

## Geometric scaling laws for quantum reflection at singular corners

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Quantum reflection is usually described as an effectively one-dimensional phenomenon governed by motion along the surface normal. However, when translational symmetry is broken by sharp geometric features, scattering becomes intrinsically multidimensional. Here, we show that singular corners generate new geometric scaling laws for quantum reflection. In a minimal two-dimensional corner model with an attractive quadrant interaction, the reflected scattering width scales as  $\sigma_{\text{refl}} \propto E^{-2}$  in the intermediate Born regime, followed at lower energies by a crossover to the unitarity-limited threshold law  $\sigma_{\text{refl}} \propto E^{-1/2}$ . The  $E^{-2}$  exponent arises from two geometric ingredients: half-space support of the interaction and specular boundary-layer kinematics. More generally, for corner-supported potentials of the form  $V(x, z) = W(x, z)\Theta(x)\Theta(z)$ , we show that the Born exponent is universal under mild regularity assumptions. The leading coefficient is asymptotically determined by a specular boundary layer and depends only on the boundary trace  $W(x, 0)$ , while microscopic details away from the boundary enter only through subleading structure. These results identify corners as the simplest surface singularities that support genuinely multidimensional geometric scaling laws, showing that geometry alone can control reflection across different energy regimes beyond the standard one-dimensional framework.

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## I. INTRODUCTION

Quantum reflection is a wave phenomenon in which particles are reflected by an attractive interaction without ever reaching a classical turning point. In its standard form, the effect is understood within an effectively one-dimensional description: For a planar surface, the interaction depends only on the distance normal to the interface, momentum parallel to the surface is conserved, and the scattering problem reduces to a single reaction coordinate [1–3]. Within this framework, the threshold behavior is governed by the long-range tail of the interaction potential [4–8].

This picture successfully describes a wide range of atom-surface phenomena, including specular reflection of slow metastable atoms [9,10], helium atom reflection from solid surfaces [11–14], reflection of ultracold atoms and Bose-Einstein condensates from material interfaces and atom-chip potentials [15–19], and diffraction from microstructured surfaces [20–22]. In all these cases, translational invariance along the surface renders the problem effectively one dimensional.

Many realistic surfaces, however, are not translationally invariant. Steps, edges, corners, trenches, and patterned structures couple longitudinal and transverse motion, thereby

invalidating the reduction to a single coordinate. Once separability is lost, familiar one-dimensional notions such as a badlands region, a reflection distance, or a turning-point interpretation cease to be fundamental. This raises a natural question: Can broken geometry do more than merely complicate the scattering problem and instead generate new universal scaling laws?

In this work, we show that it does. We identify a corner as the simplest surface singularity for which translational invariance is broken in two independent directions, so that the scattering problem is intrinsically nonseparable. In this geometry, the reflected signal is no longer controlled solely by the asymptotic tail of the interaction, but instead acquires a genuinely geometric scaling law.

It is important to distinguish the present mechanism from the classical theory of wave scattering by wedges and corners, which has a long history beginning with Sommerfeld's exact solution of diffraction by a half plane [23] and subsequent developments such as the Malyuzhinets technique [24], the geometrical theory of diffraction [25,26], and its modern formulations [27–29]. In these approaches, the scatterer is represented by a hard boundary imposing Dirichlet, Neumann, or impedance boundary conditions, and the resulting asymptotics are governed by the geometry of the contour itself.

In contrast, the present problem concerns a smooth attractive volume interaction that remains penetrable everywhere. Reflection occurs without a classical turning point and is generated by wave scattering from an attractive corner-supported potential rather than by diffraction from an impenetrable wedge. As shown below, the leading geometric scaling law is controlled by the boundary trace of the interaction potential and the associated specular boundary layer. Since hard-wall

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wedge models possess neither a Born regime nor an analogous boundary-trace mechanism, the scaling law derived here belongs to a different universality class from conventional wedge diffraction.

We first analyze a minimal two-dimensional model in which an attractive interaction is confined to a quadrant. In this setting, the reflected scattering width exhibits an extended intermediate-energy Born regime with  $\sigma_{\text{refl}} \propto E^{-2}$ , followed at lower energy by a crossover to a unitarity-limited threshold regime,  $\sigma_{\text{refl}} \propto E^{-1/2}$ .

The  $E^{-2}$  exponent arises from two geometric ingredients: the half-space support of the interaction and the  $k^{-1}$  angular compression of a narrow specular boundary layer that carries the dominant reflected flux.

We then show that the Born exponent is not specific to the minimal model, but applies to a broad class of corner-supported interactions of the form

$$V(x, z) = W(x, z)\Theta(x)\Theta(z), \quad (1)$$

provided the boundary trace  $W(x, 0)$  is sufficiently regular. In this class, the leading reflected contribution is asymptotically concentrated in a specular boundary layer, so that the corresponding coefficient is determined solely by  $W(x, 0)$ , while microscopic details away from the boundary affect only subleading structure. The detailed derivations are deferred to the Appendixes.

The central message is therefore simple: Once separability is lost, low-energy quantum reflection can be governed by geometry itself. Corners define a new universality class of multidimensional scattering in which the reflected flux is controlled by boundary-layer kinematics rather than by the one-dimensional semiclassical mechanism familiar from planar surfaces.

The paper is organized as follows. Section II briefly recalls the one-dimensional picture and why it breaks down once translational symmetry is lost. Section III introduces the corner model and the reflected scattering width. Section IV identifies the geometric mechanism behind the Born scaling and its eventual unitarity-limited breakdown. Section V states the general result for corner-supported potentials and summarizes its physical content, with derivations given in the Appendixes. Numerical results are presented in Sec. VI, followed by a comparison with conventional quantum reflection in Sec. VII and experimental implications in Sec. VIII.

## II. FROM ONE-DIMENSIONAL QUANTUM REFLECTION TO NONSEPARABILITY

For a planar surface, translational invariance conserves momentum parallel to the interface and reduces the scattering problem to motion along the surface normal. In this effectively one-dimensional setting, quantum reflection is commonly associated with the breakdown of semiclassical propagation in the long-range tail of the potential [1–3]. A standard diagnostic is the badlands function [30–32], whose growth signals the failure of the WKB approximation.

