{S}_{\text{exp1}} = |I_1(6, 3)|^{\frac{1}{3}} + |I_2(6, 3)|^{\frac{1}{3}} \leq 1, \quad (9)$$

where $I_1(6, 3) = \frac{1}{2^3} \langle (A_2^1 - A_1^1)A_7^2(A_2^3 - A_1^3)A_4^4(A_2^5 - A_1^5)A_6^6 \rangle$ and $I_2(6, 3) = \frac{1}{2^3} \langle (A_1^1 + A_2^1)A_2^2(A_1^3 + A_2^3)A_2^4(A_1^5 + A_2^5)A_2^6 \rangle$. For the network, sharing six mixed states $\rho_1, \rho_2, \dots, \rho_6$ provided by six independent sources, by applying Eq. (6), the maximal quantum value of $\mathcal{S}_{\text{exp1}}$ is

$$\mathcal{S}_{\text{exp1}}^{\max}(\rho) = \prod_{i \in \mathcal{I}} [\sqrt{t_{i,0}^2 + t_{i,1}^2}]^{\frac{1}{3}} \prod_{u \in [M] \setminus \mathcal{I}} t_{u,0}^{\frac{1}{3}}, \quad (10)$$

with $\mathcal{I} = \{1, 3, 5\}$ and $[M] \setminus \mathcal{I} = \{2, 4, 6\}$, which is achievable by letting

$$\begin{aligned} A_{x_i, i \in \{1,3,5\}}^i &= (x_i - 2)^2 \sigma_1 + (x_i - 1) \sigma_3, \\ A_{x_2}^2 &= \frac{t_{1,0} \sigma_3 + t_{1,1} (-1)^{x_2} \sigma_1}{\sqrt{t_{1,0}^2 + t_{1,1}^2}} \otimes \sigma_3 \otimes \sigma_3, \\ A_{x_4}^4 &= \sigma_3 \otimes \frac{t_{3,0} \sigma_3 + t_{3,1} (-1)^{x_4} \sigma_1}{\sqrt{t_{3,0}^2 + t_{3,1}^2}} \\ &\quad \otimes \sigma_3 \otimes \frac{t_{5,0} \sigma_3 + t_{5,1} (-1)^{x_4} \sigma_1}{\sqrt{t_{5,0}^2 + t_{5,1}^2}}, \\ A_{x_6}^6 &= \sigma_3 \otimes \sigma_3, \end{aligned} \quad (11)$$

where $x_i \in \{1, 2\}$ and $i = 1, 2, 3, 4, 5, 6$. Equation (5) implies that the quantum value of inequality (9) is at most $\sqrt{2}$, attained by the measurements in Eq. (11) with $t_{i,0} = t_{i,1} = 1$ ($i \in \mathcal{I}$) on maximally entangled states.

Example 2. (k-input case) Each party has $k > 2$ inputs. For each peripheral source, we consider the chained Bell inequality, whose classical bound and optimal quantum value have been derived as $k - 1$ and $k \cos \frac{\pi}{2k}$, respectively [34]. Then, from Eq. (2), we obtain the following inequality:

$$\mathcal{S}_{\text{exp2}} = \sum_{j=1}^k |I_j(6, 3)|^{\frac{1}{3}} \leq k - 1, \quad (12)$$

where $I_j(6, 3) = \frac{1}{2^3} \langle \prod_{u \in \{2,4,6\}} A_j^u \prod_{i \in \{1,3,5\}} (A_j^i + A_{j+1}^i) \rangle$, with $A_{k+1}^i = -A_1^i$. For any ensemble of two-qubit mixed states $\rho = \otimes_{i=1}^6 \rho_i$, by applying Eq. (6), we have

$$\mathcal{S}_{\text{exp2}}^{\max}(\rho) \leq k \cos \frac{\pi}{2k} \prod_{i \in [M]} t_{i,0}^{\frac{1}{3}}, \quad (13)$$

with saturation achieved when the largest singular value of the correlation matrices for ρ_1, ρ_3, ρ_5 is degenerate, and the parties perform the following observables:

$$\begin{aligned} A_{x_i, i \in \{1,3,5\}}^i &= \sin \frac{(x_i-1)\pi}{k} \sigma_1 + \cos \frac{(x_i-1)\pi}{k} \sigma_3, \\ A_{x_2}^2 &= \left(\sin \frac{(2x_2-1)\pi}{2k} \sigma_1 + \cos \frac{(2x_2-1)\pi}{2k} \sigma_3 \right)^{\otimes 3}, \\ A_{x_4}^4 &= \left(\sin \frac{(2x_4-1)\pi}{2k} \sigma_1 + \cos \frac{(2x_4-1)\pi}{2k} \sigma_3 \right)^{\otimes 4}, \\ A_{x_6}^6 &= \left(\sin \frac{(2x_6-1)\pi}{2k} \sigma_1 + \cos \frac{(2x_6-1)\pi}{2k} \sigma_3 \right)^{\otimes 2}, \end{aligned}$$

with $x_i \in [k]$ and $i = 1, 2, 3, 4, 5, 6$. Employing the above observables, we find that the optimal quantum value $(\mathcal{S}_{\text{exp2}})_Q^{\text{opt}} = k \cos \frac{\pi}{2k}$ can be achieved when all sources emit two-qubit maximal entangled states.

Example 3. (Asymmetry-input case) Consider the measurement configuration where the leaf nodes $\mathcal{A}_1$ and $\mathcal{A}_3$, each

has three inputs, while leaf node $\mathcal{A}_5$ and the intermediate nodes, each has four inputs. In this setting, the EBI is considered for the peripheral sources $\mathcal{S}_1$ and $\mathcal{S}_3$, with classical bound $\beta_i = 6$ and optimal quantum value $(\mathcal{B}_i)_Q^{\text{opt}} = 4\sqrt{3}$ [35]. The four-input chained Bell inequality is considered for peripheral source $\mathcal{S}_5$, with classical bound $\beta_i = 3$ and optimal quantum value $(\mathcal{B}_i)_Q^{\text{opt}} = 4 \cos \frac{\pi}{8}$ [35]. Substituting these values into Eq. (2), we obtain the nonlinear network inequality of the form

$$\mathcal{S}_{\text{exp3}} = |I_1(6, 3)|^{\frac{1}{3}} + |I_2(6, 3)|^{\frac{1}{3}} + |I_3(6, 3)|^{\frac{1}{3}} + |I_4(6, 3)|^{\frac{1}{3}} \leq 6 \left(\frac{1}{2} \right)^{\frac{1}{3}}, \quad (14)$$

where $I_j(6, 3) = \langle \prod_{u \in \{2, 4, 6\}} A_u^i \prod_{i \in \{1, 3, 5\}} \Delta_j^i \rangle$. For $i \in \{1, 3\}$, $\Delta_1^i = A_1^i + A_2^i + A_3^i$, $\Delta_2^i = A_1^i + A_2^i - A_3^i$, $\Delta_3^i = A_1^i - A_2^i + A_3^i$, $\Delta_4^i = A_1^i - A_2^i - A_3^i$, and $\Delta_j^5 = \frac{1}{2}(A_j^5 + A_{j+1}^5)$ satisfying $A_5^5 = -A_1^5$.

