

Pre-Floquet states facilitating coherent subharmonic response of periodically driven many-body systems

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We demonstrate longtime coherent subharmonic motion of a many-boson system subjected to an external time-periodic driving force. The underlying mechanism is exemplified numerically through analysis of a periodically driven Bose-Hubbard dimer, and clarified conceptually by semiclassical requantization of invariant tubes pertaining to the system's mean-field description. In this way, one arrives at pre-Floquet states that relate to the actual many-body Floquet states in a manner similar to the relation of site-localized Wannier states to lattice-extended Bloch states in solid-state physics. It is argued that even high-order subharmonic response can be systematically engineered, and be observed experimentally, with weakly interacting Floquet condensates comprising a sufficiently large number of particles.

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I. INTRODUCTION: TIME CRYSTALS AND SUBHARMONIC RESPONSE

While the thought-provoking question whether *continuous* time-translation symmetry could be spontaneously broken in the ground state of a closed quantum mechanical system [1] soon was given a negative answer [2,3], breaking of the *discrete* time-translational symmetry inherent to systems exposed to an external time-periodic stimulus has actually been observed in pioneering experiments with interacting spin chains of trapped atomic ions [4], or with dipolar spin impurities in diamond [5]. The hallmark of such discrete time crystals is a subharmonic response to the periodic drive. Considering, for instance, 1:2 clocking, that is, a response signal that occurs with strict periodicity only once during every two drive cycles, that signal could show up either in the first or in the second cycle of each two-cycle interval. This leaves us with two possible states, akin to the two ground states located in either well of a symmetric double-well potential when the tunneling contact between the two wells is closed. The documented existence of this nonequilibrium state of matter has catalyzed a multitude of further intense research, spanning, among others, Anderson localization in the time domain, ergodicity breaking, and lack of thermalization due to many-body localization, altogether disclosing far-reaching new perspectives for nonequilibrium statistical physics [6–15].

The purpose of the present contribution is to specify conditions under which long-lasting subharmonic response of many-body quantum systems occurs in an elementary manner not involving these demanding concepts. For the sake of

demonstration, we resort to the model of a periodically driven Bose-Hubbard dimer, describing a large number N of Bose particles that occupy two sites coupled by a tunneling matrix element $\hbar\Omega$, and experience a repulsive on-site interaction of strength $\hbar\kappa$. Instead of periodic modulation of the tunneling contact [16] or deltalike kicking [17], here we consider sinusoidal shaking of the site potentials with amplitude $\hbar\mu$ and angular frequency ω . In terms of bosonic operators a_j and $a_j^\dagger$ that annihilate and create, respectively, a particle at the site labeled j , its Hamiltonian takes the form [18,19]

$$H(t) = -\frac{\hbar\Omega}{2}(a_2^\dagger a_1 + a_1^\dagger a_2) + \hbar\kappa(a_1^\dagger a_1^\dagger a_1 a_1 + a_2^\dagger a_2^\dagger a_2 a_2) + \hbar\mu \sin(\omega t)(a_1^\dagger a_1 - a_2^\dagger a_2). \quad (1)$$

Model (1) is closely related to the driven one-dimensional spin chain with long-range interactions more recently studied in Ref. [14].

Having solved the system's Schrödinger equation numerically with $N = 2000$ particles in order to obtain the time-dependent states $|\psi(t)\rangle$, we depict in Fig. 1 the return probability $P_r(t) = |\langle\psi(0)|\psi(t)\rangle|^2$ versus time t to a particular initial state $|\psi(0)\rangle$ under conditions of strong driving, $\mu/\Omega = 0.4$ for $\omega/\Omega = 1.9$ and $N\kappa/\Omega = 0.92$. Evidently, the system features almost perfect 1:3 clocking, at least on the time scale of few driving cycles considered here. In the following sections, we will analyze the underlying mechanism that may be considerably more general than its realization in the idealistic model (1). To this end, we will review key attributes of the Floquet approach to periodically driven quantum systems in Sec. II, emphasizing the close connection between Sambe's extended Hilbert space and the Brillouin-zone-like structure of the quasienergy spectrum. In Sec. III, we outline how this structure is recovered by a generalization of the well-known Bohr-Sommerfeld quantization rules to fully integrable, periodically driven classical Hamiltonian

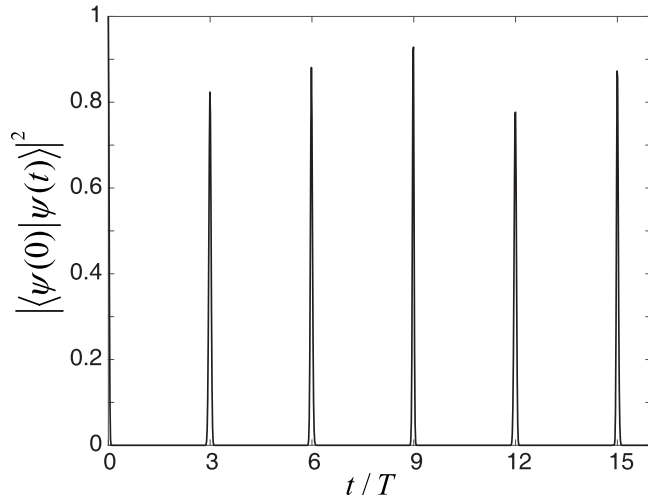


FIG. 1. Return probability $P_r(t) = |\langle \psi(0) | \psi(t) \rangle|^2$ for a state $|\psi(t)\rangle$ of the periodically driven Bose-Hubbard dimer (1) with $N = 2000$ particles. Here, the time t is scaled with respect to the cycle duration $T = 2\pi/\omega$, revealing 1:3 subharmonic clocking. Dimensionless system parameters are $N\kappa/\Omega = 0.92$, $\mu/\Omega = 0.4$, and $\omega/\Omega = 1.9$.

systems. In Sec. IV, we combine these two strands, applying the quantization rules to almost integrable mean-field motion prevailing in resonance zones surrounded by a chaotic sea. This leads to the concept of pre-Floquet states with a period other than the driving period that are capable of sustaining longtime subharmonic quantum evolution. Section V suggests the possibility to engineer high-order subharmonic response experimentally, employing Floquet condensates. In the Appendix, we argue that there exists an inevitable residual hybridization of regular and chaotic Floquet states, effectuating gradual dephasing of coherent subharmonic dynamics.

II. PRELIMINARIES: HISTORICAL DEVELOPMENT OF THE FLOQUET PICTURE AND SOME OF ITS SUBTLETIES

As is common practice by now, the investigation of the quantum dynamics generated by some periodically time-dependent Hamiltonian $H(t) = H(t + T)$ on its Hilbert space $\mathcal{H}$ proceeds by means of the Floquet approach, as successively developed in, among others, Refs. [20–26], presently providing the theoretical backbone for the discussion of Floquet time crystals [6–15]. Essentially, Floquet states constitute solutions of the time-dependent Schrödinger equation that have the temporal Bloch form

$$|\psi_\ell(t)\rangle = |u_{\ell,0}(t)\rangle \exp(-i\varepsilon_{\ell,0}t/\hbar) \quad (2)$$

with T -periodic Floquet function $|u_{\ell,0}(t)\rangle = |u_{\ell,0}(t + T)\rangle$ and quasienergy $\varepsilon_{\ell,0}$. Here, we assume that the set of all these functions with discrete state index $\ell = 1, 2, 3, \dots$ provides a complete orthonormal system in $\mathcal{H}$ at each instant t , as is automatically guaranteed if $\mathcal{H}$ is of finite dimension. In particular, this holds for the driven Bose-Hubbard dimer (1) when working with a fixed number N of particles, since then the dimension of its state space $\mathcal{H}_N$ amounts to $\dim \mathcal{H}_N = N + 1$. The rationale for attaching the additional index “0” to both the

Floquet functions and the quasienergies, but not to the Floquet states (2) themselves, will become evident below.

