

Thermodynamic criteria for signaling in quantum channels

Yutong Luo^{1,2,*}, Simon Milz^{3,†}, and Felix C. Binder^{1,2,‡}

¹*School of Physics, Trinity College Dublin, Dublin 2, Ireland*

²*Trinity Quantum Alliance, Unit 16, Trinity Technology and Enterprise Centre, Pearse Street, Dublin 2, Ireland*

³*Institute of Photonics and Quantum Sciences, Heriot-Watt University, Edinburgh EH14 4AS, United Kingdom*



(Received 4 July 2025; accepted 23 November 2025; published 24 December 2025)

Signaling quantum channels are fundamental to quantum communication, enabling the transfer of information from input to output states. In contrast, thermalization erases information about the initial state. This raises a crucial question: How does the thermalizing tendency of a quantum channel constrain its signaling power and vice versa? In this work, we address this question by considering three thermodynamic tasks associated with a quantum channel: the generation, preservation, and transmission of athermality. We provide faithful measures for athermality generation and athermality preservation of quantum channels, and prove that their difference quantifies athermality transmission. Analyzing these thermodynamic tasks, we find that the signaling ability of a quantum channel is upper-bounded by its athermality preservation and lower-bounded by its athermality transmission, thereby establishing a fundamental relationship between signaling and thermodynamic properties of channels for quantum communication. We demonstrate this interplay for the example of the quantum switch, revealing an explicit trade-off between the signaling ability and athermality of the quantum channels it can implement.

DOI: [10.1103/vjfk-vsmw](https://doi.org/10.1103/vjfk-vsmw)

I. INTRODUCTION

Quantum communication fundamentally relies on the ability of quantum channels to transmit information. It requires channels that are *signaling*, such that changes in the input state lead to distinguishable changes at the output. In practice, quantum communication devices are subject to energetic and thermodynamic constraints [1–6] and unavoidably exposed to environmental interactions that induce thermal noise [7,8], driving systems toward thermal equilibrium and erasing information about the initial state. This introduces a fundamental tension between two competing effects: the signaling power of a channel and its thermalizing tendency. While the thermodynamic cost of *implementing* a quantum channel is well understood [9–11], the *interplay* between signaling and thermodynamic resources of a given quantum channel is physically distinct and remains to be fully characterized. Understanding this trade-off is essential for assessing the limitations of quantum communication under realistic, thermal noise. For incoherent states and dynamics, it was recently shown that a quantum channel can transmit n bits of classical information if and only if it can transmit n units of energy [12].

In this work, we consider the fully quantum case and reveal the interrelation between a quantum channel's signaling and thermodynamic capabilities by analyzing three essential thermodynamic tasks: the generation, preservation, and transmission of athermality. Relating these thermodynamic tasks to resource-theoretic quantities [13–17], we prove that the ability of a quantum channel to signal is thermodynamically constrained from below (above) by how much athermality it can transmit (preserve). We then derive an operational interpretation of these resources via a channel simulation problem, yielding the resource cost of implementing a channel and extending the work-cost framework discussed in Refs. [10,11]. An area where thermodynamic, signaling, and causality considerations concurrently come to the fore is the quantum switch [18], which has attracted attention in both quantum communication and thermodynamics [19–23]. Using the thermodynamic resourcefulness of coherence [24–26] to control the causal order of two quantum channels, it is known to enhance their signaling [19,27,28] and athermal properties [20,29,30], at a thermodynamic cost. Our bounds extend these results by specifying the explicit trade-off between signaling power and athermality that the induced channel of a quantum switch exhibits in terms of invested resources. Together, our results allow us to gauge the thermodynamic and signaling abilities of quantum channels in a unified way.

The remainder of the paper is organized as follows. In Sec. II, we introduce quantitative measures for the athermality and signaling ability of quantum channels, and define a joint measure that captures both resources collectively. In Sec. III, we identify three thermodynamic tasks associated with a quantum channel—generation, preservation and transmission of athermality, and show that these tasks are

*Contact author: yuluo@tcd.ie

†Contact author: s.milz@hw.ac.uk

‡Contact author: felix.binder@tcd.ie

characterized by the defined measures. In Sec. IV, we formulate our thermodynamic criteria for signaling: Signaling is enabled by athermality transmission but constrained by athermality preservation. To give further operational meaning for our measures, Sec. V considers a channel simulation scenario and demonstrates that our measures quantify the minimal thermodynamic resources required to simulate an arbitrary quantum channel. In Sec. VI, we apply the criteria for signaling to the quantum switch, showing that it trades between thermodynamic and signaling resources via an explicit trade-off relation. Finally, in Sec. VII, we summarize our results and discuss possible future directions.

Throughout, we denote the set of quantum states of system X as $\mathcal{S}(X)$, and the set of quantum channels (i.e., CPTP maps) between input system X and output system Y as $\mathcal{O}(X \rightarrow Y)$. X' denotes a copy of system X with the same thermal state γ_X .

II. RESOURCE MEASURES

A. Robustness of athermality and signaling

In the language of resource theories, quantum states constitute static resources, whereas quantum operations between them provide dynamical resources. To quantify the amount of resource in a general state/operation, we adopt the robustness measure of resources, which are widely used in the resource theories of entanglement [31–33] and coherence [34,35], and in general resource theories [36–38]. Intuitively, robustness-based measures quantify how much noise one needs to add before the resource possessed by a quantum state/operation disappears. Let $\mathcal{D}$ be the set of quantum states/operations and $\mathcal{F} \subset \mathcal{D}$ a free set inside $\mathcal{D}$ corresponding to some type of resource. For an object $X \in \mathcal{D}$, its robustness of resource is given by

$$R(X) := \min \left\{ s \left| \frac{X + sY}{1+s} = Z \in \mathcal{F}, Y \in \mathcal{D} \right. \right\} \quad (1)$$

$$= \min \{ s | X \leq (1+s)Z, Z \in \mathcal{F} \}, \quad (2)$$

where $Y \in \mathcal{D}$ is a general “noise” to X , and in the second line $X \leq (1+s)Z$ means that $(1+s)Z - X$ is a positive semidefinite operator or a completely positive map, depending on whether $\mathcal{D}$ is the set of quantum states or the set of quantum channels. Geometrically, $R(X)$ can be interpreted as the “distance” between X and the free set $\mathcal{F}$, as shown in Fig. 1. Besides, robustness measures can also be expressed in terms of the max-relative entropy $D_{\max}(X||Z) := \ln \min\{\lambda | X \leq \lambda Z\}$ [39].

Here, we adopt the robustness of athermality R_T to measure the thermodynamic resource inherent to quantum states and channels. For a general state ρ_A , the robustness of athermality $R_T(\rho_A)$ is defined as

$$R_T(\rho_A) := \min \{ s | \rho_A \leq (1+s)\gamma_A \}. \quad (3)$$

Throughout, we consider full-rank thermal states γ , thus avoiding divergence of R_T . When ρ_A is incoherent, $R_T(\rho_A)$ relates to the necessary work for preparing ρ_A from γ_A (the *work of formation*) [40]. The set of athermality nongenerating operations from A to B are the *Gibbs-preserving operations*

$$\text{GPO}(A \rightarrow B) := \{ G | G(\gamma_A) = \gamma_B, G \in \mathcal{O}(A \rightarrow B) \}. \quad (4)$$

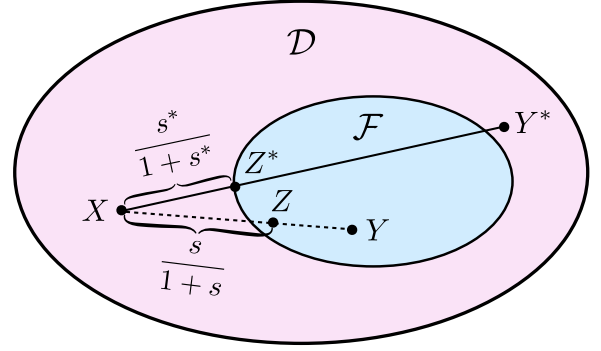


FIG. 1. Geometric interpretation of robustness measure. The distance between X and the free state Z corresponds to the factor $s/(1+s)$. Minimizing s finds the closest free state Z^* to X , and hence quantifies the distance between X and the free set $\mathcal{F}$.

For a general channel $\Lambda \in \mathcal{O}(A \rightarrow B)$, the robustness of athermality $R_T(\Lambda)$ is defined as

$$R_T(\Lambda) := \min \{ s | \Lambda \leq (1+s)G, G \in \text{GPO}(A \rightarrow B) \}. \quad (5)$$

We use R_T for both static and dynamical athermality because dynamical athermality can be reduced to static athermality (Theorem 1). The type of athermality R_T will be unambiguous from its argument.

Turning to information transmission of a general quantum channel Λ , its signaling ability can be quantified by the robustness of signaling R_S , where the free operations are nonsignaling channels [41–43]. The set of *nonsignaling operations* (NSOs) from A to B is given by trace-and-replace channels $\Phi_\sigma(\cdot) \equiv \text{Tr}\{\cdot\}\sigma$

$$\text{NSO}(A \rightarrow B) := \{ \Phi_\sigma | \sigma \in \mathcal{S}(B) \}. \quad (6)$$

For a general channel $\Lambda \in \mathcal{O}(A \rightarrow B)$, $R_S(\Lambda)$ is defined as

$$R_S(\Lambda) := \min \{ s | \Lambda \leq (1+s)\Phi, \Phi \in \text{NSO}(A \rightarrow B) \}. \quad (7)$$

In Ref. [43], $R_S(\Lambda)$ has been interpreted as a measure of the “strength” of causal loops that can be closed by contracting Λ with a general channel $\Omega \in \mathcal{O}(B \rightarrow A)$, and it has been shown that the logarithm of R_S asymptotically converges to the channel mutual information [41].

B. Joint measure of athermality and signaling of quantum channels

R_T and R_S measure athermality and signaling of a quantum channel, respectively, due to the two different free sets in their definitions, GPO and NSO. Given the input system A and the output system B , the intersection between $\text{GPO}(A \rightarrow B)$ and $\text{NSO}(A \rightarrow B)$ is the completely thermalizing channel

$$\Gamma(\cdot) \equiv \text{Tr}\{\cdot\}\gamma_B. \quad (8)$$

Note, however, that Γ is neither an extreme point in NSO nor in GPO. For NSO, this is straightforward. For GPO, see Ref. [44] for the conditions on extreme points in GPO. We thus define a joint measure that collectively gauges signaling and athermality of a general channel $\Lambda \in \mathcal{O}(A \rightarrow B)$ as

$$R(\Lambda || \Gamma) := \min \{ s | \Lambda \leq (1+s)\Gamma \}. \quad (9)$$

TABLE I. Four resource measures considered in this work.

Resources	Symbols	Types	Definitions
Athermality	R_T	Static/dynamical	Equations (3) and (5)
Signaling	R_S	Dynamical	Equation (7)
Joint resource	R	Dynamical	Equation (9)
Athermality preservability for GPOs	P_T	Dynamical	Equation (11)

Given the close connection between robustness measures and the max-relative entropy, $R(\Lambda||\Gamma)$ can also be studied as a specific instance of a generalized channel divergence with respect to the completely thermalizing channel [45,46]. Operationally, $R(\Lambda||\Gamma)$ measures the “distance” between Λ and the completely thermalizing channel Γ , and therefore it follows from its definition that

$$R(\Lambda||\Gamma) \geq \max \{R_T(\Lambda), R_S(\Lambda)\}. \quad (10)$$

We note that equality holds when $\Phi \in \text{NSO}(A \rightarrow B)$, such that $R_S(\Phi) = 0$ and $R(\Phi||\Gamma) = R_T(\Phi)$. However, when $G \in \text{GPO}(A \rightarrow B)$, $R(G||\Gamma) \geq R_S(G)$ in general. Following the framework of resource preservability, where some free operations in a resource theory are considered as resourceful with respect to resource-destroying operations, owing to their ability to preserve resources [47], we set

$$P_T(G) := R(G||\Gamma) \quad \text{for } G \in \text{GPO}(A \rightarrow B), \quad (11)$$

which is the athermality preservability of the GPO G . To conclude this section, in Table I, we summarize the four types of resources included in this work. For more properties of these resource measures, we refer to Appendix A 1.

III. GENERATION, PRESERVATION, AND TRANSMISSION OF ATHERMALITY

We now introduce three thermodynamic tasks associated with a general quantum channel Λ and show that they are characterized by the resource measures defined above. Considering a purification of γ_A as input, for simplicity, we start with the thermofield double state [48] $\tilde{\gamma}_{AA'} \equiv |\tilde{\gamma}\rangle\langle\tilde{\gamma}|_{AA'}$ with $|\tilde{\gamma}\rangle_{AA'} \equiv \sum_i \sqrt{g_i} |i\rangle_A \otimes |i\rangle_{A'}$. Here, g_i is the i th population of γ_A , corresponding to energy eigenstate $|i\rangle_A$. We have $\text{Tr}_{\{A,A'\} \setminus X} \{\tilde{\gamma}_{AA'}\} = \gamma_X$ for $X \in \{A, A'\}$. A generalization to all purified thermal states on AA' is given in Appendices A 3 and A 4.

We consider the following three thermodynamic tasks and related resources (see Fig. 2):

(I) *Athermality generation* (R_T): The amount of athermality a quantum channel $\Lambda \in \mathcal{O}(A \rightarrow B)$ generates from the thermal state γ_A .

(II) *Athermality preservation* (R): The amount of athermality a local quantum channel $\Lambda \in \mathcal{O}(A' \rightarrow B)$ can preserve when acting on $\tilde{\gamma}_{AA'}$.

(III) *Athermality transmission* ($R - R_T$): The amount of athermality a local quantum channel $\Lambda \in \mathcal{O}(A' \rightarrow B)$ can transmit when acting on $\tilde{\gamma}_{AA'}$.

The athermality generating ability of a quantum channel Λ is equivalent to its dynamical athermality $R_T(\Lambda)$:

Theorem 1. For a general channel $\Lambda \in \mathcal{O}(A \rightarrow B)$, the following equality holds:

$$R_T(\Lambda) = R_T[\Lambda(\gamma_A)]. \quad (12)$$

The proof is provided in Appendix A 2. Hence, Task I is quantified by $R_T(\Lambda)$. We remark that similar relations between static and dynamic resources have been found for coherence [35] and entanglement [33]. Analogously, the joint dynamical resource measure R can also be reduced to a static resource:

Theorem 2. For a general channel $\Lambda \in \mathcal{O}(A' \rightarrow B)$, the following equality holds:

$$R(\Lambda||\Gamma) = R_T[\mathcal{J} \otimes \Lambda(\tilde{\gamma}_{AA'})], \quad (13)$$

where $\mathcal{J}$ is the identity channel.

A proof of a stronger version of this theorem is provided in Appendix A 3. Therefore, $R(\Lambda||\Gamma)$ is equivalent to the output athermality when Λ is locally applied on one part of the thermofield double state $\tilde{\gamma}_{AA'}$. To see that R quantifies athermality preservation (Task II)—and not athermality generation, like R_T —consider the following theorem:

Theorem 3. For a general channel $\Lambda \in \mathcal{O}(A \rightarrow B)$ with $\gamma_A = \gamma_B \equiv \gamma$, the following inequality holds:

$$R(\Lambda||\Gamma) \leq R_T(\tilde{\gamma}_{AA'}) = \text{Tr}\{\gamma^{-1}\} - 1, \quad (14)$$

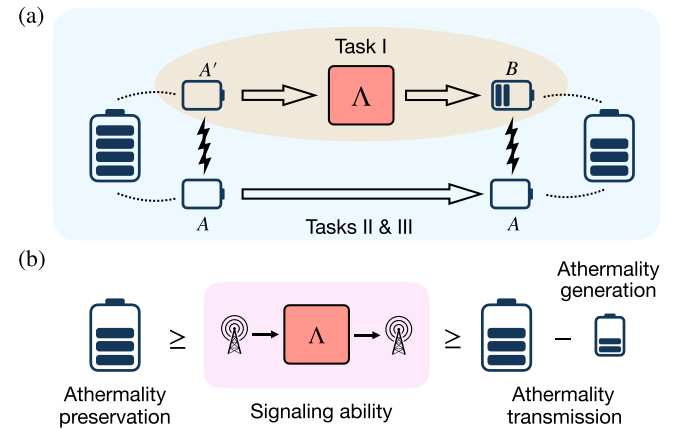


FIG. 2. Main results. (a) Three thermodynamic tasks associated with a quantum channel $\Lambda \in \mathcal{O}(A' \rightarrow B)$ correspond to three quantifiable resources: Task I (athermality generation)—how much athermality can Λ generate from a thermal input? Task II (athermality preservation)—how much input athermality does Λ preserve if A' is part of a pure, locally thermal state? Task III (athermality transmission)—how much of the initial global athermality from correlations between A and A' does Λ transfer? Athermality is denoted by charged batteries and empty batteries indicate thermal states. (b) The signaling ability of the quantum channel Λ is bounded by its thermodynamic capacities (Theorem 4).

where $\text{Tr}\{\gamma^{-1}\}$ denotes the sum over the inverses of the eigenvalues of γ and the equality holds when Λ is unitary.

A proof of a stronger version of this theorem is provided in Appendix A 4. Combined with Theorem 2, this yields a no-go relation:

$$R_T[\mathcal{J} \otimes \Lambda(\tilde{\gamma}_{AA'})] \leq R_T(\tilde{\gamma}_{AA'}), \quad \forall \Lambda \in \mathcal{O}(A' \rightarrow B). \quad (15)$$

That is, no local channel can increase the overall athermality of the thermofield double state $\tilde{\gamma}_{AA'}$, rendering $R(\Lambda||\Gamma)$ a quantifier of athermality *preservation*.

An important property of $\tilde{\gamma}_{AA'}$ is that it is locally thermal in both subsystems with all athermality residing solely in the correlations between A and A' . The transmission of this type of athermality is encapsulated in Task III.

By Theorems 1 and 2, $R(\Lambda||\Gamma)$ quantifies the total athermality in $\tilde{\gamma}_{AA'}$ that is preserved under the action of Λ , while $R_T(\Lambda)$ quantifies the athermality that Λ generates in the local system A' . The difference $R(\Lambda||\Gamma) - R_T(\Lambda)$ thus quantifies the athermality contained in $\tilde{\gamma}_{AA'}$ transmitted via the local channel Λ . We conclude that Task III is measured by $R - R_T$.

IV. THERMODYNAMIC CRITERIA FOR SIGNALING

We now show how the signaling capability of a general quantum channel is limited by its thermodynamic properties.

Theorem 4. Given a general channel $\Lambda \in \mathcal{O}(A \rightarrow B)$,

$$R(\Lambda||\Gamma) \geq R_S(\Lambda) \geq 2(g_{\min}^{(AB)})^2 [R(\Lambda||\Gamma) - R_T(\Lambda)]^2, \quad (16)$$

where $g_{\min}^{(AB)}$ is the smallest population in the energy eigenbasis of the thermal state $\gamma_{AB} \equiv \gamma_A \otimes \gamma_B$.

The proof is provided in Appendix B 1. The upper and lower bounds on $R_S(\Lambda)$ can be viewed as necessary and sufficient conditions for signaling channels, respectively:

$$\text{athermality preservation}(R > 0) \Leftarrow R_S > 0,$$

$$\text{athermality transmission}(R - R_T > 0) \Rightarrow R_S > 0.$$

For GPOs, athermality transmittability is equivalent to athermality preservability. Recall the definition of $P_T(G) := R(G||\Gamma)$ for $G \in \text{GPO}$ [Eq. (11)]. For GPOs, Eq. (16) becomes

$$P_T(G) \geq R_S(G) \geq 2(g_{\min}^{(AB)})^2 P_T(G)^2. \quad (17)$$

That is, nonzero athermality preservability guarantees nonzero signaling ability, and vice versa. According to the bounds, the completely thermalizing channel Γ is the only GPO can never signal, as expected.

Rearranging the lower bound in Eq. (16), we obtain

$$R(\Lambda||\Gamma) \leq R_T(\Lambda) + (\sqrt{2} g_{\min}^{(AB)})^{-1} \sqrt{R_S(\Lambda)}. \quad (18)$$

Equation (18) together with Eq. (10) bounds the joint measure $R(\Lambda||\Gamma)$ from both sides in terms of the two resources $R_T(\Lambda)$ and $R_S(\Lambda)$. In addition, R and R_T are operationally directly connected through a channel simulation task, as we now detail.

V. GPO DILATION OF A QUANTUM CHANNEL

We now show that the joint measure R not only is a generalization of P_T to general quantum channels, but also admits an operational interpretation in terms of P_T . We consider a

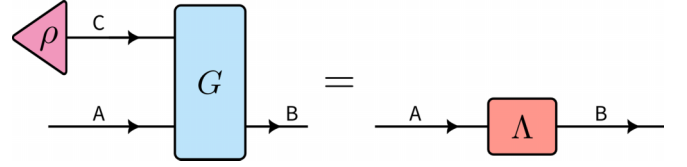


FIG. 3. GPO dilation. A general channel Λ can be dilated by a channel $G \in \text{GPO}(CA \rightarrow B)$ with some state ρ_C .

channel simulation scenario that we dub as *GPO dilation* of quantum channels. A GPO dilation of a general channel $\Lambda \in \mathcal{O}(A \rightarrow B)$, as depicted in Fig. 3, can be viewed as the combination of a GPO $G \in \text{GPO}(CA \rightarrow B)$ and an athermal state $\rho_C \in \mathcal{S}(C)$, such that $\Lambda(\cdot) = G(\rho_C \otimes \cdot)$. We emphasize that in a GPO dilation, the GPO G needs not to be unitary, which is different from the Stinespring's representation of quantum channels [49]. Now, the following theorem illustrates that $R(\Lambda||\Gamma)$ bounds the minimal athermality preservability of the GPO dilation of Λ .

Theorem 5. Given a general quantum channel $\Lambda \in \mathcal{O}(A \rightarrow B)$, we define the set of GPOs that can simulate this channel with some state ρ_C as $\mathcal{O}_{\text{GPO}}^{\text{sim}}(\Lambda)$. The following inequalities hold:

$$R(\Lambda||\Gamma) \leq \min_{G \in \mathcal{O}_{\text{GPO}}^{\text{sim}}(\Lambda)} P_T(G) \leq \max\{1/R_T(\Lambda), R(\Lambda||\Gamma)\}. \quad (19)$$

Especially, whenever $R_T(\Lambda)R(\Lambda||\Gamma) \geq 1$, we have

$$R(\Lambda||\Gamma) = \min_{G \in \mathcal{O}_{\text{GPO}}^{\text{sim}}(\Lambda)} P_T(G). \quad (20)$$

The proof is provided in Appendix C. We remark that when $\Lambda \equiv G \in \text{GPO}(A \rightarrow B)$, although the upper bound diverges, the lower bound becomes tight. Besides, since $R(\Lambda||\Gamma) \geq R_T(\Lambda)$, $R_T(\Lambda)R(\Lambda||\Gamma) \geq 1$, $\forall R_T(\Lambda) \geq 1$. Thus, Eq. (20) holds when $R_T(\Lambda) \notin (0, 1)$.

In Fig. 4, we numerically investigate the typicality of the condition $R_T(\Lambda)R(\Lambda||\Gamma) \geq 1$ for a random qubit channel Λ , where 20 000 qubit channels are randomly sampled via the protocol in Ref. [50], which gives a flat probability measure on the set of quantum channels. We found that this condition is satisfied by 17 643/20 000 $\approx 88.2\%$ channels, implying that $R(\Lambda||\Gamma) = \min_{G \in \mathcal{O}_{\text{GPO}}^{\text{sim}}(\Lambda)} P_T(G)$ universally holds for most channels. However, whether the equality holds even for all channels with $1/R_T > R$ remains as an open question for future investigation.

In accordance with Fig. 4, we report in Table II the channels identified as the most resourceful under the three measures R_T , R_S , and R . The most energetic channels are defined as the channels mapping the thermal state to the highest-energy eigenstate, which has the lowest thermal population when the thermal state is defined at a positive temperature.

Additionally, we note that $R_T(\Lambda)$ can also be interpreted via the GPO dilation of Λ . In Appendix C, we show that

$$R_T(\Lambda) = \min_{\rho_C \in \mathcal{S}_{\text{GPO}}^{\text{sim}}(\Lambda)} R_T(\rho_C), \quad (21)$$

where $\mathcal{S}_{\text{GPO}}^{\text{sim}}(\Lambda)$ is the set of states that can simulate Λ with a GPO using the protocol in Fig. 3. This result can be interpreted

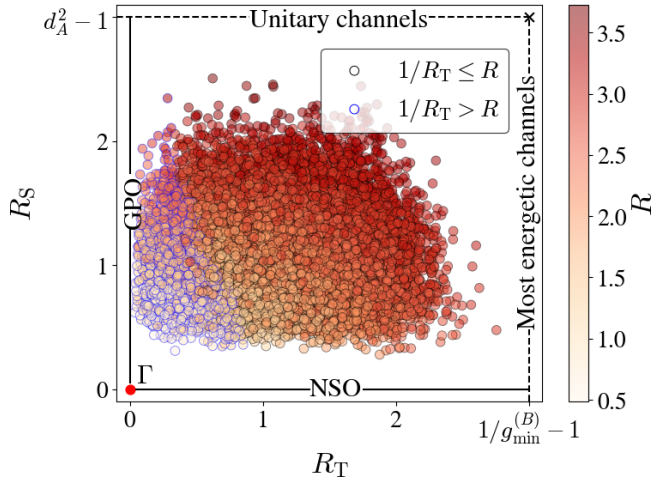


FIG. 4. Resource measures R_T , R_S , and R of random qubit channels. In total, 20 000 qubit channels in $\mathcal{O}(A \rightarrow B)$ with $d_A = d_B = 2$ and $\gamma_A = \gamma_B = 0.75 |0\rangle\langle 0| + 0.25 |1\rangle\langle 1|$ are randomly generated. Channels with $1/R_T \leq (>) R$ are marked by black (blue) edgecolor. The set GPO lies in $[0, d_A^2 - 1]$ on the line of $R_T = 0$, while the set NSO lies in $[0, 1/g_{\min}^{(B)} - 1]$ on the line of $R_S = 0$. The completely thermalizing channel Γ is at the origin. The highest R_T and R_S are achieved by the most energetic channels (sending γ_A to $|1\rangle\langle 1|_B$) and unitary channels, respectively. The symbol $\times$ denotes that there is no intersection between the most energetic channels and unitary channels.

as another manifestation of the equivalence between static and dynamical athermality.