For any ensemble of two-qubit mixed states $\rho = \otimes_{i=1}^6 \rho_i$, Eq. (6) gives the upper bound on maximal quantum violation,

$$\mathcal{S}_{\text{exp3}}^{\text{max}}(\rho) \leq 2[4\sqrt{2 + \sqrt{2}} \prod_{i \in \{1, 3\}} \sqrt{t_{i,0}^2 + t_{i,1}^2 + t_{i,2}^2} \times \prod_{u \in \{2, 4, 5, 6\}} t_{u,0}]^{\frac{1}{3}}, \quad (15)$$

where the bound is saturated if the degeneracy of the largest singular value of correlation matrix T_{ρ_s} exceeds 1. The observables required to achieve this bound are given by

$$\begin{aligned} A_{x_i, i \in \{1, 3\}}^i &= \frac{1}{2}x_i(2 - x_i)(3 - x_i)\sigma_1 + (1 - x_i)^2(3 - x_i)\sigma_2 \\ &\quad + \frac{1}{2}(1 - x_i)(2 - x_i)\sigma_3, \\ A_{x_2}^2 &= B_{x_2}^{(1)} \otimes \sigma_3 \otimes \sigma_3, \\ A_{x_4}^4 &= B_{x_4}^{(3)} \otimes B_{x_4}^{(5)} \otimes \sigma_3 \otimes \sigma_3, \\ A_{x_5}^5 &= \sin \frac{(x_5 - 1)\pi}{4} \sigma_1 + \cos \frac{(x_5 - 1)\pi}{4} \sigma_3, \\ A_{x_6}^6 &= \sigma_3 \otimes \sigma_3, \end{aligned} \quad (16)$$

with $x_1, x_3 \in \{1, 2, 3\}$, $x_2, x_4, x_5, x_6 \in \{1, 2, 3, 4\}$,

$$\begin{aligned} B_1^{(i)} &= \frac{t_{i,0}\sigma_1 + t_{i,1}\sigma_2 + t_{i,2}\sigma_3}{\sqrt{t_{i,0}^2 + t_{i,1}^2 + t_{i,2}^2}}, \\ B_2^{(i)} &= \frac{t_{i,0}\sigma_1 + t_{i,1}\sigma_2 - t_{i,2}\sigma_3}{\sqrt{t_{i,0}^2 + t_{i,1}^2 + t_{i,2}^2}}, \\ B_3^{(i)} &= \frac{t_{i,0}\sigma_1 - t_{i,1}\sigma_2 + t_{i,2}\sigma_3}{\sqrt{t_{i,0}^2 + t_{i,1}^2 + t_{i,2}^2}}, \\ B_4^{(i)} &= \frac{t_{i,0}\sigma_1 - t_{i,1}\sigma_2 - t_{i,2}\sigma_3}{\sqrt{t_{i,0}^2 + t_{i,1}^2 + t_{i,2}^2}}, \quad i = 1, 3, \\ \text{and } B_{x_4}^{(5)} &= \sin \frac{(2x_4 - 1)\pi}{8} \sigma_1 + \cos \frac{(2x_4 - 1)\pi}{8} \sigma_3. \end{aligned}$$

From Eq. (5), the quantum value of $\mathcal{S}_{\text{exp3}}$ is upper bounded by $2(12\sqrt{2 + \sqrt{2}})^{\frac{1}{3}}$, which is achieved by applying the operators in Eq. (16) with $t_{i,0} = t_{i,1} = 1$ ($i \in \mathcal{I}$) on ensembles of maximally entangled states.

In a similar manner, we can construct inequalities for general N -party networks with l ($l \geq 2$) leaf nodes. In fact, the network inequalities for specific network topologies, including star shaped [13–16], chain shaped [17,18], and tree shaped [19–21], can be recovered as special cases by specifying parameters in our inequality (2). The parameter l can be easily identified. For instance, in the chain-shaped network, $l = 2$. In the star-shaped network, we have $l = N - 1$, where N denotes the total number of parties. For the p -layer q -forked tree-shaped network, the number of leaf nodes is given by $l = q^{p-1}$.

VI. DISTINGUISHING GENERAL NETWORK TOPOLOGIES

One can also exploit our inequalities as a tool to distinguish the network topologies. Consider two networks $\mathcal{N}_1$ and $\mathcal{N}_2$ with the same size and different number of leaf nodes $l_{\mathcal{N}_1}$ and $l_{\mathcal{N}_2}$. Without loss of generality, we apply inequalities (2) and (5) under the assumption of two inputs per party. In this case, the left-hand side of both inequalities reduces to the form

$$\mathcal{S} = |I_1(N, l)|^{\frac{1}{l}} + |I_2(N, l)|^{\frac{1}{l}}, \quad (17)$$

where $I_j(N, l) = \langle \prod_{u \in \Gamma} A_u^i \prod_{i \in \Gamma} \Delta_j^i \rangle$, with $\Delta_j^i = \frac{1}{2}(A_2^i + (-1)^j A_1^i)$. The classical bound of $\mathcal{S}$ is 1, while the optimal quantum value $(\mathcal{S})_Q^{\text{opt}}$ is $\sqrt{2}$. Quantum correlations in network $\mathcal{N}_1$ can be revealed by violating $\mathcal{S}_{\mathcal{N}_1} \leq 1$ and bounded by the inequality $\mathcal{S}_{\mathcal{N}_1} \leq \sqrt{2}$. The same holds for network $\mathcal{N}_2$.

For two same-size networks with different number of leaf nodes, $l_{\mathcal{N}_1} > l_{\mathcal{N}_2} \geq 2$, the network $\mathcal{N}_2$ can be distinguished from $\mathcal{N}_1$ when quantum correlations achievable in network $\mathcal{N}_2$ violate the inequality $\mathcal{S}_{\mathcal{N}_1} \leq \sqrt{2}$. See Appendix C for the proof.

These networks can generate distinct sets of nonlocal correlations through local measurements, implying that network structures can be distinguished in a device-independent way. Next, we consider the noisy networks where each source emits Werner state, $\rho = v|\phi\rangle\langle\phi| + \frac{1-v}{4}\mathbb{I}$, with $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. From Sec. IV, the nonlocality can be revealed if

$$v > \left(\frac{1}{\sqrt{2}} \right)^{\frac{l}{M}}. \quad (18)$$

Equation (18) shows that the number of sources and the leaf nodes in different network topologies influence the network's ability to generate quantum correlations. Specifically, the larger the ratio $\frac{l}{M}$, the more readily the network can produce quantum nonlocal correlations. This suggests that, among networks of the same size with at least two leaf nodes, those with larger leaf nodes exhibit a greater capacity to generate quantum nonlocal correlations.