Inserting such a Floquet state into the time-dependent Schrödinger equation, its Floquet function is seen to obey the equation

$$\left(H(t) - i\hbar \frac{d}{dt}\right) |u_{\ell,0}(t)\rangle = \varepsilon_{\ell,0} |u_{\ell,0}(t)\rangle. \quad (3)$$

This appears to be an eigenvalue equation for the quasienergy $\varepsilon_{\ell,0}$, but within standard quantum mechanics terminology it is not, since the temporal derivative $-i\hbar d/dt$ is not a proper self-adjoint operator on $\mathcal{H}$. This seemingly formal problem that actually has profound implications for physics is resolved by “lifting” Eq. (3) from $\mathcal{H}$ to an extended Hilbert space that has been introduced into physics by Sambe [24], and now commonly is denoted as $\mathcal{K} = L^2[0, T] \otimes \mathcal{H}$ in the mathematical literature [27,28]. This extended space is comprised *only* of square-integrable functions that *a priori* are T -periodic in time. The extension consists in the requirement that the time t plays the role of an additional *coordinate* in $\mathcal{K}$, rather than a variable that parametrizes the flow as it does in $\mathcal{H}$. Consequently, the scalar product on $\mathcal{K}$ naturally involves the integration over the time coordinate. Following Sambe, this scalar product is indicated by double brackets [24],

$$\langle\langle u | v \rangle\rangle = \frac{1}{T} \int_0^T dt \langle u(t) | v(t) \rangle, \quad (4)$$

where $|u(t)\rangle$ and $|v(t)\rangle$ are T -periodic functions, not necessarily Floquet functions in the sense of Eq. (3), and $\langle u(t) | v(t) \rangle$ are their scalar products on $\mathcal{H}$, evaluated at the instants t . In accordance with this notation, such functions $|u(t)\rangle$ and $|v(t)\rangle$, when regarded as elements of $\mathcal{K}$, are written as $|u\rangle\rangle$ and $|v\rangle\rangle$, respectively.

The suggestive equation (3) then translates into a well-formulated quantum mechanical eigenvalue problem on $\mathcal{K}$ in the form

$$K |u\rangle\rangle = \varepsilon |u\rangle\rangle, \quad (5)$$

where

$$K = H(t) + p_t \quad (6)$$

is the quasienergy operator acting on $\mathcal{K}$, and

$$p_t = \frac{\hbar}{i} \frac{d}{dt} \quad (7)$$

actually serves as a proper momentum operator on $\mathcal{K}$ that is conjugate to the new coordinate t , because of the periodic boundary condition imposed on all elements of $\mathcal{K}$.

Now, there is an important observation to be made. With $\omega = 2\pi/T$ and integer $m = 0, \pm 1, \pm 2, \dots$, the functions

$$|u_{\ell,m}(t)\rangle = |u_{\ell,0}(t)\rangle \exp(im\omega t) \equiv |u_{\ell,m}\rangle \quad (8)$$

likewise are T -periodic and solve Eq. (3), or the eigenvalue problem (5), with quasienergy

$$\varepsilon_{\ell,m} = \varepsilon_{\ell,0} + m\hbar\omega; \quad (9)$$

for obvious reasons, we refer to the index m as the “photon index.” Clearly, such solutions (8) with the same state index ℓ and arbitrary photon index m produce *the same* Floquet state (2), but they constitute *different* solutions of Eq. (5),

all of them being individually required for the completeness relation on $\mathcal{K}$. Hence, a quasienergy with state index ℓ is not a mere number, but a set of equivalent representatives spaced by $\hbar\omega$. The quasienergy spectrum therefore extends itself periodically over the whole energy axis from $-\infty$ to $+\infty$ even if $\mathcal{H}$ is of finite dimension, with one representative of the quasienergy of each state located in each interval of width $\hbar\omega$; with wording borrowed from solid-state physics, such an interval is termed Brillouin zone. The fact that the quasienergy spectrum is unbounded not only from above, but also from below, ties in with the observation that the momentum operator p_t appears only linearly in the quasienergy operator K , reminiscent of eigenvalue equations in relativistic quantum mechanics. Moreover, this unboundedness implies that, strictly speaking, there is no Floquet ground state.

A further noteworthy feature arises for systems with infinitely many states, each of them placing one quasienergy representative in each Brillouin zone when driven periodically in time, so that the quasienergies may cover these zones densely. It is then a fairly difficult task to decide whether or not the quasienergy spectrum has an absolutely continuous component, giving rise to diffusive energy growth [27,28]. Taking up the path-directing mathematical analysis by Bellissard [29], this question has become known as the quantum stability problem [30]. This needs to be kept in mind when considering the driven Bose-Hubbard dimer (1) in the formal limit $N \rightarrow \infty$. Some particular ramifications of this stability problem for asymptotically large, but still finite N will be elucidated in the Appendix.

Barring these subtleties, the underpinning for the line of reasoning adopted in the following stems from the observation that the quasienergy eigenvalue equation (5) is a *bona fide* conceptual equivalent of the time-independent Schrödinger equation for energy eigenvalues and eigenstates. This implies that one can transfer many techniques known from *time-independent* quantum mechanical problems, such as steady-state perturbation theory, to *periodically time-dependent* systems [24]. Thus, a general strategy for dealing with Floquet-type systems in an analytical manner requires to (1) first lift the problem of interest to the extended Hilbert space $\mathcal{K}$, (2) apply known techniques there, and then (3) project back to the physical space $\mathcal{H}$.

III. INTERMEDIATES: SEMICLASSICAL QUANTIZATION OF INTEGRABLE FLOQUET SYSTEMS

We will now follow an analogous route—lifting to an extended classical phase space, applying standard quantization procedures there, and then projecting back to the physical space—in order to obtain a semiclassical approximation to quasienergies for fully integrable periodically time-dependent systems [31]. As a reminder, let us recall the semiclassical Einstein-Brillouin-Keller (EBK) quantization procedure [32] of an integrable classical time-independent system with f degrees of freedom deriving from a Hamiltonian function $H_{\text{cl}}(p, q)$, where we write p for the momentum variables $p_1, \dots, p_f$, likewise q for their conjugate coordinates: Integrability implies that the system's phase space $\mathcal{P}$ is completely stratified into f -tori $\mathbb{T}_f$ that are invariant under the Hamiltonian flow [32–34]. Those tori that can “carry” a quantum

energy eigenstate are singled out by the Bohr-Sommerfeld-like conditions

$$\oint_{\gamma_k} p dq = 2\pi\hbar \left(n_k + \frac{\text{ind } \gamma_k}{4} \right), \quad (10)$$

where γ_k ($k = 1, \dots, f$) denote the topologically inequivalent contours around such a torus, n_k are integer quantum numbers, and $\text{ind } \gamma_k$ is a Maslov index that accounts for the turning points of the respective contour [32]. After transforming $H_{\text{cl}}(p, q)$ to action variables, and inserting the actions of the tori selected by conditions (10) into the transformed function, one thus obtains semiclassical approximations to the energy eigenvalues of the classical system's quantum counterpart.

