

Higher-order topological parity anomaly and half-integer quantum Hall effect in high-dimensional synthetic lattices

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Recent advances in constructing synthetic dimension provide a powerful tool for exploring exotic topological states of matter in high dimensions. Here, we report that the parity anomaly and associated *half-integer* quantized Hall conductance, arising in $2j + 1$ (space-time) dimensions with a single or odd number of Dirac cones, can be realized by the boundary states of n th-order topological insulators in $(2j+n)$ -dimensional synthetic lattices. We establish a general bulk-boundary correspondence by integrating the nested Wilson loop theory with the time-reversal polarization at highly symmetric momenta; a set of $\mathbb{Z}_2$ topological invariants are extracted, which determines the number and location of higher-order-boundary Dirac cones that are protected by time-reversal and reflection symmetries. We develop a general construction procedure for Hamiltonians supporting such higher-order topological parity anomaly. Moreover, we propose an experimental implementation scheme based on photonic synthetic dimensions and provide a method for probing the associated half-integer Hall conductance by the transmission spectra. Our work offers the realization and characterization of parity anomaly in general high-dimensional higher-order topological insulators and opens an avenue for exploring fundamental physics and possible photonic device applications enabled by manipulating Dirac cones.

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I. INTRODUCTION

Dimension is a fundamental concept in physics and critically determines the topological characteristics of quantum systems [1,2], and higher-dimensional systems may host more exotic topological phases beyond lower-dimensional analogs. This profound influence becomes particularly evident when one contrasts the two-dimensional (2D) topological insulator (TI) [3,4] exhibiting *integer* Hall effect, with the three-dimensional (3D) TI [5] possessing single Dirac cone on each 2D boundary that could cause *half-integer* quantized Hall conductance under time-reversal symmetry-breaking perturbations—a manifestation of the parity anomaly in quantum field theory [6–12]. Because of this, the 3D TI also serves as axion insulator [13–16] and has garnered significant attention. Generally, parity anomaly and associated half-integer Hall response may emerge in any $2j$ -dimensional systems with single or odd number of Dirac cones [7,8]. Recent

advances in constructing synthetic dimensions utilizing atomic or photonic internal degrees of freedom [17–42] provide the possibility of studying more general parity anomaly and other exotic topological phases [such as four-dimensional (4D) quantum Hall effect [43–50], five-dimensional (5D) Weyl surfaces [51,52], and tensor and Yang monopoles [53–56], among others [57,58]] in dimensions beyond three.

Meanwhile, the exploration of topological states of matter has extended to higher-order systems supporting boundary states on corners or hinges with unconventional bulk-boundary correspondence [59–62]. In general, a d -dimensional n th-order TI can host $(d - n)$ -dimensional boundary modes. To date, despite extensive theoretical and experimental investigations [63–93], most research on higher-order TIs has revolved around the boundary modes on one-dimensional hinge and zero-dimensional corner in 3D and 2D systems, leaving higher-dimensional generalizations underexplored due to the reliance on additional dimensions exceeding the capabilities of solid-state materials. Recent studies showed that d -dimensional n th-order topological insulators can be constructed by stacking lower-dimensional ordinary TIs in extra dimensions [86], where the topological classification and characterization can be reduced to $(d - n + 1)$ -dimensional ordinary TIs. Generally, high-dimensional higher-order topological insulators cannot be universally reduced to such simple stacking. Therefore, the following fundamental questions arise: Whether the

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parity anomaly—manifested as a $2j$ -dimensional Dirac cone and half-integer Hall response—occurs in more general higher-dimensional higher-order TIs? If so, what is the bulk-boundary correspondence and how to realize it experimentally?

In this article, we address these important questions by utilizing synthetic lattices. We first take a 4D lattice model as an example and demonstrate that the 2D parity anomaly can be realized by the boundary states of the system in the second-order topological phase. Each corner can host a single Dirac cone that contributes half-integer Hall conductance under time-reversal breaking corner perturbations, as confirmed by layer-resolved Chern numbers. Then, we establish a general bulk-boundary correspondence by innovatively integrating the nested Wilson loop theory with the time-reversal polarization at highly symmetric momenta and develop a construction procedure for general d -dimensional n th-order topological Hamiltonians supporting general parity anomaly. While the nested Wilson loop method has succeeded in characterizing certain higher-order topological insulators (e.g., d -dimensional d th-order TIs), its generalization remains to be explored. Based on the combination of nested Wilson loop theory and the time-reversal polarization at highly symmetric momenta, we introduce a set of Z_2 topological invariants that determine the number of Dirac cones on the corresponding corner and their positions in momentum space. Such topological characterization is applicable to any systems with time-reversal and reflection symmetric bulk. Finally, we propose experimental realization schemes based on coupled cavity arrays with photonic internal degrees of freedom [e.g., orbital angular momentum (OAM), frequency, etc.] as additional synthetic dimensions, we suggest that the half-integer Hall conductance can be measured by the transmission spectra that capture the mode density and allow the extraction of layer-resolved Chern numbers via the Streda formula.