To illustrate our results, we consider the two small-scale networks shown in Fig. 3. These networks can be distinguished by violating $\mathcal{S}_{(a)} \leq \sqrt{2}$ using quantum correlations

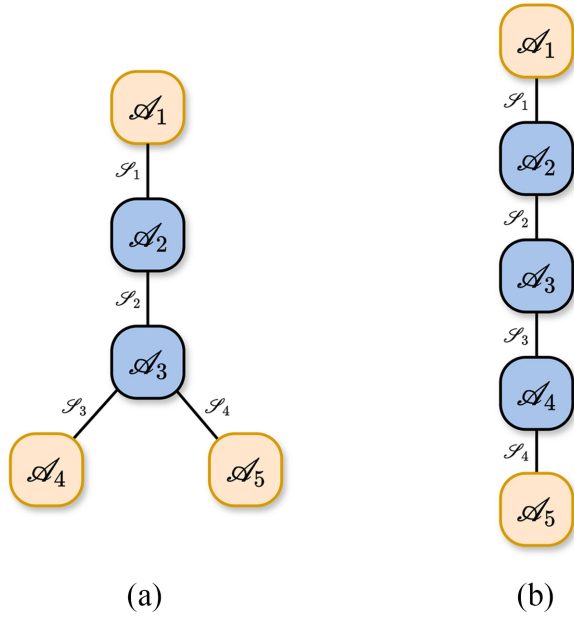


FIG. 3. Two five-party, four-source networks ($N = 5, M = 4$). (a). Tree network with three leaf nodes ($l = 3$). (b). Chain network with two leaf nodes ($l = 2$).

generated in the network of Fig. 3(b) (i.e., $\mathcal{S}_{(b)} \leq \sqrt{2}$). We also consider that all sources produce Werner states. Eq. (18) implies that for $v \in [0.8123, 0.8706]$, quantum nonlocal correlations can be detected in the network of Fig. 3(a), but not in that of Fig. 3(b). Therefore, the two networks can be distinguished using their nonlocal correlations.

Note that the two networks in Fig. 3 cannot be distinguished by the inequality proposed in Ref. [26], as they have the same number of mutually unconnected parties [27]. Similarly, our inequalities are not applicable for distinguishing networks with the same number of leaf nodes. This observation suggests that effectively distinguishing network structures may require the combination of different types of inequalities.

VII. CONCLUSIONS

In summary, we provided a family of explicit nonlinear network Bell inequalities with multiple inputs, exploiting their structural connection to bipartite FCBI. The construction relies on the number of leaf nodes, a graph-theoretic quantity computable in linear time. Since the number of leaf nodes never exceeds the maximum number of mutually unconnected parties, our inequalities possess a simpler structural form than those in Ref. [26], substantially reducing the experimental effort required to observe their violation. Using the elegant SOS approach, we analytically determined the optimal quantum violation of these network inequalities, explicitly expressed as a function of the quantum bounds of the FCBI associated with peripheral sources. Our results showed that measurement strategies achieving optimal FCBI violations can be extended to general networks. The optimal quantum value remains achievable even when intermediate sources emit classical states, provided that the largest singular value of their

correlation matrix equals 1. Moreover, we derived the upper bound on maximal quantum violation for arbitrary ensembles of two-qubit mixed states under separable measurements, and identified the conditions required to achieve it.

The main strengths of our inequalities lie in their scalability to networks with increasing numbers of nodes and measurement inputs per party, as well as in the possibility of achieving maximal quantum violation using separable measurements, which offers a significant experimental advantage. Moreover, our inequalities complement existing topological witness schemes by distinguishing same-sized networks with equal numbers of mutually unconnected parties, which cannot be distinguished by the method in Ref. [27].

Our nonlinear inequalities are proposed to characterize the network nonlocality in general quantum networks, where all sources are bipartite, and there are at least two leaf nodes. They cannot characterize the full network nonlocality (FNN), a stronger form of network nonlocality. Munshi and Pan [36] introduced a set of inequalities to demonstrate the FNN in star-shaped and chain-shaped networks. Exploring inequalities that characterize FNN in general networks is a meaningful direction for further research.

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of China (Grants No. 62571060, No. 62171056, and No. 62220106012).

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: PROOF OF THE INEQUALITIES (2) AND (5)

In this section, we present detailed proofs of the classical bounds and optimal quantum violations for the network Bell inequalities presented in the main text. We start by introducing bipartite full-correlation Bell inequalities and their quantum optimal values, which serve as the foundation for analyzing the corresponding quantities in the proposed nonlinear network Bell inequalities.

1. Bipartite full-correlation Bell inequalities

In the standard bipartite Bell scenario, a single source distributes systems to two spatially separated parties, $\mathcal{A}$ and $\mathcal{B}$. Each party performs a dichotomic measurement, denoted by A_x for $\mathcal{A}$ and B_y for $\mathcal{B}$, corresponding to inputs $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, respectively. For convenience, the measurement outcomes are denoted as $A_x, B_y = \pm 1$. Any bipartite FCBI takes the form [14]

$$\mathcal{B}_{\mathbf{M}} = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} M_{x,y} \langle A_x B_y \rangle \leq \beta, \quad (\text{A1})$$

where β is the classical bound, and $M_{x,y}$ are real numbers. The term $\langle A_x B_y \rangle$ represents the expectation value of the measurement outcomes of parties $\mathcal{A}$ and $\mathcal{B}$. The Bell inequality $\mathcal{B}_{\mathbf{M}}$ is fully characterized by a coefficient matrix $\mathbf{M} = (M_{x,y}) \in \mathbb{R}^{|\mathcal{X}| \times |\mathcal{Y}|}$. As a specific example, the CHSH inequality $\mathcal{B}_{\text{CHSH}}$ is obtained by setting $M_{x,y} = \frac{1}{2}(-1)^{xy}$, where $x, y \in \{1, 2\}$. To

compute the classical bound β , we rewrite Eq. (A1) to

$$\mathcal{B}_M = \sum_{y \in \mathcal{Y}} \langle \Delta_y B_y \rangle, \quad (\text{A2})$$

where $\Delta_y = \sum_{x \in \mathcal{X}} M_{x,y} A_x$. Setting $B_y = \text{sign}(\Delta_y)$, we have

$$\beta = \max_{\{A_x = \pm 1\}_{x \in \mathcal{X}}} \sum_{y \in \mathcal{Y}} |\Delta_y|. \quad (\text{A3})$$

To derive the optimal quantum value of the Bell inequality $\mathcal{B}_M$, we use the elegant SOS approach introduced in Ref. [15]. Consider that $(\mathcal{B}_M)_Q \leq \delta_M$, where δ_M is the upper bound of $(\mathcal{B}_M)_Q$. Equivalently, there exists a positive semidefinite operator $\gamma_{\mathcal{B}_M} \geq 0$ such that $\langle \gamma_{\mathcal{B}_M} \rangle_Q = -(\mathcal{B}_M)_Q + \delta_M$. The optimal quantum value of $(\mathcal{B}_M)_Q$ is attained when $\langle \gamma_{\mathcal{B}_M} \rangle_Q = 0$. The existence of such operators $\gamma_{\mathcal{B}_M}$ can be proved by considering a set of positive operators $\{L_y, y \in \mathcal{Y}\}$, which are linear functions of the observables $\{A_x\}$, $\{B_y\}$, such that

$$\gamma_{\mathcal{B}_M} = \sum_{y \in \mathcal{Y}} \frac{\omega_y}{2} L_y^\dagger L_y, \quad (\text{A4})$$

where the operators L_y ($\forall y$) and the quantities ω_y are given as follows:

$$L_y |\psi\rangle = \frac{\Delta_y}{\omega_y} |\psi\rangle - B_y |\psi\rangle, \quad (\text{A5})$$

$$\omega_y = \|\Delta_y |\psi\rangle\|_2, \quad (\text{A6})$$

where $\|\cdot\|_2$ is the Frobenious norm, given by $\|\mathcal{O}|\psi\rangle\|_2 = \sqrt{\langle \psi | \mathcal{O}^\dagger \mathcal{O} | \psi \rangle}$.