As a consequence, $R_T(\Lambda)$ and $R(\Lambda||\Gamma)$ are connected via the GPO dilation of Λ , yet they stem from distinct types of resources in the dilation: $R_T(\Lambda)$ derives the requisite static athermality $R_T(\rho_C)$, whereas $R(\Lambda||\Gamma)$ reflects the minimal athermality preservability $P_T(G)$.

We now demonstrate the interplay between the three resources R_T , R_S , and R , and showcase the application of the derived bounds via a channel simulation scenario involving the quantum switch [18].

VI. APPLICATION: RESOURCE ANALYSIS OF QUANTUM SWITCH

The quantum switch $\mathcal{S}$ realizes the superposition of causal orders between quantum operations [18]. Intuitively, it transforms two quantum channels and the state of a control qubit into a channel Λ_S , where the input quantum channels are implemented in a superposition of their possible orderings if the control is in a superposition of $|0\rangle$ and $|1\rangle$ (see Fig. 5). Formally, given a general quantum channel $\Omega \in \mathcal{O}(A \rightarrow B)$

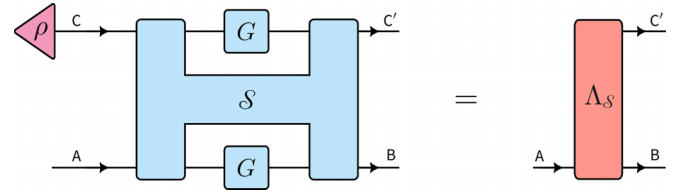


FIG. 5. Quantum switch. A quantum switch $\mathcal{S}$ with a control qubit ρ_C and two GPOs as input can be considered as a channel simulation of Λ_S .

with the Kraus representation $\Omega(\cdot) \equiv \sum_n K_n(\cdot) K_n^\dagger$, the Kraus representation of the channel implemented by the quantum switch $\mathcal{S}[\Omega, \Omega] \in \mathcal{O}(AC \rightarrow BC')$, where C' is a copy of C , can be written as $\mathcal{S}[\Omega, \Omega](\cdot) \equiv \sum_m \sum_n S_{mn}(\cdot) S_{mn}^\dagger$, where the Kraus operator S_{mn} is given by

$$S_{mn} \equiv |0\rangle\langle 0|_C \otimes K_n K_m + |1\rangle\langle 1|_C \otimes K_m K_n. \quad (22)$$

It is known that, with coherent control and two GPOs, the induced channel Λ_S is both signaling and athermal [20]. However, a comprehensive analysis of how the thermodynamic and signaling properties of Λ_S depend on input control and channels remains lacking. Considering the switch as a channel simulation task in Fig. 5, the framework developed above is ideally suited to address this challenge.

Taking two identical signaling GPOs as input channels, we demonstrate that the joint resource R of the induced channel Λ_S is constrained by the supplied resources, while R_T and R_S exhibit a trade-off relation.

Like Ref. [20], we choose the initial state of the control qubit as $\rho_C = |\psi\rangle\langle\psi|_C$, where

$$|\psi\rangle_C \equiv \sqrt{\alpha} |0\rangle_C + e^{i\varphi} \sqrt{1-\alpha} |1\rangle_C, \quad (23)$$

with $\alpha \in [0, 1]$ and $\varphi \in [0, 2\pi]$ (general states ρ_C are considered in Appendix D3). The thermal state of the control qubit is $\gamma_C = \mathbb{1}_C/2$, while all other systems have identical thermal states γ . To demonstrate the key features of how the quantum switch transforms signaling-athermality resources, we consider the following signaling GPO:

$$G \equiv s\Gamma + (1-s)\mathcal{J}, \quad (24)$$

where $\mathcal{J}$ is the identity channel and $s \in [0, 1]$ is the thermalizing strength: At $s = 0$, G acts trivially as the identity channel (i.e., it cannot thermalize any athermal input), while at $s = 1$, G is completely thermalizing (i.e., it thermalizes any input state). Define $\Lambda_S(\cdot) \equiv \mathcal{S}[G, G](\rho_C \otimes \cdot) \in \mathcal{O}(A \rightarrow BC')$. We can then prove the following:

TABLE II. Most resourceful channels in $\mathcal{O}(A \rightarrow B)$ with $d_A = d_B$ according to the three measures.

Resources	Symbols	Most resourceful channels	Highest values
Athermality	R_T	Most energetic channels	$1/g_{\min}^{(B)} - 1$ (Theorems A3 and A1)
Signaling	R_S	Unitary channels	$d_A^2 - 1$ (Ref. [42])
Joint resource	R	Unitary channels	$\text{Tr}\{\gamma_B^{-1}\} - 1$ (Theorem A5)

Theorem 6. Given the control state $\rho_C = |\psi\rangle\langle\psi|_C$ in Eq. (23) and two input channels G defined as in Eq. (24), the following bounds hold:

$$R(\Lambda_S||\Gamma) \leq R_T(\rho_C) + 2P_T^2(G)/P_T(J), \quad (25)$$

$$R(\Lambda_S||\Gamma) \geq R_S(\Lambda_S) \geq (g_{\min}^4/2)[R(\Lambda_S||\Gamma) - R_T(\Lambda_S)]^2, \quad (26)$$

where $R_T(\rho_C) = 1$, $\forall \alpha \in [0, 1]$, $\varphi \in [0, 2\pi]$ and

$$R_T(\Lambda_S) = \sqrt{1 - 4(1 - g_{\max}^2)[2 - (1 - g_{\max}^2)s^2]s^2\alpha(1 - \alpha)}, \quad (27)$$

for $\alpha, s \in [0, 1]$, $\varphi \in [0, 2\pi]$, and $g_{\min(\max)}$ is the minimal (maximal) population in the energy eigenbasis of γ . The upper bound of $R(\Lambda_S||\Gamma)$ is saturated for either α or $s = 0, 1$.

The proof is provided in Appendix D. The upper bound [Eq. (25)] shows explicitly that the joint resource R , and thus the signaling robustness R_S , of the induced channel Λ_S is bounded by the thermodynamic resource input into the switch, i.e., the athermality ρ_C and the athermality preservability of the GPO G (Fig. 5). This coincides with our intuition, because to activate the switch, the control qubit should be coherent, and therefore athermal, and the athermality preserving G with nonvanishing signaling ability [Eq. (17)] should contribute to the signaling ability of the induced channel Λ_S . Therefore, $R(\Lambda_S||\Gamma)$ is the correct characterization of the resource converted by the switch from input to output. Notably, when $s = 0$ (1), $R(\Lambda_S||\Gamma) = R_T(\rho_C) + 2P_T(J) (R_T(\rho_C))$ —precisely the resource inputs. On the other hand, the lower bound [Eq. (26)] is a direct application of Theorem 4 and can be understood as a trade-off between $R_S(\Lambda_S)$ and $R_T(\Lambda_S)$ (see below). In addition, Theorem 6 provides an analytical expression for $R_T(\Lambda_S)$. Consequently, all three types of resources— R , R_S , and R_T —are either bounded or expressed analytically, providing a complete resource analysis of the quantum switch.

To show the trade-off between R_T and R_S more clearly, we now consider the case when both input GPOs $G = \Gamma$ (i.e., $s = 1$), such that ρ_C constitutes the only resource input. This case is a canonical example in previous demonstrations of the communication/thermodynamic capabilities of quantum switches with NSOs/GPOs [19,29]. By Theorem 6, we have $R(\Lambda_S||\Gamma) = R_T(\rho_C)$. Equation (26) reduces to the following form:

$$R_T(\rho_C) \geq R_S(\Lambda_S) \geq (g_{\min}^4/2)[R_T(\rho_C) - R_T(\Lambda_S)]^2, \quad (28)$$

for all $\alpha \in [0, 1]$. Consistently, the resource gains, $R_S(\Lambda_S)$ and $R_T(\Lambda_S)$, are both upper-bounded by $R_T(\rho_C)$. The lower bound in Eq. (28) hence indicates a trade-off between $R_S(\Lambda_S)$ and $R_T(\Lambda_S)$ constrained by $R_T(\rho_C)$. Since the channel $S[G, G]$ is Gibbs-preserving [20], the control state ρ_C and the channel $S[G, G]$ can be regarded as a GPO dilation of Λ_S . Recall that by Eq. (21), optimally, simulating Λ_S only requires the amount of athermality as same as $R_T(\Lambda_S)$. The excess term $R_T(\rho_C) - R_T(\Lambda_S)$ thus is the *athermality waste* in the channel simulation realized by the switch. The lower bound in Eq. (28) shows that the signaling ability of the induced channel Λ_S is actually activated by a nonvanishing athermality waste, rather than the athermality of ρ_C .

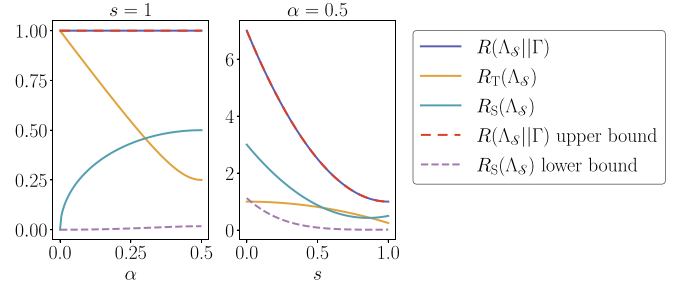


FIG. 6. Resource analysis. R , R_T , and R_S of the induced channel Λ_S with their bounds/analytical expression given in Theorem 6 (for qubit systems A and B with $\gamma_A = \gamma_B = \mathbb{1}/2$).

In Fig. 6, we exemplify Theorem 6 by considering qubits A and B with $\gamma_A = \gamma_B = \mathbb{1}/2$, and plotting the resources of Λ_S as functions of α and s , respectively. We find that the upper bound of $R(\Lambda_S||\Gamma)$ [Eq. (25)] remains tight in both cases while the lower bound of $R_S(\Lambda_S)$ [Eq. (26)] is only relatively tighter for more signaling input GPOs (smaller s). Despite the potential looseness of the lower bound, it provides an interpretation of the phenomenon that increased $R_S(\Lambda_S)$ comes at the expense of $R_T(\Lambda_S)$, aligning with their trends in both plots.

Finally, we remark that the above resource analysis is not specific to quantum switches. In Appendix E, we provide a similar analysis to coherent control of quantum channels, where instead of the order of two input channels, the implementation of one of the two channels is conditioned on the state of a control qubit [51]. In this setup, R_T and R_S are found to exhibit a similar trade-off.

VII. DISCUSSION AND CONCLUSION

In this work, we have provided a full-fledged characterization of the signaling and thermodynamic properties of quantum channels and demonstrated how they mutually bound and condition each other. Our main results are the precise identification of the thermodynamic resources for the generation, preservation, and transmission of athermality (Theorems 1–3), the characterization of a quantum channel's signaling ability in terms of its capacity for athermality preservation and transmission (Theorem 4), and an operational interpretation of the resulting resource measures in terms of channel dilatability by a GPO (Theorem 5). We finally used our results for an encompassing characterization of resources in the quantum switch (Theorem 6).

We note the relation of our findings to those of Ref. [12], which demonstrates that, in the asymptotic limit, information and energy transmission in specific incoherent setups are equivalent. Here, in contrast, we do not restrict ourselves to the classical case—in particular, the considered states $\tilde{\gamma}_{AA'}$ are entangled. On the other hand, the thermodynamic resource we analyzed is athermality, a joint measure of work and coherence, while Ref. [12] investigated extractable work. Finally, according to our measure of athermality transmittability, a signaling channel ($R_S > 0$) may have zero athermality transmittability ($R - R_T = 0$) (see Appendix B 2 for examples).

Using the trade-off relations of athermality and signaling, we demonstrated the interplay of these properties for the paradigmatic example of the quantum switch. This resolves the open question posed in Ref. [20], showing that only limited activation of information capacity is possible when the switch is composed with GPOs, as the output resources are bounded by those at the input.

A natural next step is to analyze how the bounds we provide behave when thermal operations or related subsets of GPOs [40,52–58] are considered as thermodynamically resourceless channels instead of GPOs. Since these channels are all contained within GPOs, our core bounds Eq. (16) still hold—albeit possibly in a looser sense. Another promising research direction is the study of general quantum supermaps as a complete framework for manipulating dynamical resources and their conversion of different types of resources [59–65]. Concretely, one could ask: What kind of supermaps do not increase the total resources $R_T + R_S$ but can exchange them? Noticing that $2R(\Lambda||\Gamma) \geq R_T(\Lambda) + R_S(\Lambda)$, one may think Γ -preserving supermaps are the direct answer, but we find that there exist counterexamples where Γ -preserving supermaps increase the sum $R_T + R_S$.

Having combined signaling and athermality, another possible direction is to study other resource theories where two distinct free sets have an intersection. In this case, the results from Ref. [66] may be applied to resource measures defined via distance metrics [67].

ACKNOWLEDGMENTS

This publication has emanated from research conducted with the financial support of Taighde Éireann—Research Ireland under Grant No. IRCLA/2022/3922. This publication was made possible through the support of Grant No. 62423 from the John Templeton Foundation. Y.L. was supported by China Scholarship Council (Grant No. 202408060137). S.M. acknowledges funding from the European Union’s Horizon Europe research and innovation program under the Marie Skłodowska-Curie Grant Agreement No. 101068332. The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: RESOURCE MEASURES

In this work, we focus on two quantum resources: athermality and signaling, both of which can be quantified in terms of robustness monotones [33,34,36]. In Table III, we summarize the specific resource quantifiers considered in this paper. We remark that all these measures can be expressed as semidefinite programs (SDPs) [68]. As in the main body, we use $\mathcal{S}(X)$ to denote the set of quantum states of system X , and $\mathcal{O}(X \rightarrow Y)$ to denote the set of quantum operations (completely positive and trace-preserving maps) between the input system X and the output system Y . X' denotes a copy of system X with the same thermal state γ_X . The dimension of the system X is denoted as d_X .

The standard forms of the robustness of athermality R_T for states and channels are given by

$$R_T(\rho_A) := \min \left\{ s \left| \frac{\rho_A + s\sigma_A}{1+s} = \gamma_A, \sigma_A \in \mathcal{S}(A) \right. \right\}, \quad (\text{A1})$$

$$R_T(\Lambda) := \min \left\{ s \left| \frac{\Lambda + s\Omega}{1+s} = G, \Omega \in \mathcal{O}(A \rightarrow B), \right. \right. \\ \left. \left. G \in \text{GPO}(A \rightarrow B) \right\}, \quad (\text{A2})$$

where $\text{GPO}(A \rightarrow B) := \{G | G(\gamma_A) = \gamma_B, G \in \mathcal{O}(A \rightarrow B)\}$ is the set of Gibbs-preserving operations (GPOs) from system A to system B . We use R_T for both static and dynamical athermality because dynamical athermality can be reduced to static athermality (Theorem A3). The type of athermality will remain unambiguous from the argument of R_T . Following a similar argument to Sec. III.A in Ref. [34], we can rewrite R_T as

$$1 + R_T(\rho_A) := \min \{ \lambda | \rho_A \leq \lambda \gamma_A \}, \quad (\text{A3})$$

$$1 + R_T(\Lambda) := \min \{ \lambda | \Lambda \leq \lambda G, G \in \text{GPO}(A \rightarrow B) \}, \quad (\text{A4})$$

where “ $X \leq Y$ ” means that $Y - X$ is positive semidefinite for states and completely positive (CP) for channels. Throughout, we consider full-rank thermal states γ , thus avoiding divergence of R_T .

The robustness of signaling R_S for channels is defined as

$$1 + R_S(\Lambda) := \min \{ \lambda | \Lambda \leq \lambda \Phi, \Phi \in \text{NSO}(A \rightarrow B) \}, \quad (\text{A5})$$

where $\text{NSO}(A \rightarrow B) := \{ \Phi_\sigma \equiv \text{Tr}\{\cdot\}\sigma | \sigma \in \mathcal{S}(B) \}$ is the set of nonsignalling operations (NSOs) between systems A and B .

Given the input system A and the output system B , the only intersection between $\text{GPO}(A \rightarrow B)$ and $\text{NSO}(A \rightarrow B)$ is the completely thermalizing channel: $\Gamma(\cdot) \equiv \text{Tr}\{\cdot\}\gamma_B$. Note, however, that Γ is neither an extreme point in NSO nor in GPO . For NSO , this is straightforward. For GPO , see Ref. [44]

TABLE III. Resource measures considered in this paper.

Resources	Symbols	Types	Definitions
Athermality	R_T	Static/dynamical	Equations (A3) and (A4)
Signaling	R_S	Dynamical	Equation (A5)
Joint resource	R	Dynamical	Equation (A6)
Athermality preservability for GPOs ^a	P_T	Dynamical	Equation (A7)

^aGPOs: Gibbs-preserving operations.

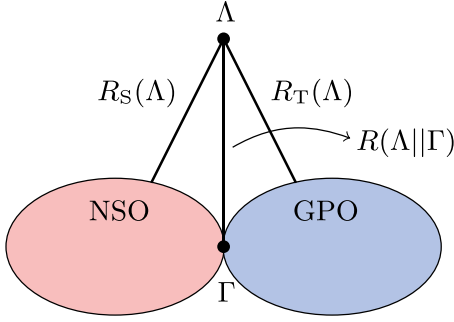


FIG. 7. Geometric sketch of the three resource monotones: $R_T(\Lambda)$, $R_S(\Lambda)$, and $R(\Lambda||\Gamma)$.

for the conditions on extreme points in GPO. We thus define a joint robustness measure that collectively gauges signaling and athermality of a general channel:

$$1 + R(\Lambda||\Gamma) := \min\{\lambda | \Lambda \leq \lambda \Gamma\}. \quad (\text{A6})$$

The relation among the three measures R_T , R_S , and R is informally represented in Fig. 7. For $\Phi \in \text{NSO}(A \rightarrow B)$, $R_S(\Phi) = 0$ and $R(\Phi||\Gamma) = R_T(\Phi)$, while for $G \in \text{GPO}(A \rightarrow B)$, $R(G||\Gamma) \geq R_S(G)$ in general. For GPOs, the joint measure R reduces to the robustness of athermality preservability mentioned in Ref. [47]:

$$P_T(G) := R(G||\Gamma). \quad (\text{A7})$$

1. Basic properties

The dual SDP of R_T for states is given as [69]

$$1 + R_T(\rho_A) = \max\{\text{Tr}\{S\rho_A\} | S \geq 0, \text{Tr}\{S\gamma_A\} = 1\}, \quad (\text{A8})$$

for which strong duality holds; i.e., primal and dual SDPs yield the same optimal values [68]. As a resource monotone, R_T satisfies desired properties. Here, we show them explicitly, by following similar proofs as Ref. [35], where they are for the robustness of coherence.

- (1) (Faithfulness) $R_T(\rho) = 0 \Leftrightarrow \rho = \gamma$.

Proof. Direct result from Eq. (A3). ■

- (2) (Monotonicity) $R_T(\rho_A) \geq R_T[G(\rho_A)]$, $\forall G \in \text{GPO}(A \rightarrow B)$.

Proof. Let $\lambda^* \equiv 1 + R_T(\rho_A)$ be the optimal value that achieves the minimum. Since $G \in \text{GPO}(A \rightarrow B)$ is completely positive, $\lambda^*\gamma_A - \rho_A \geq 0 \Rightarrow G(\lambda^*\gamma_A - \rho_A) = \lambda^*\gamma_B - G(\rho_A) \geq 0$. Consequently, λ^* is in the feasible set of the minimization defining $R_T[G(\rho_A)]$, such that $\lambda^* \geq 1 + R_T[G(\rho_A)]$. Hence, $R_T(\rho_A) \geq R_T[G(\rho_A)]$. ■

- (3) (Convexity) $R_T(\sum_i p_i \rho_i) \leq \sum_i p_i R_T(\rho_i)$, for $\sum_i p_i = 1$ and $p_i \in [0, 1]$, $\forall i$.

Proof. Let $\lambda_i^* \equiv 1 + R_T(\rho_i)$ be the optimal value for each i . $\lambda_i^*\gamma - \rho_i \geq 0, \forall i \Rightarrow \sum_i p_i \lambda_i^*\gamma - \sum_i p_i \rho_i \geq 0$. Consequently, $\sum_i p_i \lambda_i^*$ is a feasible for the minimization defining $R_T(\sum_i p_i \rho_i)$, such that $\sum_i p_i \lambda_i^* \geq 1 + R_T(\sum_i p_i \rho_i)$. Hence, $R_T(\sum_i p_i \rho_i) \leq \sum_i p_i R_T(\rho_i)$. ■

- (4) (Multiplicity) $1 + R_T(\rho_A \otimes \rho_B) = [1 + R_T(\rho_A)][1 + R_T(\rho_B)]$.

Proof. Let λ_X^* be the optimal value such that $1 + R_T(\rho_X) \equiv \lambda_X^*$, where $X = A, B$. $\lambda_X^*\gamma_X - \rho_X \geq 0, \forall X \Rightarrow \lambda_A^*\lambda_B^*\gamma_A \otimes \gamma_B \geq \rho_A \otimes \rho_B$. $\lambda_A^*\lambda_B^*$ is feasible

for the minimization defining $R_T(\rho_A \otimes \rho_B)$, such that $\lambda_A^*\lambda_B^* = [1 + R_T(\rho_A)][1 + R_T(\rho_B)] \geq 1 + R_T(\rho_A \otimes \rho_B)$.

Let S_X^* be the optimal operator of the dual SDP [Eq. (A8)] for $X = A, B$, such that $1 + R_T(\rho_X) = \text{Tr}\{S_X^*\rho_X\}$. Now $S \equiv S_A^* \otimes S_B^*$ is feasible for the dual SDP of $R_T(\rho_A \otimes \rho_B)$, so that $\text{Tr}\{S\rho_A \otimes \rho_B\} = \text{Tr}\{S_A^*\rho_A\}\text{Tr}\{S_B^*\rho_B\} = [1 + R_T(\rho_A)][1 + R_T(\rho_B)] \leq 1 + R_T(\rho_A \otimes \rho_B)$. Hence, $1 + R_T(\rho_A \otimes \rho_B) = [1 + R_T(\rho_A)][1 + R_T(\rho_B)]$. ■

By Theorem A3 (see below), $R_T(\Lambda) = R_T[\Lambda(\gamma_A)]$, $\forall \Lambda \in \mathcal{O}(A \rightarrow B)$. Thus, R_T , as a measure of dynamical resource, satisfies the following properties:

- (1) (Faithfulness) $R_T(\Lambda) = 0 \Leftrightarrow \Lambda$ is a GPO.

- (2) (Monotonicity) $R_T(\Lambda) \geq R_T(G \circ \Lambda \circ G')$, $\forall G$ and G' are GPOs.

- (3) (Convexity) $R_T(\sum_i p_i \Lambda_i) \leq \sum_i p_i R_T(\Lambda_i)$, for $\sum_i p_i = 1$ and $p_i \in [0, 1]$, $\forall i$.

- (4) (Multiplicity) $1 + R_T(\Lambda \otimes \Omega) = [1 + R_T(\Lambda)][1 + R_T(\Omega)]$.

In addition, the joint measure R defined in Eq. (A6) can be written in terms of the Choi states of channels [70]:

$$1 + R(\Lambda||\Gamma) := \min\{\lambda | J_\Lambda \leq \lambda J_\Gamma\}, \quad (\text{A9})$$

where $J_X \equiv \mathcal{I} \otimes X(|\phi^+\rangle\langle\phi^+|)$ with $|\phi^+\rangle \equiv \sum_{i=0}^{d_A-1} |i\rangle \otimes |i\rangle$ is defined as the Choi state of the channel $X \in \mathcal{O}(A \rightarrow B)$. Comparing with Eq. (A3), R can be regarded as the R_T of J_Λ with respect to the “thermal state” J_Γ . Therefore, R , as well as P_T defined in Eq. (A7), inherits the same properties as R_T :

- (1) (Faithfulness) $R(\Lambda||\Gamma) = 0 \Leftrightarrow \Lambda = \Gamma$.

- (2) (Monotonicity) $R(\Lambda||\Gamma) \geq R(G \circ \Lambda \circ G'||\Gamma)$, $\forall G$ and G' are GPOs.

- (3) (Convexity) $R(\sum_i p_i \Lambda_i||\Gamma) \leq \sum_i p_i R(\Lambda_i||\Gamma)$, for $\sum_i p_i = 1$ and $p_i \in [0, 1]$, $\forall i$.

- (4) (Multiplicity) $1 + R(\Lambda \otimes \Omega||\Gamma) = [1 + R(\Lambda||\Gamma)][1 + R(\Omega||\Gamma)]$.

To show more properties of R_T , we prove the following lemma first.

Lemma A1. The optimal S^* that maximizes the dual SDP of the R_T of states [Eq. (A8)] can always be chosen to be rank 1.