A more global interpretation is provided by exact WKB theory [33]. In this framework, reflection is governed by Stokes geometry: Crossing a Stokes curve reorganizes the asymptotic decomposition of the wave function and trans-

fers weight between semiclassical branches. The reflected amplitude arises from coupling between the incident branch and exponentially subdominant contributions associated with complex turning points [33,34]. In both formulations, however, the essential prerequisite is the same. The problem must first be reducible to an effective one-dimensional Schrödinger equation [3,4].

This assumption fails once translational symmetry is broken. In the presence of edges, corners, or structured surfaces, momentum components are no longer conserved and longitudinal and transverse motion become coupled. The scattering problem is then intrinsically nonseparable, and reflection can no longer be attributed to a single reaction coordinate or described in terms of one-dimensional badlands or Stokes structures.

Nevertheless, the underlying mechanism remains semiclassical. In multidimensional scattering, the wave field can be expressed as a superposition of saddle-point contributions whose relative weights are controlled by the imaginary part of the action. Transitions between dominant saddles occur across extended regions in configuration space, providing a higher-dimensional analog of Stokes phenomena, as described in hyperasymptotic analyses [35,36].

From this perspective, geometric nonseparability induces momentum mixing between scattering channels, redistributing flux across outgoing directions and opening pathways absent in strictly one-dimensional models [37,38]. The key question is therefore not how one-dimensional reflection is modified, but whether nonseparability introduces a genuinely new scaling mechanism. We show below that for a corner geometry, it does, leading to a geometric scaling behavior with no one-dimensional analog.

## III. MINIMAL CORNER GEOMETRY

We consider two-dimensional scattering in coordinates  $\mathbf{r} = (x, z)$  with an attractive interaction confined to the quadrant  $x > 0, z > 0$ :

$$W(x, z) = -\frac{C_n}{(x^2 + z^2 + a^2)^{n/2}}. \quad (2)$$

Here,  $C_n > 0$  sets the interaction strength,  $n$  fixes the asymptotic tail, and  $a$  regularizes the short-distance behavior near the apex.

The essential ingredient of the potential (1) is the factor  $\Theta(x)\Theta(z)$ , which breaks translational symmetry in both coordinates and makes the scattering intrinsically nonseparable. The specific radial form in Eq. (2) is chosen only as a minimal model.

We take an incident plane wave with momentum

$$k_x = k \sin \theta, \quad k_z = k \cos \theta, \quad k = \sqrt{2mE}/\hbar, \quad (3)$$

and define reflection as scattering into states with  $k'_z < 0$ , as illustrated in Fig. 1. Because the interaction is strictly attractive, a classical particle would never be reflected but would instead be drawn into the surface. This backscattering is therefore a purely wave-mechanical phenomenon, representing a genuinely multidimensional form of quantum reflection.

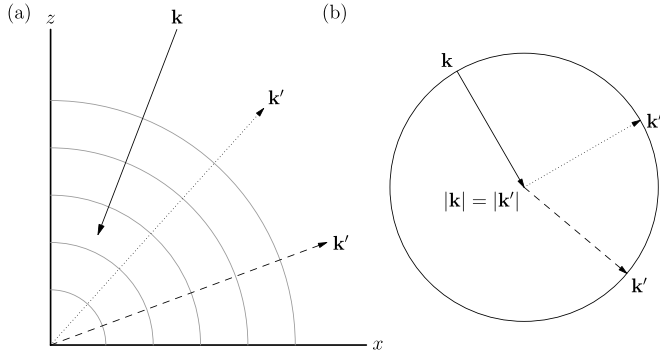


FIG. 1. Minimal corner geometry. (a) The interaction is confined to the quadrant  $x > 0, z > 0$ . (b) Because the corner breaks translational invariance in both directions, an incoming momentum  $\mathbf{k}$  can scatter into a continuum of outgoing directions.

Formally, the exact scattering wave function  $\psi(\mathbf{r})$  is the solution to the stationary Schrödinger equation

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \right] \psi(\mathbf{r}) = E \psi(\mathbf{r}), \quad (4)$$

subject to the standard asymptotic boundary condition  $\psi(\mathbf{r}) \sim e^{i\mathbf{k} \cdot \mathbf{r}} + \psi_{\text{scat}}(\mathbf{r})$  far from the corner interaction. The exact scattering amplitude  $F(\mathbf{k}' \leftarrow \mathbf{k})$  governs the far-field behavior of the outgoing cylindrical waves in the two-dimensional continuum.

While this exact formulation dictates the complete dynamics, our primary analytical tool for the intermediate-energy regime is perturbation theory. Within the first Born approximation, the scattering amplitude is the Fourier transform of the potential over the quadrant support,

$$F_B(\mathbf{k}' \leftarrow \mathbf{k}) = -\frac{m}{2\pi\hbar^2} \int d^2r e^{-i\mathbf{q} \cdot \mathbf{r}} V(\mathbf{r}), \quad (5)$$

with  $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ . This already makes clear how the corner differs from a planar interface: The relevant object is a half-space Fourier transform, not the momentum-conserving planar transform of the translationally invariant case.

For two-dimensional scattering, the asymptotic wave function defines the differential scattering width

$$\frac{d\sigma}{d\theta'} = \frac{|F(\theta')|^2}{k}, \quad (6)$$

and we focus on its reflected part,

$$\sigma_{\text{refl}} = \int_{\cos \theta' < 0} \frac{|F(\theta')|^2}{k} d\theta'. \quad (7)$$

For a finite incoming wave packet of transverse width  $W$ , the reflected fraction is  $R = \sigma_{\text{refl}}/W$ . For plane-wave scattering, however,  $\sigma_{\text{refl}}$  is the natural observable.

For the numerical calculations, it is useful to employ an explicit representation of the half-space Fourier transform for the model in Eq. (2); this is given in Appendix A.