Substituting Eqs. (A5) and (A6) into Eq. (A4) and noting that $A_i^\dagger A_i = B_i^\dagger B_i = \mathbb{I}$, we get $\langle \gamma_{\mathcal{B}_M} \rangle_Q = \sum_y \omega_y - (\mathcal{B}_M)_Q$. Since $\langle \gamma_{\mathcal{B}_M} \rangle_Q \geq 0$, we have the optimal value of $(\mathcal{B}_M)_Q$ when $\langle \gamma_{\mathcal{B}_M} \rangle_Q = 0$. Then we obtain that

$$(\mathcal{B}_M)_Q^{\text{opt}} = \max \left(\sum_{y \in \mathcal{Y}} \omega_y \right). \quad (\text{A7})$$

2. Proof of the inequality (2)

In this subsection, we provide the full proof of the classical bound presented in Eq. (2).

Assume that Eq. (1) holds. As stated in the main text, the correlator in Eq. (3) can be rewritten as

$$I_j(N, l) = \left\langle \prod_{u \in \bar{\Gamma}} A_j^u \prod_{i \in \Gamma} \Delta_j^i \right\rangle, \quad (\text{A8})$$

where $\Delta_j^i = \sum_{x_i} M_{x_i, j} A_{x_i}^i$ with $i \in \Gamma$, and A_j^u represents the measurement of party $\mathcal{A}_u$ with input j . Moreover, we write $\langle A_j^u \rangle_{\Lambda_u} = \sum_{a_u=0}^1 (-1)^{a_u} p(a_u | x_u, \Lambda_u)$. Since $|\langle A_j^u \rangle_{\Lambda_u}| \leq 1$, by Eqs. (1) and (A8), we have

$$\begin{aligned} |I_j(N, l)| &= \left| \int \prod_{v=1}^M d\lambda_v q(\lambda_v) \prod_{u \in \bar{\Gamma}} \langle A_j^u \rangle_{\Lambda_u} \prod_{i \in \Gamma} \langle \Delta_j^i \rangle_{\Lambda_i} \right|, \quad (\text{A9}) \\ &\leq \int \prod_{v=1}^M d\lambda_v q(\lambda_v) \prod_{u \in \bar{\Gamma}} |\langle A_j^u \rangle_{\Lambda_u}| \cdot \prod_{i \in \Gamma} |\langle \Delta_j^i \rangle_{\Lambda_i}|, \quad (\text{A10}) \end{aligned}$$

$$\leq \int \prod_{v=1}^M d\lambda_v q(\lambda_v) \prod_{i \in \Gamma} |\langle \Delta_j^i \rangle_{\Lambda_i}|, \quad (\text{A11})$$

$$\leq \prod_{i \in \Gamma} \int d\Lambda_i q(\Lambda_i) |\langle \Delta_j^i \rangle_{\Lambda_i}|. \quad (\text{A12})$$

According to the definition in the main text, the leaf node $\mathcal{A}_i$ ($i \in \Gamma$) is associated only with the peripheral source $\mathcal{S}_i$, and its corresponding hidden variable set is defined as $\Lambda_i = \{\lambda_i\}$. Thus, we have

$$(\mathcal{S})_L = \sum_{j=1}^k |I_j(N, l)|^{\frac{1}{l}}, \quad (\text{A13})$$

$$\leq \sum_{j=1}^k \left| \prod_{i \in \Gamma} \int d\lambda_i q(\lambda_i) |\langle \Delta_j^i \rangle_{\Lambda_i}| \right|^{\frac{1}{l}}, \quad (\text{A14})$$

$$\leq \left(\prod_{i \in \Gamma} \int d\lambda_i q(\lambda_i) \sum_{j=1}^k |\langle \Delta_j^i \rangle_{\Lambda_i}| \right)^{\frac{1}{l}}, \quad (\text{A15})$$

where the last line follows from the important inequality (4). Substituting the classical bound (A3) of the bipartite FCBI associated with $\mathbf{M}^{(i)}$ into the integrand, and noting that $\forall i : \int d\lambda_i q(\lambda_i) = 1$, we obtain

$$(\mathcal{S})_L \leq \left(\prod_{i \in \Gamma} \beta_i \right)^{\frac{1}{l}}, \quad (\text{A16})$$

where β_i is the classical bound of bipartite FCBI $\mathcal{B}_i$ associated with peripheral source $\mathcal{S}_i$, which is fully characterized by a real coefficient matrix $\mathbf{M}^{(i)} = (M_{x_i, j})$.

3. Proof of the inequality (5)

In this subsection, we prove the optimal quantum value presented in Eq. (5).

In quantum theory, each independent source $\mathcal{S}_i$ emits a bipartite quantum state $\rho_i = |\psi\rangle_i \langle \psi|_i$. The overall quantum state of network is thus given by $\rho = |\psi\rangle \langle \psi| = \bigotimes_{i=1}^M \rho_i$, where $|\psi\rangle = \bigotimes_{i=1}^M |\psi\rangle_i$. To derive the optimal quantum value of $(\mathcal{S})_Q$, we use the similar elegant SOS approach as stated in Sec. A 1.

Let us consider that $(\mathcal{S})_Q \leq \delta$, where δ is clearly the upper bound of $(\mathcal{S})_Q$. We show that there exists a positive semidefinite operator $\gamma_{\mathcal{S}}$ satisfying $\langle \gamma_{\mathcal{S}} \rangle_Q = -(\mathcal{S})_Q + \delta$. The operator is constructed from a set of positive operators $\{L_j, j = 1, 2, \dots, k\}$, each being a polynomial function of the observables $A_{x_1}^1, A_{x_2}^2, \dots, A_{x_N}^N$, such that

$$\langle \gamma_{\mathcal{S}} \rangle_Q = \sum_{j=1}^k \frac{\omega_j}{2} \langle \psi | L_j^\dagger L_j | \psi \rangle, \quad (\text{A17})$$

where ω_j is a positive number with $\omega_j = \prod_{i \in \Gamma} \omega_j^{A_i}$. The operators L_j are defined as

$$L_j |\psi\rangle = \left| \prod_{i \in \Gamma} \frac{\Delta_j^i}{\omega_j^{A_i}} |\psi\rangle \right|^{\frac{1}{l}} - \left| \prod_{u \in \bar{\Gamma}} A_j^u |\psi\rangle \right|^{\frac{1}{l}}, \quad (\text{A18})$$

$$\omega_j^{A_i} = \|\Delta_j^i |\psi\rangle\|_2, \quad (\text{A19})$$

where $\Delta_j^i = \sum_{x_i} M_{x_i,j} A_{x_i}^i$. From this construction, it follows that

$$\langle \psi | L_j^\dagger L_j | \psi \rangle = 2 - \frac{2}{(\omega_j)^{\frac{1}{t}}} \left| \langle \psi | \prod_{i \in \Gamma} \Delta_j^i \prod_{u \in \bar{\Gamma}} A_j^u | \psi \rangle \right|^{\frac{1}{t}}, \quad (\text{A20})$$

where we have used Eqs. (A18) and (A19), together with $(A_j)^2 = \mathbb{I}$. Substituting Eq. (A20) into Eq. (A17), we have

$$\langle \gamma_S \rangle_Q = \sum_j \omega_j^{\frac{1}{t}} - (S)_Q. \quad (\text{A21})$$