When adapting this procedure to integrable periodically time-dependent systems governed by a Hamiltonian function $H_{\text{cl}}(p, q, t) = H_{\text{cl}}(p, q, t + T)$, one again requires an even-dimensional phase space with pairs of canonically conjugate momentum and position variables. Therefore, the time t that merely parametrizes the flow in the system's actual phase space $\mathcal{P}$ is being considered as a coordinate and augmented by a canonically conjugate momentum variable p_t , in precise analogy to the viewpoint adopted in quantum mechanics when proceeding from Eq. (3) to the eigenvalue equation (5), altogether providing an even-dimensional extended phase space denoted here as $\mathcal{T} = T \otimes \mathcal{P}$. Consequently, the correspondent of the quasienergy operator (6) now is the classical quasienergy function $K_{\text{cl}} = H_{\text{cl}} + p_t$. With the original time t being a coordinate on equal footing with q , one is forced to introduce a new quasitime τ in order to parametrize the flow generated by K_{cl} in $\mathcal{T}$, so that the Hamiltonian equations read

$$\begin{aligned} \frac{dq}{d\tau} &= \frac{\partial K_{\text{cl}}}{\partial p} = \frac{\partial H_{\text{cl}}}{\partial p}, \\ \frac{dp}{d\tau} &= -\frac{\partial K_{\text{cl}}}{\partial q} = -\frac{\partial H_{\text{cl}}}{\partial q} \end{aligned} \quad (11)$$

for the old pairs of positions and momenta, and

$$\begin{aligned} \frac{dt}{d\tau} &= \frac{\partial K_{\text{cl}}}{\partial p_t} = 1, \\ \frac{dp_t}{d\tau} &= -\frac{\partial K_{\text{cl}}}{\partial t} = -\frac{\partial H_{\text{cl}}}{\partial t} \end{aligned} \quad (12)$$

for the new one. With respect to the original dynamics generated by H_{cl} in $\mathcal{P}$, this system possesses a straightforward interpretation: The first of the strange-looking equations (12) will allow one to identify the auxiliary time τ with the actual time t , so that the proper Hamiltonian equations pertaining to H_{cl} are recovered from the set (11). The second of the equations (12) then implies that K_{cl} is a constant of motion in $\mathcal{T}$ that we designate as ε ,

$$K_{\text{cl}}(p, q, p_t, t) = \varepsilon, \quad (13)$$

constituting the classical analog of the quasienergy. Indeed, the conservation of K_{cl} is already evident from the fact that the latter is autonomous, that is, it does not depend on τ . Observe, however, that the unphysical classical ε can be given any value by fixing an arbitrary initial value for p_t .

Proposing mere integrability of this extended system does not suffice. We also have to postulate periodic boundary conditions in t for the invariant manifolds in $\mathcal{T}$, again in analogy

to the periodic boundary conditions imposed on the elements $|u\rangle$ of the extended Hilbert space $\mathcal{K}$. Here, we presuppose that the manifolds inherit the period T of their Hamiltonian, and therefore identify the coordinate $t = 0$ with $t = T$. Note that it is this seemingly natural assumption that will be relaxed in Sec. IV. With this proviso, we obtain $(f + 1)$ -tori $\mathbb{T}_{f+1}$ as required for EBK quantization, so that the extensions of the standard conditions (10) take the form

$$\oint_{\gamma_k} (pdq + p_t dt) = 2\pi\hbar \left(n_k + \frac{\text{ind } \gamma_k}{4} \right) \quad (14)$$

with $k = 1, \dots, f + 1$ in order to account for the added degree of freedom. From here we return to the physical phase space $\mathcal{P}$ with time t as a flow parameter. This means to identify τ with t , to get rid of p_t , and to cut the $(f + 1)$ -dimensional tori in such a way that they reduce to f -tori that flow in time, termed $\mathbb{T}_f^+$. To these purposes we shift the contours γ_k with $k = 1, \dots, f$ into a hyperplane $t = \text{const}$. This implies $dt = 0$, so that the first k conditions (14) for the semiclassical Floquet states re-acquire the familiar form (10). The remaining condition is brought back to $\mathcal{P}$ by exploiting the insight that the quasienergy function K_{cl} is a constant of motion in $\mathcal{T}$, as expressed by Eq. (13), giving $p_t = \varepsilon - H_{\text{cl}}(p, q, t)$. We then denote the periodic contour γ_{f+1} that is led along $\mathbb{T}_f^+$ in time as γ_t and observe $\text{ind } \gamma_t = 0$, since there are no “turning points in time.” Renaming the corresponding integer quantum number n_{f+1} as m , we now have

$$\int_{\gamma_t} (pdq - H_{\text{cl}} dt) + \varepsilon T = 2\pi\hbar m, \quad (15)$$

yielding

$$\varepsilon = -\frac{1}{T} \int_{\gamma_t} (pdq - H_{\text{cl}} dt) + m\hbar\omega \quad (16)$$

with $\omega = 2\pi/T$. This finally is the reward of the painstakingly tedious above reasoning: Besides the standard conditions (10), there is the additional rule (16) that provides a semiclassical approximation to the quasienergies. Remarkably, this rule already accounts for the familiar arrangement (9) of the quasienergy spectrum in Brillouin zones of width $\hbar\omega$, with m serving as photon index. These combined quantization conditions indeed furnish the correct quantum mechanical quasienergies of the periodically driven harmonic oscillator [31]. From here on, the construction of the semiclassical Floquet states in a WKB-type manner parallels the construction of semiclassical energy eigenstates [31, 35], but the technical details of this procedure are not needed for our present purposes.

IV. SYNTHESIS: PRE-FLOQUET STATES AND DYNAMICAL TUNNELING

The above semiclassical approach to quasienergies and Floquet wave functions will now be employed to investigate the subharmonic response of the Bose-Hubbard dimer to periodic driving. To this end, we will first consider the system’s classical-like mean-field dynamics, and then “requantize” the almost integrable component of the latter by means of relations (10) and (16).

Following Refs. [36, 37], the mean-field approximation to system (1) is obtained by replacing operator products $a_i^\dagger a_j$, when acting on an N -particle space $\mathcal{H}_N$, by $Nc_i^* c_j$, and decomposing the c -number amplitudes c_j into absolute values and phases according to

$$c_j(\tau) = |c_j(\tau)| \exp(i\theta_j(\tau)). \quad (17)$$

Defining the population imbalance

$$p = |c_1|^2 - |c_2|^2 \quad (18)$$

together with the relative phase

$$\varphi = \theta_2 - \theta_1, \quad (19)$$

the mean-field equations of motion then are equivalent to the equations of motion furnished by the dimensionless classical single-particle Hamiltonian function

$$H_{\text{mf}}(\tau) = \alpha p^2 - \sqrt{1 - p^2} \cos(\varphi) + 2\frac{\mu}{\Omega} p \sin\left(\frac{\omega}{\Omega} \tau\right) \quad (20)$$

that conforms to a periodically driven pendulum with momentum-shortened length [36, 37]. Here, we use the time variable $\tau = \Omega t$, and invoke the parameter

$$\alpha = N\kappa/\Omega \quad (21)$$

that is inversely proportional to the pendulum mass. Multiplication of H_{mf} by $N\hbar\Omega$ then procures approximate energies pertaining to the actual N -particle quantum system (1). Importantly, when the mean-field dynamics are compared to those of the N -particle system for various N , the ratio κ/Ω has to be adjusted such that the numerical value of α remains unchanged. Hence, when the strength $\hbar\Omega$ of the tunneling contact is kept constant while the particle number N is made large, as conducted in the present work, the interparticle interaction strength $\hbar\kappa$ has to be reduced accordingly. With this proviso, the very same phase-space structure can be juxtaposed to the quantum N -particle dynamics for *all* N , pictorially meaning that the phase space can be “filled” with an arbitrary number of particles.