#### IV. GEOMETRIC SCALING LAWS IN THE MINIMAL CORNER MODEL

The central physical mechanism is most transparent near the specular direction  $\theta'_s = \pi - \theta$ . In that region, the reflected signal is carried by a narrow specular boundary layer of angular width  $O(k^{-1})$ . At the same time, the momentum transfer is strongly anisotropic: The normal component remains of order  $k$ , while the tangential component is compressed into the boundary-layer scale.

This anisotropy fixes the scaling. For corner-supported interactions, the Born amplitude in the reflected sector takes the form

$$F_B(\theta'; k) \sim \frac{1}{k} \mathcal{A}(k\delta; \theta), \quad (8)$$

where  $\delta = \theta' - \theta'_s$  and  $\mathcal{A}$  is a scaling function. The factor  $k^{-1}$  arises from the large- $q_z$  asymptotics of the half-space Fourier transform, while the dependence on  $k\delta$  reflects the angular compression of the reflected sector. Inserting Eq. (8) into Eq. (7) and changing variables  $u = k\delta$  immediately gives

$$\sigma_{\text{refl}} \sim k^{-4} \sim E^{-2}, \quad (9)$$

provided the scaling integral converges.

The exponent is therefore geometric in origin. One factor of  $k^{-1}$  comes from the half-space support of the interaction, and the second from the shrinking specular boundary layer. This mechanism has no one-dimensional analog: The scaling is fixed by boundary geometry rather than by a badlands region or a turning-point structure along a single coordinate.

At sufficiently low energy, however, this growth must break down. In the exact scattering problem, unitarity constrains the on-shell amplitude and prevents arbitrarily large reflection probabilities. The strongest divergence compatible with these constraints is purely kinematic,

$$\sigma_{\text{refl}} \propto k^{-1} \sim E^{-1/2}. \quad (10)$$

This is the scaling expected when the angular amplitude saturates over a finite range of outgoing directions.

The low-energy behavior must therefore cross over from the intermediate-energy geometric Born regime in Eq. (9) to a unitarity-limited threshold regime in Eq. (10). In Sec. V, we show that unitarity indeed imposes a rigorous bound of this form, so that the breakdown of the  $k^{-4}$  scaling is not a feature of the minimal model but a general consequence of the scattering framework.

For numerical illustration, we employ the standard  $K$ -matrix (or Heitler) unitarization scheme,

$$F \rightarrow \frac{F}{1 - iF}. \quad (11)$$

While this correctly reproduces the Born limit for weak scattering and enforces saturation at strong scattering, we strongly emphasize that it is used here solely as a phenomenological visualization tool to exhibit the expected crossover, rather than as a rigorous derivation of the underlying strong-scattering dynamics.

## V. GENERAL CONSTRAINTS AND UNIVERSALITY FOR CORNER-SUPPORTED POTENTIALS

The minimal model isolates the geometric mechanism responsible for the Born scaling. We now turn to statements that do not depend on the specific radial form of the interaction. Two general features emerge: a nonperturbative constraint imposed by unitarity in the arbitrarily low-energy limit and a universal geometric scaling law in the intermediate-energy Born regime.

### A. Unitarity constraint

The reflected scattering width cannot grow arbitrarily fast at ultralow energies. For regularized corner interactions of finite effective range, unitarity of the on-shell scattering operator implies that the reflected angular amplitude remains uniformly bounded in  $L^2$  for any normalized incoming state. Using Eq. (7), this yields the bound

$$\sigma_{\text{refl}}(k) \leq \frac{C}{k}, \quad (12)$$

where  $C$  depends on the interaction and on the incoming state, but is independent of  $k$ . A derivation is given in Appendix B.

Equation (12) shows that the geometric Born scaling  $\sigma_{\text{refl}} \sim k^{-4}$  cannot persist down to arbitrarily low energy. Its eventual breakdown is therefore not model dependent, but a direct consequence of unitarity.

### B. Universal Born scaling law

We now consider a general corner-supported interaction of the form (1) and denote by  $W_0(x) = W(x, 0)$  its boundary trace. We assume that  $W$  is sufficiently regular in the normal direction and that  $W_0$  is square integrable and has one integrable derivative. These conditions justify the reduction of the Born amplitude to the boundary trace and ensure sufficient decay of its tangential Fourier transform. The precise mathematical assumptions are detailed in Appendix C, while their physical scope and applicability to realistic atom-surface interactions are discussed in Sec. V C.

*Proposition (specular saturation).* Under these assumptions, for fixed  $0 < \theta < \pi/2$  the leading  $k^{-4}$  coefficient of the reflected Born width is asymptotically saturated by a specular boundary layer of angular width  $O(k^{-1})$  around  $\theta'_s = \pi - \theta$ .

The proof is given in Appendix C.

*Theorem (universal geometric Born scaling law).* Under the same assumptions, the reflected scattering width in the intermediate Born regime satisfies

$$\sigma_{\text{refl}}(k) = D(\theta) k^{-4} + o(k^{-4}), \quad (13)$$

for fixed incidence angle  $0 < \theta < \pi/2$ , with

$$D(\theta) = \frac{m^2}{8\pi\hbar^4 \cos^3 \theta} \int_0^\infty |W(x, 0)|^2 dx. \quad (14)$$

Equivalently,  $\sigma_{\text{refl}}(E) \sim C(\theta)E^{-2}$ .

*Derivation.* Starting from the Born amplitude,

$$F_B(\mathbf{k}' \leftarrow \mathbf{k}) = -\frac{m}{2\pi\hbar^2} \int_0^\infty dx \int_0^\infty dz e^{-i(q_x x + q_z z)} W(x, z), \quad (15)$$

integration by parts in the normal direction gives, for large  $|q_z|$ ,

$$F_B(\mathbf{k}' \leftarrow \mathbf{k}) = -\frac{m}{2\pi\hbar^2} \frac{1}{iq_z} \int_0^\infty dx e^{-iq_x x} W_0(x) + O(q_z^{-2}). \quad (16)$$

In the specular region  $\theta' = \theta'_s + \delta$ , with  $|\delta| \lesssim k^{-1}$ , one has  $q_z \sim k$  and  $q_x \sim k\delta$ . The amplitude therefore takes the scaling form

$$F_B(\theta'; k) = \frac{1}{k} \mathcal{A}(k\delta; \theta) + O(k^{-2}), \quad (17)$$

where  $\mathcal{A}$  is determined by the Fourier transform of  $W_0$ .