Since $\langle \gamma_S \rangle_Q \geq 0$, the optimal quantum value of $(S)_Q$ is achieved when $\langle \gamma_S \rangle_Q = 0$, which yields

$$(S)_Q^{\text{opt}} = \max \left(\sum_{j=1}^k \omega_j^{\frac{1}{t}} \right). \quad (\text{A22})$$

Noting that $\omega_j = \prod_{i \in \Gamma} \omega_j^{A_i}$ and combining Eq. (4), we obtain

$$(S)_Q^{\text{opt}} \leq \prod_{i \in \Gamma} \left(\sum_{j=1}^k \omega_j^{A_i} \right)^{\frac{1}{t}}, \quad (\text{A23})$$

$$\leq \left(\prod_{i \in \Gamma} (\mathcal{B}_i)_Q^{\text{opt}} \right)^{\frac{1}{t}}, \quad (\text{A24})$$

where the second inequality follows from Eq. (A7), which gives the optimal quantum value of the bipartite FCBI $\mathcal{B}$ characterized by coefficient matrix $\mathbf{M} = (M_{x,y})$. The quantity $(\mathcal{B}_i)_Q^{\text{opt}}$ denotes the optimal quantum violation of the inequality $\mathcal{B}_i$, fully characterized by $\mathbf{M}^{(i)} = (M_{x_i,j})$. The upper bound in Eq. (A24) can be achieved if and only if the operators realizing $(\mathcal{B}_i)_Q^{\text{opt}}$ satisfy the equality condition of Eq. (4) with $x_{j,i} = \omega_j^{A_i}$. In addition, for every j , the condition $\langle \psi | L_j^\dagger L_j | \psi \rangle = 0$ must hold, which is equivalent to $\langle \prod_{i \in \Gamma} \frac{\Delta_j^i}{\omega_j^{A_i}} \prod_{u \in \bar{\Gamma}} A_j^u \rangle_\rho = 1$, as follows from Eq. (A20).

4. Conditions for achieving upper bound in inequality (5) with separable measurements

In the following, we show that the upper bound can be attained using separable measurements. In our framework, each intermediate party receives multiple particles, while each leaf node receives a single particle. Intermediate parties perform separable measurements on each of their subsystems. Let $B_{x_i}^{(w)}$ denote the local observable performed by the intermediate node $\mathcal{A}_i (i \in \Gamma)$ on the subsystem from state ρ_w with input x_i . Under this assumption, the measurement of an intermediate node $\mathcal{A}_i (i \in \bar{\Gamma})$ with input x_i factorizes as

$$A_{x_i}^i = \prod_{w \in \mathcal{I}_i} B_{x_i}^{(w)}, \quad (\text{A25})$$

where $\mathcal{I}_i$ denotes the set of indices of sources associated with party $\mathcal{A}_i$. For a leaf node $\mathcal{A}_i (i \in \Gamma)$, the measurement $A_{x_i}^i$ with input x_i acts only on a single subsystem from the peripheral state ρ_i . Since no decomposition is involved, there is no need to relabel them, i.e., $A_{x_i}^i = A_{x_i}^i$ for $i \in \Gamma$.

We now investigate the conditions required to achieve the upper bound (A24).

First, to make the equality in (A24) hold, the local observables $\{A_{x_i}^i, x_i \in |\mathcal{X}_i|\}$ and $\{B_j^{(i)}, j \in [k]\}$ acting on the peripheral source $\mathcal{S}_i (i \in \mathcal{I})$ should optimize the value of $(\mathcal{B}_i)_Q^{\text{opt}}$.

Then, to saturate inequality (A23), the equality condition of Eq. (4) must be satisfied by the matrix $\mathbf{X} = (x_{j,i}) \in \mathbb{R}^{k \times |\mathcal{I}|}$, where the condition is either $\text{Rank}(\mathbf{X}) = 1$ or, for some $i \in \mathcal{I}$, $x_{j,i} = 0$ for all $j \in [k]$. Here, $\mathbf{X}$ is a matrix with elements $x_{j,i} = \omega_j^{A_i} = \|\Delta_j^i | \psi \rangle\|_2 = \sqrt{\text{Tr}[(\Delta_j^i)^\dagger \Delta_j^i \rho_i]} \geq 0$, where $i \in \mathcal{I}$, $j \in [k]$, and $\Delta_j^i = \sum_{x_i} M_{x_i,j} A_{x_i}^i$.

Finally, the optimal quantum value in Eq. (A22) will occur under the optimization condition $\langle \gamma_S \rangle_Q = 0$. From Eqs. (A17) and (A20), this is equivalent to $\langle \prod_{i \in \Gamma} \frac{\Delta_j^i}{\omega_j^{A_i}} \prod_{u \in \bar{\Gamma}} A_j^u \rangle_\rho = 1$, for all $j \in [k]$. According to Eq. (A25), $A_j^u = A_{x_u=j}^u = \prod_{w \in \mathcal{I}_u} B_{x_u=j}^{(w)} = \prod_{w \in \mathcal{I}_u} B_j^{(w)}$. This condition becomes

$$\begin{aligned} \left\langle \prod_{i \in \Gamma} \frac{\Delta_j^i}{\omega_j^{A_i}} \prod_{u \in \bar{\Gamma}} A_j^u \right\rangle_\rho &= \prod_{i \in \Gamma} \left\langle \frac{\Delta_j^i}{\omega_j^{A_i}} B_j^{(i)} \right\rangle_{\rho_i} \prod_{w \in \bar{\mathcal{I}}} \langle B_{x_\square=j}^{(w)} B_{x_\Delta=j}^{(w)} \rangle_{\rho_w} \\ &= 1, \end{aligned} \quad (\text{A26})$$

where $x_\square$ and x_Δ represent the inputs of intermediate parties $\mathcal{A}_\square$ and $\mathcal{A}_\Delta$ connected via the intermediate source ρ_w , with $\square, \Delta \in \bar{\Gamma}$ denoting the indices of intermediate parties. $\mathcal{I}$ represents the set of indices of all peripheral sources, and $\bar{\mathcal{I}} = [M] \setminus \mathcal{I}$ denotes the set of indices of all intermediate sources.

From the derivation of optimal quantum violations for the bipartite FCBIs (see Appendix A 1), it follows that $(\mathcal{B}_i)_Q^{\text{opt}}$ will occur under the condition $\langle \gamma_{\mathcal{B}_i} \rangle_Q = 0$, which is equivalent to $\langle \frac{\Delta_j^i}{\omega_j^{A_i}} B_j^{(i)} \rangle_{\rho_i} = 1$ for all $i \in \mathcal{I}$. Therefore, to satisfy Eq. (A26), it suffices to ensure that for every intermediate source $\mathcal{S}_i (i \in \bar{\mathcal{I}})$, the condition $\langle B_{x_\square=j}^{(i)} B_{x_\Delta=j}^{(i)} \rangle_{\rho_i} = 1$ holds for all inputs.