In Fig. 2, we depict a Poincaré map for this driven pendulum (20) that visualizes the intersection of the Hamiltonian flow at time $t = 0$ with the phase-space plane [32, 38]. This map has been produced in the usual manner by integrating Hamilton’s equations for a set of appropriately placed initial conditions over a large number of driving periods $T = 2\pi/\omega$, and recording the image points once per period at $t = 0 \bmod T$. In addition, we have given each sequence of successors originating from one of the initial phase-plane points its own color [39]. Parameters employed here are $\alpha = 0.92$, $\mu/\Omega = 0.40$, and $\omega/\Omega = 1.90$, the same as in Fig. 1. As expected, this map features the coexistence of regular and chaotic motion that is typical for nonlinear Hamiltonian systems [38]. The large island of close-to-regular motion observed here stems from the principal resonance that occurs when the time required for one oscillation of the undriven pendulum about matches one driving period, so that the elliptic fixed point in its center indicates a stable T -periodic orbit. Given the pendulum’s single degree of freedom, the invariant curves surrounding this fixed point represent 1-tori $\mathbb{T}_1$ in the language of the preceding section; when continued

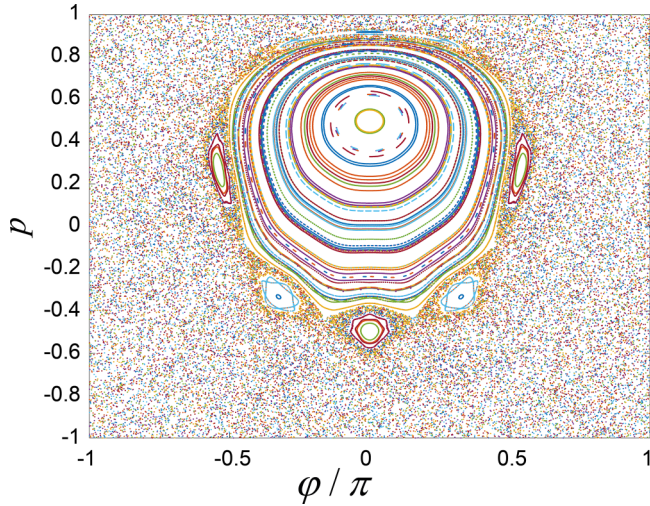


FIG. 2. Poincaré map generated at time $t = 0$ by the periodically driven pendulum (20), providing the mean-field description of the many-body system (1), with parameters $\alpha = 0.92$, $\mu/\Omega = 0.40$, and $\omega/\Omega = 1.90$.

in time, these flowing curves generate the T -periodic tubes $\mathbb{T}_1^+$ required by the semiclassical rules (10) and (16).

We now set out to verify that these rules, mathematically designed for fully integrable systems, also capture the exact N -particle quantum dynamics under the pseudo-integrable conditions prevailing in this island. Expressed in terms of the variables p, φ, τ appearing in the dimensionless Hamiltonian function (20), they take the forms

$$\oint_{\gamma_1} p d\varphi = 2\pi \hbar_{\text{eff}} \left(n + \frac{1}{2} \right),$$

$$\frac{\varepsilon}{N \hbar \Omega} = -\frac{1}{\Delta \tau} \int_{\gamma_1} (p d\varphi - H_{\text{mf}} d\tau) + m \hbar_{\text{eff}} \frac{2\pi}{\Delta \tau}. \quad (22)$$

The effective Planck constant $\hbar_{\text{eff}}$ introduced here is determined by the requirement that the total area $\Delta p \times \Delta \varphi$ of the phase-space plane has to accommodate the $N + 1$ Floquet states of the quantum system (1), each of these occupying an area $2\pi \hbar_{\text{eff}}$, thus demanding $2 \times 2\pi = 2\pi \hbar_{\text{eff}}(N + 1)$, or

$$\hbar_{\text{eff}} = \frac{2}{N} \quad (23)$$

for large N . Moreover, we have inserted in $\gamma_1 = 2$ for the two turning points of a contour γ_1 around $\mathbb{T}_1^+$, and have written $\Delta \tau = 2\pi \Omega/\omega$ for the scaled cycle duration.

Next, we utilize the coherent N -particle states [40]

$$|\vartheta, \varphi\rangle_N = \frac{1}{\sqrt{N!}} (A^\dagger(\vartheta, \varphi))^N |\text{vac}\rangle, \quad (24)$$

where the creation operators

$$A^\dagger(\vartheta, \varphi) = \cos \frac{\vartheta}{2} a_1^\dagger + \sin \frac{\vartheta}{2} e^{i\varphi} a_2^\dagger \quad (25)$$

act on the empty-dimer state $|\text{vac}\rangle$, so that the specific population imbalance (18) of such a state is given by $p = \cos^2(\vartheta/2) - \sin^2(\vartheta/2) = \cos \vartheta$, while φ coincides with the

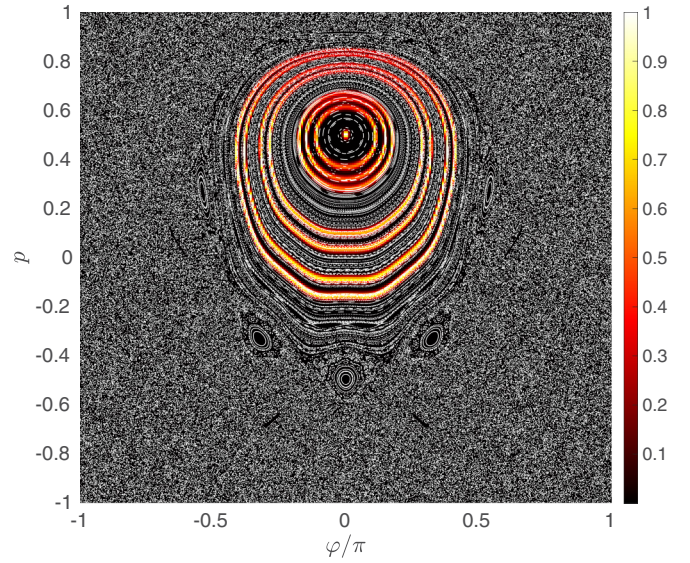


FIG. 3. Color-coded Husimi projections (26) of eight Floquet states $|\psi\rangle = |u(0)\rangle$ for $N = 10\,000$ particles, intersected at time $t = 0$, onto the surface of section shown in Fig. 2, here depicted in black and white. Observe that each of these states is localized on its respective closed contour γ_1 selected by the upper of conditions (22). Their semiclassical quantum numbers are $n = 0, 109, 193, 275, 767, 971, 1414$, and 1672 (inner to outer).

relative phase (19). Hence, the squared scalar product

$$Q_{|\psi\rangle}^{(N)}(p, \varphi) = |\langle \psi | \vartheta, \varphi \rangle_N|^2 \quad (26)$$

reveals how strongly a given N -particle state $|\psi\rangle$ is associated with the phase-space point $(p = \cos \vartheta, \varphi)$; computation of this quantity (26) for all $-1 \leq p \leq +1$ and $-\pi \leq \varphi \leq +\pi$ provides a Husimi projection of that quantum state onto the classical phase-space plane.