Proof. Since the Gibbs state γ is full rank by assumption, we can define a bijective map, $X \equiv \gamma^{1/2} S \gamma^{1/2}$, such that Eq. (A8) can be rewritten as

$$1 + R_T(\rho) = \max_X \{\text{Tr}\{\tilde{\rho} X\} | X \geq 0, \text{Tr}\{X\} = 1\}, \quad (\text{A10})$$

where we denote $\tilde{\rho} \equiv \gamma^{-1/2} \rho \gamma^{-1/2}$. The feasible set $\{X | X \geq 0, \text{Tr}\{X\} = 1\}$ is the set of quantum states, meaning that any X in the set can be written as a convex combination of rank 1 states. Let $X^* = \sum_i p_i |\phi_i\rangle\langle\phi_i|$ for $\sum_i p_i = 1$ with $p_i \in [0, 1]$, $\forall i$, be the optimal operator, such that

$$\begin{aligned} 1 + R_T(\rho) &= \text{Tr}\{\tilde{\rho} X^*\} \\ &= \sum_i p_i \text{Tr}\{\tilde{\rho} |\phi_i\rangle\langle\phi_i|\} \leq \max_i \text{Tr}\{\tilde{\rho} |\phi_i\rangle\langle\phi_i|\}. \end{aligned} \quad (\text{A11})$$

However, by Eq. (A10), $1 + R_T(\rho) \geq \max_i \text{Tr}\{\tilde{\rho} |\phi_i\rangle\langle\phi_i|\}$. We thus have $1 + R_T(\rho) = \max_i \text{Tr}\{\tilde{\rho} |\phi_i\rangle\langle\phi_i|\}$ and X^* can be chosen to be rank 1. Note that X^* does not have to be rank

1 since for some $\tilde{\rho}$, $\text{Tr}\{\tilde{\rho}|\phi_i\rangle\langle\phi_i|\}$ can be independent of i ; i.e., any convex combination of $\{|\phi_i\rangle\langle\phi_i|\}_i$ is an optimal X^* .

The optimal S^* in Eq. (A8) is given by $S^* \equiv \gamma^{-1/2} X^* \gamma^{-1/2}$. Since X^* can always be chosen to be rank 1, such that $X^* \equiv |\phi\rangle\langle\phi|$, $S^* \equiv \gamma^{-1/2} X^* \gamma^{-1/2} = \gamma^{-1/2} |\phi\rangle\langle\phi| \gamma^{-1/2}$ is also rank 1. ■

We now find the state with highest R_T , i.e., the most athermal state.

Theorem A1 (The most athermal state). Given a nondegenerate, full-rank Gibbs state $\gamma \equiv \sum_{i=0}^{d-1} g_i |i\rangle\langle i|$ with the populations $g_{d-1} < g_{d-2} < \dots < g_0$, the unique state with the maximal R_T is the state: $\rho^* \equiv |d-1\rangle\langle d-1|$, with $1 + R_T(\rho^*) = g_{d-1}^{-1}$.

Proof. We define $\lambda(\rho) \equiv 1 + R_T(\rho)$. From the dual form of $1 + R_T(\rho)$ [Eq. (A8)] and Lemma A1, we have

$$\lambda(\rho) = \max_{|\phi\rangle} \left\{ \text{Tr}\{\rho S\} \middle| S = \frac{|\phi\rangle\langle\phi|}{\text{Tr}\{\gamma|\phi\rangle\langle\phi|\}} \right\} = \max_{|\phi\rangle} \frac{\langle\phi|\rho|\phi\rangle}{\langle\phi|\gamma|\phi\rangle}. \quad (\text{A12})$$

We denote the value of λ corresponding to the maximal $1 + R_T$ as $\lambda_{\max}$, such that

$$\begin{aligned} \lambda_{\max} &\equiv \max_{\rho} \lambda(\rho) = \max_{\rho} \max_{|\phi\rangle} \frac{\langle\phi|\rho|\phi\rangle}{\langle\phi|\gamma|\phi\rangle} \\ &\leq \frac{\max_{\rho} \max_{|\phi\rangle} \langle\phi|\rho|\phi\rangle}{\min_{|\phi\rangle} \langle\phi|\gamma|\phi\rangle}. \end{aligned} \quad (\text{A13})$$

It is clear that $\max_{\rho} \max_{|\phi\rangle} \langle\phi|\rho|\phi\rangle = 1$. For the denominator, since $\langle\phi|\gamma - g_{d-1}\mathbb{1}|\phi\rangle \geq 0, \forall |\phi\rangle$ and the equality holds as $|\phi\rangle = |d-1\rangle$, we have $\min_{|\phi\rangle} \langle\phi|\gamma|\phi\rangle = g_{d-1}$. Therefore,

$$\lambda_{\max} = \max_{\rho} \max_{|\phi\rangle} \frac{\langle\phi|\rho|\phi\rangle}{\langle\phi|\gamma|\phi\rangle} \leq \frac{1}{g_{d-1}}. \quad (\text{A14})$$

Equality holds for $\rho = \rho^* \equiv |d-1\rangle\langle d-1|$ and $|\phi\rangle = |d-1\rangle$; i.e., ρ^* is the most athermal state with $1 + R_T(\rho^*) = \lambda_{\max} = g_{d-1}^{-1}$.

Now we prove $\rho = |d-1\rangle\langle d-1|$ is the only most athermal state. Due to the convexity of R_T , the most athermal state can only be mixed if it is a convex combination of pure states with the same R_T . Therefore, to prove that ρ is unique, it is enough to assume that there is another pure state $\sigma \equiv |\psi\rangle\langle\psi|$ with $\lambda(\sigma) = g_{d-1}^{-1}$ and to show that σ does not exist. Since $\sigma \neq \rho$, $|\psi\rangle \neq |d-1\rangle$. Let $|\phi'\rangle$ be the optimal state that gives

$$\frac{\langle\phi'|\sigma|\phi'\rangle}{\langle\phi'|\gamma|\phi'\rangle} = \frac{|\langle\phi'|\psi\rangle|^2}{\langle\phi'|\gamma|\phi'\rangle} = \frac{1}{g_{d-1}}. \quad (\text{A15})$$

We note that $|\phi'\rangle \neq |d-1\rangle$, since $\frac{|\langle d-1|\psi\rangle|^2}{\langle d-1|\gamma|d-1\rangle} = \frac{|\langle d-1|\psi\rangle|^2}{g_{d-1}} < \frac{1}{g_{d-1}}$. However, since $|\phi'\rangle \neq |d-1\rangle$, we have $\langle\phi'|\gamma|\phi'\rangle > g_{d-1}$ ($|d-1\rangle$ is the only state satisfying $\langle d-1|\gamma|d-1\rangle = g_{d-1}$ because γ is nondegenerate). Equation (A15) therefore requires $|\langle\phi'|\psi\rangle| > 1$. Because such a state σ cannot exist, $\rho = |d-1\rangle\langle d-1|$ is the only state with the maximal R_T , for which $1 + R_T(\rho) = g_{d-1}^{-1}$. ■

We remark that when the Gibbs state γ is defined at positive temperatures, the most athermal state $\rho^* \equiv |d-1\rangle\langle d-1|$ is the most energetic state corresponding to the highest energy level. However, some systems can equilibrate at negative temperatures (see, e.g., Ref. [71]), in which the most athermal state ρ^* will be the ground state.

We further show that R_T can be related with commonly used distances, which is given by a continuity property. In the context of resource theories, the continuity of resource monotone refers to the property of the monotone guaranteeing that any two states that are close in some chosen distance are close in terms of their resources according to this monotone. The continuity of robustness measures has been studied in Ref. [72] and the continuity of R_T has been explicitly proven in Lemma F3 in Ref. [47]. Here, we provide a self-contained proof for the continuity of R_T , which improves the bound of Ref. [47] by a factor $1/2$.

Theorem A2 (Continuity of R_T). For any two states ρ, σ and a given Gibbs state γ , the following inequality holds:

$$|R_T(\rho) - R_T(\sigma)| \leq \frac{1}{2g_{\min}} \|\rho - \sigma\|_1, \quad (\text{A16})$$

where $\|A\|_1 := \text{Tr}\{\sqrt{A^\dagger A}\}$ is the trace norm and $g_{\min}(\max)$ is the minimal (maximal) population of energy eigenbasis of γ .

Proof. Consider the dual SDP of R_T of states [Eq. (A8)]. We denote the optimal operators as S_ρ^* and S_σ^* , such that

$$\text{Tr}\{\rho S_\rho^*\} = 1 + R_T(\rho), \quad (\text{A17})$$

$$\text{Tr}\{\sigma S_\sigma^*\} = 1 + R_T(\sigma). \quad (\text{A18})$$

Without loss of generality, we assume that $R_T(\rho) \geq R_T(\sigma)$, i.e., $\text{Tr}\{\rho S_\rho^*\} \geq \text{Tr}\{\sigma S_\sigma^*\}$. We then have

$$\text{Tr}\{\rho S_\rho^*\} - \text{Tr}\{\sigma S_\sigma^*\} \leq \text{Tr}\{(\rho - \sigma) S_\rho^*\}. \quad (\text{A19})$$

By assumption, both sides of the above inequality are non-negative. We can take the absolute values of both sides and obtain

$$\begin{aligned} |\text{Tr}\{\rho S_\rho^*\} - \text{Tr}\{\sigma S_\sigma^*\}| &\leq |\text{Tr}\{(\rho - \sigma) S_\rho^*\}| \\ &= |\langle\psi_\rho|(\rho - \sigma)|\psi_\rho\rangle| \cdot \|S_\rho^*\|_\infty \\ &\leq |\langle\psi_\rho|(\rho - \sigma)|\psi_\rho\rangle| \cdot g_{\min}^{-1} \\ &\leq \frac{1}{2} \|\rho - \sigma\|_1 \cdot g_{\min}^{-1}. \end{aligned} \quad (\text{A20})$$

Here, in the second line, we used that, according to Lemma A1, S_ρ^* can be chosen as rank 1, i.e., $S_\rho^* \equiv \|S_\rho^*\|_\infty |\psi_\rho\rangle\langle\psi_\rho|$, where $|\psi_\rho\rangle\langle\psi_\rho|$ is a (normalized) state and $\|X\|_\infty$ is the operator norm of X that equals to the largest eigenvalue of X . In the third line, according to the dual SDP of R_T [Eq. (A8)], $\text{Tr}\{\gamma S_\rho^*\} = \|S_\rho^*\|_\infty \langle\psi_\rho|\gamma|\psi_\rho\rangle = 1$. Since $\langle\psi_\rho|\gamma|\psi_\rho\rangle \geq g_{\min}$, we have $\|S_\rho^*\|_\infty \leq g_{\min}^{-1}$. In the forth line, we used the fact that

$$\begin{aligned} |\langle\psi_\rho|(\rho - \sigma)|\psi_\rho\rangle| &\leq \max_k |\mu_k| \leq \sum_{k \in \{\ell|\mu_\ell \geq 0\}} \mu_k \\ &= \sum_{k \in \{\ell|\mu_\ell \leq 0\}} |\mu_k| = \frac{1}{2} \|\rho - \sigma\|_1, \end{aligned} \quad (\text{A21})$$

where μ_k is the k th eigenvalue of $(\rho - \sigma)$ satisfying $\sum_k \mu_k = 0$ due to $\text{Tr}\{\rho - \sigma\} = 0$. Referring to Eqs. (A17) and (A18), we obtain $|R_T(\rho) - R_T(\sigma)| \leq (g_{\min}^{-1}/2) \|\rho - \sigma\|_1$. ■

2. Proof of Theorem 1

In this subsection, we prove Theorem 1, which states that the robustness of athermality R_T of a general quantum channel

Λ is equal to the robustness of athermality $R_T[\Lambda(\gamma)]$ of its output state with the input state being the Gibbs state. In other words, the dynamical athermality of a channel is equivalent to its static athermality generating power:

Theorem A3 [Equivalence between dynamical and static R_T (Theorem 1)]. For a general channel $\Lambda \in \mathcal{O}(A \rightarrow B)$, the following equality holds:

$$R_T(\Lambda) = R_T[\Lambda(\gamma_A)]. \quad (\text{A22})$$

Proof. We first show that $R_T(\Lambda) \geq R_T[\Lambda(\gamma_A)]$. Let $\lambda^* \equiv 1 + R_T(\Lambda)$ and $G^* \in \text{GPO}(A \rightarrow B)$ be the optimal variable and GPO in the definition of $R_T(\Lambda)$ [Eq. (A4)], respectively. We have

$$\lambda^* G^* - \Lambda \geq 0 \Rightarrow (\lambda^* G^* - \Lambda)(\gamma_A) = \lambda^* \gamma_B - \Lambda(\gamma_A) \geq 0. \quad (\text{A23})$$

Therefore, λ^* is a feasible variable for the SDP of $R_T[\Lambda(\gamma_A)]$ in Eq. (A3), which gives us $1 + R_T(\Lambda) \equiv \lambda^* \geq 1 + R_T[\Lambda(\gamma_A)]$, i.e., $R_T(\Lambda) \geq R_T[\Lambda(\gamma_A)]$.

To prove that $R_T(\Lambda) \leq R_T[\Lambda(\gamma_A)]$, we consider the standard form of $R_T[\Lambda(\gamma_A)]$ [Eq. (A1)] and denote the optimal state as σ_B^* , which satisfies

$$\frac{1}{1 + R_T[\Lambda(\gamma_A)]} \Lambda(\gamma_A) + \frac{R_T[\Lambda(\gamma_A)]}{1 + R_T[\Lambda(\gamma_A)]} \sigma_B^* = \gamma_B. \quad (\text{A24})$$

Define the NSO $\Phi_{\sigma_B^*}(\cdot) := \text{Tr}\{\cdot\} \sigma_B^* \in \text{NSO}(A \rightarrow B)$. By Eq. (A24), we have

$$\frac{1}{1 + R_T[\Lambda(\gamma_A)]} \Lambda + \frac{R_T[\Lambda(\gamma_A)]}{1 + R_T[\Lambda(\gamma_A)]} \Phi_{\sigma_B^*} \in \text{GPO}(A \rightarrow B). \quad (\text{A25})$$

According to the standard form of $R_T(\Lambda)$ in Eq. (A2), we obtain $R_T(\Lambda) \leq R_T[\Lambda(\gamma_A)]$. Hence, Eq. (A22) holds. ■

3. Proof of Theorem 2

In this subsection, we prove Theorem 2, which states that the joint resource measure R of a general quantum channel is equal to the R_T of its output state when it is locally applied on the thermofield double state $\tilde{\gamma}_{AA'}$ [48] defined as $\tilde{\gamma}_{AA'} \equiv |\tilde{\gamma}\rangle\langle\tilde{\gamma}|_{AA'}$ with $|\tilde{\gamma}\rangle_{AA'} \equiv \sum_i \sqrt{g_i} |i\rangle_A \otimes |i\rangle_{A'}$ where g_i is the population of γ_A of the i th energy eigenstate $|i\rangle_A$, satisfying $\text{Tr}_{[A,A'] \setminus X} \{\tilde{\gamma}_{AA'}\} = \gamma_X$ for $X \in \{A, A'\}$. Specifically, we have the following more general theorem:

Theorem A4 [Equivalence between the dynamical resource R and static R_T]. For a general channel $\Lambda \in \mathcal{O}(A' \rightarrow B)$, the following equality holds:

$$R(\Lambda||\Gamma) = R_T[\mathcal{J} \otimes \Lambda(\tau_{AA'})], \quad (\text{A26})$$

where A' is a copy of system A and $\tau_{AA'}$ is a general purification of γ_A on system AA' satisfying $\text{Tr}_{A'}\{\tau_{AA'}\} = \gamma_A$.

Proof. Consider the Choi operator J_Λ and note that the Choi operator of Γ is $J_\Gamma \equiv \mathbb{1}_A \otimes \gamma_B$. We have

$$\begin{aligned} J_\Lambda &\leq \lambda J_\Gamma \Leftrightarrow (\gamma_A^{1/2} U_A \otimes \mathbb{1}_B) J_\Lambda (U_A^\dagger \gamma_A^{1/2} \otimes \mathbb{1}_B) \\ &\leq \lambda (\gamma_A^{1/2} U_A \otimes \mathbb{1}_B) J_\Gamma (U_A^\dagger \gamma_A^{1/2} \otimes \mathbb{1}_B) \end{aligned} \quad (\text{A27})$$

$$\Leftrightarrow \mathcal{J}_A \otimes \Lambda[(\gamma_A^{1/2} U_A \otimes \mathbb{1}_{A'}) |\phi^+\rangle\langle\phi^+|_{AA'} (U_A^\dagger \gamma_A^{1/2} \otimes \mathbb{1}_{A'})]$$

$$\leq \lambda \gamma_A \otimes \gamma_B \quad (\text{A28})$$

$$\Leftrightarrow \mathcal{J}_A \otimes \Lambda[\mathcal{J}_A \otimes \mathcal{U}_{A'}(\tilde{\gamma}_{AA'})] \leq \lambda \gamma_A \otimes \gamma_B \quad (\text{A29})$$

$$\Leftrightarrow \mathcal{J}_A \otimes \Lambda(\tau_{AA'}) \leq \lambda \gamma_A \otimes \gamma_B, \quad (\text{A30})$$

where U_A is a general unitary operator on system A , $\tilde{\gamma}_{AA'} \equiv |\tilde{\gamma}\rangle\langle\tilde{\gamma}|_{AA'}$ with $|\tilde{\gamma}\rangle_{AA'} = (\gamma_A^{1/2} \otimes \mathbb{1}_{A'}) |\phi^+\rangle_{AA'} = (\mathbb{1}_A \otimes \gamma_A^{1/2}) |\phi^+\rangle_{AA'}$ is the thermofield double state, $\mathcal{U}_{A'}(\cdot) \equiv U_{A'}^\dagger(\cdot)(U_{A'}^\dagger)^\dagger$ is a unitary channel acting on system A' and $\tau_{AA'} \equiv \mathcal{J}_A \otimes \mathcal{U}_{A'}(\tilde{\gamma}_{AA'})$. The third equivalence relation is from the fact that $(X_A \otimes \mathbb{1}_{A'}) |\phi^+\rangle_{AA'} = (\mathbb{1}_A \otimes X_{A'}^\dagger) |\phi^+\rangle_{AA'}$, for any operator X on system A (or A'). Since any purification of γ_A on system AA' can be written as $\mathcal{J}_A \otimes \mathcal{V}_{A'}(\tilde{\gamma}_{AA'})$ with a unitary channel $\mathcal{V}_{A'}$ [73], $\tau_{AA'}$ is a general purification of γ_A satisfying $\text{Tr}_{A'}\{\tau_{AA'}\} = \gamma_A$. According to the definitions of R [Eq. (A6)] and R_T of states [Eq. (A3)], Eq. (A26) holds. ■

Theorem 2 directly follows from Theorem A4 by setting $\tau_{AA'} = \tilde{\gamma}_{AA'}$. Furthermore, two important implications follow from Theorem A4:

Corollary A4.1. When systems A and B have the same dimension, all purifications of γ_A on system AA' have the same athermality with respect to the thermal state $\gamma_{AB} \equiv \gamma_A \otimes \gamma_B$, which is equal to $R(\mathcal{J}||\Gamma)$.

Proof. When systems A and B have the same dimension, the identity map $\mathcal{J} \in \mathcal{O}(A' \rightarrow B)$. The statement directly follows from Theorem A4 by setting $\Lambda = \mathcal{J}$.

Corollary A4.2. When systems A and B have the same dimension, all unitary channels from A' to B have the same R equal to $R(\mathcal{J}||\Gamma)$.

Proof. By Theorem A4, $R(\mathcal{U}||\Gamma) = R_T[\mathcal{J} \otimes \mathcal{U}(\tau_{AA'})]$, for all unitary channels $\mathcal{U} \in \mathcal{O}(A' \rightarrow B)$. Since $\mathcal{J} \otimes \mathcal{U}(\tau_{AA'})$ is a purification of γ_A on AA' , $R_T[\mathcal{J} \otimes \mathcal{U}(\tau_{AA'})] = R(\mathcal{J}||\Gamma)$ due to Corollary A4.1. Thus, $R(\mathcal{U}||\Gamma) = R(\mathcal{J}||\Gamma)$, for all unitary channels $\mathcal{U} \in \mathcal{O}(A' \rightarrow B)$.

4. The most resourceful channels measured by R (proof of Theorem 3)

In this section, we find the most resourceful channels according to the measure R , which will have the largest combined resource of signaling and athermality. Before we present the result, we prove the following lemma first:

Lemma A2. Given two operators X and Y where Y is positive, the following statements are equivalent:

- (1) $\min\{f|X \leq fY\} = 1$.
- (2) $\exists |\psi\rangle, \langle\psi|Y - X|\psi\rangle = 0$ and $Y - X \geq 0$.

Proof. For the direction (1 $\Rightarrow$ 2), it is straightforward that $\min\{f|X \leq fY\} = 1 \Rightarrow Y - X \geq 0$. To show the existence of such a state $|\psi\rangle$, we assume that $\forall |\psi\rangle, \langle\psi|Y - X|\psi\rangle > 0$. Define $|\psi_{\min}\rangle \equiv \arg \min_{|\psi\rangle} \langle\psi|Y - X|\psi\rangle$ and $|\psi_{\max}\rangle \equiv \arg \max_{|\psi\rangle} \langle\psi|Y|\psi\rangle$. Then, $\forall |\psi\rangle$,

$$\begin{aligned} \langle\psi| \left[\left(1 - \frac{\langle\psi_{\min}|Y - X|\psi_{\min}\rangle}{\langle\psi_{\max}|Y|\psi_{\max}\rangle} \right) Y - X \right] |\psi\rangle \\ = \langle\psi|Y - X|\psi\rangle - \frac{\langle\psi|Y|\psi\rangle}{\langle\psi_{\max}|Y|\psi_{\max}\rangle} \langle\psi_{\min}|Y - X|\psi_{\min}\rangle \end{aligned} \quad (\text{A31})$$

$$\geq \langle\psi|Y - X|\psi\rangle - \langle\psi_{\min}|Y - X|\psi_{\min}\rangle \geq 0. \quad (\text{A32})$$

Therefore,

$$\min\{f|X \leq fY\} \leq 1 - \frac{\langle \psi_{\min} | Y - X | \psi_{\min} \rangle}{\langle \psi_{\max} | Y | \psi_{\max} \rangle} < 1, \quad (\text{A33})$$

which contradicts $\min\{f|X \leq fY\} = 1$. The direction $(1 \Rightarrow 2)$ is proven.

For the direction $(2 \Rightarrow 1)$, since $Y - X \geq 0$, we have $\min\{f|X \leq fY\} \leq 1$. Assume that $f^* \equiv \min\{f|X \leq fY\} < 1$. For the state $|\psi\rangle$ satisfying $\langle \psi | Y - X | \psi \rangle = 0$, we have

$$\begin{aligned} \langle \psi | f^* Y - X | \psi \rangle &= \langle \psi | [(f^* - 1)Y + Y - X] | \psi \rangle \\ &= (f^* - 1) \langle \psi | Y | \psi \rangle < 0, \end{aligned} \quad (\text{A34})$$

because $\langle \psi | Y | \psi \rangle > 0$ for a positive Y . This contradicts the assumption that $\langle \psi | f^* Y - X | \psi \rangle \geq 0, \forall |\psi\rangle$. Therefore, f^* must be 1. The direction $(2 \Rightarrow 1)$ is proven. ■

Theorem A5 (Channels with the highest R). For a general channel $\Lambda \in \mathcal{O}(A \rightarrow B)$, the following inequality holds:

$$R(\Lambda || \Gamma) \leq \text{Tr}\{\gamma_B^{-1}\} - 1, \quad (\text{A35})$$

where the equality holds when $d_A = d_B$ and Λ is unitary.

Proof. The idea of the proof is to firstly show that when $d_A = d_B \equiv d$, for any unitary channel $\mathcal{U} \in \mathcal{O}(A \rightarrow B)$, $R(\mathcal{U} || \Gamma) = \text{Tr}\{\gamma_B^{-1}\} - 1$, and then to show that in general $R(\Lambda || \Gamma) \leq \text{Tr}\{\gamma_B^{-1}\} - 1, \forall \Lambda \in \mathcal{O}(A \rightarrow B)$.