For many FCBIs, such as the CHSH inequality [37], the chained Bell inequality [38], Gisin's elegant Bell inequality (EBI) [35], and the Bell inequalities derived from communication games [39], self-testing protocols show that the quantum state and measurements yielding the maximal violation $(\mathcal{B}_i)_Q^{\text{opt}}$ ensure that $\sqrt{\text{Tr}[(\Delta_j^i)^\dagger \Delta_j^i \rho_i]}$ takes the same value for all j [34,40–42]. The condition $\text{Rank}(\mathbf{X}) = 1$ required to achieve $(\prod_{i \in \Gamma} (\mathcal{B}_i)_Q^{\text{opt}})^{\frac{1}{t}}$ is satisfied. The condition $\langle B_{x_\square=j}^{(i)} B_{x_\Delta=j}^{(i)} \rangle_{\rho_i} = 1$ ($i \in \bar{\mathcal{I}}, j \in [k]$) can be fulfilled by measuring all systems associated with intermediate sources using fixed local observables along σ_3 and by assuming that the correlation matrix of each intermediate state has unit largest singular value [29]. This further implies that the optimal quantum violation of inequality (2) can be achieved even when intermediate sources emit classical states ρ_u with correlation matrix $\text{diag}\{0, 0, 1\}$ (e.g., $\rho_u = |00\rangle\langle 00|$).

APPENDIX B: PROOF OF THE EQ. (6)

In this section, we derive the maximal quantum violations of Eq. (2) obtained by arbitrary ensembles of two-qubit states, assuming that each party's measurements are separable across the qubit systems. Let all sources in Fig. 1 emit any two-qubit mixed states $\rho_1, \rho_2, \dots, \rho_M$, which can be expressed in terms

of Pauli matrices as

$$\rho_i = \frac{1}{4} \left(\mathbb{I} \otimes \mathbb{I} + \sum_{u=1}^3 a_u^{(i)} \sigma_u \otimes \mathbb{I} + \sum_{v=1}^3 b_v^{(i)} \mathbb{I} \otimes \sigma_v + \sum_{u,v=1}^3 \mathbf{t}_{uv}^{(i)} \sigma_u \otimes \sigma_v \right), \quad (\text{B1})$$

where $T_{\rho_i} = (\mathbf{t}_{uv}^{(i)})$ with $\mathbf{t}_{uv}^{(i)} = \text{Tr}[\rho_i(\sigma_u \otimes \sigma_v)]$ is the correlation matrix of the state ρ_i , and $\sigma_u (u = 1, 2, 3)$ are the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$, respectively. Denote $1 \geq t_{i,0} \geq t_{i,1} \geq t_{i,2} \geq 0$ the three singular values of T_{ρ_i} . As discussed in Ref. [29], by performing local qubit unitaries U_A and $U_B \in \text{SU}(2)$ on the state ρ , i.e., $\rho'_i = (U_A \otimes U_B) \rho_i (U_A \otimes U_B)^\dagger$, the correlation matrix $T_{\rho'_i} \in \mathbb{R}^{3 \times 3}$ can be diagonalized and ordered as $\text{diag}\{t_{i,1}, t_{i,2}, t_{i,0}\}$. Since local unitary operations can be freely applied to the state, we can, without loss of generality, assume that the correlation matrix T_{ρ_i} of the state ρ_i is $\text{diag}\{t_{i,1}, t_{i,2}, t_{i,0}\}$. Therefore, it is sufficient to measure the subsystems of ρ_i in σ_3 basis to achieve $\langle \sigma_3 \otimes \sigma_3 \rangle_{\rho_i} = t_{i,0}$.

We consider the measurements that are separable across the qubit systems. According to Eq. (A25), Eq. (A8) can be rewritten as

$$I_j(N, l) = \prod_{i \in \mathcal{I}} \langle B_j^{(i)} \Delta_j^i \rangle_{\rho_i} \cdot \prod_{u \in \bar{\mathcal{I}}} \langle B_{x_{\square=j}}^{(u)} B_{x_{\Delta=j}}^{(u)} \rangle_{\rho_u}, \quad (\text{B2})$$

where $\langle B_j^{(i)} \Delta_j^i \rangle$ are the full-correlation observables defined in Eq. (A2), and $\bar{\mathcal{I}} = [M] \setminus \mathcal{I}$ denotes the set of indices of all intermediate sources. Substituting Eq. (B2) into Eq. (2), and noting that $\langle B_{x_{\square=j}}^{(u)} B_{x_{\Delta=j}}^{(u)} \rangle_{\rho_u} \leq t_{u,0}$ for all $j \in [k]$ (with saturation when $B_{x_{\square=j}}^{(u)} = B_{x_{\Delta=j}}^{(u)} = \sigma_3$), we obtain

$$S^{\max}(\rho) \leq \sum_{j=1}^k \prod_{i \in \mathcal{I}} |\langle B_j^{(i)} \Delta_j^i \rangle_{\rho_i}|^{\frac{1}{l}} \prod_{u \in \bar{\mathcal{I}}} t_{u,0}^{\frac{1}{l}}, \quad (\text{B3})$$

$$\leq \prod_{i \in \mathcal{I}} [\mathcal{B}_i^{\max}(\rho_i)]^{\frac{1}{l}} \prod_{u \in \bar{\mathcal{I}}} t_{u,0}^{\frac{1}{l}}, \quad (\text{B4})$$

where $\mathcal{B}_i(\rho_i) = \sum_{j=1}^k \langle B_j^{(i)} \Delta_j^i \rangle_{\rho_i}$, and we assume that $\langle B_j^{(i)} \Delta_j^i \rangle_{\rho_i} \geq 0$ because the non-negativity condition can be ensured through appropriate orientation adjustments of operator $B_j^{(i)}$. The set of observables is not guaranteed to maximize $\mathcal{B}_i(\rho_i)$, which explains the inequality in Eq. (B4). The equalities in Eqs. (B3) and (B4) hold when $\langle B_{x_{\square=j}}^{(u)} B_{x_{\Delta=j}}^{(u)} \rangle_{\rho_u} = t_{u,0}$ for all $u \in \bar{\mathcal{I}}$ and $j \in [k]$, and the operators that maximize $\mathcal{B}_i(\rho_i)$ (where $i \in \mathcal{I}$) also satisfy the equality condition of Eq. (4), with $x_{j,i} = \langle B_j^{(i)} \Delta_j^i \rangle_{\rho_i} \geq 0$.

APPENDIX C: DISTINGUISHING GENERAL NETWORK TOPOLOGIES

In this section, we prove that the same-size networks with different number of leaf nodes can be distinguished using our inequalities. Without loss of generality, we assume that each party has two inputs. From Eq. (5), the quantum correlations in network are bounded by $\sqrt{2}$. For clarity and completeness, we formally restate the main result below.

For two same-size networks with different number of leaf nodes, $l_{\mathcal{N}_1} > l_{\mathcal{N}_2} \geq 2$, the network $\mathcal{N}_2$ can be distinguished from $\mathcal{N}_1$ when quantum correlations achievable in $\mathcal{N}_2$ lead to a violation of $\mathcal{S}_{\mathcal{N}_1} \leq \sqrt{2}$. Specifically, we demonstrate that the inequality $\mathcal{S}_{\mathcal{N}_1} \leq \sqrt{2}$ can be violated when $\mathcal{S}_{\mathcal{N}_2} \leq \sqrt{2}$ holds true.