In Fig. 3, we display such color-coded Husimi projections (26) of eight numerically computed Floquet states $|\psi\rangle = |u(0)\rangle$ for $N = 10\,000$ particles. One of these states emerges as the light spot in the middle of the central island, right around the elliptic fixed point, each of the others as one of the surrounding light closed curves covering their respective contour γ_1 as selected by the first of conditions (22). These states are superimposed on the mean-field Poincaré section, indicated here in black and white only. While Floquet states normally cannot be ordered with respect to the magnitude of their quasienergies, because of the unbounded Brillouin-zone-like quasienergy spectrum, Floquet states that are semiclassically associated with an island of almost regular mean-field motion can be well ordered with respect to their semiclassical quantum numbers [41]. Referring to the quantum number $n = n_1$ employed in the upper of the scaled conditions (22), the states portrayed in Fig. 3 carry the labels $n = 0, 109, 193, 275, 767, 971, 1414$, and 1672 , respectively (inner to outer). Hence, the Floquet state $n = 0$ that adheres most closely to the elliptic periodic orbit constitutes the resonance-induced effective ground state of the main regular island. The excited states, viewed here at $t = 0$ only, likewise cling to their respective invariant circles $\mathbb{T}_1$; when continued in time, they stick to the emanating tubes

$\mathbb{T}_1^+$. Thus, Fig. 3 provides a visible testimony of the fact that exact Floquet states that occupy predominantly regular regions of phase space are attached to their mean-field tubes in a semiclassical manner. Having descended before from the quantum mechanical N -particle level to a classical-like mean-field description, we refer to the return from that description to the full N -particle dynamics by means of conditions (10) and (16) as requantization [41].

Still, from the viewpoint of rigorous mathematics, there is an objection that should be taken into account at this place. If the mean-field motion within the almost regular islands indeed were fully integrable, the Bohr-Sommerfeld-like rules would always find their exact contours γ_1 . Actually, however, there is fine-scale chaos even within the seemingly regular islands, due to the destruction of all those tubes for which the ratio between the driving frequency and the unperturbed pendulum frequency is rational, giving way to chains of alternating elliptic and hyperbolic fixed points with the associated homoclinic tangles, in accordance with the Poincaré-Birkhoff theorem [32,34,38]. Thus, there are chaotic dynamics “squeezed” between the remaining preserved tubes, self-similar on all (infinitesimally fine) scales, although these are not accessible to finite-precision arithmetics. Hence, those tubes which fulfill the quantization condition identically (in the absolute mathematical sense) may not exist, even if there are others that come arbitrarily close. As long as N remains finite, so that each quantum state occupies a finite area $2\pi\hbar_{\text{eff}}$, the N -particle system will be unable to resolve such hyperfine phase-space structures, which therefore remain practically undetectable. On the other hand, this implies that the formal limit $N \rightarrow \infty$ is not well controlled.

For driven nonlinear oscillators with only one degree of freedom, such as those considered here, the preserved tubes represent impenetrable barriers for each trajectory, because of the uniqueness theorem for ordinary differential equations. Therefore, a mean-field trajectory starting within a mainly regular island, chaotic or not, remains within this island forever. This is different for quantum mechanical wave packets, as discussed in the Appendix.

Around the main regular island, Fig. 2 also reveals a chain of six smaller secondary islands, one of them squeezed at the top. The alternating coloring within this chain indicates that it disaggregates into two separate subchains, each consisting of three islands, so that the elliptic fixed point in the center of each island stems from a periodic orbit that closes in itself after three driving periods. Therefore, the invariant curves surrounding them represent sections of two disconnected sets of $3T$ -periodic tubes with the Poincaré plane. To make the underlying geometry more clear, we sketch in Fig. 4 the hypothetical case of a single 1:2 resonance. The large central tube depicted there visualizes a T -periodic flowing contour $\mathbb{T}_1^+$ as considered before. Winding around it there are two narrower tubes that are displaced against each other by one driving period T , otherwise being identical, and therefore provide two intersections with the Poincaré plane. The simultaneous existence of tubes with different periods is possible in nonintegrable systems only, since such tubes necessarily have to be separated by zones of chaotic motion within which no invariant manifolds exist. Disregarding these zones, one can now apply the semiclassical rules (22) to each of these

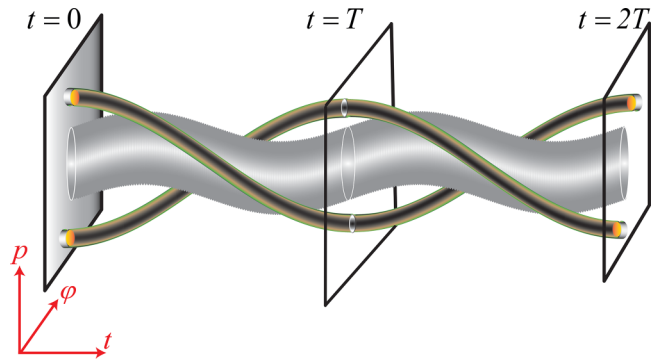


FIG. 4. Phase-space geometry pertaining to a hypothetical 1:2 resonance (schematically). The central T -periodic mean-field tube provides a proper approximate N -particle Floquet state upon semiclassical “requantization.” The two $2T$ -periodic tubes winding around it yield two $2T$ -periodic pre-Floquet states. Taking even and odd superpositions of these, thus accounting for dynamical tunneling between them, gives two approximate Floquet states.

secondary tubes individually, in the nonrigorous sense discussed above, replacing the scaled driving period $\Delta\tau$ by the tube period $2\Delta\tau$. As required by the second of these rules, the width of the entailing quasienergy Brillouin zone then shrinks from $\hbar\omega$ to $\hbar\omega/2$. Moreover, since there are two equivalent tubes, the semiclassical states obtained by single-tube quantization appear in doublets with still identical quasienergies. But evidently each of these $2T$ -periodic semiclassical states alone cannot yet approximate a proper quantum mechanical Floquet state, since Floquet functions inevitably are strictly T -periodic. For this reason, we denominate states constructed by semiclassical quantization of single tubes possessing a period other than T pre-Floquet states.

The situation encountered here closely parallels the double-well paradigm alluded to in the Introduction. Semiclassical quantization of the motion in each well of a symmetric double-well potential, disregarding tunneling through the barrier, provides pre-eigenstates with identical energies. However, these states do not respect the actual reflection symmetry of the system. This symmetry is restored when accounting for quantum tunneling through the barrier by taking even and odd superpositions of the pre-eigenstates, thereby introducing a tiny energy splitting between the members of each doublet [42]. By analogy, even or odd superpositions of the pre-Floquet states derived from a 1:2 resonance acquire the proper translational symmetry in time. Here, we do not find tunneling through a barrier, but dynamical tunneling through a chaotic zone of phase space between symmetry-related islands, akin to the quantum dynamical tunneling in bound states pioneered by Davis and Heller [43]. By the same token, such even or odd superpositions of pre-Floquet states also give rise to a tiny quasienergy splitting. This implies that initial states $|\psi(0)\rangle$ prepared in a single pre-Floquet state will tunnel from one tube to the other on a rather long timescale that is inversely proportional to that splitting. For the paradigmatic case of a 1:2 resonance, the splitting of each individual Floquet-state doublet can approximately be expressed analytically in terms of characteristic values of the Mathieu equation [44–46]; higher resonances are treated accordingly. In particular,