By the specular saturation proposition, the leading  $k^{-4}$  contribution to  $\sigma_{\text{refl}}$  is obtained by restricting to this boundary layer. Performing the change of variables  $u = k \cos \theta \delta$  and using Parseval's identity then yields Eqs. (13) and (14). ■

### C. Physical interpretation

Physically, the regularity assumptions require the boundary trace  $W(x, 0)$  to be nonsingular at the exact corner origin and to decay sufficiently fast along the interface. Standard neutral atom-surface interactions, such as Casimir-van der Waals potentials ( $n \geq 3$ ), naturally satisfy the long-range decay requirement. The short-distance regularity is ensured by any physical microscopic cutoff. For instance, the regularization parameter  $a > 0$  in our minimal model [Eq. (2)] accounts for the finite physical sharpness of realistic corners.

Conversely, an idealized mathematical singularity (e.g., a pure  $1/r^n$  divergence extending all the way to the origin) would violate these conditions. Its unphysical infinite divergence would artificially dominate the Born amplitude, obscuring the extended geometric boundary-layer mechanism that produces the robust  $E^{-2}$  scaling.

The universality of the exponent follows from two geometric ingredients. One factor of  $k^{-1}$  arises from the large- $q_z$  asymptotics of the half-space Fourier transform, while a second factor of  $k^{-1}$  comes from the angular compression of the specular boundary layer. The coefficient  $D(\theta)$  depends only on the boundary trace  $W(x, 0)$ , whereas microscopic details away from the boundary enter only through subleading structure.

The result above characterizes the intermediate-energy Born regime and must be read together with the unitarity constraint in Eq. (12), which enforces a crossover to the kinematic scaling  $\sigma_{\text{refl}} \sim k^{-1}$  at arbitrarily low energies.

Finally, we note that the coefficient  $D(\theta)$  in Eq. (14) contains a  $\cos^{-3} \theta$  factor, which diverges at grazing incidence ( $\theta \rightarrow \pi/2$ ). This physical divergence indicates that for grazing angles, the momentum transfer into the surface normal becomes vanishingly small, causing the Born description to break down at increasingly higher energies. This is consistent with the well-known extreme enhancement of quantum reflection under grazing incidence conditions, as observed in fast atom diffraction [12–14].

### D. Validity of the Born approximation

To assess whether the geometric scaling law is merely a first-order artifact or persists within a controlled perturbative

regime, it is useful to estimate the second-order Born contribution's scaling. The corresponding amplitude is obtained from an integration over intermediate scattering states and contains two insertions of the corner-supported potential.

In the specular boundary layer, the dominant momentum transfer remains of order  $q \sim k$ . As discussed above, the half-space Fourier transform of the potential contributes a factor proportional to  $\lambda/k$  for each large momentum transfer, where  $\lambda$  denotes the characteristic momentum scale associated with the interaction strength. Furthermore, in two dimensions, the density of states is energy independent, so the integration over the intermediate free Green's function does not introduce an additional power of  $k$ . Consequently, the leading kinematic scaling of the second-order contribution is

$$F_{B2} \sim \left(\frac{\lambda}{k}\right)^2, \quad (18)$$

whereas the first-order amplitude scales as

$$F_{B1} \sim \frac{\lambda}{k}. \quad (19)$$

The ratio between successive Born contributions therefore behaves as

$$\frac{F_{B2}}{F_{B1}} \sim \frac{\lambda}{k}. \quad (20)$$

The perturbative expansion is thus controlled by the dimensionless parameter  $\lambda/k$ . In the regime  $k \gg \lambda$ , higher-order multiple-scattering processes remain parametrically suppressed, and the first Born contribution dominates. The geometric scaling law  $\sigma_{\text{refl}} \propto E^{-2}$  should therefore be interpreted as a controlled intermediate-energy asymptotic regime rather than as a purely first-order artifact.

As the incident momentum approaches the interaction scale,  $k \sim \lambda$ , higher-order terms become comparable to the leading contribution and the Born expansion ceases to be controlled. In this regime, multiple scattering and unitarity corrections become essential, ultimately leading to the crossover toward the unitarity-limited behavior discussed above. While this scaling estimate does not constitute a proof of the exact asymptotics, it provides evidence that the geometric  $E^{-2}$  scaling remains stable against higher-order Born corrections throughout the regime in which the perturbative expansion is valid.

### E. Separable corner potentials

For separable interactions of the form

$$V(x, z) = -\lambda g(x)h(z)\Theta(x)\Theta(z), \quad (21)$$

one has  $W_0(x) = -\lambda h(0)g(x)$ . Under the same regularity conditions, the general result immediately yields

$$\sigma_{\text{refl}} \propto k^{-4} \propto E^{-2}. \quad (22)$$

Thus, separable models provide explicit realisations of the general boundary-trace mechanism, but do not exhaust the universality class. Details are given in Appendix D.

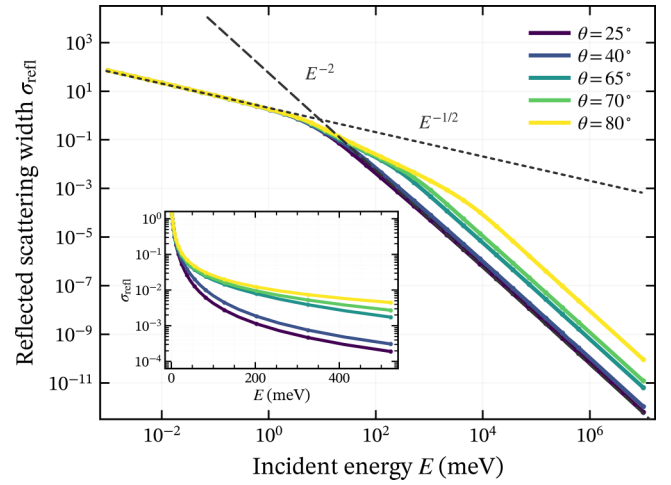


FIG. 2. Reflected scattering width as a function of energy for several incidence angles. The intermediate-energy regime exhibits robust  $E^{-2}$  scaling, whereas at low energies all curves bend toward a weaker, unitarity-limited behavior.