By Corollary A4.2, we have $R(\mathcal{U} || \Gamma) = R(\mathcal{J} || \Gamma)$, for all unitary channels $\mathcal{U} \in \mathcal{O}(A' \rightarrow B)$. Thus, for the first step, when $d_A = d_B \equiv d$, we choose the specific unitary channel $\mathcal{V}$ such that $\mathcal{J}_A \otimes \mathcal{V}(|\phi^+\rangle\langle\phi^+|_{AA'}) = |\phi^+\rangle\langle\phi^+|_{AB} \equiv \sum_{i,j=0}^{d-1} |i\rangle\langle j|_A \otimes |i\rangle\langle j|_B$ where $\{|k\rangle_A\}_k$ and $\{|k\rangle_B\}_k$ are eigenbases of γ_A and γ_B , respectively. We aim to show that $R(\mathcal{V} || \Gamma) = \text{Tr}\{\gamma_B^{-1}\} - 1$. According to the definition of R [Eq. (A6)], it means that

$$\begin{aligned} &\min \left\{ \lambda \left| \frac{1}{\text{Tr}\{\gamma_B^{-1}\}} \mathcal{J}_\mathcal{V} \leq \lambda \mathcal{J}_\Gamma \right. \right\} \\ &= \min \left\{ \lambda \left| \frac{1}{\text{Tr}\{\gamma_B^{-1}\}} |\phi^+\rangle\langle\phi^+|_{AB} \leq \lambda \mathbb{1}_A \otimes \gamma_B \right. \right\} = 1, \end{aligned} \quad (\text{A36})$$

By Lemma A2, it is equivalent to show that

$$\begin{aligned} &\exists |\psi\rangle, \langle \psi | \left(\mathbb{1}_A \otimes \gamma_B - \frac{1}{\text{Tr}\{\gamma_B^{-1}\}} |\phi^+\rangle\langle\phi^+|_{AB} \right) |\psi\rangle = 0 \text{ and} \\ &\mathbb{1}_A \otimes \gamma_B - \frac{1}{\text{Tr}\{\gamma_B^{-1}\}} |\phi^+\rangle\langle\phi^+|_{AB} \geq 0. \end{aligned}$$

Consider the (unnormalized) state $|\gamma_B^{-1}\rangle \equiv \sum_{i=0}^{d-1} 1/g_i^{(B)} |i\rangle_A \otimes |i\rangle_B$, where $g_k^{(B)}$ is the population of the k th eigenstate of γ_B . We have

$$\begin{aligned} &\langle \gamma_B^{-1} | \left(\mathbb{1}_A \otimes \gamma_B - \frac{1}{\text{Tr}\{\gamma_B^{-1}\}} |\phi^+\rangle\langle\phi^+|_{AB} \right) | \gamma_B^{-1} \rangle \\ &= \text{Tr}\{\gamma_B^{-1}\} - \frac{1}{\text{Tr}\{\gamma_B^{-1}\}} (\text{Tr}\{\gamma_B^{-1}\})^2 = 0. \end{aligned} \quad (\text{A37})$$

Besides, defining $|\gamma_B^{-1/2}\rangle \equiv \sum_{i=0}^{d-1} 1/\sqrt{g_i^{(B)}} |i\rangle_A \otimes |i\rangle_B$, we have

$$\begin{aligned} &\mathbb{1}_A \otimes \gamma_B - \frac{1}{\text{Tr}\{\gamma_B^{-1}\}} |\phi^+\rangle\langle\phi^+|_{AB} \\ &= (\mathbb{1}_A \otimes \gamma_B^{1/2}) \left(\mathbb{1}_A \otimes \mathbb{1}_B - \frac{1}{\text{Tr}\{\gamma_B^{-1}\}} |\gamma_B^{-1/2}\rangle\langle\gamma_B^{-1/2}| \right) \\ &\quad \times (\mathbb{1}_A \otimes \gamma_B^{1/2}) \geq 0, \end{aligned} \quad (\text{A38})$$

because $\mathbb{1}_A \otimes \mathbb{1}_B - |\gamma_B^{-1/2}\rangle\langle\gamma_B^{-1/2}| / \text{Tr}\{\gamma_B^{-1}\} \geq 0$ following from the fact that $|\gamma_B^{-1/2}\rangle\langle\gamma_B^{-1/2}| / \text{Tr}\{\gamma_B^{-1}\} \in \mathcal{S}(AB)$ is a pure quantum state. Hence, using Lemma A2, we conclude that $R(\mathcal{V} || \Gamma) = \text{Tr}\{\gamma_B^{-1}\} - 1$, and therefore, $R(\mathcal{U} || \Gamma) = \text{Tr}\{\gamma_B^{-1}\} - 1$ for all unitary channels $\mathcal{U} \in \mathcal{O}(A \rightarrow B)$.

We now prove that, without conditioning on dimensions, $R(\Lambda || \Gamma) \leq \text{Tr}\{\gamma_B^{-1}\} - 1$ for any channel Λ in general. By the definition of R in Eq. (A6), this is equivalent to proving that for any channel Λ we have $\text{Tr}\{\gamma_B^{-1}\}(\mathbb{1}_A \otimes \gamma_B) \geq J_\Lambda$. Note that $J_\Lambda \in \mathcal{L}(A \otimes B)$ where $\mathcal{L}(X)$ denotes the set of linear operators on the Hilbert space X . We consider the map $\Omega : \mathcal{L}(A \otimes B) \rightarrow \mathcal{L}(A \otimes B)$ defined as

$$\Omega(J) := \text{Tr}_B\{J\} \otimes \text{Tr}\{\gamma_B^{-1}\} \gamma_B - J. \quad (\text{A39})$$

If the map Ω is CP, we have $\Omega(J_\Lambda) = \mathbb{1}_A \otimes \text{Tr}\{\gamma_B^{-1}\} \gamma_B - J_\Lambda \geq 0$ and thus complete our proof. To show that Ω is CP, we consider the Choi operator of Ω given by $J_\Omega \equiv \mathcal{J} \otimes \Omega(\Phi^+) \in \mathcal{L}(A \otimes B \otimes A' \otimes B')$ where $\Phi^+ \equiv \sum_{i,k=0}^{d_A-1} \sum_{j,\ell=0}^{d_B-1} |i\rangle\langle k|_A \otimes |j\rangle\langle\ell|_B$. Noticing that

$$\mathcal{J} \otimes \text{Tr}_{B'}\{\Phi^+\} \otimes \text{Tr}\{\gamma_{B'}^{-1}\} \gamma_{B'} = \sum_{i,k=0}^{d_A-1} |i\rangle\langle k|_A \otimes \mathbb{1}_B \otimes |i\rangle\langle k|_{A'} \otimes \text{Tr}\{\gamma_{B'}^{-1}\} \gamma_{B'}, \quad (\text{A40})$$

we have

$$J_\Omega = \mathcal{J} \otimes \text{Tr}_{B'}\{\Phi^+\} \otimes \text{Tr}\{\gamma_{B'}^{-1}\} \gamma_{B'} - \Phi^+ \quad (\text{A41})$$

$$= \sum_{i,k=0}^{d_A-1} |i\rangle\langle k|_A \otimes \mathbb{1}_B \otimes |i\rangle\langle k|_{A'} \otimes \text{Tr}\{\gamma_{B'}^{-1}\} \gamma_{B'} - \sum_{i,k=0}^{d_A-1} \sum_{j,\ell=0}^{d_B-1} |i\rangle\langle k|_A \otimes |j\rangle\langle\ell|_B \otimes |i\rangle\langle k|_{A'} \otimes |j\rangle\langle\ell|_{B'} \quad (\text{A42})$$

$$= \sum_{i,k=0}^{d_A-1} |i\rangle\langle k|_A \otimes |i\rangle\langle k|_{A'} \otimes \mathbb{1}_B \otimes \text{Tr}\{\gamma_{B'}^{-1}\} \gamma_{B'} - \sum_{i,k=0}^{d_A-1} \sum_{j,\ell=0}^{d_B-1} |i\rangle\langle k|_A \otimes |i\rangle\langle k|_{A'} \otimes |j\rangle\langle \ell|_B \otimes |j\rangle\langle \ell|_{B'} \quad (\text{A43})$$

$$= \sum_{i,k=0}^{d_A-1} |i\rangle\langle k|_A \otimes |i\rangle\langle k|_{A'} \otimes \left[\text{Tr}\{\gamma_{B'}^{-1}\}(\mathbb{1}_B \otimes \gamma_{B'}) - \sum_{j,\ell=0}^{d_B-1} |j\rangle\langle \ell|_B \otimes |j\rangle\langle \ell|_{B'} \right] \quad (\text{A44})$$

$$= |\phi^+\rangle\langle\phi^+|_{AA'} \otimes [\text{Tr}\{\gamma_{B'}^{-1}\}(\mathbb{1}_B \otimes \gamma_{B'}) - |\phi^+\rangle\langle\phi^+|_{BB'}]. \quad (\text{A45})$$

Since we have shown that $R(\mathcal{J}||\Gamma) = \text{Tr}\{\gamma_B^{-1}\} - 1$, the term $\text{Tr}\{\gamma_{B'}^{-1}\}(\mathbb{1}_B \otimes \gamma_{B'}) - |\phi^+\rangle\langle\phi^+|_{BB'} \geq 0$. Therefore $J_\Omega \geq 0$ and Ω is CP. We thus prove that $\Omega(J_\Lambda) = \mathbb{1}_A \otimes \text{Tr}\{\gamma_B^{-1}\} \gamma_B - J_\Lambda \geq 0$ for any channel Λ , which means that $R(\Lambda||\Gamma) \leq \text{Tr}\{\gamma_B^{-1}\} - 1$, $\forall \Lambda \in \mathcal{O}(A \rightarrow B)$.

Theorem 3 is a special case of Theorem A5 with the additional conditions $d_A = d_B$ and $\gamma_A = \gamma_B \equiv \gamma$.

APPENDIX B: CHANNEL MUTUAL INFORMATION, R_T AND R_S

In this Appendix, we consider the case shown in Fig. 8. Given system B and system A with its copy A' , the mutual information of the channel $\Lambda \in \mathcal{O}(A' \rightarrow B)$ is defined as [70]

$$I(\Lambda) := \max_{\sigma_{AA'} \in \mathcal{S}(AA')} I(A : B)_{\mathcal{J} \otimes \Lambda(\sigma_{AA'})} \\ = \max_{\sigma_{AA'} \in \mathcal{S}(AA')} D(\sigma_{AB}^{\text{out}} || \sigma_A^{\text{out}} \otimes \sigma_B^{\text{out}}), \quad (\text{B1})$$

where $\sigma_{AB}^{\text{out}} \equiv \mathcal{J} \otimes \Lambda(\sigma_{AA'})$ is the output state, $\sigma_X^{\text{out}} \equiv \text{Tr}_{\{A,B\} \setminus X} \{\sigma_{AB}^{\text{out}}\}$ with $X \in \{A, B\}$ and $D(\rho_1 || \rho_2) \equiv \text{Tr}\{\rho_1 \ln \rho_1 - \rho_1 \ln \rho_2\}$ is the relative entropy between ρ_1 and ρ_2 .

Theorem B1 (Lower bound on channel mutual information by R and R_T). Given a general channel $\Lambda \in \mathcal{O}(A' \rightarrow B)$, the following bounds hold:

$$I(\Lambda) \geq I(A : B)_{\mathcal{J} \otimes \Lambda(\tilde{\gamma}_{AA'})} \geq 2(g_{\min}^{(AB)})^2 [R(\Lambda||\Gamma) - R_T(\Lambda)]^2. \quad (\text{B2})$$

Here, $\tilde{\gamma}_{AA'} \equiv |\tilde{\gamma}\rangle\langle\tilde{\gamma}|_{AA'}$ is the thermofield double state with $|\tilde{\gamma}\rangle_{AA'} \equiv \sum_i \sqrt{g_i} |i\rangle_A \otimes |i\rangle_{A'}$, where g_i is the population of γ_A of the i th energy level $|i\rangle_A$, and $g_{\min}^{(AB)}$ is the smallest population for the thermal state $\gamma_{AB} \equiv \gamma_A \otimes \gamma_B$.

Proof. The first inequality directly follows from the definition of the channel mutual information [Eq. (B1)]. To prove the second inequality, we denote $\rho_{AB} \equiv \mathcal{J} \otimes \Lambda(\tilde{\gamma}_{AA'})$. Therefore,

$$I(A : B)_{\rho_{AB}} = D(\rho_{AB} || \rho_A \otimes \rho_B) \quad (\text{B3})$$

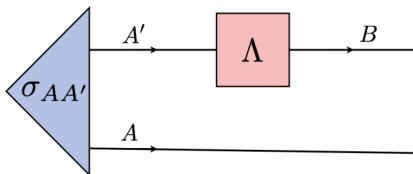


FIG. 8. The scenario in the definition of the channel mutual information.

$$= D(\rho_{AB} || \gamma_A \otimes \rho_B) \quad (\text{B4})$$

$$\geq \frac{1}{2} \|\rho_{AB} - \gamma_A \otimes \rho_B\|_1^2 \quad (\text{B5})$$

$$= \frac{1}{2} \|\mathcal{J} \otimes \Lambda(\tilde{\gamma}_{AA'}) - \gamma_A \otimes \rho_B\|_1^2 \quad (\text{B6})$$

$$\geq 2(g_{\min}^{(AB)})^2 |R_T[\mathcal{J} \otimes \Lambda(\tilde{\gamma}_{AA'})] - R_T[\gamma_A \otimes \Lambda(\gamma_{A'})]|^2 \quad (\text{B7})$$

$$= 2(g_{\min}^{(AB)})^2 [R(\Lambda||\Gamma) - R_T(\Lambda)]^2. \quad (\text{B8})$$

where the first inequality comes from the Pinsker's inequality (Theorem 11.9.1 in Ref. [70]), the second inequality follows from the continuity of R_T (Theorem A2) and the last line follows from the multiplicity of R_T , Theorems A3 and A4. We hence obtain the lower bound in Eq. (B2). ■

Theorem B2 (Upper bound of channel mutual information by R_S). Given a general channel $\Lambda \in \mathcal{O}(A' \rightarrow B)$, the following bound holds:

$$R_S(\Lambda) \geq I(A : B)_{\mathcal{J} \otimes \Lambda(\tilde{\gamma}_{AA'})}. \quad (\text{B9})$$

Proof. We consider the max-relative entropy [39]

$$D_{\max}(\rho || \sigma) := \ln \min\{\lambda | \rho \leq \lambda \sigma\}. \quad (\text{B10})$$

Define $\rho_{AB} \equiv \mathcal{J} \otimes \Lambda(\tilde{\gamma}_{AA'})$. We then have

$$I(A : B)_{\mathcal{J} \otimes \Lambda(\tilde{\gamma}_{AA'})} = D(\rho_{AB} || \rho_A \otimes \rho_B) \quad (\text{B11})$$

$$= \min_{\sigma_B} D(\rho_{AB} || \rho_A \otimes \sigma_B) \quad (\text{B12})$$

$$= \min_{\sigma_B} D(\rho_{AB} || \gamma_A \otimes \sigma_B) \quad (\text{B13})$$

$$\leq \min_{\sigma_B} D_{\max}(\rho_{AB} || \gamma_A \otimes \sigma_B) \quad (\text{B14})$$

$$= \min_{\sigma_B} D_{\max}(J_\Lambda || \mathbb{1}_A \otimes \sigma_B) \quad (\text{B15})$$

$$= \ln[1 + R_S(\Lambda)], \quad (\text{B16})$$

where the second line follows from

$$D(\rho_{AB} || \rho_A \otimes \rho_B) = \min_{\sigma_B} [D(\rho_{AB} || \rho_A \otimes \rho_B) + D(\rho_B || \sigma_B)] \\ = \min_{\sigma_B} D(\rho_{AB} || \rho_A \otimes \sigma_B), \quad (\text{B17})$$

the inequality is due to $D_{\max}(\rho || \sigma) \geq D(\rho || \sigma)$, $\forall \rho, \sigma$ [39], the fifth line is from the fact that

$$J_\Lambda \leq \lambda \mathbb{1}_A \otimes \sigma_B \Leftrightarrow (\gamma_A^{1/2} \otimes \mathbb{1}_B) J_\Lambda (\gamma_A^{1/2} \otimes \mathbb{1}_B) \\ \leq \lambda (\gamma_A^{1/2} \otimes \mathbb{1}_B) \mathbb{1}_A \otimes \sigma_B (\gamma_A^{1/2} \otimes \mathbb{1}_B) \Leftrightarrow \rho_{AB} \\ \leq \lambda \gamma_A \otimes \sigma_B, \quad (\text{B18})$$

and the last line follows from the definition of $R_S(\Lambda)$ [Eq. (A5)] with noticing that $J_{\Phi_\sigma} \equiv \mathbb{1} \otimes \sigma$. Because $R_S(\Lambda) \geq 0$, we have $R_S(\Lambda) \geq \ln[1 + R_S(\Lambda)] \geq I(A : B)_{\mathcal{J} \otimes \Lambda(\tilde{\gamma}_{AA'})}$. ■

1. Proof of Theorem 4

The upper bound in Theorem 4 is straightforward and has been shown in the main text. For the lower bound, according to Theorems B1 and B2, for a general channel $\Lambda \in \mathcal{O}(A' \rightarrow B)$, we have

$$R_S(\Lambda) \geq 2(g_{\min}^{(AB)})^2 [R(\Lambda||\Gamma) - R_T(\Lambda)]^2. \quad (\text{B19})$$

Since A' is a copy of system A , the above also holds for the channel $\Lambda \in \mathcal{O}(A \rightarrow B)$. Therefore, we proved the lower bound in Theorem 4.

2. Signaling channels with zero athermality transmittability

Here we exemplify that a signaling channel $\Lambda \in \mathcal{O}(A' \rightarrow B)$ with $R_S(\Lambda) > 0$ can have zero athermality transmittability, i.e., $R(\Lambda||\Gamma) - R_T(\Lambda) = 0$. Define $\rho_{AB} \equiv \mathcal{J} \otimes \Lambda(\tilde{\gamma}_{AA'})$. By Theorems A4 and A3, we have $R(\Lambda||\Gamma) = R_T(\rho_{AB})$ and $R_T(\Lambda) = R_T(\rho_B)$ where $\rho_B \equiv \text{Tr}_A\{\rho_{AB}\}$. According to Ref. [39], $1 + R_T(\rho) = \|\gamma^{-1/2} \rho \gamma^{-1/2}\|_\infty$. Therefore,

$R_T(\rho_{AB}) = R_T(\rho_B)$ is equivalent to

$$\|(\gamma_A \otimes \gamma_B)^{-1/2} \rho_{AB} (\gamma_A \otimes \gamma_B)^{-1/2}\|_\infty = \|\gamma_B^{-1/2} \rho_B \gamma_B^{-1/2}\|_\infty. \quad (\text{B20})$$

We consider the infinite temperature case, where $\gamma_{A(B)} = \mathbb{1}/d_{A(B)}$. When Λ is entanglement-breaking [74], ρ_{AB} is separable. We consider a special ρ_{AB} , which can be written as

$$\rho_{AB} = \sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} p_{ij} |i\rangle\langle i|_A \otimes |\psi_j\rangle\langle \psi_j|_B, \quad (\text{B21})$$

where $p_{ij} \geq 0$, $\forall i, j$, $\sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} p_{ij} = 1$, and $\{|i\rangle_A\}_{i=0}^{d_A-1}$ is the eigenbasis of γ_A while $\{|\psi_j\rangle_B\}_{j=0}^{d_B-1}$ is a general orthonormal basis for system B . By requiring $\rho_A \equiv \text{Tr}_B\{\rho_{AB}\} = \gamma_A$, we have $\sum_{j=0}^{d_B-1} p_{ij} = d_A^{-1}$, $\forall i$. Also,

$$\begin{aligned} \|(\gamma_A \otimes \gamma_B)^{-1/2} \rho_{AB} (\gamma_A \otimes \gamma_B)^{-1/2}\|_\infty &= d_A d_B \|\rho_{AB}\|_\infty \\ &= d_A d_B \max_{i,j} p_{ij}, \end{aligned} \quad (\text{B22})$$

$$\|\gamma_B^{-1/2} \rho_B \gamma_B^{-1/2}\|_\infty = d_B \|\rho_B\|_\infty = d_B \max_j \sum_{i=0}^{d_A-1} p_{ij}. \quad (\text{B23})$$

Thus, Eq. (B20) can hold when $\exists j$, $p_{ij} = \max_{i,j} p_{ij}$, $\forall i$. For example, when $d_A = d_B = 3$, we consider the following channel:

$$\begin{aligned} \Lambda(\cdot) &:= \text{Tr}\{|0\rangle\langle 0|_A (\cdot)\} \left(\frac{1}{2} |0\rangle\langle 0|_B + \frac{1}{4} |1\rangle\langle 1|_B + \frac{1}{4} |2\rangle\langle 2|_B \right) + \text{Tr}\{|1\rangle\langle 1|_A (\cdot)\} \left(\frac{1}{2} |0\rangle\langle 0|_B + \frac{3}{10} |1\rangle\langle 1|_B + \frac{1}{5} |2\rangle\langle 2|_B \right) \\ &\quad + \text{Tr}\{|2\rangle\langle 2|_A (\cdot)\} \left(\frac{1}{2} |0\rangle\langle 0|_B + \frac{1}{5} |1\rangle\langle 1|_B + \frac{3}{10} |2\rangle\langle 2|_B \right) \end{aligned} \quad (\text{B24})$$

The corresponding ρ_{AB} is given by

$$\begin{aligned} \rho_{AB} &= \frac{1}{6} |0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + \frac{1}{12} |0\rangle\langle 0|_A \otimes |1\rangle\langle 1|_B + \frac{1}{12} |0\rangle\langle 0|_A \otimes |2\rangle\langle 2|_B + \frac{1}{6} |1\rangle\langle 1|_A \otimes |0\rangle\langle 0|_B + \frac{1}{10} |1\rangle\langle 1|_A \otimes |1\rangle\langle 1|_B \\ &\quad + \frac{1}{15} |1\rangle\langle 1|_A \otimes |2\rangle\langle 2|_B + \frac{1}{6} |2\rangle\langle 2|_A \otimes |0\rangle\langle 0|_B + \frac{1}{15} |2\rangle\langle 2|_A \otimes |1\rangle\langle 1|_B + \frac{1}{10} |2\rangle\langle 2|_A \otimes |2\rangle\langle 2|_B \end{aligned} \quad (\text{B25})$$

which leads to $\|(\gamma_A \otimes \gamma_B)^{-1/2} \rho_{AB} (\gamma_A \otimes \gamma_B)^{-1/2}\|_\infty = \|\gamma_B^{-1/2} \rho_B \gamma_B^{-1/2}\|_\infty = 1.5$. Thus, for the channel Λ defined in Eq. (B24), despite of $R_S(\Lambda) > 0$, $R(\Lambda||\Gamma) - R_T(\Lambda) = 0$.

Finally, we note that according to the standard definition of $R_T(\Lambda)$ [Eq. (A2)]:

$$\begin{aligned} R_T(\Lambda) &:= \min \left\{ s \left| \frac{\Lambda + s\Omega}{1+s} = G, \Omega \in \mathcal{O}(A' \rightarrow B), \right. \right. \\ &\quad \left. \left. G \in \text{GPO}(A' \rightarrow B) \right\}, \end{aligned} \quad (\text{B26})$$

$R_T(\Lambda)$ characterizes the “distance” of Λ to the set $\text{GPO}(A' \rightarrow B)$ with the closest GPO to Λ being the optimal channel G^* of the minimization in Eq. (B26) (see Fig. 9). Therefore, channels for which $R(\Lambda||\Gamma) - R_T(\Lambda) = 0$ have the completely thermalizing channel Γ as their closest GPOs under this measure. This offers a heuristic interpretation of the vanishing athermality transmittability of such channels.

APPENDIX C: GPO DILATION OF A GENERAL QUANTUM CHANNEL (PROOF OF THEOREM 5)

In this Appendix, we provide operational meanings of the measures $R_T(\Lambda)$ and $R(\Lambda||\Gamma)$ by considering the GPO dilation of Λ . A GPO dilation of a general channel $\Lambda \in \mathcal{O}(A \rightarrow B)$ can be viewed as the combination of a GPO $G \in \text{GPO}(CA \rightarrow B)$ and an athermal state $\rho_C \in \mathcal{S}(C)$, such

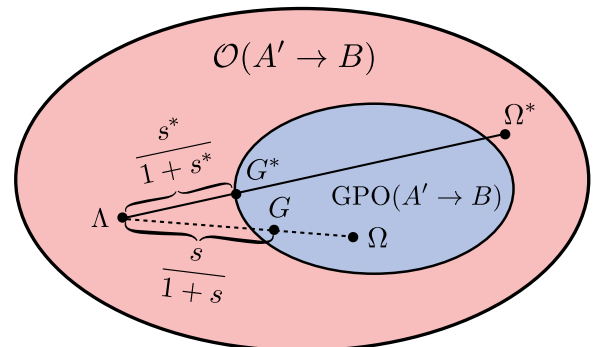


FIG. 9. Geometric illustration of $R_T(\Lambda)$.

that $\Lambda(\cdot) = G(\rho_C \otimes \cdot)$. We emphasize that in a GPO dilation, the GPO G needs not to be unitary, which is different from the Stinespring's representation of quantum channels [49].

The following theorem relates this simulation task to $R_T(\Lambda)$:

Theorem C1 (Operational meaning of $R_T(\Lambda)$). Given a general quantum channel $\Lambda \in \mathcal{O}(A \rightarrow B)$, we define the set of states which can simulate this channel with a GPO as $\mathcal{S}_{\text{GPO}}^{\text{sim}}(\Lambda)$. The states in $\mathcal{S}_{\text{GPO}}^{\text{sim}}(\Lambda)$ can use different GPOs for simulation. The following equality holds:

$$R_T(\Lambda) = \min_{\rho_C \in \mathcal{S}_{\text{GPO}}^{\text{sim}}(\Lambda)} R_T(\rho_C). \quad (\text{C1})$$

Proof. First, we prove that $R_T(\Lambda) \leq \min_{\rho_C \in \mathcal{S}_{\text{GPO}}^{\text{sim}}(\Lambda)} R_T(\rho_C)$. Assume that $\exists \rho_C \in \mathcal{S}(C)$, $G \in \text{GPO}(CA \rightarrow B)$ such that $G(\rho_C \otimes \cdot) = \Lambda(\cdot)$. We have

$$\begin{aligned} R_T(\Lambda) &= R_T[\Lambda(\gamma_A)] = R_T[G(\rho_C \otimes \gamma_A)] \leq R_T(\rho_C \otimes \gamma_A) \\ &= R_T(\rho_C), \end{aligned} \quad (\text{C2})$$

where the first equality follows from Theorem A3, the inequality is due to the monotonicity of R_T and in the final equality we used the multiplicativity of R_T . Thus, $R_T(\Lambda) \leq R_T(\rho_C)$ holds for all ρ_C and G that simulate Λ . We conclude that $R_T(\Lambda) \leq \min_{\rho_C \in \mathcal{S}_{\text{GPO}}^{\text{sim}}(\Lambda)} R_T(\rho_C)$.