II. THE MODEL

We first consider a 4D eight-band lattice model with Bloch Hamiltonian:

$$\begin{aligned} H(\mathbf{k}) = & \lambda_{\text{SO}}(\sin k_x \Gamma_1 + \sin k_y \Gamma_2 + \sin k_z \Gamma_3 + \sin k_w \Gamma_4) \\ & + (m_1 + 2t_x \cos k_x + 2t_y \cos k_y) \Gamma_5 \\ & + (m_2 + 2t_z \cos k_z + 2t_w \cos k_w) \Gamma_6, \end{aligned} \quad (1)$$

where Γ_j are the Dirac matrices satisfying Clifford algebra $\{\Gamma_j, \Gamma_{j'}\} = 2\delta_{jj'}$. For our eight-band model with three Paulis, the Dirac matrices read (different choices are equivalent) $\{\Gamma_{(1)-(7)} = (\sigma_z \tau_z s_x, \sigma_y, \sigma_z \tau_z s_y, \sigma_z \tau_y, \sigma_x, \sigma_z \tau_x, \sigma_z \tau_z s_z)\}$, with σ_j, τ_j, s_j ($j = x, y, z$) the Pauli matrices representing different (pseudo)spins or sublattice sites. The first line on the right-hand side of Eq. (1) can be viewed as the spin-orbit coupling, while the last two lines as kinetic energy. Here, we consider isotropic tunneling $t_x = t_y = t_z = t_w \equiv t$ for simplicity and set $2t = 1$ as the energy unit. The eigenenergies are $E(\mathbf{k}) = \pm \sqrt{\sum_j h_j(\mathbf{k})^2}$, with $h_j(\mathbf{k})$ the coefficient of Γ_j in the Hamiltonian, each of the two bands has fourfold degeneracy, and the band gap closes only at the nine points on the m_1 - m_2 plane with $m_1 = 0, \pm 2$ and $m_2 = 0, \pm 2$. Note that these gap closing points are obtained by solving $h_j(k) = 0$

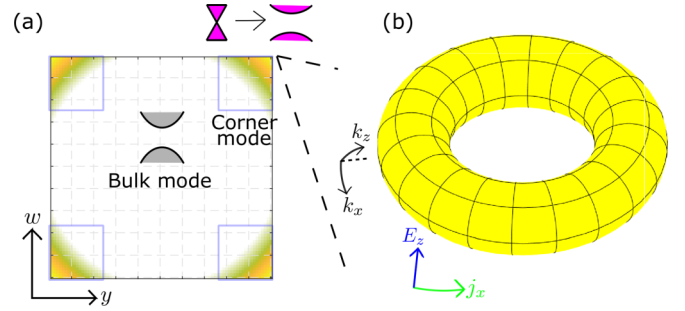


FIG. 1. Schematic of 4D second-order topological insulator for the model equation (1), with open (periodic) boundary condition along y, w (x, z). The topology of gapped bulk is responsible for the gapless Dirac cone at each corner; time-reversal breaking perturbation can lead to a corner gap and half-integer quantized Hall conductance.