Assume both networks have N nodes. Let $\Gamma_{\mathcal{N}_1}$ and $\Gamma_{\mathcal{N}_2}$ denote the sets of indices of leaf nodes in networks $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively, with $\Gamma_{\mathcal{N}_1} \supset \Gamma_{\mathcal{N}_2} \neq \emptyset$. From Eq. (5), the Bell inequalities for networks $\mathcal{N}_1$ and $\mathcal{N}_2$ take the form

$$\mathcal{S}_{\mathcal{N}_1} = \left| \left\langle \prod_{u \in \bar{\Gamma}_{\mathcal{N}_1}} A_0^u \prod_{i \in \Gamma_{\mathcal{N}_1}} \Delta_0^i \right\rangle \right|^{\frac{1}{l_{\mathcal{N}_1}}} + \left| \left\langle \prod_{u \in \bar{\Gamma}_{\mathcal{N}_1}} A_1^u \prod_{i \in \Gamma_{\mathcal{N}_1}} \Delta_1^i \right\rangle \right|^{\frac{1}{l_{\mathcal{N}_1}}} \leq \sqrt{2}, \quad (\text{C1})$$

$$\mathcal{S}_{\mathcal{N}_2} = \left| \left\langle \prod_{u \in \bar{\Gamma}_{\mathcal{N}_2}} A_0^u \prod_{i \in \Gamma_{\mathcal{N}_2}} \Delta_0^i \right\rangle \right|^{\frac{1}{l_{\mathcal{N}_2}}} + \left| \left\langle \prod_{u \in \bar{\Gamma}_{\mathcal{N}_2}} A_1^u \prod_{i \in \Gamma_{\mathcal{N}_2}} \Delta_1^i \right\rangle \right|^{\frac{1}{l_{\mathcal{N}_2}}} \leq \sqrt{2}. \quad (\text{C2})$$

Equation (C2) can be alternatively expressed as

$$\mathcal{S}_{\mathcal{N}_2} = \left| \left\langle \prod_{u \in \bar{\Gamma}_{\mathcal{N}_1}} A_0^u \prod_{v \in \bar{\Gamma}_{\mathcal{N}_2} \setminus \bar{\Gamma}_{\mathcal{N}_1}} A_1^v \prod_{i \in \Gamma_{\mathcal{N}_2}} \Delta_0^i \right\rangle \right|^{\frac{1}{l_{\mathcal{N}_2}}} + \left| \left\langle \prod_{u \in \bar{\Gamma}_{\mathcal{N}_1}} A_1^u \prod_{v \in \bar{\Gamma}_{\mathcal{N}_2} \setminus \bar{\Gamma}_{\mathcal{N}_1}} A_0^v \prod_{i \in \Gamma_{\mathcal{N}_2}} \Delta_1^i \right\rangle \right|^{\frac{1}{l_{\mathcal{N}_2}}} \leq \sqrt{2}. \quad (\text{C3})$$

Promising that $\mathcal{S}_{\mathcal{N}_2} \leq \sqrt{2}$ holds true. We should prove that $\mathcal{S}_{\mathcal{N}_1} \leq \sqrt{2}$ can be violated. Write

$$p_0 = 2^{l_{\mathcal{N}_2}} \left\langle \prod_{u \in \bar{\Gamma}_{\mathcal{N}_2}} A_0^u \prod_{i \in \Gamma_{\mathcal{N}_2}} \Delta_0^i \right\rangle, \quad (\text{C4})$$

$$p_1 = 2^{l_{\mathcal{N}_2}} \left\langle \prod_{u \in \bar{\Gamma}_{\mathcal{N}_2}} A_1^u \prod_{i \in \Gamma_{\mathcal{N}_2}} \Delta_1^i \right\rangle, \quad (\text{C5})$$

$$q_0 = 2^{l_{\mathcal{N}_2}} \left\langle \prod_{u \in \bar{\Gamma}_{\mathcal{N}_1}} A_0^u \prod_{v \in \bar{\Gamma}_{\mathcal{N}_2} \setminus \bar{\Gamma}_{\mathcal{N}_1}} A_1^v \prod_{i \in \Gamma_{\mathcal{N}_2}} \Delta_0^i \right\rangle, \quad (\text{C6})$$

$$q_1 = 2^{l_{\mathcal{N}_2}} \left\langle \prod_{u \in \bar{\Gamma}_{\mathcal{N}_1}} A_1^u \prod_{v \in \bar{\Gamma}_{\mathcal{N}_2} \setminus \bar{\Gamma}_{\mathcal{N}_1}} A_0^v \prod_{i \in \Gamma_{\mathcal{N}_2}} \Delta_1^i \right\rangle. \quad (\text{C7})$$

Equations (C2) and (C3) can be further rewritten as $\frac{1}{2}(|p_0|^{\frac{1}{l_{\mathcal{N}_2}}} + |p_1|^{\frac{1}{l_{\mathcal{N}_2}}}) \leq \sqrt{2}$ and $\frac{1}{2}(|q_0|^{\frac{1}{l_{\mathcal{N}_2}}} + |q_1|^{\frac{1}{l_{\mathcal{N}_2}}}) \leq \sqrt{2}$, respectively. Thus, there exist $\epsilon_1, \epsilon_2 \geq 0$ such that $|p_0|^{\frac{1}{l_{\mathcal{N}_2}}} + |p_1|^{\frac{1}{l_{\mathcal{N}_2}}} = 2\sqrt{2} - \epsilon_1$ and $|q_0|^{\frac{1}{l_{\mathcal{N}_2}}} + |q_1|^{\frac{1}{l_{\mathcal{N}_2}}} = 2\sqrt{2} - \epsilon_2$. By parametrizing with angles $\theta_1, \theta_2 \in [0, \pi/2]$, one obtains $|p_0|^{\frac{1}{l_{\mathcal{N}_2}}} = (2\sqrt{2} - \epsilon_1) \cos^2 \theta_1$, $|p_1|^{\frac{1}{l_{\mathcal{N}_2}}} = (2\sqrt{2} - \epsilon_1) \sin^2 \theta_1$, $|q_0|^{\frac{1}{l_{\mathcal{N}_2}}} = (2\sqrt{2} - \epsilon_2) \cos^2 \theta_2$, $|q_1|^{\frac{1}{l_{\mathcal{N}_2}}} = (2\sqrt{2} - \epsilon_2) \sin^2 \theta_2$.