To prove that $R_T(\Lambda) \geq \min_{\rho_C \in \mathcal{S}_{\text{GPO}}^{\text{sim}}(\Lambda)} R_T(\rho_C)$, we consider a specific channel simulation protocol using a qubit system C , with the thermal state γ_C given by

$$\gamma_C = \frac{1}{1 + R_T(\Lambda)} |0\rangle\langle 0|_C + \frac{R_T(\Lambda)}{1 + R_T(\Lambda)} |1\rangle\langle 1|_C. \quad (\text{C3})$$

The joint GPO for simulation is constructed in the following:

$$\tilde{G}(\tau_C \otimes \rho_A) := \text{Tr}\{|0\rangle\langle 0|_C \tau_C\} \Lambda(\rho_A) + \text{Tr}\{|1\rangle\langle 1|_C \tau_C\} \sigma_B, \quad (\text{C4})$$

where σ_B is determined from

$$\frac{1}{1 + R_T(\Lambda)} \Lambda(\gamma_A) + \frac{R_T(\Lambda)}{1 + R_T(\Lambda)} \sigma_B = \gamma_B. \quad (\text{C5})$$

According to the equivalence relation between the static and dynamical R_T (Theorem A3), i.e., $R_T(\Lambda) = R_T[\Lambda(\gamma_A)]$, we note that the state σ_B exists and is the optimal state in the standard form of $R_T[\Lambda(\gamma_A)]$ [Eq. (A1)]. It is straightforward to check that using the state $\rho_C \equiv |0\rangle\langle 0|_C$, the channel $\tilde{G}$ in Eq. (C4) is a valid GPO that simulates Λ :

$$\tilde{G}(|0\rangle\langle 0|_C \otimes \cdot) = \Lambda(\cdot), \quad (\text{C6})$$

$$\tilde{G}(\gamma_C \otimes \gamma_A) = \gamma_B. \quad (\text{C7})$$

Since $R_T(\rho_C) = R_T(|0\rangle\langle 0|_C) = R_T(\Lambda)$, we have $R_T(\Lambda) = R_T(\rho_C) \geq \min_{\rho_C \in \mathcal{S}_{\text{GPO}}^{\text{sim}}(\Lambda)} R_T(\rho_C)$. Hence, we have

$$R_T(\Lambda) = \min_{\rho_C \in \mathcal{S}_{\text{GPO}}^{\text{sim}}(\Lambda)} R_T(\rho_C). \quad (\text{C8})$$

■

In a similar way, we can provide an operational meaning for $R(\Lambda||\Gamma)$ in terms of the athermality preservability P_T :

Theorem C2 [Operational meaning of $R(\Lambda||\Gamma)$ (Theorem 5)]. Given a general quantum channel $\Lambda \in \mathcal{O}(A \rightarrow B)$, we define

the set of GPOs which can simulate this channel with a state as $\mathcal{O}_{\text{GPO}}^{\text{sim}}(\Lambda)$. The GPOs in $\mathcal{O}_{\text{GPO}}^{\text{sim}}(\Lambda)$ can use different static resources for simulation. The following inequalities hold:

$$R(\Lambda||\Gamma) \leq \min_{G \in \mathcal{O}_{\text{GPO}}^{\text{sim}}(\Lambda)} P_T(G) \leq \max\{1/R_T(\Lambda), R(\Lambda||\Gamma)\}. \quad (\text{C9})$$

Especially, whenever $R_T(\Lambda)R(\Lambda||\Gamma) \geq 1$, we have

$$R(\Lambda||\Gamma) = \min_{G \in \mathcal{O}_{\text{GPO}}^{\text{sim}}(\Lambda)} P_T(G). \quad (\text{C10})$$

Proof. Firstly, we prove the lower bound. Assume that $\exists \rho_C \in \mathcal{S}(C)$, $G \in \text{GPO}(CA \rightarrow B)$ and $R_T(\rho_C) \geq R_T(\Lambda)$ that give $G(\rho_C \otimes \cdot) = \Lambda(\cdot)$. Recalling the definition of P_T in Eq. (A7), we then have

$$1 + P_T(G) \equiv \min\{\lambda | J_G \leq \lambda(\mathbb{1}_C \otimes \mathbb{1}_A \otimes \gamma_B)\} \quad (\text{C11})$$

$$\begin{aligned} &\geq \min\{\lambda | \text{Tr}_C\{(\rho_C^T \otimes \mathbb{1}_A \otimes \mathbb{1}_B) J_G\} \\ &\leq \lambda \text{Tr}_C\{(\rho_C^T \otimes \mathbb{1}_A \otimes \mathbb{1}_B)(\mathbb{1}_C \otimes \mathbb{1}_A \otimes \gamma_B)\}\} \end{aligned} \quad (\text{C12})$$

$$= \min\{\lambda | J_\Lambda \leq \lambda(\mathbb{1}_A \otimes \gamma_B)\} \quad (\text{C13})$$

$$= 1 + R(\Lambda||\Gamma), \quad (\text{C14})$$

where, in the second line we have used that multiplication with ρ_C^T followed by a trace over C is a CP map. Hence, $R(\Lambda||\Gamma) \leq P_T(G)$ for all ρ_C and G used to simulate Λ . We conclude that $R(\Lambda||\Gamma) \leq \min_{G \in \mathcal{O}_{\text{GPO}}^{\text{sim}}(\Lambda)} P_T(G)$.

To prove the upper bound, we consider a specific channel simulation protocol using a qubit system C , with the thermal state γ_C given by

$$\gamma_C = \frac{1}{1 + R_T(\Lambda)} |0\rangle\langle 0|_C + \frac{R_T(\Lambda)}{1 + R_T(\Lambda)} |1\rangle\langle 1|_C. \quad (\text{C15})$$

The joint GPO for simulation is constructed in the following:

$$\tilde{G}(\tau_C \otimes \rho_A) := \text{Tr}\{|0\rangle\langle 0|_C \tau_C\} \Lambda(\rho_A) + \text{Tr}\{|1\rangle\langle 1|_C \tau_C\} \sigma_B, \quad (\text{C16})$$

where σ_B is determined from

$$\frac{1}{1 + R_T(\Lambda)} \Lambda(\gamma_A) + \frac{R_T(\Lambda)}{1 + R_T(\Lambda)} \sigma_B = \gamma_B. \quad (\text{C17})$$

According to the equivalence relation between the static and dynamical R_T (Theorem A3), i.e., $R_T(\Lambda) = R_T[\Lambda(\gamma_A)]$, we note that σ_B is the optimal state in the standard form of $R_T[\Lambda(\gamma_A)]$ [Eq. (A1)]. It is straightforward to check that using the state $\rho_C \equiv |0\rangle\langle 0|_C$, the channel $\tilde{G}$ in Eq. (C16) is a valid GPO that simulates Λ :

$$\tilde{G}(|0\rangle\langle 0|_C \otimes \cdot) = \Lambda(\cdot), \quad (\text{C18})$$

$$\tilde{G}(\gamma_C \otimes \gamma_A) = \gamma_B. \quad (\text{C19})$$

Consider the Choi operator of $\tilde{G}$. We have

$$J_{\tilde{G}} = |0\rangle\langle 0|_C \otimes J_\Lambda + |1\rangle\langle 1|_C \otimes \mathbb{1}_A \otimes \sigma_B \quad (\text{C20})$$

$$\leq [1 + R(\Lambda||\Gamma)] |0\rangle\langle 0|_C \otimes \mathbb{1}_A \otimes \gamma_B + |1\rangle\langle 1|_C \otimes \mathbb{1}_A \otimes \sigma_B \quad (\text{C21})$$

$$\leq [1 + R(\Lambda||\Gamma)] |0\rangle\langle 0|_C \otimes \mathbb{1}_A \otimes \gamma_B + [1 + 1/R_T(\Lambda)] |1\rangle\langle 1|_C \otimes \mathbb{1}_A \otimes \gamma_B \quad (\text{C22})$$

$$\leq [1 + \max \{1/R_T(\Lambda), R(\Lambda||\Gamma)\}] \mathbb{1}_C \otimes \mathbb{1}_A \otimes \gamma_B, \quad (\text{C23})$$

where the first inequality is from the definition of $R(\Lambda||\Gamma)$ and the second inequality is from the definition of σ_B in Eq. (C17). Recalling the definition of $P_T(G)$, we then have

$$P_T(\tilde{G}) \leq \max \{1/R_T(\Lambda), R(\Lambda||\Gamma)\}. \quad (\text{C24})$$

Therefore,

$$\min_{G \in \mathcal{O}_{\text{GPO}}^{\text{sim}}(\Lambda)} P_T(G) \leq P_T(\tilde{G}) \leq \max \{1/R_T(\Lambda), R(\Lambda||\Gamma)\}. \quad (\text{C25})$$

■

APPENDIX D: RESOURCE ANALYSIS FOR QUANTUM SWITCH

1. Proof of the upper bound in Theorem 6

The lower bound [Eq. (26)] in Theorem 6 is a direct application of Theorem 4. In the rest of this subsection, we prove the upper bound [Eq. (25)] in Theorem 6.

Given a general quantum channel $\Omega \in \mathcal{O}(A \rightarrow B)$ with the Kraus representation $\Omega(\cdot) \equiv \sum_n K_n(\cdot) K_n^\dagger$, the Kraus representation of the quantum switch $\mathcal{S}[\Omega, \Omega] \in \mathcal{O}(AC \rightarrow BC')$, where C' is a copy of C , can be written as $\mathcal{S}[\Omega, \Omega](\cdot) \equiv \sum_m \sum_n S_{mn}(\cdot) S_{mn}^\dagger$, where the Kraus operator S_{mn} is given by

$$S_{mn} \equiv |0\rangle\langle 0|_C \otimes K_n K_m + |1\rangle\langle 1|_C \otimes K_m K_n. \quad (\text{D1})$$

Consider the case when systems A and B have the same thermal state, i.e., $\gamma_A = \gamma_B \equiv \gamma$ with the dimension of d . The completely thermalizing channel $\Gamma \in \text{GPO}(A \rightarrow B)$ has the Kraus representation:

$$\begin{aligned} \Gamma(\cdot) &= \text{Tr}\{\cdot\} \gamma = \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \sqrt{g_j} |j\rangle\langle i| \cdot |i\rangle\langle j| \sqrt{g_j} \\ &\equiv \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \tilde{K}_{ij}(\cdot) \tilde{K}_{ij}^\dagger, \end{aligned} \quad (\text{D2})$$

where g_j is the population of γ on the j th energy level and $\tilde{K}_{ij} \equiv \sqrt{g_j} |j\rangle\langle i|$. According to Eq. (D1), the Kraus operator of the switch $\mathcal{S}[\Gamma, \Gamma]$ is given by

$$\begin{aligned} S_{ijkl}^{(\Gamma)} &\equiv |0\rangle\langle 0|_C \otimes \tilde{K}_{kl} \tilde{K}_{ij} + |1\rangle\langle 1|_C \otimes \tilde{K}_{ij} \tilde{K}_{kl}, \\ \forall i, j, k, \ell &= 0, 1, \dots, d-1, \end{aligned} \quad (\text{D3})$$

such that

$$\mathcal{S}[\Gamma, \Gamma](\cdot) = \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \sum_{k=0}^{d-1} \sum_{\ell=0}^{d-1} S_{ijkl}^{(\Gamma)}(\cdot) (S_{ijkl}^{(\Gamma)})^\dagger. \quad (\text{D4})$$

The signaling GPO input to the switch is defined as

$$G(\cdot) \equiv s\Gamma + (1-s)\mathcal{J} \quad (\text{D5})$$

$$= s \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \tilde{K}_{ij}(\cdot) \tilde{K}_{ij}^\dagger + (1-s) \mathbb{1}(\cdot) \mathbb{1} \quad (\text{D6})$$

$$\equiv \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} K_{ij}(\cdot) K_{ij}^\dagger + K_{dd}(\cdot) K_{dd}^\dagger, \quad (\text{D7})$$

where $s \in [0, 1]$ is the thermalizing strength and we defined the Kraus operators:

$$K_{ij} \equiv \sqrt{s} \tilde{K}_{ij} = \sqrt{s} \sqrt{g_j} |j\rangle\langle i|, \quad \forall i, j = 0, 1, \dots, d-1, \quad (\text{D8})$$

$$K_{dd} \equiv \sqrt{1-s} \mathbb{1}. \quad (\text{D9})$$

According to Eq. (D1), the Kraus operators of the switch $\mathcal{S}[G, G]$ are given by

$$S_{ijkl}^{(G)} \equiv |0\rangle\langle 0|_C \otimes K_{kl} K_{ij} + |1\rangle\langle 1|_C \otimes K_{ij} K_{kl} \quad (\text{D10})$$

$$= s(|0\rangle\langle 0|_C \otimes \tilde{K}_{kl} \tilde{K}_{ij} + |1\rangle\langle 1|_C \otimes \tilde{K}_{ij} \tilde{K}_{kl}) \quad (\text{D11})$$

$$= s S_{ijkl}^{(\Gamma)}, \quad (\text{D12})$$

$$S_{ijdd}^{(G)} \equiv |0\rangle\langle 0|_C \otimes K_{dd} K_{ij} + |1\rangle\langle 1|_C \otimes K_{ij} K_{dd} \quad (\text{D13})$$

$$= |0\rangle\langle 0|_C \otimes \sqrt{1-s} \mathbb{1} \sqrt{s} \tilde{K}_{ij} + |1\rangle\langle 1|_C \otimes \sqrt{s} \tilde{K}_{ij} \sqrt{1-s} \mathbb{1} \quad (\text{D14})$$

$$= \sqrt{s(1-s)} \mathbb{1}_C \otimes \tilde{K}_{ij}, \quad (\text{D15})$$

$$S_{ddkl}^{(G)} \equiv |0\rangle\langle 0|_C \otimes K_{kl} K_{dd} + |1\rangle\langle 1|_C \otimes K_{dd} K_{kl} \quad (\text{D16})$$

$$= |0\rangle\langle 0|_C \otimes \sqrt{s} \tilde{K}_{kl} \sqrt{1-s} \mathbb{1} + |1\rangle\langle 1|_C \otimes \sqrt{1-s} \mathbb{1} \sqrt{s} \tilde{K}_{kl} \quad (\text{D17})$$

$$= \sqrt{s(1-s)} \mathbb{1}_C \otimes \tilde{K}_{kl}, \quad (\text{D18})$$

$$S_{dddd}^{(G)} \equiv |0\rangle\langle 0|_C \otimes K_{dd} K_{dd} + |1\rangle\langle 1|_C \otimes K_{dd} K_{dd} \quad (\text{D19})$$

$$= (1-s) \mathbb{1}_C \otimes \mathbb{1}, \quad (\text{D20})$$

$\forall i, j, k, \ell = 0, 1, \dots, d-1$. The channel $\mathcal{S}[G, G]$ can therefore be written as

$$\mathcal{S}[G, G](\cdot) = \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \sum_{k=0}^{d-1} \sum_{\ell=0}^{d-1} S_{ijkl}^{(G)}(\cdot) (S_{ijkl}^{(G)})^\dagger + \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} S_{ijdd}^{(G)}(\cdot) (S_{ijdd}^{(G)})^\dagger + \sum_{k=0}^{d-1} \sum_{\ell=0}^{d-1} S_{ddkl}^{(G)}(\cdot) (S_{ddkl}^{(G)})^\dagger + S_{dddd}^{(G)}(\cdot) (S_{dddd}^{(G)})^\dagger \quad (\text{D21})$$

$$\begin{aligned} &= s^2 \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \sum_{k=0}^{d-1} \sum_{\ell=0}^{d-1} S_{ijkl}^{(\Gamma)}(\cdot) (S_{ijkl}^{(\Gamma)})^\dagger + s(1-s) \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} (\mathbb{1}_C \otimes \tilde{K}_{ij})(\cdot) (\mathbb{1}_C \otimes \tilde{K}_{ij}^\dagger) \\ &\quad + s(1-s) \sum_{k=0}^{d-1} \sum_{\ell=0}^{d-1} (\mathbb{1}_C \otimes \tilde{K}_{kl})(\cdot) (\mathbb{1}_C \otimes \tilde{K}_{kl}^\dagger) + (1-s)^2 (\mathbb{1}_C \otimes \mathbb{1})(\cdot) (\mathbb{1}_C \otimes \mathbb{1}) \end{aligned} \quad (\text{D22})$$

$$= s^2 \mathcal{S}[\Gamma, \Gamma](\cdot) + 2s(1-s) \mathcal{J}_C \otimes \Gamma(\cdot) + (1-s)^2 \mathcal{J}_C \otimes \mathcal{J}(\cdot), \quad (\text{D23})$$

where in the last line we used the Kraus representations for $\mathcal{S}[\Gamma, \Gamma]$ [Eq. (D4)] and Γ [Eq. (D2)]. From Eq. (D23), the induced channel $\Lambda_s^{(G)}(\cdot) \equiv \mathcal{S}[G, G](\rho_C \otimes \cdot) \in \mathcal{O}(A \rightarrow BC')$ can be written as

$$\Lambda_s^{(G)} = s^2 \Lambda_s^{(\Gamma)} + 2s(1-s)\rho_C \otimes \Gamma + (1-s)^2 \rho_C \otimes \mathcal{J}, \quad (\text{D24})$$

where $\Lambda_s^{(\Gamma)}(\cdot) \equiv \mathcal{S}[\Gamma, \Gamma](\rho_C \otimes \cdot) \in \mathcal{O}(A \rightarrow BC')$ is the channel induced by $\mathcal{S}[\Gamma, \Gamma]$. We then consider the control qubit in the state:

$$\rho_C \equiv |\psi\rangle\langle\psi|_C \quad \text{with} \quad |\psi\rangle_C \equiv \sqrt{\alpha}|0\rangle + e^{i\varphi}\sqrt{1-\alpha}|1\rangle, \quad (\text{D25})$$

for $\alpha \in [0, 1]$ and $\varphi \in [0, 2\pi]$. The thermal state of the control qubit is $\gamma_C \equiv \mathbb{1}_C/2$. Since γ_C can be rewritten as $\gamma_C =$

$(|\psi\rangle\langle\psi|_C + |\psi^\perp\rangle\langle\psi^\perp|_C)/2$, where $\langle\psi|\psi^\perp\rangle_C = 0$, it is easy to check that $R_T(\rho_C) = 1$, $\forall \alpha \in [0, 1]$ and $\varphi \in [0, 2\pi]$.

In order to prove the upper bound of $R(\Lambda_s^{(G)}||\Gamma_{A \rightarrow BC'})$ in Eq. (25), we first prove the following lemma.

Lemma D1. Given the control qubit in Eq. (D25), the following equality holds:

$$R(\Lambda_s^{(\Gamma)}||\Gamma_{A \rightarrow BC'}) = 1, \quad \forall \alpha \in [0, 1], \varphi \in [0, 2\pi]. \quad (\text{D26})$$

Proof. By Theorem A4, we have

$$R(\Lambda_s^{(\Gamma)}||\Gamma_{A \rightarrow BC'}) = R_T[\mathcal{J} \otimes \Lambda_s^{(\Gamma)}(\tilde{\gamma}_{AA'})], \quad (\text{D27})$$

where $\tilde{\gamma}_{AA'} = |\tilde{\gamma}\rangle\langle\tilde{\gamma}|_{AA'}$ with $|\tilde{\gamma}\rangle_{AA'} \equiv \sum_{i=0}^{d-1} \sqrt{g_i}|i\rangle_A \otimes |i\rangle_{A'}$. Define the output state

$$\rho_{ABC'}^{\text{out}} \equiv \mathcal{J} \otimes \Lambda_s^{(\Gamma)}(\tilde{\gamma}_{AA'}) \quad (\text{D28})$$

$$= \mathcal{J} \otimes \mathcal{S}[\Gamma, \Gamma](\rho_C \otimes \tilde{\gamma}_{AA'}) \quad (\text{D29})$$

$$= \sum_{k, \ell=0}^{d-1} \sum_{m, n=0}^{d-1} (S_{k\ell mn}^{(\Gamma)} \otimes \mathbb{1}_A)(\rho_C \otimes \tilde{\gamma}_{AA'})[(S_{k\ell mn}^{(\Gamma)})^\dagger \otimes \mathbb{1}_A] \quad (\text{D30})$$

$$= \sum_{i, j=0}^{d-1} \sum_{k, \ell=0}^{d-1} \sum_{m, n=0}^{d-1} \sqrt{g_i} \sqrt{g_j} [\alpha |0\rangle\langle 0|_C \otimes \tilde{K}_{mn} \tilde{K}_{k\ell} (|i\rangle\langle j|) \tilde{K}_{k\ell}^\dagger \tilde{K}_{mn}^\dagger + (1-\alpha) |1\rangle\langle 1|_C \otimes \tilde{K}_{k\ell} \tilde{K}_{mn} (|i\rangle\langle j|) \tilde{K}_{mn}^\dagger \tilde{K}_{k\ell}^\dagger + e^{-i\varphi} \sqrt{\alpha(1-\alpha)} |0\rangle\langle 1|_C \otimes \tilde{K}_{mn} \tilde{K}_{k\ell} (|i\rangle\langle j|) \tilde{K}_{k\ell}^\dagger \tilde{K}_{mn}^\dagger + e^{i\varphi} \sqrt{\alpha(1-\alpha)} |1\rangle\langle 0|_C \otimes \tilde{K}_{k\ell} \tilde{K}_{mn} (|i\rangle\langle j|) \tilde{K}_{k\ell}^\dagger \tilde{K}_{mn}^\dagger] \otimes |i\rangle\langle j|_A \quad (\text{D31})$$

$$= (\alpha |0\rangle\langle 0|_C + (1-\alpha) |1\rangle\langle 1|_C) \otimes \gamma_B \otimes \gamma_A + \sqrt{\alpha(1-\alpha)} (e^{-i\varphi} |0\rangle\langle 1|_C + e^{i\varphi} |1\rangle\langle 0|_C) \otimes \sigma_{BA}, \quad (\text{D32})$$

where for the convenience of writing, we permute the order of the subsystems but use subscripts to avoid potential ambiguity. The state σ_{BA} in Eq. (D32) is given by

$$\begin{aligned} \sigma_{BA} &\equiv (\gamma_B \otimes \mathbb{1}_A) \tilde{\gamma}_{A'A} (\gamma_B \otimes \mathbb{1}_A) \\ &= \sum_{i, j=0}^{d-1} g_i^{3/2} g_j^{3/2} |i\rangle\langle j|_B \otimes |i\rangle\langle j|_A, \end{aligned} \quad (\text{D33})$$

where we note that $\gamma_A = \gamma_B = \gamma_{A'} \equiv \gamma$. Recalling the definition in Eq. (A3), we have

$$1 + R_T[\mathcal{J} \otimes \Lambda_s^{(\Gamma)}(\tilde{\gamma}_{AA'})] = \min \{ \lambda | \rho_{ABC'}^{\text{out}} \leq \lambda \gamma_C \otimes \gamma_B \otimes \gamma_A \}. \quad (\text{D34})$$

To find the minimal λ , we write $(\lambda \gamma_C \otimes \gamma_B \otimes \gamma_A - \rho_{ABC'}^{\text{out}})$ in the energy eigenbasis of C' as a block matrix M (from now we ignore the labels of systems):

$$M = \begin{pmatrix} (\lambda/2 - \alpha)\gamma \otimes \gamma & -e^{-i\varphi} \sqrt{\alpha(1-\alpha)}\sigma \\ -e^{i\varphi} \sqrt{\alpha(1-\alpha)}\sigma & (\lambda/2 + \alpha - 1)\gamma \otimes \gamma \end{pmatrix}, \quad (\text{D35})$$

where $\lambda \geq 1$. When $\alpha = 0$ or 1 , it is straightforward that the minimal λ ensuring $M \geq 0$ is 2 , and therefore we have $R_T[\mathcal{J} \otimes \Lambda_s^{(\Gamma)}(\tilde{\gamma}_{AA'})] = 2 - 1 = 1$. For $\alpha \in (0, 1)$, to let $M \geq 0$, we need $\lambda/2 > \max\{\alpha, 1-\alpha\}$, which renders the top-left and bottom-right blocks in M [Eq. (D35)] be positive definite. By Schur complement (Theorem 1.12 in Ref. [75]), we know

that

$$\begin{aligned} M \geq 0 &\Leftrightarrow (\lambda/2 + \alpha - 1)\gamma \otimes \gamma - \frac{\alpha(1-\alpha)}{\lambda/2 - \alpha} \sigma (\gamma^{-1} \otimes \gamma^{-1}) \sigma \\ &\geq 0. \end{aligned} \quad (\text{D36})$$

The condition can be rewritten as

$$\sigma \leq \frac{(\lambda/2 + \alpha - 1)(\lambda/2 - \alpha)}{\alpha(1-\alpha)} \gamma \otimes \gamma, \quad (\text{D37})$$

where we used the fact that $\sigma(\gamma^{-1} \otimes \gamma^{-1})\sigma = \sigma$ which can be seen from the following:

$$\begin{aligned} &(\gamma^{1/2} \otimes \gamma^{1/2}) \tilde{\gamma} (\gamma^{1/2} \otimes \gamma^{1/2}) \\ &= (\gamma^{1/2} \otimes \gamma^{1/2}) (\gamma^{1/2} \otimes \mathbb{1}) |\phi^+\rangle \langle \phi^+| (\gamma^{1/2} \otimes \mathbb{1}) \\ &\quad \times (\gamma^{1/2} \otimes \gamma^{1/2}) \end{aligned} \quad (\text{D38})$$

$$\begin{aligned} &= (\gamma \otimes \mathbb{1}) (\mathbb{1} \otimes \gamma^{1/2}) |\phi^+\rangle \langle \phi^+| (\mathbb{1} \otimes \gamma^{1/2}) \\ &\quad \times (\gamma \otimes \mathbb{1}) \end{aligned} \quad (\text{D39})$$

$$= (\gamma \otimes \mathbb{1}) \tilde{\gamma} (\gamma \otimes \mathbb{1}) \quad (\text{D40})$$

$$= \sigma, \quad (\text{D41})$$

where $|\phi^+\rangle \equiv \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle$ and we used Eq. (D33) and the fact that $|\tilde{\gamma}\rangle = (\gamma^{1/2} \otimes \mathbb{1}) |\phi^+\rangle = (\mathbb{1} \otimes \gamma^{1/2}) |\phi^+\rangle$.