for all j , that is, $\lambda_{\text{SO}} \sin k_j = 0$ for the spin-orbit coupling term, leading to solutions independent of λ_{SO} . Without loss of generality, we will set $\lambda_{\text{SO}} = 1$ in the following. Our model preserves time-reversal symmetry $TH(\mathbf{k})T^\dagger = H(-\mathbf{k})$, with $T = s_y K$ (K is the complex conjugation), and reflection symmetry $M_i H(\mathbf{k}) M_i = H(-k_i, k_{j \neq i})$, with $M_x = s_y$, $M_y = \sigma_x \tau_z s_z$, $M_z = s_x$, and $M_w = \tau_x s_z$. For isotropic tunneling, the model also has C_4 rotational symmetries in the x - y and z - w planes with $C_4^{x,y} = \frac{I - \Gamma_1 \Gamma_2}{\sqrt{2}}$ and $C_4^{z,w} = \frac{I - \Gamma_3 \Gamma_4}{\sqrt{2}}$, as well as a combined rotational symmetry $C_4^{xy,zw} H(k_x, k_y, k_z, k_w) (C_4^{xy,zw})^{-1} = H(-k_z, -k_w, k_x, k_y)$ when $m_1 = m_2$, with $C_4^{xy,zw} = \frac{I - \Gamma_1 \Gamma_3}{\sqrt{2}} \frac{I - \Gamma_2 \Gamma_4}{\sqrt{2}} \frac{\Gamma_5 + \Gamma_6}{\sqrt{2}}$ (similarly for $m_1 = -m_2$). We want to emphasize that the reflection and time-reversal symmetries are crucial for the emergence of corner Dirac cones and the corresponding topological characterization. In addition, we note that our model cannot be considered as stacking of 3D TIs along the fourth dimension, which goes beyond previous study in Ref. [86].

We find the model defines a 4D second-order topological insulator with massless Dirac cones formed by topological corner modes, one for each corner, leading to higher-order topological parity anomaly. To show this, we consider open-boundary (period-boundary) conditions along y and w (x and z) directions, as shown in Fig. 1. The corner modes can be obtained by assuming the ansatz [68,94] $|\Psi_{\text{corner}}\rangle \propto \sum_{y,w \geq 0} (\kappa_1)^y (\kappa_2)^w |y, w\rangle \otimes |\xi\rangle$ and solving the eigenequation $H|\Psi_{\text{corner}}\rangle = E_{\text{corner}}|\Psi_{\text{corner}}\rangle$, with $|\xi\rangle$ the spin state. For the corner around $y, w = 0$, the two solutions read $|\xi\rangle = \{|\uparrow \uparrow \uparrow\rangle_{\sigma \tau s}, |\uparrow \uparrow \downarrow\rangle_{\sigma \tau s}\}$, with $\kappa_1 = -m_1 - \cos k_x$, $\kappa_2 = -m_2 - \cos k_z$. Within the subspace spanned by the two corner modes, the effective Hamiltonian can be written as

$$H_{\text{corner}}(k_x, k_z) = s_x \sin k_x + s_y \sin k_z, \quad (2)$$

which forms a Dirac cone in the x - z space as illustrated in Fig. 1. Similar results can be obtained for other three corners; the numerical band structure is shown in Fig. 2(a), where the gapped bulk corresponds to the gray bands, while the corner Dirac cones are given by the colored bands with color bar showing the populations on the four corner sites. These cor-

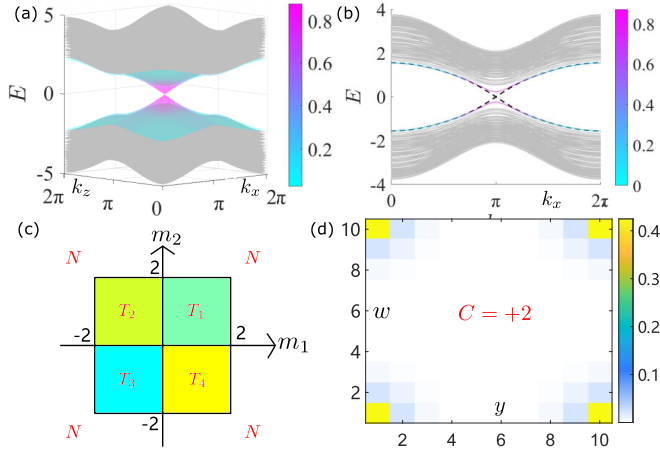


FIG. 2. (a) Bulk (gray) and corner (colored) bands of the model equation (1) with boundary condition shown in Fig. 1. Color bar shows the populations on the four corner sites. (b) Energy bands at $k_z = \pi$ with additional mass term $H_M = -Ms_z$ on the four corner sites. Dashed lines represent the gapless corner modes in panel (a). (c) Phase diagram in the m_1 - m_2 plane, with N the normal phase and T_i the topological phases. (d) Layer-resolved Chern number with half-quantized corner contribution. We set $m_1 = m_2 = 1.5$ with system size $N_y = N_w = 10$, $N_x = N_z = 36$ in panels (a), (b), and (d) and $M = 0.4$ in panels (b) and (d).

ner Dirac cones are protected by reflection and time-reversal symmetries.