## VI. NUMERICAL RESULTS AND CROSSOVER

We now illustrate the two scaling regimes for the minimal corner model. The purpose of the numerics is not to establish the asymptotics rigorously, but to test the geometric scaling picture in the Born regime and to exhibit the crossover expected once unitarity becomes relevant. The numerical calculations presented here are not intended as an independent verification of the exact asymptotic behavior beyond the Born framework. Their purpose is to illustrate the geometric scaling regime and the crossover mechanism within the minimal corner model.

### A. Energy dependence of the reflected width

Figure 2 shows the reflected scattering width as a function of energy for several incidence angles. Over an extended intermediate-energy window, all curves exhibit a clear power-law behavior consistent with the geometric Born prediction  $\sigma_{\text{refl}} \sim E^{-2}$ .

Two features are worth emphasizing. First, the scaling regime is robust with respect to the incidence angle: Although the overall magnitude and the onset of the scaling window depend on  $\theta$ , the slope remains essentially unchanged. This supports the interpretation that the exponent is fixed by geometry rather than by microscopic details of the interaction. Second, the extent of the scaling regime shifts with angle, reflecting the dependence of the boundary-layer contribution on  $\cos \theta$  through the kinematics of the specular sector.

At lower energies, all curves bend systematically away from the  $E^{-2}$  behavior and toward a weaker divergence. The breakdown occurs earlier for larger angles, where the effective normal momentum transfer is reduced. This behavior is consistent with the expectation that the Born description must eventually fail and give way to a unitarity-limited regime.

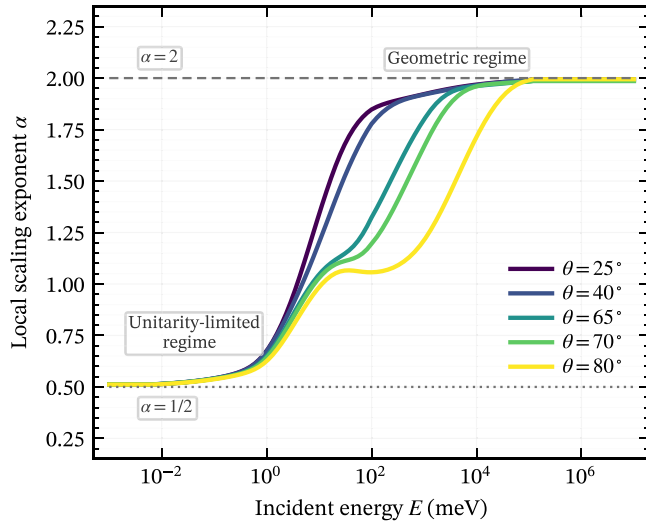


FIG. 3. Local scaling exponent for different decrease angles. The intermediate regime approaches the geometric value  $\alpha = 2$ , while the lowest energies approach the unitarity-limited value  $\alpha = 1/2$ . The smooth crossover reflects the gradual breakdown of the Born approximation.

### B. Local scaling exponent and crossover

To make the crossover more transparent, Fig. 3 shows the local scaling exponent

$$\alpha(E) = -\frac{d \ln \sigma_{\text{refl}}}{d \ln E}. \quad (23)$$

In the intermediate regime,  $\alpha(E)$  approaches the geometric value  $\alpha = 2$ , confirming that the observed scaling is not an artifact of a limited fitting window. At lower energies, the exponent decreases toward  $\alpha = 1/2$ , reflecting the gradual breakdown of the Born approximation and the onset of unitarity-limited behavior. The limiting value  $\alpha = 1/2$  is precisely the kinematic scaling associated with the bound  $\sigma_{\text{refl}} \lesssim C/k$  derived in Sec. V.

The figure therefore provides a direct visualization of the scaling crossover from a geometry-dominated Born regime to a unitarity-dominated low-energy regime.

### C. Born versus unitarized behavior

Figure 4 compares the pure Born result with the unitarized interpolation introduced in Eq. (11). The Born approximation accurately captures the  $E^{-2}$  regime, but continues to grow at low energies in a way that is incompatible with the exact unitarity bound.

In contrast, the unitarized amplitude reproduces the Born behavior at intermediate energies, while saturating at low energies. The resulting curve follows the same qualitative trend as the numerical data, in particular, the crossover toward  $E^{-1/2}$  scaling.

This comparison clarifies the role of the unitarization scheme. As emphasized earlier, it is not intended as a microscopic description of the strong-scattering regime, but strictly as a minimal visualization tool that enforces the correct

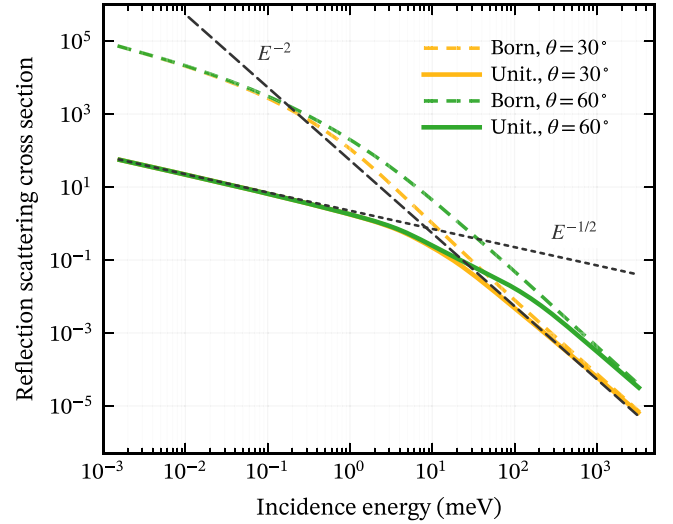


FIG. 4. Comparison between the Born result and the unitarized interpolation  $F \rightarrow F/(1 - iF)$ . The Born approximation captures the geometric regime but violates the exact low-energy constraint, while the unitarized form provides a simple interpolation consistent with the expected crossover.

low-energy constraint while preserving the geometric Born regime.