Define the function $f(\lambda) \equiv \frac{(\lambda/2 + \alpha - 1)(\lambda/2 - \alpha)}{\alpha(1-\alpha)}$. $f(\lambda)$ is monotonically increasing with λ . In the following, we will show that for Eq. (D37) to hold, $f(\lambda) \geq 1$ such that the minimum $\lambda_{\min} =$

2 (note that $\lambda \geq 1$) satisfying $f(\lambda_{\min}) = 1$. We then conclude that $M \geq 0 \Leftrightarrow \lambda \geq \lambda_{\min} = 2$. By Eq. (D34), we have $R_T[\mathcal{J} \otimes \Lambda_s^{(\Gamma)}(\tilde{\gamma}_{AA'})] = \lambda_{\min} - 1 = 1$, $\forall \alpha \in (0, 1)$, $\varphi \in [0, 2\pi]$. Since $R_T[\mathcal{J} \otimes \Lambda_s^{(\Gamma)}(\tilde{\gamma}_{AA'})] = 1$ also for $\alpha = 0$ or 1 , $\varphi \in [0, 2\pi]$, using Eq. (D27), we therefore obtain $R(\Lambda_s^{(\Gamma)} || \Gamma_{A \rightarrow BC'}) = 1$, $\forall \alpha \in [0, 1]$, $\varphi \in [0, 2\pi]$.

Now the only thing left is to show that

$$\min\{f|\sigma \leq f\gamma \otimes \gamma\} = 1. \quad (\text{D42})$$

By Lemma A2, we note that Eq. (D42) $\Leftrightarrow \exists |\phi\rangle$, $\langle\phi| \gamma \otimes \gamma - \sigma |\phi\rangle = 0$ and $\gamma \otimes \gamma - \sigma \geq 0$. The (unnormalized) state $|\phi\rangle$ can be found as $|\phi\rangle = (\gamma^{-1/2} \otimes \mathbb{1}) |\phi^+\rangle$. We have

$$\begin{aligned} \langle\phi| \gamma \otimes \gamma - \sigma |\phi\rangle &= \langle\phi^+| (\gamma^{-1/2} \otimes \mathbb{1}) (\gamma \otimes \gamma - \sigma) \\ &\quad \times (\gamma^{-1/2} \otimes \mathbb{1}) |\phi^+\rangle \end{aligned} \quad (\text{D43})$$

$$\begin{aligned} &= \langle\phi^+| \mathbb{1} \otimes \gamma |\phi^+\rangle - \langle\phi^+| \gamma \otimes \mathbb{1} |\phi^+\rangle \\ &\quad \times \langle\phi^+| \gamma \otimes \mathbb{1} |\phi^+\rangle \end{aligned} \quad (\text{D44})$$

$$= \text{Tr}\{\gamma\} - \text{Tr}\{\gamma\} \text{Tr}\{\gamma\} \quad (\text{D45})$$

$$= 0, \quad (\text{D46})$$

where in the second line we used the fact that $\sigma = (\gamma^{3/2} \otimes \mathbb{1}) |\phi^+\rangle \langle\phi^+| (\gamma^{3/2} \otimes \mathbb{1})$.

Now we show that $\gamma \otimes \gamma - \sigma \geq 0$. Using Eq. (D41), we have

$$\gamma \otimes \gamma - \sigma = (\gamma^{1/2} \otimes \gamma^{1/2})(\mathbb{1} \otimes \mathbb{1} - \tilde{\gamma})(\gamma^{1/2} \otimes \gamma^{1/2}). \quad (\text{D47})$$

Since $\mathbb{1} \otimes \mathbb{1} - \tilde{\gamma} \geq 0$ and γ is full-rank, we obtain $\gamma \otimes \gamma - \sigma \geq 0$. Hence, Eq. (D42) is proven and as we have argued above, this leads us to $R(\Lambda_s^{(\Gamma)} || \Gamma_{A \rightarrow BC'}) = 1$, $\forall \alpha \in [0, 1]$, $\varphi \in [0, 2\pi]$. ■

Now we prove the upper bound [Eq. (25)] in Theorem 6. According to Eq. (D24), we can rewrite $\Lambda_s^{(G)}$ as

$$\Lambda_s^{(G)} = s^2 \Lambda_s^{(\Gamma)} + 2s(1-s)\rho_C \otimes \Gamma + (1-s)^2 \rho_C \otimes \mathcal{J} \quad (\text{D48})$$

$$= s^2 \Lambda_s^{(\Gamma)} + (1-s^2)\rho_C \otimes G', \quad (\text{D49})$$

where $G' \in \text{GPO}(A \rightarrow B)$ is defined as

$$G' \equiv \frac{2s(1-s)}{1-s^2} \Gamma + \frac{(1-s)^2}{1-s^2} \mathcal{J}. \quad (\text{D50})$$

We then have

$$\begin{aligned} R(\Lambda_s^{(G)} || \Gamma_{A \rightarrow BC'}) &\leq s^2 R(\Lambda_s^{(\Gamma)} || \Gamma_{A \rightarrow BC'}) + (1-s^2) R(\rho_C \otimes G' || \Gamma_{A \rightarrow BC'}) \\ &= s^2 + (1-s^2)[(1 + R(G' || \Gamma))(1 + R_T(\rho_C)) - 1] \end{aligned} \quad (\text{D51})$$

$$\begin{aligned} &= s^2 + (1-s^2)[2(1 + P_T(G')) - 1] \\ &= s^2 + (1-s^2) \left[1 + \frac{2(1-s)^2}{1-s^2} P_T(\mathcal{J}) \right] \end{aligned} \quad (\text{D52})$$

$$\begin{aligned} &= 1 + 2(1-s)^2 P_T(\mathcal{J}) \\ &= R_T(\rho_C) + 2P_T(G)^2 / P_T(\mathcal{J}), \end{aligned} \quad (\text{D53})$$

$$= 1 + 2(1-s)^2 P_T(\mathcal{J}) \quad (\text{D54})$$

$$= R_T(\rho_C) + 2P_T(G)^2 / P_T(\mathcal{J}), \quad (\text{D55})$$

$$= R_T(\rho_C) + 2P_T(G)^2 / P_T(\mathcal{J}), \quad (\text{D56})$$

where the inequality is from the convexity of $R(\Lambda || \Gamma)$, the second line is from Lemma D1 and the multiplicity of $R(\Lambda || \Gamma)$ by considering ρ_C as a state-preparing map, and in the third and the last line we used the fact that $R_T(\rho_C) = 1$, $\forall \alpha \in [0, 1]$, $\varphi \in [0, 2\pi]$. Moreover, in the forth line and the last line we used Lemma D2:

Lemma D2. Given a GPO $G \equiv q\Gamma + (1-q)\mathcal{J}$ for $q \in [0, 1]$, the following equality holds:

$$P_T(G) = (1-q)P_T(\mathcal{J}). \quad (\text{D57})$$

Proof. It is straightforward that the equality holds when $q = 0$ or 1 . For $q \in (0, 1)$, define $\lambda^* \equiv 1 + P_T(G) \equiv 1 + R(G || \Gamma)$. From the definition of $R(G || \Gamma)$ [Eq. (A6)], we have $G \leq \lambda^* \Gamma \Rightarrow \mathcal{J} \leq \frac{\lambda^* - q}{1-q} \Gamma \Rightarrow (1-q)P_T(\mathcal{J}) \leq \lambda^* - 1 \equiv P_T(G)$. On the other hand, due to the convexity of $R(G || \Gamma)$, we have $P_T(G) \equiv R(G || \Gamma) \leq (1-q)P_T(\mathcal{J})$. Hence, $P_T(G) = (1-q)P_T(\mathcal{J})$. ■

We remark that the upper bound Eq. (D56) is saturated as either α or $s = 0$ or 1 . To see this, we note that when $\alpha = 0$ or 1 , the switch is fully off and according to Eq. (D48), the induced channel $\Lambda_s^{(G)} = s^2 \rho_C \otimes \Gamma + 2s(1-s)\rho_C \otimes \Gamma + (1-s)^2 \rho_C \otimes \mathcal{J} = \rho_C \otimes [(1 - (1-s)^2)\Gamma + (1-s)^2\mathcal{J}]$. Thus, using the multiplicity of R , $R_T(\rho_C) = 1$ and Lemma D2, we have $R(\Lambda_s^{(G)} || \Gamma_{A \rightarrow BC'}) = R_T(\rho_C) + 2P_T(G)^2 / P_T(\mathcal{J})$. The upper bound in Eq. (D56) is achieved as $\alpha = 0$ or 1 . When $s = 0$, by Eq. (D48), we have $\Lambda_s^{(G)} = \rho_C \otimes \mathcal{J}$ with $R(\Lambda_s^{(G)} || \Gamma_{A \rightarrow BC'}) = R_T(\rho_C) + 2P_T(\mathcal{J})$. When $s = 1$, $\Lambda_s^{(G)} = \Lambda_s^{(\Gamma)}$ such that $R(\Lambda_s^{(G)} || \Gamma_{A \rightarrow BC'}) = R_T(\rho_C)$ by Lemma D1. Hence, the upper bound in Eq. (D56) is saturated when $s = 0$ or 1 .

The proof of the upper bound [Eq. (25)] in Theorem 6 is completed.

2. Derivation of Eq. (27) in Theorem 6

In this subsection, we derive the analytical expression of $R_T(\Lambda_s^{(G)})$ in Eq. (27). By Theorem A3, we know that $R_T(\Lambda_s^{(G)}) = R_T[\Lambda_s^{(G)}(\gamma_A)]$. To find $\Lambda_s^{(G)}(\gamma_A)$, we recall Eq. (D24) and obtain

$$\begin{aligned} \Lambda_s^{(G)}(\gamma_A) &= s^2 \Lambda_s^{(\Gamma)}(\gamma_A) + 2s(1-s)\rho_C \otimes \Gamma(\gamma_A) \\ &\quad + (1-s)^2 \rho_C \otimes \gamma_B \end{aligned} \quad (\text{D58})$$

$$= s^2 \text{Tr}_A\{\rho_{ABC'}^{\text{out}}\} + (1-s^2)\rho_C \otimes \gamma_B, \quad (\text{D59})$$

where $\rho_{ABC'}^{\text{out}}$ given in Eq. (D32) is the output of $\mathcal{J} \otimes \Lambda_s^{(\Gamma)}(\tilde{\gamma}_{AA'})$. The last line follows from the fact that $\text{Tr}_A\{\rho_{ABC'}^{\text{out}}\} = \text{Tr}_A\{\mathcal{J} \otimes \Lambda_s^{(\Gamma)}(\tilde{\gamma}_{AA'})\} = \Lambda_s^{(\Gamma)}(\text{Tr}_A\{\tilde{\gamma}_{AA'}\}) = \Lambda_s^{(\Gamma)}(\gamma_A) = \Lambda_s^{(\Gamma)}(\gamma_A)$ (note that A' is a copy of A). Given the ρ_C in Eq. (D25) and $\lambda \geq 1$, we write $[\lambda \gamma_C \otimes \gamma_B - \Lambda_s^{(G)}(\gamma_A)]$ in the energy eigenbasis of C' as a block matrix N (from now

we ignore the labels of systems):

$$N = \lambda \begin{pmatrix} \gamma/2 & 0 \\ 0 & \gamma/2 \end{pmatrix} - s^2 \begin{pmatrix} \alpha\gamma & e^{-i\varphi} \sqrt{\alpha(1-\alpha)}\gamma^3 \\ e^{i\varphi} \sqrt{\alpha(1-\alpha)}\gamma^3 & (1-\alpha)\gamma \end{pmatrix} - (1-s^2) \begin{pmatrix} \alpha\gamma & e^{-i\varphi} \sqrt{\alpha(1-\alpha)}\gamma \\ e^{i\varphi} \sqrt{\alpha(1-\alpha)}\gamma & (1-\alpha)\gamma \end{pmatrix} \quad (\text{D60})$$

$$= \begin{pmatrix} (\lambda/2 - \alpha)\gamma & -e^{-i\varphi} \sqrt{\alpha(1-\alpha)}[s^2\gamma^3 + (1-s^2)\gamma] \\ -e^{i\varphi} \sqrt{\alpha(1-\alpha)}[s^2\gamma^3 + (1-s^2)\gamma] & (\lambda/2 + \alpha - 1)\gamma \end{pmatrix}. \quad (\text{D61})$$

When $\alpha = 0$ or 1 , it is easy to find that the minimal λ ensuring $N \geq 0$ is 2 , and thus $R_T(\Lambda_s^{(G)}) = R_T[\Lambda_s^{(G)}(\gamma_A)] = 2 - 1 = 1$. For $\alpha \in (0, 1)$, to let $N \geq 0$, we need $\lambda/2 > \max\{\alpha, 1-\alpha\}$, which renders the top-left and bottom-right blocks in N [Eq. (D61)] be positive definite. By Schur complement (Theorem 1.12 in Ref. [75]), we have

$$N \geq 0 \Leftrightarrow (\lambda/2 + \alpha - 1)\gamma - \frac{\alpha(1-\alpha)}{\lambda/2 - \alpha} [s^2\gamma^3 + (1-s^2)\gamma]\gamma^{-1} [s^2\gamma^3 + (1-s^2)\gamma] \geq 0. \quad (\text{D62})$$

The condition can be rewritten as

$$\gamma \geq \frac{\alpha(1-\alpha)}{(\lambda/2 + \alpha - 1)(\lambda/2 - \alpha)} [s^2\gamma^3 + (1-s^2)\gamma]\gamma^{-1} [s^2\gamma^3 + (1-s^2)\gamma]. \quad (\text{D63})$$

Since both sides are diagonal in the same basis, it is equivalent to require

$$g_i \geq \frac{\alpha(1-\alpha)}{(\lambda/2 + \alpha - 1)(\lambda/2 - \alpha)} [s^2g_i^3 + (1-s^2)g_i]g_i^{-1} [s^2g_i^3 + (1-s^2)g_i], \quad \forall i = 0, 1, \dots, d-1, \quad (\text{D64})$$

$$\Rightarrow \lambda \geq 1 + \sqrt{1 - 4(1 - g_i^2)[2 - (1 - g_i^2)s^2]s^2\alpha(1-\alpha)}, \quad \forall i = 0, 1, \dots, d-1, \quad (\text{D65})$$

$$\Rightarrow \lambda \geq 1 + \sqrt{1 - 4(1 - g_{\max}^2)[2 - (1 - g_{\max}^2)s^2]s^2\alpha(1-\alpha)}, \quad (\text{D66})$$

where $g_{\max} \equiv \max_i g_i$ is the maximal diagonal of γ . In the second line we have used $\alpha \in (0, 1)$, $\lambda \geq 1$ and $\lambda/2 > \max\{\alpha, 1-\alpha\}$ to neglect other ranges of λ and the last line follows from that the right-hand side of the second line is a monotonically increasing function of g_i . Therefore, we have

$$\begin{aligned} R_T(\Lambda_s^{(G)}) &= R_T[\Lambda_s^{(G)}(\gamma_A)] = \lambda_{\min} - 1 \\ &= \sqrt{1 - 4(1 - g_{\max}^2)[2 - (1 - g_{\max}^2)s^2]s^2\alpha(1-\alpha)}, \end{aligned} \quad (\text{D67})$$

$\forall \alpha \in [0, 1]$, $\varphi \in [0, 2\pi]$, $s \in [0, 1]$, where we have included the case of $\alpha = 0$ or 1 because the expression correctly gives $R_T(\Lambda_s^{(G)}) = 1$ as $\alpha = 0$ or 1 . Equation (27) is hence derived.

3. Generalization of Theorem 6 for a general ρ_C

In previous subsections, Theorem 6 is proved for a pure state ρ_C defined in Eq. (D25). In this subsection, we extend Theorem 6 for a fully general ρ_C , which can always be written as

$$\begin{aligned} \rho_C &\equiv r|\psi\rangle\langle\psi|_C + (1-r)\gamma_C \text{ with } |\psi\rangle_C \\ &\equiv \sqrt{\alpha}|0\rangle + e^{i\varphi}\sqrt{1-\alpha}|1\rangle, \end{aligned} \quad (\text{D68})$$

for $r \in [0, 1]$, $\alpha \in [0, 1]$ and $\varphi \in [0, 2\pi]$. Since $\gamma_C \equiv \mathbb{1}_C/2$ is the center of Bloch sphere while $|\psi\rangle\langle\psi|_C$ parametrizes all points on the Bloch sphere, ρ_C as a convex combination of γ_C and $|\psi\rangle\langle\psi|_C$ covers all possible quantum states of the control

qubit C . Following the same idea of the proof of Lemma D2, it is easy to show that $R_T(\rho_C) = rR_T(|\psi\rangle\langle\psi|_C) = r$, $\forall \alpha \in [0, 1]$, $\varphi \in [0, 2\pi]$. In this case,

$$\begin{aligned} \Lambda_s^{(\Gamma)}(\cdot) &\equiv \mathcal{S}[\Gamma, \Gamma](\rho_C \otimes \cdot) = r\mathcal{S}[\Gamma, \Gamma](|\psi\rangle\langle\psi|_C \otimes \cdot) \\ &\quad + (1-r)\mathcal{S}[\Gamma, \Gamma](\gamma_C \otimes \cdot) \\ &= r\tilde{\Lambda}_s^{(\Gamma)}(\cdot) + (1-r)\gamma_C \otimes \Gamma(\cdot), \end{aligned} \quad (\text{D69})$$

where $\tilde{\Lambda}_s^{(\Gamma)}$ denotes $\Lambda_s^{(\Gamma)}$ in Section D 1, which is the induced channel with the pure control qubit $|\psi\rangle\langle\psi|_C$. By Eq. (D24), we can write $\Lambda_s^{(G)}(\cdot) \equiv \mathcal{S}[G, G](\rho_C \otimes \cdot)$ as

$$\begin{aligned} \Lambda_s^{(G)} &= s^2\Lambda_s^{(\Gamma)} + 2s(1-s)\rho_C \otimes \Gamma + (1-s)^2\rho_C \otimes \mathcal{J} \\ &= rs^2\tilde{\Lambda}_s^{(\Gamma)} + (1-r)s^2\gamma_C \otimes \Gamma + (1-s^2)\rho_C \otimes G', \end{aligned} \quad (\text{D70})$$

$$(\text{D71})$$

where $G' \in \text{GPO}(A \rightarrow B)$ is defined as

$$G' \equiv \frac{2s(1-s)}{1-s^2}\Gamma + \frac{(1-s)^2}{1-s^2}\mathcal{J}. \quad (\text{D72})$$

We then have

$$\begin{aligned} R(\Lambda_s^{(G)}||\Gamma_{A \rightarrow BC'}) &\leq rs^2R(\tilde{\Lambda}_s^{(\Gamma)}||\Gamma_{A \rightarrow BC'}) + (1-r)s^2R(\gamma_C \otimes \Gamma||\Gamma_{A \rightarrow BC'}) \\ &\quad + (1-s^2)R(\rho_C \otimes G'||\Gamma_{A \rightarrow BC'}) \end{aligned} \quad (\text{D73})$$

$$= rs^2 + (1-s^2)[(1 + R(G'||\Gamma))(1 + R_T(\rho_C)) - 1] \quad (\text{D74})$$

$$= rs^2 + (1 - s^2)[(1 + P_T(G'))(1 + r) - 1] \quad (\text{D75})$$

$$= rs^2 + (1 - s^2) \left[r + \frac{(1 + r)(1 - s)^2}{1 - s^2} P_T(J) \right] \quad (\text{D76})$$

$$= r + (1 + r)(1 - s)^2 P_T(J) \quad (\text{D77})$$

$$= R_T(\rho_C) + [1 + R_T(\rho_C)] P_T(G)^2 / P_T(J), \quad (\text{D78})$$

where the inequality is from the convexity of $R(\Lambda||\Gamma)$, the second line is from Lemma D1 and the multiplicity of $R(\Lambda||\Gamma)$ by considering ρ_C as a state-preparing map, and in the third and the last line we used the fact that $R_T(\rho_C) = r$, $\forall \alpha \in [0, 1]$, $\varphi \in [0, 2\pi]$. Moreover, in the fourth line and the last line we used Lemma D2. By similar arguments around Eq. (D56), we find that the upper bound Eq. (D78) is saturated as either α or $s = 0$ or 1 .

To derive the analytical expression of $R_T(\Lambda_S^{(G)})$ for a general ρ_C , by Theorem A3, we know that $R_T(\Lambda_S^{(G)}) = R_T[\Lambda_S^{(G)}(\gamma_A)]$. Using Eqs. (D68) and (D71), we rewrite $\Lambda_S^{(G)}(\gamma_A)$ as

$$\Lambda_S^{(G)}(\gamma_A) = rs^2 \tilde{\Lambda}_S^{(\Gamma)}(\gamma_A) + (1 - r)s^2 \gamma_C \otimes \gamma_B + (1 - s^2) \rho_C \otimes \gamma_B \quad (\text{D79})$$

$$= rs^2 \tilde{\Lambda}_S^{(\Gamma)}(\gamma_A) + r(1 - s^2) |\psi\rangle\langle\psi|_C \otimes \gamma_B + (1 - r) \gamma_C \otimes \gamma_B \quad (\text{D80})$$

$$= r \tilde{\Lambda}_S^{(G)}(\gamma_A) + (1 - r) \gamma_C \otimes \gamma_B, \quad (\text{D81})$$

where

$$\tilde{\Lambda}_S^{(G)}(\gamma_A) \equiv s^2 \tilde{\Lambda}_S^{(\Gamma)}(\gamma_A) + (1 - s^2) |\psi\rangle\langle\psi|_C \otimes \gamma_B. \quad (\text{D82})$$

Following the same idea of the proof of Lemma D2, it is straightforward to show that $R_T[\Lambda_S^{(G)}(\gamma_A)] = r R_T[\tilde{\Lambda}_S^{(G)}(\gamma_A)]$. Noticing that $\tilde{\Lambda}_S^{(G)}(\gamma_A)$ denotes $\Lambda_S^{(G)}(\gamma_A)$ in Eq. (D59) in Section D2, corresponding to the induced channel with the pure control qubit $|\psi\rangle\langle\psi|_C$. Using the expression of $R_T[\tilde{\Lambda}_S^{(G)}(\gamma_A)]$ in Eq. (D67), we have

$$\begin{aligned} R_T[\Lambda_S^{(G)}(\gamma_A)] &= r R_T[\tilde{\Lambda}_S^{(G)}(\gamma_A)] \\ &= r \sqrt{1 - 4(1 - g_{\max}^2)[2 - (1 - g_{\max}^2)s^2]s^2\alpha(1 - \alpha)}, \end{aligned} \quad (\text{D83})$$

$\forall r \in [0, 1]$, $\alpha \in [0, 1]$, $\varphi \in [0, 2\pi]$, $s \in [0, 1]$.

Comparing the results with a general ρ_C [Eqs. (D78) and (D83)] with those with a pure ρ_C [Eqs. (D56) and (D67)], we note that the mixedness of ρ_C only changes our results by a prefactor r , which relates to the purity of ρ_C as $\mathcal{P}(\rho_C) \equiv \text{Tr}\{\rho_C^2\} = (1 + r^2)/2$.

APPENDIX E: RESOURCE ANALYSIS FOR COHERENT CONTROL

An alternative way to activate the signaling ability of a quantum channel by multiple using it is the coherent control of quantum channels [51]. In this setup, the implementation of one of two quantum channels is conditioned on the state of a control qubit, either $|0\rangle$ or $|1\rangle$. By preparing the control qubit in the superposition state $|+\rangle$, the two channel

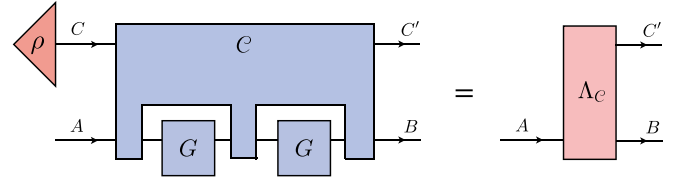


FIG. 10. A coherent control of two GPOs with a control qubit ρ_C can be considered as a channel simulation of Λ_C .

implementations can be coherently superposed. This differs from the case of the quantum switch, where it is the causal order of the two channels—not their implementations—that is placed in a quantum superposition. Due to the similar structures, the performances of coherent control and quantum switch have been compared in the literature [51,76]. In this section, we perform a similar resource analysis as we did for the quantum switch and show how the two objects are similar.

For a fair comparison, we choose the same control qubit ρ_C [Eq. (D25)], systems A and B , and the signaling GPO [Eq. (D5)] as those for the quantum switch. The channel induced by the coherent control is denoted as $\Lambda_C \equiv \mathcal{C}[G, G](\rho_C \otimes \cdot) \in \mathcal{O}(A \rightarrow BC')$ (see Fig. 10). We have the following theorem:

Theorem E1. Given the coherent control with the two input channels as G in Eq. (D5), the following bounds hold:

$$R(\Lambda_C||\Gamma) \leq R_T(\rho_C) + 2P_T(G) + \sqrt{\alpha(1 - \alpha)s(1 - s)}f(\gamma), \quad (\text{E1})$$

$$R(\Lambda_C||\Gamma) \geq R_S(\Lambda_C) \geq (g_{\min}^4/2)[R(\Lambda_C||\Gamma) - R_T(\Lambda_C)]^2, \quad (\text{E2})$$

where $R_T(\rho_C) = 1$, $\forall \alpha \in [0, 1]$, $\varphi \in [0, 2\pi]$ and $f(\gamma) > 0$ is a function of γ (see Sec. E2). When $s = 1$, i.e., the input GPOs are $G \equiv \Gamma$,

$$\begin{aligned} R_T(\Lambda_C) &= \sqrt{(1 - 2\alpha)^2 + 4d^{-2}\alpha(1 - \alpha)}, \quad \forall \alpha \in [0, 1], \\ \varphi &\in [0, 2\pi]. \end{aligned} \quad (\text{E3})$$

The upper bound of $R(\Lambda_C||\Gamma)$ is saturated for $s = 0$ or 1 .

When $s = 1$, by Theorem E1, we have $R(\Lambda_C||\Gamma) = R_T(\rho_C)$. Equation (E2) reduces to

$$R_T(\rho_C) \geq R_S(\Lambda_C) \geq (g_{\min}^4/2)[R_T(\rho_C) - R_T(\Lambda_C)]^2. \quad (\text{E4})$$

$R_T(\Lambda_C)$ and $R_S(\Lambda_C)$ thus exhibit a similar trade-off as in the case of quantum switch.