The above corner modes (around $y, w = 0$) require that $|\kappa_1|, |\kappa_2| < 1$, so the existence and momentum position of the Dirac cone depend on m_1, m_2 . In particular, the corner Dirac cone exists and the system is topological in the region $|m_1|, |m_2| < 2$, and is trivial otherwise. The Dirac cone is located at $k_x = \pi(0)$ for positive (negative) m_1 , along with $k_z = \pi(0)$ for positive (negative) m_2 , and thus the system has four topological phases as shown in Fig. 2(b). Recall that the bulk gap closes only at certain points in the m_1 - m_2 plane. Thus, instead of the bulk bands, the edge spectra (states located on $x = 0$ and $z = 0$ edges) close the gap across the phase boundary and lead to the phase transition of the corner modes, a typical phenomenon for higher-order TIs [59–62].

By adding T -breaking perturbation mass term (analogous to gauge-invariant regularization process in quantum field theory), the parity symmetry (i.e., reflection with respect to the Dirac point) is also broken, leading to half-integer Hall conductivity [6–11]. We introduce the mass term by a Zeeman field $H_M = -Ms_z$ applied on the four corner sites in the y - w space (all sites in the x - z space), it opens a gap in the corner Dirac cone as shown in Fig. 2(c). Then, each corner contributes $1/2$ -quantized Hall current $j_x = \frac{1}{2} \frac{e^2}{h} E_z$ along x in the presence of the gradient field E_z along z , a typical consequence of parity anomaly, as illustrated in Fig. 1. This is verified by calculating the layer-resolved Chern number [95,96] $C(y, w) = \frac{1}{2\pi} \int dk_x dk_z \text{Tr}[\Omega_{k_x, k_z} \rho_{k_x, k_z}(y, w)]$, with Ω_{k_x, k_z} and $\rho_{k_x, k_z}(y, w)$ the non-Abelian Berry connection and local density-of-state matrix of the occupied bands Supplemental Material [94]. We can see that the Hall conductance of each corner state is half-quantized, as shown in Fig. 2(d). The total Chern number is always quantized to integers. We

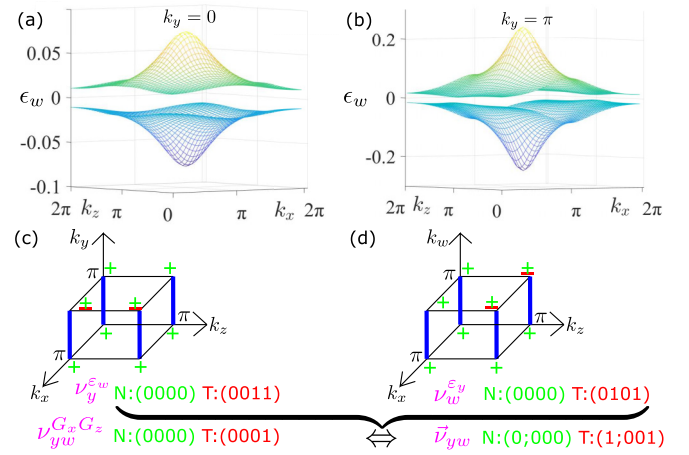


FIG. 3. (a), (b) The Wannier-band structures $\varepsilon_{w,\mathbf{k}}$ with $k_y = 0$ and $k_y = \pi$, respectively. $m_1 = m_2 = 1.5$. (c), (d) The parity of the Wannier-band basis (i.e., eigenvalue of the reflection operator) $\eta^{\varepsilon_w}(\mathbf{G})$ and $\eta^{\varepsilon_y}(\mathbf{G})$, respectively, for Wannier sector $\varepsilon \in (0, 1/2)$. The time-reversal polarizations $\nu_y^{\varepsilon_w}(G_x, G_z)$ and $\nu_w^{\varepsilon_y}(G_x, G_z)$, as well as topological invariants $\nu_{yw}^{G_x, G_z}$ and $\bar{\nu}_{yw}^{G_x, G_z}$, are also shown for both the normal and topological phases.

have $C = 2$ and -2 in phases $T_{2,4}$ and $T_{1,3}$, respectively (since their Dirac cone chiralities are different). It is worth noticing that the sign of the mass and the direction of the Hall current can be controlled independently for each corner. For general open boundaries along directions q_1, q_2 , the above results of the corner Dirac cones and half-integer Hall effect of corner state also apply if $q_1 \in (x, y)$ and $q_2 \in (z, w)$ Supplemental Material [94].