## VII. COMPARISON WITH ONE-DIMENSIONAL QUANTUM REFLECTION

In conventional quantum reflection, the scaling law is controlled by the one-dimensional semiclassical structure of the potential. In the corner problem studied here, the scattering is instead intrinsically multidimensional, and the reflected flux arises from momentum mixing between channels that are absent at a planar interface.

The leading reflected contribution is captured by a specular boundary layer with anisotropic momentum transfer: One component remains of order  $k$ , while the other is compressed into an angular window of width  $k^{-1}$ . This mechanism has no one-dimensional analog. The resulting exponent is therefore geometric: It is fixed by support on a corner and by the associated boundary-layer kinematics, not by a badlands region or by a turning-point structure along a single coordinate.

This distinction also clarifies the relation between the present mechanism and the standard exact-WKB picture. In one dimension, reflection is organized by Stokes coupling between semiclassical branches associated with complex turning points. In the corner problem, the relevant phenomenon is instead the redistribution of scattering weight across angular channels when translational symmetry is broken. The scaling law is therefore a property of multidimensional geometry rather than a reformulation of the conventional one-dimensional reflection mechanism.

## VIII. EXPERIMENTAL IMPLICATIONS

The mechanism discussed here should be relevant wherever matter waves probe attractive surface interactions in the presence of sharp geometric features. Natural candidates

include atom-surface scattering from lithographically patterned substrates, nanostructured gratings, atom-chip architectures, and surface steps or etched trenches whose local geometry effectively contains cornerlike regions. Closely related geometries already arise in matter-wave diffraction experiments, where atom-surface interactions have been shown to significantly influence transmission through nanostructured gratings and can be measured with high precision [39,40].

A particularly clear signature is the crossover in the energy dependence of the reflected signal. In a regime where the incoming de Broglie wavelength is large enough for long-range attraction to matter, yet not so small that the Born description fully breaks down, the reflected width is predicted to scale as  $\sigma_{\text{refl}} \propto E^{-2}$ . At still lower energy, this growth must soften toward the unitarity-limited behavior  $\sigma_{\text{refl}} \propto E^{-1/2}$ . Measuring the reflected fraction as a function of incident velocity would therefore provide a direct test of the geometric scaling law.

A second signature is angular redistribution. Because the corner breaks translational symmetry in both directions, the reflected flux is not restricted to the narrow specular channel familiar from planar reflection. Instead, the corner mixes longitudinal and transverse momentum, producing a reflected distribution over a finite angular sector. Angle-resolved detection could therefore help distinguish geometric reflection from conventional planar quantum reflection.

More broadly, the results suggest that surface structuring can be used not only to diffract matter waves but also to control their threshold reflection properties. In this sense, sharp geometric features may provide a route toward geometry-engineered matter-wave mirrors and reflection-based atom-optical elements beyond the planar paradigm.

## IX. CONCLUSIONS

We have shown that singular corners induce a mechanism of quantum reflection that is intrinsically multidimensional. For a minimal corner model, the reflected scattering width exhibits a geometric Born regime with

$$\sigma_{\text{refl}} \propto E^{-2}, \quad (24)$$

followed at lower energy by a crossover to the unitarity-limited behavior

$$\sigma_{\text{refl}} \propto E^{-1/2}. \quad (25)$$

The intermediate scaling is controlled by boundary geometry, while the low-energy behavior is constrained by unitarity.

Our main theoretical result is a universal geometric scaling law for corner-supported potentials of the form  $V(x, z) = W(x, z)\Theta(x)\Theta(z)$ . Under explicit boundary regularity assumptions, the reflected width obeys

$$\sigma_{\text{refl}}(k) = D(\theta)k^{-4} + o(k^{-4}), \quad (26)$$

with a coefficient determined solely by the boundary trace  $W(x, 0)$ . The leading contribution is asymptotically concentrated in a specular boundary layer, which saturates the  $k^{-4}$  coefficient, while microscopic details away from the boundary enter only through subleading structure.

Together with the exact unitarity constraint, this establishes a complete scaling picture: a geometry-controlled Born

regime at intermediate energies and a universal, unitarity-limited regime at the lowest energies.

These results identify corners as the simplest surface singularities that support a geometric threshold law for quantum reflection and suggest that other singular interfaces may define further universality classes of multidimensional threshold scattering. More broadly, they show that once separability is lost, low-energy reflection can be governed by geometry itself rather than by the one-dimensional mechanisms familiar from planar surfaces.

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## DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

## APPENDIX A: HALF-SPACE FOURIER TRANSFORM FOR THE MINIMAL CORNER MODEL

For the numerical evaluation of the Born amplitude in the minimal corner model, it is convenient to rewrite the half-space Fourier transform of the potential in a form that factorizes into one-dimensional integrals.

We consider the model potential

$$V(x, z) = -\frac{C_n}{(x^2 + z^2 + a^2)^{n/2}} \Theta(x)\Theta(z), \quad (A1)$$

with  $C_n > 0$ ,  $a > 0$ , and  $n > 0$ . Its half-space Fourier transform is

$$\tilde{V}_c(q_x, q_z) = \int_0^\infty dx \int_0^\infty dz e^{-i(q_x x + q_z z)} V(x, z). \quad (A2)$$

Substituting Eq. (A1) gives

$$\begin{aligned} \tilde{V}_c(q_x, q_z) &= -C_n \int_0^\infty dx \int_0^\infty dz \frac{e^{-i(q_x x + q_z z)}}{(x^2 + z^2 + a^2)^{n/2}}. \end{aligned} \quad (A3)$$

To evaluate this expression, we use the Laplace representation

$$\begin{aligned} \frac{1}{(x^2 + z^2 + a^2)^{n/2}} &= \frac{1}{\Gamma(n/2)} \int_0^\infty dt t^{n/2-1} e^{-t(x^2 + z^2 + a^2)}. \end{aligned} \quad (A4)$$