1. Proof of the upper bound in Theorem E1

Given a general quantum channel $\Omega \in \mathcal{O}(A \rightarrow B)$ with the Kraus representation $\Omega(\cdot) \equiv \sum_{n=0}^{N_{\text{Kraus}}-1} K_n(\cdot)K_n^\dagger$ where N_{Kraus} is the number of Kraus operators, the Kraus representation of the coherent control $\mathcal{C}[\Omega, \Omega] \in \mathcal{O}(AC \rightarrow BC')$, where C' is a copy of C , can be written as $\mathcal{C}[\Omega, \Omega](\cdot) \equiv \sum_{m=0}^{N_{\text{Kraus}}-1} \sum_{n=0}^{N_{\text{Kraus}}-1} C_{mn}(\cdot)C_{mn}^\dagger$, where the Kraus operator C_{mn}

is given by

$$C_{mn} \equiv \frac{1}{\sqrt{N_{\text{Kraus}}}} (|0\rangle\langle 0|_C \otimes K_m + |1\rangle\langle 1|_C \otimes K_n), \quad (\text{E5})$$

where the normalized factor $1/\sqrt{N_{\text{Kraus}}}$ ensures $\sum_{m=0}^{N_{\text{Kraus}}-1} \sum_{n=0}^{N_{\text{Kraus}}-1} C_{mn}^\dagger C_{mn} = \mathbb{1}$. As pointed in Ref. [51], the action of coherent control can depend on the implementation of the controlled channels. By considering the Stinespring's dilation of the channel $\Omega(\cdot) = \text{Tr}_E\{U(\cdot \otimes |\eta\rangle\langle\eta|_E)U^\dagger\}$, the Kraus operator C_{mn} in Eq. (E5) corresponds to the case when $|\eta\rangle_E = \sum_{j=0}^{N_{\text{Kraus}}-1} 1/\sqrt{N_{\text{Kraus}}} |j\rangle_E$ where $\{|j\rangle_E\}_j$ is an orthonormal basis of the environment E (see the discussion around Eq. (5) in Ref. [51]). We choose this specific form for a fair comparison with the quantum switch, whose Kraus operators are given in Eq. (D1).

Consider the case when systems A and B have the same thermal state, i.e., $\gamma_A = \gamma_B \equiv \gamma$ with the dimension of d . The coherent control of the completely thermalizing channel $\Gamma \in \text{GPO}(A \rightarrow B)$ is $\mathcal{C}[\Gamma, \Gamma]$, whose Kraus operator is given by

$$C_{ijkl}^{(\Gamma)} \equiv \frac{1}{d} (|0\rangle\langle 0|_C \otimes \tilde{K}_{ij} + |1\rangle\langle 1|_C \otimes \tilde{K}_{kl}), \quad (\text{E6})$$

where $\tilde{K}_{ij} \equiv \sqrt{g_j} |j\rangle\langle i|$ is the Kraus operator of Γ [see Eq. (D2)]. Thus,

$$\mathcal{C}[\Gamma, \Gamma](\cdot) \equiv \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \sum_{k=0}^{d-1} \sum_{\ell=0}^{d-1} C_{ijkl}^{(\Gamma)}(\cdot) (C_{ijkl}^{(\Gamma)})^\dagger. \quad (\text{E7})$$

Now consider the signaling GPO G in Eq. (D5).

$$G(\cdot) \equiv s\Gamma + (1-s)\mathcal{J} \quad (\text{E8})$$

$$= s \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \tilde{K}_{ij}(\cdot) \tilde{K}_{ij}^\dagger + (1-s) \mathbb{1}(\cdot) \mathbb{1} \quad (\text{E9})$$

$$\equiv \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} K_{ij}(\cdot) K_{ij}^\dagger + K_{dd}(\cdot) K_{dd}^\dagger, \quad (\text{E10})$$

where $s \in [0, 1]$ and the Kraus operators are given by

$$K_{ij} \equiv \sqrt{s} \tilde{K}_{ij} = \sqrt{s} \sqrt{g_j} |j\rangle\langle i|, \quad \forall i, j = 0, 1, \dots, d-1, \quad (\text{E11})$$

$$K_{dd} \equiv \sqrt{1-s} \mathbb{1}. \quad (\text{E12})$$

Now we consider the case when $s \in (0, 1)$. The channel G thus has $d^2 + 1$ Kraus operators. According to Eq. (E5), the Kraus operator of the coherent control $\mathcal{C}[G, G]$ is given by

$$C_{ijkl}^{(G)} \equiv \frac{1}{\sqrt{d^2+1}} (|0\rangle\langle 0|_C \otimes K_{ij} + |1\rangle\langle 1|_C \otimes K_{kl}) \quad (\text{E13})$$

$$= \frac{\sqrt{s}}{\sqrt{d^2+1}} (|0\rangle\langle 0|_C \otimes \tilde{K}_{ij} + |1\rangle\langle 1|_C \otimes \tilde{K}_{kl}) \quad (\text{E14})$$

$$= \frac{d\sqrt{s}}{\sqrt{d^2+1}} C_{ijkl}^{(\Gamma)}, \quad (\text{E15})$$

$$C_{ijdd}^{(G)} \equiv \frac{1}{\sqrt{d^2+1}} (|0\rangle\langle 0|_C \otimes K_{ij} + |1\rangle\langle 1|_C \otimes K_{dd}) \quad (\text{E16})$$

$$= \frac{1}{\sqrt{d^2+1}} (|0\rangle\langle 0|_C \otimes \sqrt{s} \tilde{K}_{ij} + |1\rangle\langle 1|_C \otimes \sqrt{1-s} \mathbb{1}), \quad (\text{E17})$$

$$C_{ddkl}^{(G)} \equiv \frac{1}{\sqrt{d^2+1}} (|0\rangle\langle 0|_C \otimes K_{dd} + |1\rangle\langle 1|_C \otimes K_{kl}) \quad (\text{E18})$$

$$= \frac{1}{\sqrt{d^2+1}} (|0\rangle\langle 0|_C \otimes \sqrt{1-s} \mathbb{1} + |1\rangle\langle 1|_C \otimes \sqrt{s} \tilde{K}_{kl}), \quad (\text{E19})$$

$$C_{dddd}^{(G)} \equiv \frac{1}{\sqrt{d^2+1}} (|0\rangle\langle 0|_C \otimes K_{dd} + |1\rangle\langle 1|_C \otimes K_{dd}) \quad (\text{E20})$$

$$= \frac{\sqrt{1-s}}{\sqrt{d^2+1}} \mathbb{1}_C \otimes \mathbb{1}, \quad (\text{E21})$$

$\forall i, j, k, \ell = 0, 1, \dots, d-1$. The factor $1/\sqrt{d^2+1}$ comes from the fact that G has $d^2 + 1$ Kraus operators. The channel $\mathcal{C}[G, G]$ can therefore be written as

$$\mathcal{C}[G, G](\cdot) = \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \sum_{k=0}^{d-1} \sum_{\ell=0}^{d-1} C_{ijkl}^{(G)}(\cdot) (C_{ijkl}^{(G)})^\dagger + \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} C_{ijdd}^{(G)}(\cdot) (C_{ijdd}^{(G)})^\dagger + \sum_{k=0}^{d-1} \sum_{\ell=0}^{d-1} C_{ddkl}^{(G)}(\cdot) (C_{ddkl}^{(G)})^\dagger + C_{dddd}^{(G)}(\cdot) (C_{dddd}^{(G)})^\dagger \quad (\text{E22})$$

$$= \frac{d^2 s}{d^2 + 1} \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \sum_{k=0}^{d-1} \sum_{\ell=0}^{d-1} C_{ijkl}^{(\Gamma)}(\cdot) (C_{ijkl}^{(\Gamma)})^\dagger + \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} C_{ijdd}^{(G)}(\cdot) (C_{ijdd}^{(G)})^\dagger + \sum_{k=0}^{d-1} \sum_{\ell=0}^{d-1} C_{ddkl}^{(G)}(\cdot) (C_{ddkl}^{(G)})^\dagger + \frac{1-s}{d^2+1} (\mathbb{1}_C \otimes \mathbb{1})(\cdot) (\mathbb{1}_C \otimes \mathbb{1}) \quad (\text{E23})$$

$$= \frac{d^2 s}{d^2 + 1} \mathcal{C}[\Gamma, \Gamma](\cdot) + \frac{s + (1-s)d^2}{d^2 + 1} \mathcal{M}(\cdot) + \frac{1-s}{d^2 + 1} \mathcal{J}_C \otimes \mathcal{J}(\cdot), \quad (\text{E24})$$

where in the last line we used the Kraus representation for $\mathcal{C}[\Gamma, \Gamma]$ [Eq. (E7)] and Γ [Eq. (D2)], and the channel $\mathcal{M} \in \mathcal{O}(AC \rightarrow BC')$ is defined as

$$\mathcal{M}(\cdot) := \frac{d^2 + 1}{s + (1-s)d^2} \left[\sum_{i=0}^{d-1} \sum_{j=0}^{d-1} C_{ijdd}^{(G)}(\cdot) (C_{ijdd}^{(G)})^\dagger + \sum_{k=0}^{d-1} \sum_{\ell=0}^{d-1} C_{ddkl}^{(G)}(\cdot) (C_{ddkl}^{(G)})^\dagger \right]. \quad (\text{E25})$$

$\mathcal{M}$ is trace-preserving because

$$\begin{aligned}
 & \frac{d^2 + 1}{s + (1-s)d^2} \left[\sum_{i=0}^{d-1} \sum_{j=0}^{d-1} (C_{ijdd}^{(G)})^\dagger C_{ijdd}^{(G)} \right. \\
 & \quad \left. + \sum_{k=0}^{d-1} \sum_{\ell=0}^{d-1} (C_{ddk\ell}^{(G)})^\dagger C_{ddk\ell}^{(G)} \right] \\
 &= \frac{1}{s + (1-s)d^2} \left[|0\rangle\langle 0|_C \otimes s \sum_{i,j=0}^{d-1} \tilde{K}_{ij}^\dagger \tilde{K}_{ij} + |1\rangle\langle 1|_C \right. \\
 & \quad \otimes (1-s)d^2 \mathbb{1} + |0\rangle\langle 0|_C \otimes (1-s)d^2 \mathbb{1} + |1\rangle\langle 1|_C \\
 & \quad \left. \otimes s \sum_{k,\ell=0}^{d-1} \tilde{K}_{k\ell}^\dagger \tilde{K}_{k\ell} \right] \quad (\text{E26}) \\
 &= \frac{1}{s + (1-s)d^2} [|0\rangle\langle 0|_C \otimes s \mathbb{1} + |1\rangle\langle 1|_C \otimes (1-s)d^2 \mathbb{1}]
 \end{aligned}$$

$$+ |0\rangle\langle 0|_C \otimes (1-s)d^2 \mathbb{1} + |1\rangle\langle 1|_C \otimes s \mathbb{1}] \quad (\text{E27})$$

$$= \mathbb{1}_C \otimes \mathbb{1}. \quad (\text{E28})$$

From Eq. (E24), the induced channel $\Lambda_c^{(G)}(\cdot) \equiv \mathcal{C}[G, G](\rho_C \otimes \cdot) \in \mathcal{O}(A \rightarrow BC')$ can be written as

$$\Lambda_c^{(G)} = \frac{d^2 s}{d^2 + 1} \Lambda_c^{(\Gamma)} + \frac{s + (1-s)d^2}{d^2 + 1} \mathcal{N} + \frac{1-s}{d^2 + 1} \rho_C \otimes \mathcal{J}, \quad (\text{E29})$$

where $\Lambda_c^{(\Gamma)}(\cdot) \equiv \mathcal{C}[\Gamma, \Gamma](\rho_C \otimes \cdot)$ and $\mathcal{N}(\cdot) \equiv \mathcal{M}(\rho_C \otimes \cdot)$.

Lemma E1. Given the control qubit in Eq. (D25), the following equality holds:

$$R(\Lambda_c^{(\Gamma)} || \Gamma_{A \rightarrow BC'}) = 1, \quad \forall \alpha \in [0, 1], \varphi \in [0, 2\pi] \quad (\text{E30})$$

Proof. By Theorem A4, we have

$$R(\Lambda_c^{(\Gamma)} || \Gamma_{A \rightarrow BC'}) = R_T[\mathcal{J} \otimes \Lambda_c^{(\Gamma)}(\tilde{\gamma}_{AA'})], \quad (\text{E31})$$

where $\tilde{\gamma}_{AA'} = |\tilde{\gamma}\rangle\langle \tilde{\gamma}|_{AA'}$ with $|\tilde{\gamma}\rangle_{AA'} \equiv \sum_{i=0}^{d-1} \sqrt{g_i} |i\rangle_A \otimes |i\rangle_{A'}$. Define the output state

$$\rho_{ABC'}^{\text{out}} \equiv \mathcal{J} \otimes \Lambda_c^{(\Gamma)}(\tilde{\gamma}_{AA'}) \quad (\text{E32})$$

$$= \mathcal{J} \otimes \mathcal{C}[\Gamma, \Gamma](\rho_C \otimes \tilde{\gamma}_{AA'}) \quad (\text{E33})$$

$$= \sum_{k,\ell=0}^{d-1} \sum_{m,n=0}^{d-1} (C_{k\ell mn}^{(\Gamma)} \otimes \mathbb{1}_A)(\rho_C \otimes \tilde{\gamma}_{AA'}) [(C_{k\ell mn}^{(\Gamma)})^\dagger \otimes \mathbb{1}_A] \quad (\text{E34})$$

$$\begin{aligned}
 &= \sum_{i,j=0}^{d-1} \sum_{k,\ell=0}^{d-1} \sum_{m,n=0}^{d-1} \frac{\sqrt{g_i} \sqrt{g_j}}{d^2} [\alpha |0\rangle\langle 0|_C \otimes \tilde{K}_{k\ell}(|i\rangle\langle j|) \tilde{K}_{k\ell}^\dagger + (1-\alpha) |1\rangle\langle 1|_C \otimes \tilde{K}_{mn}(|i\rangle\langle j|) \tilde{K}_{mn}^\dagger \\
 & \quad + e^{-i\varphi} \sqrt{\alpha(1-\alpha)} |0\rangle\langle 1|_C \otimes \tilde{K}_{k\ell}(|i\rangle\langle j|) \tilde{K}_{mn}^\dagger + e^{i\varphi} \sqrt{\alpha(1-\alpha)} |1\rangle\langle 0|_C \otimes \tilde{K}_{mn}(|i\rangle\langle j|) \tilde{K}_{k\ell}^\dagger] \otimes |i\rangle\langle j|_A \quad (\text{E35})
 \end{aligned}$$

$$= (\alpha |0\rangle\langle 0|_C + (1-\alpha) |1\rangle\langle 1|_C) \otimes \gamma_B \otimes \gamma_A + \frac{\sqrt{\alpha(1-\alpha)}}{d^2} (e^{-i\varphi} |0\rangle\langle 1|_C + e^{i\varphi} |1\rangle\langle 0|_C) \otimes \hat{\gamma}_B \otimes \hat{\gamma}_A, \quad (\text{E36})$$

where we note that $\gamma_A = \gamma_B \equiv \gamma$ and the state $\hat{\gamma}$ is defined as

$$\hat{\gamma} \equiv \sum_{i,j=0}^{d-1} \sqrt{g_i} \sqrt{g_j} |i\rangle\langle j|. \quad (\text{E37})$$

Recalling the definition in Eq. (A3), we have

$$1 + R_T[\mathcal{J} \otimes \Lambda_c^{(\Gamma)}(\tilde{\gamma}_{AA'})] = \min \{ \lambda | \rho_{ABC'}^{\text{out}} \leq \lambda \gamma_{C'} \otimes \gamma_B \otimes \gamma_A \}. \quad (\text{E38})$$

To find the minimal λ , we write $(\lambda \gamma_{C'} \otimes \gamma_B \otimes \gamma_A - \rho_{ABC'}^{\text{out}})$ in the energy eigenbasis of C' as a block matrix M (from now we ignore the labels of systems):

$$M = \begin{pmatrix} (\lambda/2 - \alpha) \gamma \otimes \gamma & -d^{-2} e^{-i\varphi} \sqrt{\alpha(1-\alpha)} \hat{\gamma} \otimes \hat{\gamma} \\ -d^{-2} e^{i\varphi} \sqrt{\alpha(1-\alpha)} \hat{\gamma} \otimes \hat{\gamma} & (\lambda/2 + \alpha - 1) \gamma \otimes \gamma \end{pmatrix}, \quad (\text{E39})$$

with $\lambda \geq 1$. When $\alpha = 0$ or 1 , it is clear that the minimal λ guaranteeing $M \geq 0$ is 2 , and thus $R_T[\mathcal{J} \otimes \Lambda_c^{(\Gamma)}(\tilde{\gamma}_{AA'})] = 2 - 1 = 1$. For $\alpha \in (0, 1)$, to let $M \geq 0$, we need $\lambda/2 > \max\{\alpha, 1-\alpha\}$, which makes the top-left and bottom-right blocks in M [Eq. (E39)] be positive definite. By Schur complement (Theorem 1.12 in Ref. [75]), we have

$$M \geq 0 \Leftrightarrow (\lambda/2 + \alpha - 1) \gamma \otimes \gamma - \frac{\alpha(1-\alpha)}{d^4(\lambda/2 - \alpha)} (\hat{\gamma} \otimes \hat{\gamma})(\gamma^{-1} \otimes \gamma^{-1})(\hat{\gamma} \otimes \hat{\gamma}) \geq 0. \quad (\text{E40})$$

This condition is equivalent to the following:

$$\hat{\gamma} \otimes \hat{\gamma} \leq d^2 \frac{(\lambda/2 + \alpha - 1)(\lambda/2 - \alpha)}{\alpha(1-\alpha)} \gamma \otimes \gamma, \quad (\text{E41})$$

which comes from

$$\hat{\gamma} \gamma^{-1} \hat{\gamma} = \sum_{i,j=0}^{d-1} \sum_{k,\ell=0}^{d-1} \sqrt{g_i} \sqrt{g_j} \sqrt{g_k} \sqrt{g_\ell} |i\rangle \langle j| g_k^{-1} |k\rangle \langle \ell| = \sum_{i,\ell=0}^{d-1} \sum_{j=0}^{d-1} \sqrt{g_i} \sqrt{g_\ell} |i\rangle \langle \ell| = d \hat{\gamma}. \quad (\text{E42})$$

Define the function $f(\lambda) \equiv \frac{\alpha(1-\alpha)}{(\lambda/2+\alpha-1)(\lambda/2-\alpha)}$. $f(\lambda)$ is monotonically increasing with λ . In the following, we will show that for Eq. (E41) to hold, $f(\lambda) \geq 1$ such that the minimum $\lambda_{\min} = 2$ (note that $\lambda \geq 1$) satisfying $f(\lambda_{\min}) = 1$. We then conclude that $M \geq 0 \Leftrightarrow \lambda \geq \lambda_{\min} = 2$. By Eq. (E38), we have $R_T[\mathcal{J} \otimes \Lambda_c^{(\Gamma)}(\tilde{\gamma}_{AA'})] = \lambda_{\min} - 1 = 1$, $\forall \alpha \in (0, 1)$. Since $R_T[\mathcal{J} \otimes \Lambda_c^{(\Gamma)}(\tilde{\gamma}_{AA'})] = 1$ also for $\alpha = 0$ or 1 , using Eq. (E31), we therefore obtain $R(\Lambda_c^{(\Gamma)} || \Gamma_{A \rightarrow BC'}) = 1$, $\forall \alpha \in [0, 1]$, $\varphi \in [0, 2\pi]$.

Now the only thing left is to show that

$$\min\{f|\hat{\gamma} \otimes \hat{\gamma} \leq f \cdot d^2(\gamma \otimes \gamma)\} = 1. \quad (\text{E43})$$

By Lemma A2, we have: Eq. (E43) $\Leftrightarrow \exists |\phi\rangle$, $\langle \phi | d^2(\gamma \otimes \gamma) - \hat{\gamma} \otimes \hat{\gamma} | \phi \rangle = 0$ and $d^2(\gamma \otimes \gamma) - \hat{\gamma} \otimes \hat{\gamma} \geq 0$. Such a state $|\phi\rangle$ can be chosen as $|\phi\rangle \equiv \sum_{i,j=0}^{d-1} g_i^{-1/2} g_j^{-1/2} |i\rangle \otimes |j\rangle$ so that

$$\langle \phi | d^2(\gamma \otimes \gamma) | \phi \rangle = d^2 \left(\sum_{k,\ell=0}^{d-1} g_k^{-1/2} g_\ell^{-1/2} \langle k | \otimes \langle \ell | \right) \left(\sum_{i,j=0}^{d-1} g_i^{1/2} g_j^{1/2} |i\rangle \otimes |j\rangle \right) = d^2 \sum_{i,j=0}^{d-1} 1 = d^4, \quad (\text{E44})$$

$$\langle \phi | \hat{\gamma} \otimes \hat{\gamma} | \phi \rangle = \langle \phi | \sum_{i,j=0}^{d-1} \sum_{k,\ell=0}^{d-1} \sum_{m,n=0}^{d-1} g_k^{1/2} g_\ell^{1/2} g_m^{1/2} g_n^{1/2} g_i^{-1/2} g_j^{-1/2} (|k\rangle \langle \ell| \otimes |m\rangle \langle n|) (|i\rangle \otimes |j\rangle) \quad (\text{E45})$$

$$= \langle \phi | \left(\sum_{i,j=0}^{d-1} 1 \right) \sum_{k,m=0}^{d-1} g_k^{1/2} g_m^{1/2} |k\rangle \otimes |m\rangle \quad (\text{E46})$$

$$= d^2 \left(\sum_{i,j=0}^{d-1} g_i^{-1/2} g_j^{-1/2} \langle i | \otimes \langle j | \right) \left(\sum_{k,m=0}^{d-1} g_k^{1/2} g_m^{1/2} |k\rangle \otimes |m\rangle \right) \quad (\text{E47})$$

$$= d^2 \sum_{i,j=0}^{d-1} 1 \quad (\text{E48})$$

$$= d^4, \quad (\text{E49})$$

$$\Rightarrow \langle \phi | d^2(\gamma \otimes \gamma) - \hat{\gamma} \otimes \hat{\gamma} | \phi \rangle = d^4 - d^4 = 0. \quad (\text{E50})$$

To show that $d^2(\gamma \otimes \gamma) - \hat{\gamma} \otimes \hat{\gamma} \geq 0$, we note that

$$d^2(\gamma \otimes \gamma) - \hat{\gamma} \otimes \hat{\gamma} = (\gamma^{1/2} \otimes \gamma^{1/2})(d\mathbb{1} \otimes d\mathbb{1} - O_d \otimes O_d)(\gamma^{1/2} \otimes \gamma^{1/2}), \quad (\text{E51})$$

where $O_d \equiv \sum_{i,j=0}^{d-1} |i\rangle \langle j|$. Since $d\mathbb{1} \otimes d\mathbb{1} - O_d \otimes O_d \geq 0$ and γ is full-rank, we have $d^2(\gamma \otimes \gamma) - \hat{\gamma} \otimes \hat{\gamma} \geq 0$. Therefore, Eq. (E43) is proven and as we have argued above, this leads us to $R(\Lambda_c^{(\Gamma)} || \Gamma_{A \rightarrow BC'}) = 1$, $\forall \alpha \in [0, 1]$, $\varphi \in [0, 2\pi]$. ■

For later use, we calculate the action of $\mathcal{N}$ on a state that can be obtained from $\mathcal{M}$ in Eq. (E25) by using $\mathcal{N}(\cdot) \equiv \mathcal{M}(\rho_C \otimes \cdot)$. We can find

$$\begin{aligned} \mathcal{N}(\cdot) := & \frac{1}{s + (1-s)d^2} \left[(\alpha |0\rangle \langle 0|_C + (1-\alpha) |1\rangle \langle 1|_C) \otimes (s\Gamma + (1-s)d^2\mathcal{J})(\cdot) \right. \\ & \left. + \sqrt{\alpha(1-\alpha)} \sqrt{s(1-s)} (e^{-i\varphi} |0\rangle \langle 1|_C + e^{i\varphi} |1\rangle \langle 0|_C) \otimes \sum_{k,\ell=0}^{d-1} (\tilde{K}_{k\ell}(\cdot) + (\cdot) \tilde{K}_{k\ell}^\dagger) \right]. \quad (\text{E52}) \end{aligned}$$

The output state of $\mathcal{J} \otimes \mathcal{N}$ with the input $\tilde{\gamma}_{AA'}$ is thus given as

$$\rho_{ABC'}^{\text{out}(\mathcal{N})} \equiv \mathcal{J} \otimes \mathcal{N}(\tilde{\gamma}_{AA'}) \quad (\text{E53})$$

$$= (\alpha |0\rangle \langle 0|_C + (1-\alpha) |1\rangle \langle 1|_C) \otimes \frac{s\gamma \otimes \gamma + (1-s)d^2\tilde{\gamma}}{s + (1-s)d^2} + \frac{\sqrt{\alpha(1-\alpha)s(1-s)}}{s + (1-s)d^2} (e^{-i\varphi} |0\rangle \langle 1|_C + e^{i\varphi} |1\rangle \langle 0|_C) \otimes \tau, \quad (\text{E54})$$

where τ is defined as

$$\tau \equiv |\hat{\gamma}\rangle\langle\hat{\gamma}| + |\tilde{\gamma}\rangle\langle\tilde{\gamma}| = \sum_{i=0}^{d-1} \sum_{k,\ell=0}^{d-1} \sqrt{g_i} \sqrt{g_k} \sqrt{g_\ell} (|k\rangle\langle i| \otimes |\ell\rangle\langle i| + |i\rangle\langle k| \otimes |i\rangle\langle \ell|), \quad (\text{E55})$$

where $|\hat{\gamma}\rangle \equiv \sum_{k,\ell=0}^{d-1} \sqrt{g_k} \sqrt{g_\ell} |k\rangle \otimes |\ell\rangle$ is the vectorization of $\hat{\gamma}$ defined in Eq. (E37). By Theorem A4, we have

$$R(\mathcal{N}||\Gamma) = R_T[\mathcal{J} \otimes \mathcal{N}(\tilde{\gamma}_{AA'})] = R_T(\rho_{ABC'}^{\text{out}(\mathcal{N})}) \quad (\text{E56})$$

$$\begin{aligned} &\leq R_T \left[(\alpha |0\rangle\langle 0|_C + (1-\alpha) |1\rangle\langle 1|_C) \otimes \frac{s\gamma \otimes \gamma + (1-s)d^2\tilde{\gamma}}{s + (1-s)d^2} \right] + 1 \\ &\quad + R_T \left[\frac{\sqrt{\alpha(1-\alpha)s(1-s)}}{s + (1-s)d^2} (e^{-i\varphi} |0\rangle\langle 1|_C + e^{i\varphi} |1\rangle\langle 0|_C) \otimes \tau \right] \end{aligned} \quad (\text{E57})$$

$$\begin{aligned} &= [1 + R_T(\alpha |0\rangle\langle 0|_C + (1-\alpha) |1\rangle\langle 1|_C)] \left[1 + \frac{(1-s)d^2}{s + (1-s)d^2} R_T(\tilde{\gamma}) \right] - 1 \\ &\quad + \frac{\sqrt{\alpha(1-\alpha)s(1-s)}}{s + (1-s)d^2} [1 + R_T((e^{-i\varphi} |0\rangle\langle 1|_C + e^{i\varphi} |1\rangle\langle 0|_C) \otimes \tau)] \end{aligned} \quad (\text{E58})$$

$$\leq \frac{2(1-s)d^2}{s + (1-s)d^2} P_T(\mathcal{J}) + 1 + \frac{\sqrt{\alpha(1-\alpha)s(1-s)}}{s + (1-s)d^2} [1 + R_T(\chi)], \quad (\text{E59})$$

where the first inequality comes from the fact that $R_T(\rho + \sigma) \leq 1 + R_T(\rho) + R_T(\sigma)$, the second line follows from the equality $1 + R_T(c\rho) = c[1 + R_T(\rho)]$ for a constant $c \geq 0$, and the second inequality comes from the fact that $1 + R_T(\alpha |0\rangle\langle 0|_C + (1-\alpha) |1\rangle\langle 1|_C) = 2 \max\{\alpha, 1-\alpha\} \leq 2$, $\chi \equiv (e^{-i\varphi} |0\rangle\langle 1|_C + e^{i\varphi} |1\rangle\langle 0|_C) \otimes \tau$ and $P_T(\mathcal{J}) = R_T(\tilde{\gamma})$ (Corollary A4.1).