III. BULK-BOUNDARY CORRESPONDENCE

To show the topological originating of the corner parity anomaly, we combine the nested Wilson loop method [59,60] with the time-reversal polarization (a Z_2 invariant characterizing topological insulator) [97,98], which give rise to a series of Z_2 topological invariants determining the number of Dirac cones and their locations. For the 4D lattice model of Eq. (1), we introduce the 4D Wilson loop operator

$$W_{w,\mathbf{k}} = \mathcal{P} \exp \left[-i \int_0^{2\pi} A_w(\mathbf{k}) dk_w \right], \quad (3)$$

where $A_w(\mathbf{k}) = i \langle u_{m',\mathbf{k}} | \partial_{k_w} | u_{m,\mathbf{k}} \rangle$ is the non-Abelian Berry connection and $|u_{m,\mathbf{k}}\rangle$ is the m th occupied Bloch band satisfying $H(\mathbf{k})|u_{m,\mathbf{k}}\rangle = E_m(\mathbf{k})|u_{m,\mathbf{k}}\rangle$ with $\langle u_{n,\mathbf{k}} | u_{m,\mathbf{k}} \rangle = \delta_{n,m}$. $\mathcal{P}$ is the path-ordering operator. The Wilson loop operator leads to the Wannier Hamiltonian $H_{Ww}(\mathbf{k}) = -i/2\pi \ln W_{w,\mathbf{k}}$, with $H_{Ww}(\mathbf{k})|\varepsilon_{w,j,\mathbf{k}}\rangle = \varepsilon_{w,j,\mathbf{k}}|\varepsilon_{w,j,\mathbf{k}}\rangle$ and j the Wannier-band index. $|\varepsilon_{w,j,\mathbf{k}}\rangle$ satisfies $\langle \varepsilon_{w,j',\mathbf{k}} | \varepsilon_{w,j,\mathbf{k}} \rangle = \delta_{j',j}$ and $\varepsilon_{w,j,\mathbf{k}}$ obeys the identification $\varepsilon_{w,j,\mathbf{k}} = \varepsilon_{w,j,\mathbf{k}} \bmod 1$. Due to the reflection symmetry, the Wannier bands appear in $\pm\varepsilon$ pairs, or locked at the symmetric values 0, $1/2$, and gapped Wannier bands can carry topological invariants Supplemental Material [94].

The typical Wannier bands in the topological phase are shown in Figs. 3(a) and 3(b), with a gap at $\varepsilon = 0$. We will focus on the Wannier sector with $\varepsilon_j \in (0, 1/2)$ with two degenerate Wannier bands and define the Wannier-band basis

$|\phi_{j,w}(\mathbf{k})\rangle = \sum_{m=1}^{N_{\text{occ}}} [\varepsilon_{w,j,\mathbf{k}}]_m |u_{m,\mathbf{k}}\rangle$, with N_{occ} the number of occupied energy bands and $\langle \phi_{j,w}(\mathbf{k}) | \phi_{j',w}(\mathbf{k}) \rangle = \delta_{j,j'}$ [59]. In the presence of reflection symmetry, $|\phi_{j,w}(\mathbf{G})\rangle$ satisfies

$$-iM_x M_y M_z |\phi_{j,w}(\mathbf{G})\rangle = \eta^{\varepsilon_w}(\mathbf{G}) |\phi_{j,w}(\mathbf{G})\rangle, \quad (4)$$

where $\mathbf{G}$ refers to the time-reversal invariant momenta (i.e., $G_{x,y,z} = 0, \pi$). Within each Wannier sector, the Wannier-band basis forms time-reversal pairs and shares the same eigenvalue $\eta^{\varepsilon_w} = \pm 1$ Supplemental Material [94] of the reflection symmetry. There are eight η^{ε_w} 's corresponding to the eight $\mathbf{G}$ momenta (since k_w is integrated out), as shown in Figs. 3(c) and 3(d). Now we define the nested time-reversal polarization $v_y^{\varepsilon_w}(G_x, G_z)$ along the y direction through