Inserting this into Eq. (A3) and exchanging the order of integration yields

$$\begin{aligned} \tilde{V}_c(q_x, q_z) &= -\frac{C_n}{\Gamma(n/2)} \int_0^\infty dt t^{n/2-1} e^{-a^2 t} X(q_x; t) X(q_z; t), \end{aligned} \quad (A5)$$

where

$$X(q; t) := \int_0^\infty dx e^{-tx^2 - iqx}, \quad t > 0. \quad (\text{A6})$$

The remaining one-dimensional integral is a half-Gaussian Fourier transform. Completing the square gives

$$tx^2 + iqx = t \left( x + \frac{iq}{2t} \right)^2 + \frac{q^2}{4t}, \quad (\text{A7})$$

so that

$$X(q; t) = e^{-q^2/(4t)} \int_0^\infty dx \exp \left[ -t \left( x + \frac{iq}{2t} \right)^2 \right]. \quad (\text{A8})$$

After the change of variables

$$y = \sqrt{t} \left( x + \frac{iq}{2t} \right), \quad (\text{A9})$$

one obtains

$$X(q; t) = \frac{1}{\sqrt{t}} e^{-q^2/(4t)} \int_{iq/(2\sqrt{t})}^{\infty + iq/(2\sqrt{t})} e^{-y^2} dy. \quad (\text{A10})$$

Using the complementary error function, this can be written in closed form as

$$X(q; t) = \frac{1}{2} \sqrt{\frac{\pi}{t}} \exp \left( -\frac{q^2}{4t} \right) \operatorname{erfc} \left( \frac{iq}{2\sqrt{t}} \right). \quad (\text{A11})$$

Plugging Eq. (A11) into Eq. (A5) provides the explicit representation of the half-space Fourier transform. This representation is numerically advantageous because it reduces the original two-dimensional oscillatory integral over the quadrant to a single integral over  $t$ , while the dependence on  $q_x$  and  $q_z$  enters only through the analytic function  $X(q; t)$ .

## APPENDIX B: EXACT UNITARITY BOUND

In this Appendix, we derive the exact bound (12) on the reflected scattering width.

For a regularized corner interaction of finite effective range, the exact scattering state admits a standard angular-channel decomposition outside the interaction region. The on-shell scattering operator  $S(k)$  acts unitarily on the space of incoming angular amplitudes. Let

$$A(k) := S(k) - I. \quad (\text{B1})$$

Since  $S(k)$  is unitary, one has

$$\|A(k)a\| = \|(S(k) - I)a\| \leq \|S(k)a\| + \|a\| \leq 2\|a\|, \quad (\text{B2})$$

for any incoming normalized state  $a$ .

Let  $P_{\text{refl}}$  denote the projector onto the reflected sector  $\cos \theta' < 0$ . Then,

$$\|P_{\text{refl}}A(k)a\| \leq \|A(k)a\| \leq 2\|a\|. \quad (\text{B3})$$

Using Parseval's identity between the angular-channel coefficients and the angular representation of the scattering amplitude, Eq. (B3) implies that the reflected on-shell angular amplitude has uniformly bounded  $L^2$  norm. Equivalently, there exists a constant  $C_0$ , independent of  $k$ , such that

$$\int_{\cos \theta' < 0} |F(\theta'; k)|^2 d\theta' \leq C_0. \quad (\text{B4})$$

Using Eq. (6), we obtain

$$\sigma_{\text{refl}}(k) = \frac{1}{k} \int_{\cos \theta' < 0} |F(\theta'; k)|^2 d\theta' \leq \frac{C_0}{k}, \quad (\text{B5})$$

which proves Eq. (12).

## APPENDIX C: SPECULAR SATURATION AND DERIVATION OF THE UNIVERSAL BORN COEFFICIENT

In this Appendix, we derive Eq. (C16), which states that the specular boundary layer asymptotically saturates the leading Born coefficient of the reflected width, and complete the derivation of the universal coefficient in Eq. (14). We assume the following throughout this Appendix:

(1) For almost every  $x \geq 0$ , the map  $z \mapsto W(x, z)$  is absolutely continuous on  $[0, \infty)$ .

(2) The boundary trace  $W_0(x) = W(x, 0)$  satisfies  $W_0 \in L^2(0, \infty)$  and  $W_0 \not\equiv 0$ .

(3)  $\partial_z W$  is integrable on the quadrant and  $W(x, z) \rightarrow 0$  as  $z \rightarrow \infty$  for almost every  $x$ .

(4)  $W_0 \in W^{1,1}(0, \infty)$ .

Starting from the Born amplitude,

$$F_B(\mathbf{k}' \leftarrow \mathbf{k}) = -\frac{m}{2\pi\hbar^2} \int_0^\infty dx \int_0^\infty dz e^{-i(q_x x + q_z z)} W(x, z), \quad (\text{C1})$$

integration by parts in  $z$  gives, using assumptions (A1)–(A3),

$$F_B = -\frac{m}{2\pi\hbar^2} \frac{1}{iq_z} \int_0^\infty dx e^{-iq_x x} W_0(x) + O(q_z^{-2}). \quad (\text{C2})$$

In the reflected sector  $\cos \theta' < 0$  and for fixed  $0 < \theta < \pi/2$ , one has

$$|q_z| = k|\cos \theta' - \cos \theta| \geq k \cos \theta, \quad (\text{C3})$$

so the boundary-trace expansion holds uniformly for large  $k$ . Hence,

$$F_B(\theta'; k) = -\frac{m}{2\pi\hbar^2} \frac{1}{iq_z} \widehat{W}_0(q_x) + O(k^{-2}), \quad (\text{C4})$$

with

$$\widehat{W}_0(u) = \int_0^\infty e^{-iux} W_0(x) dx. \quad (\text{C5})$$

Because  $W_0 \in W^{1,1}(0, \infty)$ , integration by parts in  $x$  gives

$$|\widehat{W}_0(u)| \leq \frac{C}{|u|}, \quad |u| \rightarrow \infty. \quad (\text{C6})$$

Write

$$\theta' = \theta'_s + \delta, \quad \theta'_s = \pi - \theta.$$

Then, for  $|\delta| \ll 1$ ,

$$q_x = -k \cos \theta \delta + O(k\delta^2), \quad q_z = -2k \cos \theta + O(k\delta). \quad (\text{C7})$$

We split the reflected sector into three regions.