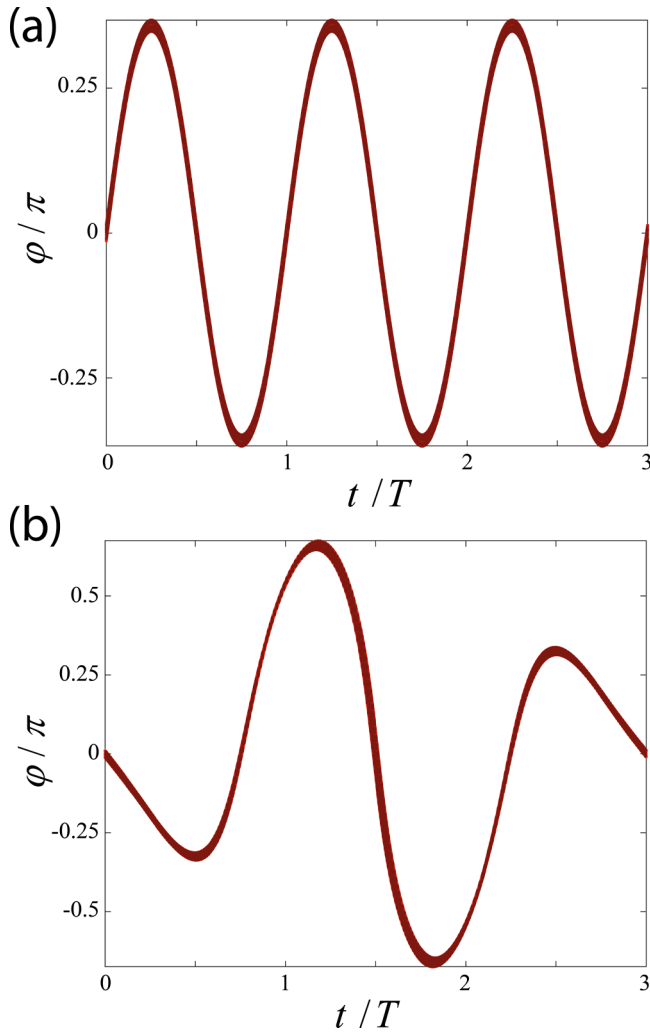


FIG. 5. (a) Projection of a tube obtained by following an invariant contour surrounding the central elliptic fixed point of the main regular island depicted in Fig. 2 in time. Such tubes are T -periodic, providing T -periodic Floquet states upon semiclassical quantization. (b) Projection of a tube generated by following a contour surrounding the central elliptic fixed point of the lowest secondary island observed in Fig. 2 in time. Such tubes are $3T$ -periodic, and therefore provide $3T$ -periodic pre-Floquet states that effectuate the 1:3 subharmonic clocking recognized in Fig. 1.

dynamical tunneling within the driven Bose-Hubbard dimer (1) has been studied in some detail both analytically and numerically for moderate particle numbers [47].

Coming back to our guiding numerical example, we display in Fig. 5(a) the projection from (p, φ, t) space to the (φ, t) plane of a tube that emanates from a contour encircling the central elliptic fixed point inside the main regular island observed in Fig. 2. Such tubes are T -periodic and thus provide the scaffolds for semiclassical Floquet states that effectuate standard 1:1 clocking, akin to the wide T -periodic tube sketched in Fig. 4. In contrast, Fig. 5(b) depicts the projection of a tube generated by a contour surrounding the central elliptic fixed point in the lowest island of the secondary chain. Evidently, this projection closes in itself after three driving periods, confirming that the ostensive chain of

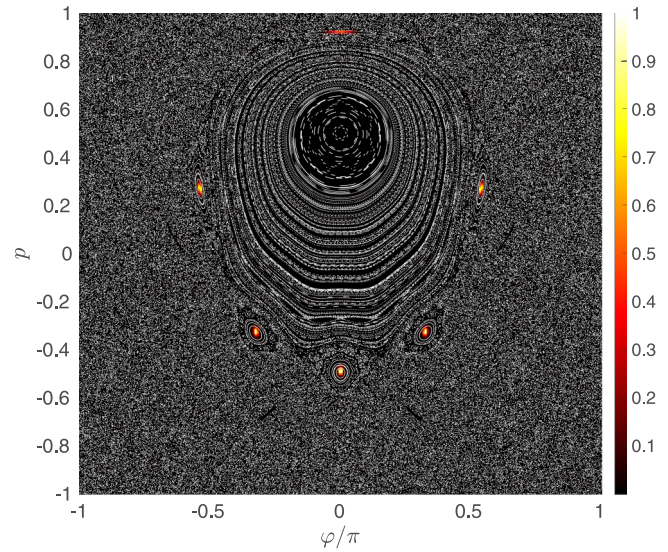


FIG. 6. Husimi projection of one representative of the six Floquet states associated with the innermost quantized tubes surrounding the elliptic periodic orbits belonging to the secondary six-island chain observed in Fig. 2. The occupation of all six islands indicates not only tunnel coupling of three $3T$ -periodic pre-Floquet states provided by one set of $3T$ -periodic tubes, but also hybridization with those obtained from the second set. The particle number is $N = 10\,000$, as in Fig. 3.

six islands actually consists of two disconnected three-island subchains. Thus, there are two differences in comparison to the pedagogical Fig. 4: The tubes showing up here possess the period $3T$, and there exist even two sets of such tubes. Each set thus provides $3T$ -periodic tunneling-coupled pre-Floquet states upon semiclassical quantization, linear superpositions of which yield proper Floquet states.

Inspecting the Husimi projection of one of the six numerically computed Floquet states with $N = 10\,000$ Bose particles and semiclassical quantum number $n = 0$ referring to the six-island chain seen in Fig. 2, we find in Fig. 6 occupation of not only one of the two disconnected subsets of three islands, but also of the other one. Therefore, these Floquet states do result not only from tunnel coupling of three $3T$ -periodic pre-Floquet states, but also from hybridization with the other three.

Notwithstanding this additional subtlety, an initial state $|\psi(0)\rangle$ placed on only one of the six islands is composed mainly of the associated $3T$ -periodic pre-Floquet states, instead of T -periodic Floquet states, and therefore will feature $3T$ -periodic time evolution on time scales which are short in comparison with the tunneling times. This is confirmed in Fig. 7 that depicts the time evolution of an initial coherent state (24) with $N = 2000$ particles placed right upon the elliptic fixed point of the lowest secondary island in Fig. 2, that is, with parameters $p = \cos \vartheta = -0.497$ and $\varphi = 0.0$. Here, we employ the Fock states $|j, N - j\rangle$ of the Bose-Hubbard dimer with j particles occupying site 1 and, accordingly, $N - j$ particles occupying site 2, and plot the color-coded evolution of their occupation probabilities

$$F(j; t) = |\langle \psi(t) | j, N - j \rangle|^2. \quad (27)$$

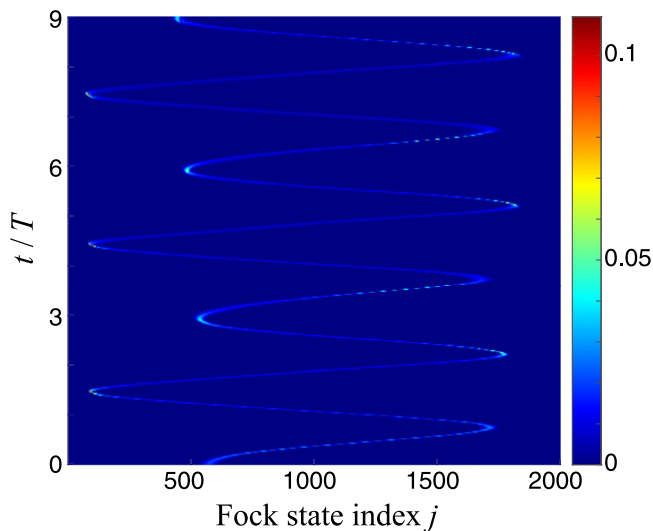


FIG. 7. Time evolution (27) of an initial coherent state (24) with $N = 2000$ particles and parameters $p = \cos \vartheta = -0.497$ and $\varphi = 0.0$ that specify the central elliptic fixed point in the lowest of the six secondary islands seen in Fig. 2, viewed in the Fock basis $|j, N - j\rangle$ of the Bose-Hubbard dimer. This initial state $|\psi(0)\rangle$ is composed mainly of the $3T$ -periodic pre-Floquet states associated with this island, giving rise to longtime coherent subharmonic motion, and causing the 1:3 subharmonic return probability $P_r(t)$ shown in Fig. 1.