$$(-1)^{v_y^{\varepsilon_w}(G_x, G_z)} = \prod_{G_y=0,\pi} \eta^{\varepsilon_w}(\mathbf{G}); \quad (5)$$

here “nested” means it is based on the Wannier-band basis. Notice that $\eta^{\varepsilon_w}(\mathbf{G})$ is associated with the Pfaffian $\eta^{\varepsilon_w}(\mathbf{G}) = \text{Pf}[S^{\varepsilon_w}(\mathbf{G})]/\sqrt{\text{Det}[S^{\varepsilon_w}(\mathbf{G})]}$ with skew matrix $S_{j'j}^{\varepsilon_w}(\mathbf{G}) \equiv \langle \phi_{j',w}(\mathbf{G}) | T | \phi_{j,w}(\mathbf{G}) \rangle$ under transverse gauge choice [97]. Similarly, we can calculate the Wannier bands along y and obtain the nested time-reversal polarization $v_w^{\varepsilon_y}(G_x, G_z)$ along w . Finally, we arrive at the Z_2 topological invariant

$$v_{yw}^{G_x G_z} = v_y^{\varepsilon_w}(G_x, G_z) v_w^{\varepsilon_y}(G_x, G_z). \quad (6)$$

$v_{yw}^{G_x G_z} = 1$ or 0 indicates the existence or absence of a Dirac cone at $k_x = G_x, k_z = G_z$, for each corner of the y - w space with open boundaries Supplemental Material [94]. It is worthy to note that these topological invariants are well defined as long as the system preserves time-reversal and reflection symmetries.

There are four independent Z_2 topological invariants $v_{yw}^{00}, v_{yw}^{0\pi}, v_{yw}^{\pi\pi}, v_{yw}^{\pi 0}$, and the even-odd property of the total Dirac cones at each corner is characterized by $v_{yw}^{\text{total}} = \sum_{G_x, G_z} v_{yw}^{G_x G_z} \bmod 2$. Similarly to 3D TIs [5], here we use $\vec{v}_{yw} = (v_{yw}^{\text{total}}, v_{yw}^{00}, v_{yw}^{0\pi}, v_{yw}^{\pi\pi})$. $v_{yw}^{\text{total}} = 1$ implies the realization of higher-order topological parity anomaly with odd number of Dirac cones at each corner, while $v_{yw}^{00}, v_{yw}^{0\pi}, v_{yw}^{\pi\pi}$ encode the location information of the Dirac cones. For example, $\vec{v}_{yw} = (0; 0, 0, 0)$ in the trivial phase and $\vec{v}_{yw} = (1; 0, 0, 1)$ in the topological T_1 phase, as shown in Fig. 3. Across the phase boundary, the Wannier-band gap closes and reopens, leading to the change of topological invariants. It is straightforward to obtain the topological invariants $\vec{v}_{q_1 q_2}$ for open boundaries along different directions.

IV. GENERALIZATION

We now explore the possibility of generalizing our results and provide a general procedure for constructing d -dimensional n th-order topological insulators, which host $(d - n)$ -dimensional Dirac cone as topological boundary modes. First, we need to generate $d + n$ Dirac matrices. Using $p = \lceil \frac{d+n-1}{2} \rceil$ Paulis, we can construct the Dirac matrix set $\Gamma^{(p)} = \{\sigma_x^{(p)}, \sigma_y^{(p)}, \sigma_z^{(p)} \otimes \Gamma^{(p-1)}\}$, with $\Gamma^{(1)} = \{\sigma_x^{(1)}, \sigma_y^{(1)}, \sigma_z^{(1)}\}$, so $\Gamma^{(p)} = \{\Gamma_1, \dots, \Gamma_{d+n}, \dots\}$ contains at least $d + n$ elements. Then, we divide the d dimensions $X = \{x_1, \dots, x_d\}$ into n

set $X = \{X_1, \dots, X_n\}$; each set X_j contains $d_j > 0$ dimensions with $\sum_j d_j = d$. The general Hamiltonian reads

$$H(\mathbf{k}) = \sum_{j=1}^d \sin k_j \Gamma_j + \sum_{j=1}^n \left(m_j + \sum_{l \in X_j} \cos k_l \right) \Gamma_{j+d}. \quad (7)$$

The system is topological in the regime $|m_j| < d_j$, supporting $(d - n)$ -dimensional Dirac cone at the intersection corner under open boundaries of $q_1, \dots, q_n$ with $q_j \in X_j$ Supplemental Material [94]. When $(d - n)$ is even (odd), the system realizes a general higher-order topological parity (chiral) anomaly. With proper mass term to open corner gaps, anomalous Hall conductivity—characterized by $(d - n)/2$ th Chern number—arises Supplemental Material [94].