(1) *Specular layer*:  $|\delta| \leq \Lambda/k$ . Here,  $q_x = O(1)$  and  $q_z \sim k$ , so

$$F_B = \frac{1}{k} A(k \cos \theta \delta) + O(k^{-2}), \quad (\text{C8})$$

with

$$A(u) = \frac{m}{4\pi\hbar^2 \cos \theta} \widehat{W}_0(u). \quad (\text{C9})$$

This region contributes at order  $k^{-4}$ .

(2) *Intermediate region*:  $\Lambda/k < |\delta| \leq \eta$ . Here,  $q_x \sim k\delta$  and  $q_z \sim k$ . Using  $u = k \cos \theta \delta$ ,

$$\begin{aligned} & \int_{\Lambda/k < |\delta| \leq \eta} \frac{|F_B|^2}{k} d\theta' \\ &= \frac{1}{k^4} \int_{\Lambda \cos \theta < |u| \leq \eta k \cos \theta} |\widehat{W}_0(u)|^2 du + o(k^{-4}). \end{aligned} \quad (\text{C10})$$

Since  $\widehat{W}_0 \in L^2(\mathbb{R})$ , for every  $\varepsilon > 0$  there exists  $\Lambda > 0$  such that

$$\int_{|u| > \Lambda \cos \theta} |\widehat{W}_0(u)|^2 du < \varepsilon. \quad (\text{C11})$$

For this fixed  $\Lambda$  and sufficiently large  $k$ ,

$$\int_{\Lambda/k < |\delta| \leq \eta} \frac{|F_B|^2}{k} d\theta' < \frac{\varepsilon}{k^4} + o(k^{-4}). \quad (\text{C12})$$

Thus, the intermediate region contributes only an arbitrarily small fraction of the leading  $k^{-4}$  coefficient.

(3) *Nonspecular region*:  $|\delta| > \eta$ . Then,  $|q_x| \geq c_\eta k$  for some  $c_\eta > 0$ , and therefore

$$|\widehat{W}_0(q_x)| \leq \frac{C}{|q_x|} \leq \frac{C_\eta}{k}. \quad (\text{C13})$$

Since also  $|q_z| \geq k \cos \theta$ ,

$$|F_B| \leq \frac{C_\eta}{k^2}, \quad (\text{C14})$$

and hence

$$\int_{|\delta| > \eta} \frac{|F_B|^2}{k} d\theta' = O(k^{-5}) = o(k^{-4}). \quad (\text{C15})$$

Combining the three regions gives

$$\lim_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow \infty} k^4 |\sigma_{\text{refl}}(k) - \sigma_{\text{spec}}^{(\Lambda)}(k)| = 0. \quad (\text{C16})$$

It remains to evaluate the leading contribution from the specular layer. Using the expression above for  $F_B$  in this region, one obtains

$$\sigma_{\text{spec}}^{(\Lambda)}(k) = \frac{1}{k^4 \cos \theta} \int_{|u| \leq \Lambda \cos \theta} |A(u)|^2 du + o(k^{-4}), \quad (\text{C17})$$

with  $u = k \cos \theta \delta$ . Letting first  $k \rightarrow \infty$  and then  $\Lambda \rightarrow \infty$ , and using Parseval's identity,

$$\int_{-\infty}^{\infty} |\widehat{W}_0(u)|^2 du = 2\pi \int_0^{\infty} |W_0(x)|^2 dx, \quad (\text{C18})$$

yields Eq. (14) and completes the derivation of the universal coefficient [Eq. (13)].

#### APPENDIX D: SEPARABLE CORNER POTENTIALS

For completeness, consider

$$V(x, z) = -\lambda g(x)h(z)\Theta(x)\Theta(z), \quad (\text{D1})$$

with  $g \in L^2(0, \infty)$  and  $h$  absolutely continuous on  $[0, \infty)$ , with  $h(0)$  well defined and  $h'(z)$  integrable on  $[0, \infty)$ . Then,

$$\tilde{V}(q_x, q_z) = -\lambda G(q_x)H(q_z), \quad (\text{D2})$$

where

$$G(q) = \int_0^{\infty} dx e^{-iqx} g(x), \quad H(q) = \int_0^{\infty} dz e^{-iqz} h(z). \quad (\text{D3})$$

Integrating  $H$  by parts gives

$$H(q) = \frac{h(0)}{iq} + O(q^{-2}), \quad |q| \rightarrow \infty. \quad (\text{D4})$$

In the reflected boundary layer,

$$F_B(\theta'; k) \simeq \frac{m\lambda h(0)}{4\pi\hbar^2 \cos \theta} \frac{1}{k} G(k \cos \theta \delta). \quad (\text{D5})$$

Hence,

$$\sigma_{\text{refl}} \simeq \frac{|C(\theta)|^2}{k^3} \int d\delta |G(k \cos \theta \delta)|^2, \quad (\text{D6})$$

with

$$C(\theta) = \frac{m\lambda h(0)}{4\pi\hbar^2 \cos \theta}. \quad (\text{D7})$$

After  $u = k \cos \theta \delta$ ,

$$\sigma_{\text{refl}} \simeq \frac{|C(\theta)|^2}{k^4 \cos \theta} \int_{-\infty}^{\infty} |G(u)|^2 du. \quad (\text{D8})$$

By Parseval,

$$\int_{-\infty}^{\infty} |G(u)|^2 du = 2\pi \int_0^{\infty} |g(x)|^2 dx < \infty, \quad (\text{D9})$$

so that

$$\sigma_{\text{refl}} \propto k^{-4} \propto E^{-2}. \quad (\text{D10})$$

This reproduces the separable corollary discussed in the main text.

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