Note that Fig. 7 has been obtained by sheer numerical computation, not taking recourse to any semiclassical approximation or reasoning. The evolving state remains coherent on the short time interval displayed here, and exhibits almost perfect subharmonic 1:3 motion; this is precisely the setting that leads to the 1:3 clocking found in Fig. 1.

V. OUTLOOK: SUBHARMONIC RESPONSE OF FLOQUET CONDENSATES

The pre-Floquet states expounded on in the present work are a semiclassical, and hence approximate concept, glossing over imperfect integrability within the mean-field resonance zones, and ignoring complications due to proliferating near-degeneracies of quasienergies. As laid out in the Appendix, the latter imply intricate hybridization of regular and chaotic states. Nonetheless, such pre-Floquet states are well suited to approach longtime, albeit not perpetual subharmonic response of T -periodically driven many-boson systems on an intuitive level, giving access to wave packets that follow periodic mean-field orbits having a period that is an integer or fractional multiple of T , to multiplets of quasienergies together with dynamical tunneling effects lifting their degeneracy, and to the ultimate decay of subharmonic motion. Being semiclassically constructed in a WKB-type way from only one of the nT -periodic invariant mean-field tubes pertaining to an island chain with n individual islands, nT -periodic pre-Floquet states are related to genuine Floquet states in a manner similar to the relation of site-localized Wannier states to lattice-extended Bloch waves in solid-state physics, although the number of “lattice” sites is relatively small here, being given by the number of islands in the chain considered. The reasoning put

forward in the Appendix suggests that pre-Floquet states span well-isolated subspaces of regular subharmonic motion that are only very weakly coupled to a “bath” of chaotic states.

While the mechanism for subharmonic generation investigated in the present case study does not involve many-body localization that would be a characteristic prerequisite for genuine discrete time crystals [4–15], it does heavily rely on coherence. The return from the mean-field level to the full many-body dynamics by semiclassical requantization with the help of conditions (10) and (16) can be meaningfully made only if the solutions to the mean-field equations of motion, effectively describing single-particle dynamics, represent macroscopically occupied single-particle Floquet states, i.e., Floquet condensates. Seen from this perspective, the scenario exemplified in the preceding section constitutes a straightforward adaptation of another prescription for subharmonic generation in single-particle quantum systems [45,46]. A distinct new twist coming into play here is the appearance of an effective many-body Planck constant (23) that is inversely proportional to the particle number N : The larger N , the smaller $\hbar_{\text{eff}}$, and the finer the details of the mean-field phase space that the quantum N -particle system is able to resolve [41]. Therefore, it is the magnitude of N that decides whether or not a requantized mean-field tube fits into an island of regular motion in accordance with the first of conditions (22), providing a semiclassical pre-Floquet state. We surmise that this feature remains decisive, with appropriate changes and extensions, also for experimental setups that are more complex than the driven Bose-Hubbard dimer (1). This is of interest insofar as the classical Hamiltonian dynamics of nonintegrable systems are self-similar on all scales [34]. With regard to our sketchy Fig. 4, this means that there actually exists an infinite hierarchy of “tubes winding around tubes that wind around tubes.” These should be detectable in principle in experiments with Floquet condensates with gradually increased numbers of particles and Feshbach-reduced inter-particle interaction strengths, varied such that the product of both quantities remains constant in order to approach the mean-field regime. It should be kept in mind, however, that in contrast to time-independent Bose-Einstein condensates, which occupy true ground states, Floquet condensates are metastable [41], as outlined in the Appendix, with lifetimes that still need to be determined.

As an extension along these lines of our numerical example, we depict in Fig. 8 a magnification of one of the second-order islands previously observed in Fig. 2: Here, one detects a surrounding third-order chain consisting of six islands that again produces sets of invariant tubes; when $\hbar_{\text{eff}}$ is made sufficiently small, these tubes in their turn host pre-Floquet states with return period $6 \times 3T = 18T$. Even when N is still not large enough, so that the mean-field regime is not yet reached to the extent that a quantized tube would fit fully into a regular island that small, signs of the perfect mean-field high-order subharmonic motion can already manifest themselves in the exact time evolution of the N -particle system. This is demonstrated in Fig. 9(a) that highlights the return probability $P_r(t)$ for an initial coherent state (24) with $N = 2000$ particles starting from the third-order island with $p = -0.4278$ and $\varphi = 0.0$. Although the strict mean-field 1:18 subharmonic clocking cannot be realized perfectly under

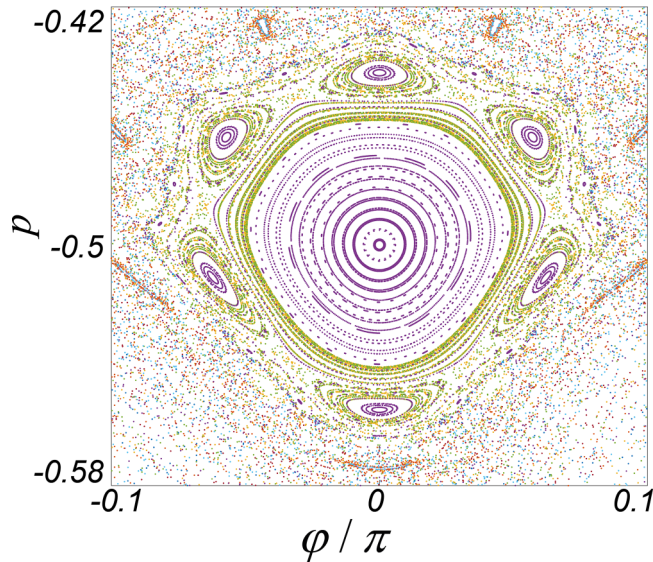


FIG. 8. Magnification of the lowest secondary resonant island observed in Fig. 2, revealing a surrounding further, third-order chain of six equivalent islands of regular motion. When $\hbar_{\text{eff}}$ is made sufficiently small, each of these islands can host pre-Floquet states with period $18T$, giving rise to 1:18 subharmonic clocking.

these conditions, and signals related to the $3T$ -periodic parent tubes still appear at most integer multiples of $3T$, indications of that expected high-order clocking are evident. Even more striking, when the particle number is increased to $N = 5000$, so that $\hbar_{\text{eff}}$ is reduced by a factor of 0.4, the side peaks are suppressed markedly and the 1:18 clocking stands out in an impressive manner, as witnessed by Fig. 9(b). This example illustrates how to engineer high-order subharmonic response in a deliberate and systematic way.

Naturally, the question emerges whether concepts derived from the investigation of the two-site Bose-Hubbard dimer can be carried over to periodically driven many-body models with more sites. This question leads to a further demanding challenge, since even only three interacting sites without periodic forcing correspond to two coupled mean-field pendula endowed with a four-dimensional phase space $\mathcal{P}$, and therefore yield the familiar mixture of regular and chaotic motion by themselves [38]. Installing time-periodic driving then adds a new feature: Namely, while a 2-torus embedded in a four-dimensional phase space divides that space into an interior and an exterior region, this is no longer the case in higher dimensions, i.e., for systems with more than two degrees of freedom. Therefore, such high-dimensional tori do no longer constitute “impenetrable diffusion barriers” for chaotic motion in their extended phase space $\mathcal{T}$, but allow for Arnold diffusion [48–50]. Thus, a driven Bose-Hubbard trimer appears to be a prime candidate for studying the implications of mean-field Arnold diffusion for exact N -particle dynamics.