It is straightforward to extend our bulk-boundary correspondence to the general case of higher-order topological parity anomaly. The nested time-reversal polarization $v_{q_j}^{\varepsilon_{q_j}}$ along q_j can be calculated using the Wannier-band basis after evaluating the nested Wilson loops along $\vec{q}_j = (q_1, \dots, q_{j-1}, q_{j+1}, q_n)$; the Z_2 topological invariants then read $v_{q_1 \dots q_n} = \prod_j v_{q_j}^{\varepsilon_{q_j}}$. We have $2^{d-n} Z_2$ invariants associated with the symmetric momenta in the $d - n$ dimensions. In fact, our topological characterization relies solely on time-reversal and reflection symmetries (with C_4 symmetry nonessential), making it applicable to any models preserving these symmetries, including 3D second-order hinge states [64] and higher-order chiral anomaly with odd $(d - n)$. In the Supplemental Material [94], we have considered another 4D model and a six-dimensional model, respectively, and demonstrated the existence of second-order parity anomaly and bulk-boundary correspondence.

V. EXPERIMENTAL CONSIDERATION

It is possible to implement the lattice model [Eq. (1)] using synthetic lattices in atomic, electric circuit, or photonic systems [29,70]. Here, we focus on the realization with photonic systems due to their unlimited internal degrees of freedom [29–31] (e.g., orbital angular momentum and frequency). We employ a 2D coupled cavity array in real space, together with two synthetic dimensions based on frequency and/or OAM to simulate the 4D lattice (photons with different OAM and frequencies can resonate simultaneously inside the main cavity [32]), as shown in Fig. 4(a). The (pseudo)spin states can be realized by two polarization modes (σ), clockwise and counterclockwise modes (τ), and two subcavities (s). The spin-orbit coupling and kinetic tunneling can be realized by properly designed auxiliary coupling cavities. For example, the tunneling phase can be controlled by the length imbalance of the two arms in the coupling cavity [25], so polarization-dependent phase leads to the σ_z -type coupling. Polarization-flip (σ_x -type) coupling can be realized by inserting a $\lambda/4$ -wave plate into the coupling cavity. For the clockwise-counterclockwise modes and subcavities, the coupling can be realized similarly [34], combining together one can obtain couplings for all Γ_j matrices (note that $\sigma_y = i\sigma_x \sigma_z$). Furthermore, multisynthetic dimensions can be realized using solely OAM or frequency, enabled by long-range tunnelings [26,28]. Therefore, photonic synthetic lattices not only enable the realization of our 4D model with high feasibility but also

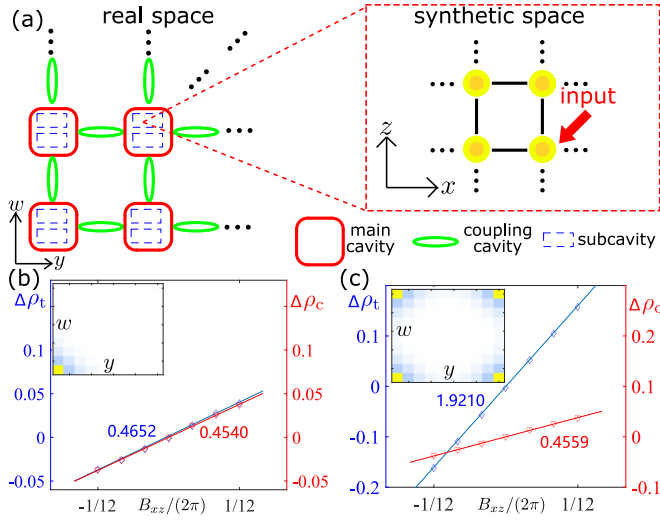


FIG. 4. (a) Experimental implementation of our model equation (1) based on 2D coupled cavity array with two additional synthetic dimensions. (b), (c) Numerical simulation of the mode density through photon transmission for different B_{xz} ; the slopes (marked by the digits) represent the corresponding Chern number. $\Delta\rho_t = \sum_{y_0, w_0} [\rho(y_0, w_0, B_{xz}) - \rho(y_0, w_0, 0)]$ is the total mode-density derivation (diamonds); similarly for the corner mode-density derivation (hearts) $\Delta\rho_c$, except the summation is taken over a single corner region. The insets show the extracted layer-resolved Chern number. The mass term is applied on one corner in panel (b) and four corners in panel (c). Other parameters are $m_1 = m_2 = 1.5$, $M = 0.4$, and $\gamma = 0.01$.

provide a suitable platform for exploring physics in even higher dimensions.