According to Eq. (E29), we have

$$R(\Lambda_e^{(G)}||\Gamma) \leq \frac{d^2s}{d^2+1} R(\Lambda_e^{(\Gamma)}||\Gamma) + \frac{s + (1-s)d^2}{d^2+1} R(\mathcal{N}||\Gamma) + \frac{1-s}{d^2+1} [(1 + R_T(\rho_C))(1 + P_T(\mathcal{J})) - 1] \quad (\text{E60})$$

$$\begin{aligned} &\leq \frac{d^2s}{d^2+1} + \frac{2(1-s)d^2}{d^2+1} P_T(\mathcal{J}) + \frac{s + (1-s)d^2}{d^2+1} + \frac{\sqrt{\alpha(1-\alpha)}\sqrt{s(1-s)}}{d^2+1} [1 + R_T(\chi)] + \frac{1-s}{d^2+1} (2P_T(\mathcal{J}) + 1) \end{aligned} \quad (\text{E61})$$

$$\begin{aligned} &= \frac{d^2s}{d^2+1} + \frac{2d^2}{d^2+1} P_T(G) + \frac{s + (1-s)d^2}{d^2+1} + \frac{\sqrt{\alpha(1-\alpha)}\sqrt{s(1-s)}}{d^2+1} [1 + R_T(\chi)] + \frac{2}{d^2+1} P_T(G) + \frac{1-s}{d^2+1} \end{aligned} \quad (\text{E62})$$

$$= 1 + 2P_T(G) + \frac{\sqrt{\alpha(1-\alpha)}\sqrt{s(1-s)}}{d^2+1} [1 + R_T(\chi)] \quad (\text{E63})$$

$$= R_T(\rho_C) + 2P_T(G) + \frac{\sqrt{\alpha(1-\alpha)}\sqrt{s(1-s)}}{d^2+1} [1 + R_T(\chi)], \quad (\text{E64})$$

where the first inequality comes from the convexity and the multiplicity of $R(\Lambda||\Gamma)$, the second inequality is obtained by substituting the upper bound of $R(\mathcal{N}||\Gamma)$ in Eq. (E59), the third line follows the Lemma D2 and the last line used the fact that $R_T(\rho_C) = 1$, $\forall \alpha \in [0, 1]$. Note that the upper bound in Eq. (E64) is derived for the case when $s \in (0, 1)$. When $s = 0$, we have $\Lambda_e^{(G)} = \rho_C \otimes \mathcal{J}$ with $R(\Lambda_e^{(G)}||\Gamma) = R_T(\rho_C) + 2P_T(\mathcal{J})$. When $s = 1$, $\Lambda_e^{(G)}$ reduces to $\Lambda_e^{(\Gamma)}$ and by Lemma E1, $R(\Lambda_e^{(\Gamma)}||\Gamma) = 1 = R_T(\rho_C)$. Therefore, Eq. (E64) is an upper bound of $R(\Lambda_e^{(G)}||\Gamma)$ for $s \in [0, 1]$ and is saturated at $s = 0$ or 1 . We eventually obtain the upper bound in Theorem E1:

$$\begin{aligned} R(\Lambda_e^{(\Gamma)}||\Gamma) &\leq R_T(\rho_C) + 2P_T(G) + \sqrt{\alpha(1-\alpha)s(1-s)} f(\gamma), \\ \forall \alpha, s &\in [0, 1], \varphi \in [0, 2\pi]. \end{aligned} \quad (\text{E65})$$

where $f(\gamma) \equiv [1 + R_T(\chi)]/(d^2 + 1)$.

2. Derivation of Eq. (E3) in Theorem E1

In this subsection, we derive the analytical expression of $R_T(\Lambda_e^{(\Gamma)})$ in Eq. (E3). By Theorem A3, we know that $R_T(\Lambda_e^{(\Gamma)}) = R_T[\Lambda_e^{(\Gamma)}(\gamma_A)]$. To find $\Lambda_e^{(\Gamma)}(\gamma_A)$, we recall $\rho_{ABC'}^{\text{out}}$ in Eq. (E36) and have $\text{Tr}_A\{\rho_{ABC'}^{\text{out}}\} = \text{Tr}_A\{\mathcal{J} \otimes \Lambda_e^{(\Gamma)}(\tilde{\gamma}_{AA'})\} = \Lambda_e^{(\Gamma)}(\text{Tr}_A\{\tilde{\gamma}_{AA'}\}) = \Lambda_e^{(\Gamma)}(\gamma_{A'}) = \Lambda_e^{(\Gamma)}(\gamma_A)$ (note that A' is a copy of A). Given $\lambda \geq 1$, we write $[\lambda\gamma_{C'} \otimes \gamma_B - \Lambda_e^{(\Gamma)}(\gamma_A)]$ in the energy eigenbasis of C

as a block matrix N (from now we ignore the labels of systems):

$$N = \lambda \begin{pmatrix} \gamma/2 & 0 \\ 0 & \gamma/2 \end{pmatrix} - \begin{pmatrix} \alpha\gamma & d^{-2}e^{-i\varphi}\sqrt{\alpha(1-\alpha)}\hat{\gamma} \\ d^{-2}e^{i\varphi}\sqrt{\alpha(1-\alpha)}\hat{\gamma} & (1-\alpha)\gamma \end{pmatrix} \quad (\text{E66})$$

$$= \begin{pmatrix} (\lambda/2 - \alpha)\gamma & -d^{-2}e^{-i\varphi}\sqrt{\alpha(1-\alpha)}\hat{\gamma} \\ -d^{-2}e^{i\varphi}\sqrt{\alpha(1-\alpha)}\hat{\gamma} & (\lambda/2 + \alpha - 1)\gamma \end{pmatrix}. \quad (\text{E67})$$

When $\alpha = 0$ or 1 , it is easy to find that the minimal λ ensuring $N \geq 0$ is 2 , and thus $R_T(\Lambda_c^{(\Gamma)}) = R_T[\Lambda_c^{(\Gamma)}(\gamma_A)] = 2 - 1 = 1$. For $\alpha \in (0, 1)$, to let $N \geq 0$, we need $\lambda/2 > \max\{\alpha, 1 - \alpha\}$, which renders the top-left and bottom-right blocks in N [Eq. (E67)] be positive definite. By Schur complement (Theorem 1.12 in Ref. [75]), we have

$$N \geq 0 \Leftrightarrow (\lambda/2 + \alpha - 1)\gamma - \frac{\alpha(1-\alpha)}{d^4(\lambda/2 - \alpha)} \hat{\gamma} \gamma^{-1} \hat{\gamma} \geq 0. \quad (\text{E68})$$

Using the equality $\hat{\gamma} \gamma^{-1} \hat{\gamma} = d \hat{\gamma}$ [Eq. (E42)], the condition can be rewritten as

$$\gamma \geq \frac{\alpha(1-\alpha)}{d^3(\lambda/2 + \alpha - 1)(\lambda/2 - \alpha)} \hat{\gamma}. \quad (\text{E69})$$

Now we show that $\min\{f|\hat{\gamma} \leq f \cdot d\gamma\} = 1$. By Lemma A2, it is equivalent to show that $\exists |\phi\rangle$, $\langle\phi|d\gamma - \hat{\gamma}|\phi\rangle = 0$ and $d\gamma - \hat{\gamma} \geq 0$. Such a state $|\phi\rangle$ can be chosen as $|\phi\rangle \equiv \sum_{i=0}^{d-1} g_i^{-1/2} |i\rangle$ so that

$$\langle\phi|d\gamma|\phi\rangle = d \left(\sum_{i=0}^{d-1} g_i^{-1/2} \langle i| \right) \gamma \left(\sum_{j=0}^{d-1} g_j^{-1/2} |j\rangle \right) = d \left(\sum_{i=0}^{d-1} g_i^{-1/2} \langle i| \right) \left(\sum_{j=0}^{d-1} g_j^{1/2} |j\rangle \right) = d \sum_{i=0}^{d-1} 1 = d^2, \quad (\text{E70})$$

$$\langle\phi|\hat{\gamma}|\phi\rangle = \langle\phi| \sum_{i,j=0}^{d-1} \sum_{k=0}^{d-1} g_i^{1/2} g_j^{1/2} g_k^{-1/2} |i\rangle\langle j| |k\rangle = d \langle\phi| \sum_{i=0}^{d-1} g_i^{1/2} |i\rangle = d \left(\sum_{\ell=0}^{d-1} g_\ell^{-1/2} \langle \ell| \right) \left(\sum_{i=0}^{d-1} g_i^{1/2} |i\rangle \right) = d \sum_{i=0}^{d-1} 1 = d^2. \quad (\text{E71})$$

Thus, $\langle\phi|d\gamma - \hat{\gamma}|\phi\rangle = d^2 - d^2 = 0$. Besides, we note that $d\gamma - \hat{\gamma} = \gamma^{1/2}(d\mathbb{1} - O_d)\gamma^{1/2}$ where $O_d \equiv \sum_{i,j=0}^{d-1} |i\rangle\langle j|$. Since $d\mathbb{1} - O_d \geq 0$ and γ is full-rank, we have $d\gamma - \hat{\gamma} \geq 0$. Therefore, for Eq. (E69) to hold, we have

$$d \leq \frac{d^3(\lambda/2 + \alpha - 1)(\lambda/2 - \alpha)}{\alpha(1-\alpha)}, \quad (\text{E72})$$

$$\Rightarrow \lambda \geq 1 + \sqrt{(1-2\alpha)^2 + 4d^{-2}\alpha(1-\alpha)}, \quad (\text{E73})$$

where in the second line we used $\alpha \in (0, 1)$, $\lambda \geq 1$ and $\lambda/2 > \max\{\alpha, 1 - \alpha\}$ to neglect other ranges of λ . We thus have

$$\begin{aligned} R_T(\Lambda_c^{(\Gamma)}) &= R_T[\Lambda_c^{(\Gamma)}(\gamma_A)] = \lambda_{\min} - 1 \\ &= \sqrt{(1-2\alpha)^2 + 4d^{-2}\alpha(1-\alpha)}, \\ \forall \alpha &\in [0, 1], \varphi \in [0, 2\pi], \end{aligned} \quad (\text{E74})$$

where we have included the case of $\alpha = 0$ or 1 because the expression correctly gives $R_T(\Lambda_c^{(\Gamma)}) = 1$ as $\alpha = 0$ or 1 . Equation (E3) is hence derived.

-
- [1] M. Jarzyna, Classical capacity per unit cost for quantum channels, *Phys. Rev. A* **96**, 032340 (2017).
 - [2] M. M. Wilde and H. Qi, Energy-constrained private and quantum capacities of quantum channels, *IEEE Trans. Inf. Theory* **64**, 7802 (2018).
 - [3] D. Ding, D. S. Pavlichin, and M. M. Wilde, Quantum channel capacities per unit cost, *IEEE Trans. Inf. Theory* **65**, 418 (2019).
 - [4] V. Narasimhachar, J. Thompson, J. Ma, G. Gour, and M. Gu, Quantifying memory capacity as a quantum thermodynamic resource, *Phys. Rev. Lett.* **122**, 060601 (2019).
 - [5] K. Korzekwa, Z. Puchała, M. Tomamichel, and K. Życzkowski, Encoding classical information into quantum resources, *IEEE Trans. Inf. Theory* **68**, 4518 (2022).
 - [6] T. Biswas, A. d. O. Junior, M. Horodecki, and K. Korzekwa, Fluctuation-dissipation relations for thermodynamic distillation processes, *Phys. Rev. E* **105**, 054127 (2022).
 - [7] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, New York, 2010).
 - [8] M. A. Schlosshauer, *Decoherence: And the Quantum-to-Classical Transition* (Springer, Berlin, 2007).
 - [9] P. Faist, F. Dupuis, J. Oppenheim, and R. Renner, The minimal work cost of information processing, *Nat. Commun.* **6**, 7669 (2015).
 - [10] P. Faist and R. Renner, Fundamental work cost of quantum processes, *Phys. Rev. X* **8**, 021011 (2018).

- [11] P. Faist, M. Berta, and F. G. Brandao, Thermodynamic implementations of quantum processes, *Commun. Math. Phys.* **384**, 1709 (2021).
- [12] C.-Y. Hsieh, Dynamical landauer principle: Quantifying information transmission by thermodynamics, *Phys. Rev. Lett.* **134**, 050404 (2025).
- [13] M. Horodecki and J. Oppenheim, (Quantumness in the context of) resource theories, *Int. J. Mod. Phys. B* **27**, 1345019 (2013).
- [14] F. G. S. L. Brandão and G. Gour, Reversible framework for quantum resource theories, *Phys. Rev. Lett.* **115**, 070503 (2015).
- [15] B. Coecke, T. Fritz, and R. W. Spekkens, A mathematical theory of resources, *Inf. Comp.* **250**, 59 (2016).
- [16] E. Chitambar and G. Gour, Quantum resource theories, *Rev. Mod. Phys.* **91**, 025001 (2019).
- [17] G. Gour, *Quantum Resource Theories* (Cambridge University Press, Cambridge, 2025).
- [18] G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron, Quantum computations without definite causal structure, *Phys. Rev. A* **88**, 022318 (2013).
- [19] D. Ebler, S. Salek, and G. Chiribella, Enhanced communication with the assistance of indefinite causal order, *Phys. Rev. Lett.* **120**, 120502 (2018).
- [20] X. Liu, D. Ebler, and O. Dahlsten, Thermodynamics of quantum switch information capacity activation, *Phys. Rev. Lett.* **129**, 230604 (2022).
- [21] C. Xi, X. Liu, H. Liu, K. Huang, X. Long, D. Ebler, X. Nie, O. Dahlsten, and D. Lu, Experimental validation of enhanced information capacity by quantum switch in accordance with thermodynamic laws, *Phys. Rev. Lett.* **133**, 040401 (2024).
- [22] M. Capela, H. Verma, F. Costa, and L. C. Céleri, Reassessing thermodynamic advantage from indefinite causal order, *Phys. Rev. A* **107**, 062208 (2023).
- [23] Z. Wu, J. Fullwood, Z. Ma, S.-Q. Zhou, Q. Zhao, and G. Chiribella, General communication enhancement via the quantum switch, *Phys. Rev. A* **111**, 012605 (2025).
- [24] M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph, Quantum coherence, time-translation symmetry, and thermodynamics, *Phys. Rev. X* **5**, 021001 (2015).
- [25] A. Streltsov, G. Adesso, and M. B. Plenio, *Colloquium: Quantum coherence as a resource*, *Rev. Mod. Phys.* **89**, 041003 (2017).
- [26] G. Francica, F. C. Binder, G. Guarnieri, M. T. Mitchison, J. Goold, and F. Plastina, Quantum coherence and ergotropy, *Phys. Rev. Lett.* **125**, 180603 (2020).
- [27] K. Goswami, Y. Cao, G. A. Paz-Silva, J. Romero, and A. G. White, Increasing communication capacity via superposition of order, *Phys. Rev. Res.* **2**, 033292 (2020).
- [28] G. Chiribella, M. Banik, S. S. Bhattacharya, T. Guha, M. Alimuddin, A. Roy, S. Saha, S. Agrawal, and G. Kar, Indefinite causal order enables perfect quantum communication with zero capacity channels, *New J. Phys.* **23**, 033039 (2021).
- [29] D. Felce and V. Vedral, Quantum refrigeration with indefinite causal order, *Phys. Rev. Lett.* **125**, 070603 (2020).
- [30] G. Zhu, Y. Chen, Y. Hasegawa, and P. Xue, Charging quantum batteries via indefinite causal order: Theory and experiment, *Phys. Rev. Lett.* **131**, 240401 (2023).
- [31] G. Vidal and R. Tarrach, Robustness of entanglement, *Phys. Rev. A* **59**, 141 (1999).
- [32] M. Steiner, Generalized robustness of entanglement, *Phys. Rev. A* **67**, 054305 (2003).
- [33] L. Lami and B. Regula, Computable lower bounds on the entanglement cost of quantum channels, *J. Phys. A: Math. Theor.* **56**, 035302 (2023).
- [34] M. Piani, M. Cianciaruso, T. R. Bromley, C. Napoli, N. Johnston, and G. Adesso, Robustness of asymmetry and coherence of quantum states, *Phys. Rev. A* **93**, 042107 (2016).
- [35] M. G. Díaz, K. Fang, X. Wang, M. Rosati, M. Skotiniotis, J. Calsamiglia, and A. Winter, Using and reusing coherence to realize quantum processes, *Quantum* **2**, 100 (2018).
- [36] Z.-W. Liu, K. Bu, and R. Takagi, One-shot operational quantum resource theory, *Phys. Rev. Lett.* **123**, 020401 (2019).
- [37] B. Regula and R. Takagi, One-shot manipulation of dynamical quantum resources, *Phys. Rev. Lett.* **127**, 060402 (2021).
- [38] R. Takagi and B. Regula, General resource theories in quantum mechanics and beyond: Operational characterization via discrimination tasks, *Phys. Rev. X* **9**, 031053 (2019).
- [39] N. Datta, Min-and max-relative entropies and a new entanglement monotone, *IEEE Trans. Inf. Theory* **55**, 2816 (2009).
- [40] M. Horodecki and J. Oppenheim, Fundamental limitations for quantum and nanoscale thermodynamics, *Nat. Commun.* **4**, 2059 (2013).
- [41] K. Fang, X. Wang, M. Tomamichel, and M. Berta, Quantum channel simulation and the channel's smooth max-information, *IEEE Trans. Inf. Theory* **66**, 2129 (2019).
- [42] R. Takagi, K. Wang, and M. Hayashi, Application of the resource theory of channels to communication scenarios, *Phys. Rev. Lett.* **124**, 120502 (2020).
- [43] S. Milz, J. Bavaresco, and G. Chiribella, Resource theory of causal connection, *Quantum* **6**, 788 (2022).
- [44] G. Chiribella, Extreme quantum states and processes, and extreme points of general spectrahedra in finite dimensional algebras, *Int. J. Quantum Inf.* **22**, 2440004 (2024).
- [45] Sohail, V. Pandey, U. Singh, and S. Das, Fundamental limitations on the recoverability of quantum processes, *Ann. Henri Poincaré* (2025).
- [46] H. Badhani, D. G. S., and S. Das, Thermodynamics of quantum processes: An operational framework for free energy and reversible athermality, *arXiv:2510.12790*.
- [47] C.-Y. Hsieh, Resource preservability, *Quantum* **4**, 244 (2020).
- [48] Y. Takahashi and H. Umezawa, Thermo field dynamics, *Int. J. Mod. Phys. B* **10**, 1755 (1996).
- [49] W. F. Stinespring, Positive functions on C^* -algebras, *Proc. Am. Math. Soc.* **6**, 211 (1955).
- [50] R. Kukulski, I. Nechita, Ł. Paweł, Z. Puchała, and K. Życzkowski, Generating random quantum channels, *J. Math. Phys.* **62**, 062201 (2021).
- [51] A. A. Abbott, J. Wechs, D. Horsman, M. Mhalla, and C. Branciard, Communication through coherent control of quantum channels, *Quantum* **4**, 333 (2020).
- [52] D. Janzing, P. Wocjan, R. Zeier, R. Geiss, and T. Beth, Thermodynamic cost of reliability and low temperatures: Tightening landauer's principle and the second law, *Int. J. Theor. Phys.* **39**, 2717 (2000).
- [53] F. G. Brandao, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, Resource theory of quantum states out of thermal equilibrium, *Phys. Rev. Lett.* **111**, 250404 (2013).

- [54] P. Ćwikliński, M. Studziński, M. Horodecki, and J. Oppenheim, Limitations on the evolution of quantum coherences: Towards fully quantum second laws of thermodynamics, *Phys. Rev. Lett.* **115**, 210403 (2015).
- [55] M. Lostaglio, Á. M. Alhambra, and C. Perry, Elementary thermal operations, *Quantum* **2**, 52 (2018).
- [56] P. Faist, J. Oppenheim, and R. Renner, Gibbs-preserving maps outperform thermal operations in the quantum regime, *New J. Phys.* **17**, 043003 (2015).
- [57] Y. Ding, F. Ding, and X. Hu, Exploring the gap between thermal operations and enhanced thermal operations, *Phys. Rev. A* **103**, 052214 (2021).
- [58] H. Tajima and R. Takagi, Gibbs-preserving operations requiring infinite amount of quantum coherence, *Phys. Rev. Lett.* **134**, 170201 (2025).
- [59] T. Theurer, D. Egloff, L. Zhang, and M. B. Plenio, Quantifying operations with an application to coherence, *Phys. Rev. Lett.* **122**, 190405 (2019).
- [60] Z.-W. Liu and A. Winter, Resource theories of quantum channels and the universal role of resource erasure, [arXiv:1904.04201](https://arxiv.org/abs/1904.04201).
- [61] Y. Liu and X. Yuan, Operational resource theory of quantum channels, *Phys. Rev. Res.* **2**, 012035 (2020).
- [62] G. Gour and C. M. Scandolo, Dynamical resources, [arXiv:2101.01552](https://arxiv.org/abs/2101.01552).
- [63] G. Gour and C. M. Scandolo, Dynamical entanglement, *Phys. Rev. Lett.* **125**, 180505 (2020).
- [64] G. Gour and C. M. Scandolo, Entanglement of a bipartite channel, *Phys. Rev. A* **103**, 062422 (2021).
- [65] G. Gour, D. Kim, T. Nateeboon, G. Shemesh, and G. Yoeli, Inevitable negativity: Additivity commands negative quantum channel entropy, *Phys. Rev. A* **111**, 052424 (2025).
- [66] A. Hoffmann, The distance to the intersection of two convex sets expressed by the distances to each of them, *Math. Nachr.* **157**, 81 (1992).
- [67] L. Li, K. Bu, and Z.-W. Liu, Quantifying the resource content of quantum channels: An operational approach, *Phys. Rev. A* **101**, 022335 (2020).
- [68] P. Skrzypczyk and D. Cavalcanti, *Semidefinite Programming in Quantum Information Science* (IOP Publishing, Bristol, 2023).
- [69] J. Watrous, Lecture 2: Max-relative entropy and conditional min-entropy, in *Advanced Topics in Quantum Information Theory, CS 798/QIC 890*, Lecture Notes (University of Waterloo, 2020).
- [70] M. M. Wilde, *Quantum Information Theory* (Cambridge University Press, Cambridge, 2013).
- [71] K. Baudin, J. Garnier, A. Fusaro, N. Berti, C. Michel, K. Krupa, G. Millot, and A. Picozzi, Observation of light thermalization to negative-temperature Rayleigh-Jeans equilibrium states in multimode optical fibers, *Phys. Rev. Lett.* **130**, 063801 (2023).
- [72] J. Schluck, G. Murta, H. Kampermann, D. Bruß, and N. Wyderka, Continuity of robustness measures in quantum resource theories, *J. Phys. A: Math. Theor.* **56**, 255303 (2023).
- [73] K. Kirkpatrick, The Schrödinger-HJW theorem, *Found. Phys. Lett.* **19**, 95 (2006).
- [74] M. Horodecki, P. W. Shor, and M. B. Ruskai, Entanglement breaking channels, *Rev. Math. Phys.* **15**, 629 (2003).
- [75] F. Zhang, *The Schur Complement and Its Applications* (Springer Science & Business Media, New York, 2006), Vol. 4.
- [76] N. Loizeau and A. Grinbaum, Channel capacity enhancement with indefinite causal order, *Phys. Rev. A* **101**, 012340 (2020).