Then, let $\theta_1 \geq \theta_2$, from Eq. (C1), we obtain

$$\begin{aligned} \mathcal{S}_{\mathcal{N}_1} &= \left| \left\langle \prod_{u \in \Gamma_{\mathcal{N}_1}} A_0^u \prod_{i \in \Gamma_{\mathcal{N}_1}} \Delta_0^i \right\rangle \right|^{\frac{1}{l_{\mathcal{N}_1}}} + \left| \left\langle \prod_{u \in \Gamma_{\mathcal{N}_1}} A_1^u \prod_{i \in \Gamma_{\mathcal{N}_1}} \Delta_1^i \right\rangle \right|^{\frac{1}{l_{\mathcal{N}_1}}} = \left| \frac{p_0 + q_0}{2^{l_{\mathcal{N}_1}}} \right|^{\frac{1}{l_{\mathcal{N}_1}}} + \left| \frac{p_1 - q_1}{2^{l_{\mathcal{N}_1}}} \right|^{\frac{1}{l_{\mathcal{N}_1}}} \\ &\geq \frac{1}{2} \left(\|p_0\| - \|q_0\| \right)^{\frac{1}{l_{\mathcal{N}_1}}} + \frac{1}{2} \left(\|p_1\| - \|q_1\| \right)^{\frac{1}{l_{\mathcal{N}_1}}} = \frac{1}{2} \left(|(2\sqrt{2} - \epsilon_1) \cos^2 \theta_1|^{l_{\mathcal{N}_2}} - ((2\sqrt{2} - \epsilon_2) \cos^2 \theta_2)^{l_{\mathcal{N}_2}} \right)^{\frac{1}{l_{\mathcal{N}_1}}} \\ &\quad + |(2\sqrt{2} - \epsilon_1) \sin^2 \theta_1|^{l_{\mathcal{N}_2}} - ((2\sqrt{2} - \epsilon_2) \sin^2 \theta_2)^{l_{\mathcal{N}_2}} \right)^{\frac{1}{l_{\mathcal{N}_1}}} \\ &= \frac{1}{2} \left([(2\sqrt{2} - \epsilon_2) \cos^2 \theta_2]^{l_{\mathcal{N}_2}} - ((2\sqrt{2} - \epsilon_1) \cos^2 \theta_1)^{l_{\mathcal{N}_2}} \right)^{\frac{1}{l_{\mathcal{N}_1}}} + [(2\sqrt{2} - \epsilon_1) \sin^2 \theta_1]^{l_{\mathcal{N}_2}} - ((2\sqrt{2} - \epsilon_2) \sin^2 \theta_2)^{l_{\mathcal{N}_2}} \right)^{\frac{1}{l_{\mathcal{N}_1}}}, \end{aligned} \quad (\text{C8})$$

where in the last line is a binary function $f(\theta_1, \theta_2)$, whose extreme value is taken at $\theta_1 = \theta_2$, and $f(\theta_1, \theta_1) = \frac{1}{2} [(2\sqrt{2} - \epsilon_2)^{l_{\mathcal{N}_2}} - (2\sqrt{2} - \epsilon_1)^{l_{\mathcal{N}_2}}] \left(\cos^{\frac{2l_{\mathcal{N}_2}}{l_{\mathcal{N}_1}}} \theta_1 - \sin^{\frac{2l_{\mathcal{N}_2}}{l_{\mathcal{N}_1}}} \theta_1 \right)$. Finally, evaluating the function at endpoints yields $f(\frac{\pi}{2}, 0) = \frac{1}{2} [(2\sqrt{2} - \epsilon_2)^{\frac{l_{\mathcal{N}_2}}{l_{\mathcal{N}_1}}} + (2\sqrt{2} - \epsilon_1)^{\frac{l_{\mathcal{N}_2}}{l_{\mathcal{N}_1}}}]$, which exceeds $\sqrt{2}$ for some $\epsilon_1, \epsilon_2 \in [0, 1)$. Therefore, $\mathcal{S}_{\mathcal{N}_1} > \sqrt{2}$ is possible, showing that inequality (C1) can be violated.

Distinguishing four-source network topologies

To further illustrate the presented approach used to distinguish network topologies, we consider two small-scale networks (see Fig. 3) with different number of leaf nodes, each consisting of five parties and four independent sources. The local correlations from the networks depicted in Fig. 3, respectively, admit distributions of the following forms:

$$\begin{aligned} P^{(a)}(a_1, \dots, a_5 | x_1, \dots, x_5) \\ &= \int d\lambda_1 d\lambda_2 d\lambda_3 d\lambda_4 q(\lambda_1) q(\lambda_2) q(\lambda_3) q(\lambda_4) p(a_1 | x_1, \lambda_1) \\ &\quad \times p(a_2 | x_2, \lambda_1, \lambda_2) p(a_3 | x_3, \lambda_2, \lambda_3, \lambda_4) p(a_4 | x_4, \lambda_3) \\ &\quad \times p(a_5 | x_5, \lambda_4) \end{aligned} \quad (\text{C9})$$

and

$$\begin{aligned} P^{(b)}(a_1, \dots, a_5 | x_1, \dots, x_5) \\ &= \int d\lambda_1 d\lambda_2 d\lambda_3 d\lambda_4 q(\lambda_1) q(\lambda_2) q(\lambda_3) q(\lambda_4) p(a_1 | x_1, \lambda_1) \\ &\quad \times p(a_2 | x_2, \lambda_1, \lambda_2) p(a_3 | x_3, \lambda_2, \lambda_3) p(a_4 | x_4, \lambda_3, \lambda_4) \\ &\quad \times p(a_5 | x_5, \lambda_4), \end{aligned} \quad (\text{C10})$$

which are the concretization of Eq. (1). It is obviously known that $N = 5, l = 3$ for the network in Fig. 3(a), and $N = 5, l = 2$ for the network in Fig. 3(b). Substituting the corresponding values of $\{N, l\}$ into Eq. (17), we find that the quantum correlations in these two networks respectively satisfy the following inequalities:

$$\mathcal{S}_{(a)} = |I_1^{(a)}(5, 3)|^{\frac{1}{3}} + |I_2^{(a)}(5, 3)|^{\frac{1}{3}} \leq \sqrt{2} \quad (\text{C11})$$

and

$$\mathcal{S}_{(b)} = |I_1^{(b)}(5, 2)|^{\frac{1}{2}} + |I_2^{(b)}(5, 2)|^{\frac{1}{2}} \leq \sqrt{2}. \quad (\text{C12})$$

Furthermore, correlations from the two networks are certified to be nonlocal by violating inequalities

$$\mathcal{S}_{(a)} = |I_1^{(a)}(5, 3)|^{\frac{1}{3}} + |I_2^{(a)}(5, 3)|^{\frac{1}{3}} \leq 1 \quad (\text{C13})$$

and

$$\mathcal{S}_{(b)} = |I_1^{(b)}(5, 2)|^{\frac{1}{2}} + |I_2^{(b)}(5, 2)|^{\frac{1}{2}} \leq 1, \quad (\text{C14})$$

respectively. The terms are defined as $I_j^{(a)}(N, l) = \langle \prod_{u \in \{2,3\}} A_j^u \prod_{i \in \{1,4,5\}} \Delta_j^i \rangle$ and $I_j^{(b)}(N, l) = \langle \prod_{u \in \{2,3,4\}} A_j^u \prod_{i \in \{1,5\}} \Delta_j^i \rangle$ with $\Delta_j^i = \frac{1}{2}(A_j^i + (-1)^j A_j^i)$, $j = 1, 2$.

Similar to the proof in Sec. C, one can show that the inequality $\mathcal{S}_{(a)} \leq \sqrt{2}$ can be violated when the inequality $\mathcal{S}_{(b)} \leq \sqrt{2}$ holds. So, two networks in Fig. 3 can be distinguished by violating Eq. (C11) using quantum correlations achievable in network in Fig. 3(b).

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