To probe the parity anomaly, we propose to measure the anomalous half-integer Hall conductivity in the presence of mass term H_M . The properties of the total Hamiltonian are fully captured by the photonic transmission operator [25,32] $\mathbb{T}_{\mathbf{r}_0, \mathbf{r}_i}(\omega) = \langle \mathbf{r}_i | \frac{-i\gamma}{\omega - H + i\gamma} | \mathbf{r}_0 \rangle$, where $\mathbf{r} = (x, y, z, w, \alpha)$ is the site-spin index and γ is the photonic decay rate. We fix the input index (x_i, z_i) and measure the total intensity transmission to sites (y_0, w_0) , that is, $\mathcal{T}_{y_0 w_0}(\omega) = \sum_{\mathbf{r}'_0, \mathbf{r}'_i} |\mathbb{T}_{\mathbf{r}_0, \mathbf{r}_i}(\omega)|^2$, with $\mathbf{r}'_0 = (x_0, z_0, \alpha_0)$ and $\mathbf{r}'_i = (y_i, w_i, \alpha_i)$. It can be shown that $\int_{-\infty}^0 \mathcal{T}_{y_0 w_0}(\omega) d\omega \simeq \pi \gamma \rho(y_0, w_0)$ for γ smaller compared to the bandwidth and band gap Supplemental Material [94], where $\rho(y_0, w_0)$ is the mode density of all occupied bands. Next, we introduce a weak gauge field in the x - z space (it can be implemented by controlling the tunneling phase [32]) and examine the variation of $\rho(y_0, w_0)$; the layer-resolved Chern number can be obtained according to the Streda formula [99] $C(y_0, w_0) = 2\pi \frac{\partial \rho(y_0, w_0)}{\partial B_{xz}}$ (which can be generalized to higher dimensions), with B_{xz} the corresponding weak magnetic field Supplemental Material [94]. Shown in Figs. 4(b) and 4(c) are our numerical simulations with mass term on only one corner and on all four corners, respectively. The former leads to 1/2-quantized total Chern number mainly distributed around the corresponding corner, while the latter lead to a total Chern number $C = 2$ with each corner contributing 1/2.

VI. CONCLUSION AND DISCUSSION

In conclusion, we have studied the realization, characterization, and detection of general parity anomaly and half-integer Hall response in high-dimensional higher-order topological insulators. Using the 4D second-order topological insulator as an example, we demonstrate that parity anomaly emerges at each corner hosting a single Dirac cone. We establish a general bulk-boundary correspondence by integrating nested Wilson loop theory with time-reversal polarization, leading to a set of Z_2 topological invariants that determine the number and momentum-space positions of Dirac cones at the corresponding corners. These results can be extended to the general case of d -dimensional n th-order topological systems with time-reversal and reflection symmetric bulk, which can be constructed following the developed procedure. Furthermore, we propose an experimental realization based on coupled cavity arrays, utilizing photonic internal degrees of freedom as synthetic dimensions. The parity-anomaly-induced half-integer Hall conductance can be measured using transmission spectra, which directly reflect the mode density and allow extracting layer-resolved Chern numbers via the Streda formula.

Higher-order TIs with parity anomaly can be viewed as a higher-order extension of the axion insulators; further studies on how axion field is distributed in such systems would be interesting. Also, our Hamiltonian construction procedure and topological characterization are applicable for higher-order chiral anomaly on odd-dimensional boundaries, where the mass term can induce the violation of chiral current conservation [7,8]. Moreover, topological edge states are very useful in robust energy and signal transport, and general high-dimensional higher-order TIs may increase the richness of transport properties and facilitate wider transport manipulation capabilities in topological photonics [13,14,100]. Therefore, our work not only advances the understanding of parity anomaly and topological phases, but also paves the way for exploring high-dimensional higher-order topological phenomena and their device applications in photonics.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [105].

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