Other than that, our technique indeed does apply to driven many-site systems: Find periodic mean-field orbits with a period other than T and check that these are surrounded by zones of near-integrable motion. Provided the particle number N is sufficiently large so that “quantizable” tubes fit into these

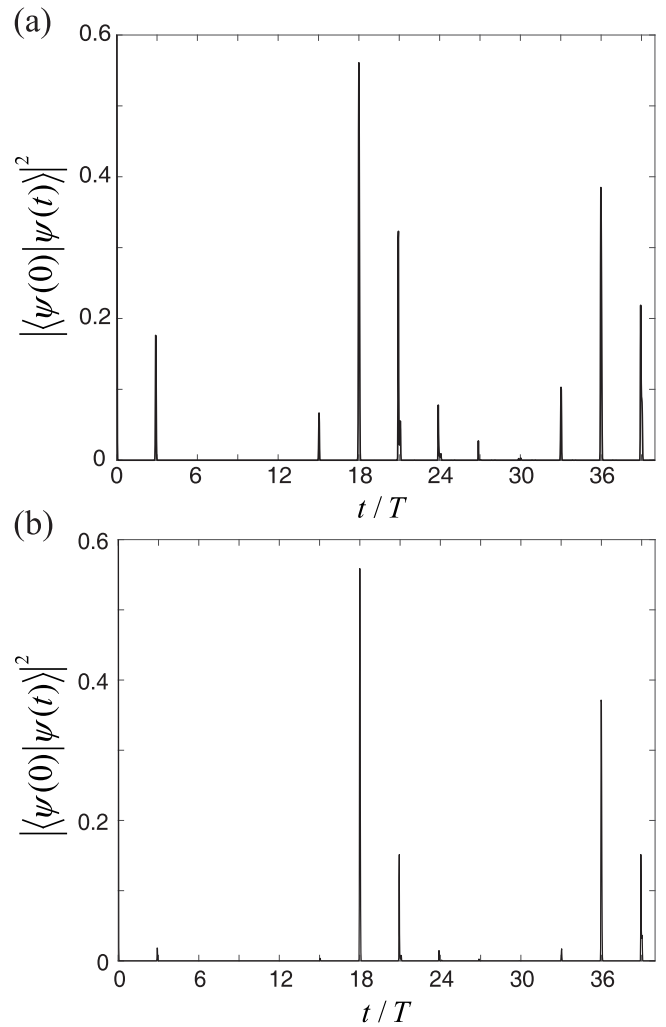


FIG. 9. (a) Return probability $P_r(t) = |\langle\psi(0)|\psi(t)\rangle|^2$ for an initial coherent state (24) with $N = 2000$ particles placed on the lowest third-order island visible in Fig. 8 with $p = -0.4278$ and $\varphi = 0.0$. Although $\hbar_{\text{eff}} = 2/N$ is not small enough to accommodate a quantized tube conforming to Eq. (22) in that island, indications of the mean-field 1:18 clocking are already showing up here. (b) As above, but with $N = 5000$ particles. With $\hbar_{\text{eff}}$ being reduced, the side peaks still present in panel (a) are substantially suppressed, and 1:18 clocking unmarks itself.

zones, there will be quantum N -particle states that exhibit almost perfect subharmonic motion for many driving periods, under conditions such that exact many-body calculations are way beyond the capabilities of present computers.

We conclude that the experimental observation of high-order subharmonic motion, and of signatures of dynamical tunneling between the pre-Floquet states involved, would constitute a novel route towards unraveling the classical-quantum correspondence, as well as its limitations. Arguably, a major challenge for future laboratory experiments with Floquet condensates aiming in this direction would be the preparation of the required initial pre-Floquet states $|\psi(0)\rangle$. This demand might potentially be matched by turning on the periodic drive in an adiabatic manner, possibly involving simultaneous variation of more than one parameter in order to guide an initial

time-independent Bose-Einstein condensate coherently into targeted pre-Floquet states, leaving ample opportunities to break genuinely new ground.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [39].

APPENDIX: PHASE-SPACE FILLING AND BEYOND

As discussed in Sec. IV, a mean-field trajectory of the driven Bose-Hubbard dimer that starts in one of the almost regular islands remains confined to its island chain forever. In this Appendix, we argue that this is not the case for the exact quantum mechanical N -particle system, for which Floquet states and pre-Floquet states semiclassically connected to an island actually remain weakly coupled to the “bath” provided by the surrounding stochastic sea, eventually decohering harmonic and subharmonic response likewise. Especially this occurs for arbitrarily large, finite N , that is, when the classical mean-field phase space is filled with an arbitrarily large number of particles.

The quasienergy operator (6) of the driven Bose-Hubbard dimer possesses a space-time symmetry: Swapping the site indices in combination with a shift in time by half a period according to

$$P : \begin{cases} 1 & \longleftrightarrow & 2, \\ t & \longrightarrow & t + \pi/\omega \end{cases} \quad (\text{A1})$$

leaves K unchanged. Clearly, this generalized parity operation P refers to Sambe’s extended Hilbert space $\mathcal{K}$ in which t is a cyclic coordinate, so that $P^2 = 1$, but not to the physical space $\mathcal{H}$. Hence, the eigensolutions $|u\rangle$ of K are even or odd under P . Notably, solutions $|u_{j,m}\rangle$ that provide the same Floquet state j while differing in photon index m by an odd integer have different generalized parity, but this does not show up in the Floquet states themselves. Now the Neumann-Wigner noncrossing rule states that eigenvalues of a Hermitian operator which belong to the same symmetry class generally do not cross when only one system parameter is varied [51], with the possible exception of certain diabolical points [52]. This applies, in particular, to the eigenvalues $\varepsilon_{j,m}$ of K :

Upon variation of, say, the scaled driving amplitude μ/Ω , quasienergy representatives $\varepsilon_{j,m}$ provided by eigensolutions $|u_{j,m}\rangle$ possessing the same generalized parity under operation (A1) generically avoid each other, no matter whether their Floquet states are linked to a regular island, or to the stochastic sea. This established fact has to be viewed against the finding that quasienergy eigenvalues tend to fill the Brillouin zones densely when N is increased without bound, so that an arbitrarily small quasienergy interval around each representative may contain an arbitrarily large number of other representatives belonging to the same parity class. This necessarily leads to a complex net of tiny avoided crossings when μ/Ω is varied, presumably such that fine anticrossings are interspersed with even finer ones, and so on. In any case, the approach to the limit $N \rightarrow \infty$ is highly nontrivial [53]. Such manifold avoided crossings, in their turn, inevitably imply intricate hybridization of the participating states, not discriminating between “regular” and “chaotic” ones. Therefore, a wave packet initially placed on an almost regular island, being composed of large- N Floquet states associated with both that island and the stochastic sea, will eventually leak out of the island, on time scales determined by the width of the hyperfine anticrossings. In this regard, the quantum dynamics remain fundamentally different from their mean-field description even when the phase space is filled with an arbitrarily large number N of Bose particles.

While this imaginary construction still does not cover the virtual limit “ $N = \infty$ ” itself that is supposed to be singular, it seems to indicate a precursor of the quantum stability problem that manifests itself here already for macroscopically large, but finite N . The latter problem is driven by the mathematical question whether a dense point spectrum turns into a continuous one under small perturbations [27–30]. In the present case, one encounters Floquet states that are distinctly categorized as either “regular” or “chaotic” when viewed semiclassically in a coarse-grained manner, but actually remain weakly coupled to each other on a more rigorous level, giving rise to long-time dynamics reminiscent of the decay of a quantum state embedded in a continuum.

The physical picture that emerges here is that of almost isolated subsystems of Floquet states or pre-Floquet states associated with the almost regular islands that are capable of sustaining longtime harmonic or subharmonic response to periodic driving, but eventually will decohere on timescales that, arguably, would considerably exceed the dynamical tunneling times of typical pre-Floquet states. Thus, such residual couplings might go unnoticed in both finite-precision numerical simulations and finite-time laboratory experiments.

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