# Exchange-symmetrized qudit Bell bases and Bell-state distinguishability

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Entanglement of qudit pairs, with single-particle Hilbert space dimension $d$, has important potential for quantum information processing, with applications in cryptography, algorithms, and error correction. For a pair of qudits of arbitrary even dimension $d$, we introduce a generalized Bell basis with definite symmetry under exchange of internal states between the two particles. We show that no complete exchange-symmetrized basis can exist for odd $d$. This framework extends prior work on exchange-symmetrized hyperentangled qubit bases, where $d$ is a power of 2. For our exchange-symmetrized basis, we show that measurement devices restricted to linear evolution and local measurement (LELM) can unambiguously distinguish $2d-1$ qudit Bell states for any even $d$. This achieves the upper bound in general for reliable Bell-state distinguishability via LELM and augments previously known limits for $d=2^n$ and $d=3$. This result is relevant to near-term realizations of quantum communication protocols.

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## I. INTRODUCTION

Entanglement is a key resource in quantum algorithms and communication protocols, from prime factorization and machine learning [1–5] to quantum secret sharing [6–8]. Most schemes employ entangled qubits, but some are based on entangled qudits—particles with $d>2$ orthogonal states each [9,10]. Although qudits can exhibit larger error channels than qubits [11], they offer several advantages, including increased information density ($\propto d^n$ rather than $2^n$ for $n$ particles), lower decoherence in the underlying physical systems [12,13], and the ability to encode redundancy in additional dimensions [12,14,15].

Experimental demonstrations of qudit-based schemes include a variety of quantum communication applications [16–19] and quantum algorithms [20–23]. Central to many of these protocols is measurement in an entangled basis, which necessarily involves unentangling logic gates for complete, deterministic results. These devices are challenging to implement, especially for photonic states, so many realizations use only linear evolution and local measurement (LELM), which is limited to noncomplete Bell-state measurement [24,25]. In an LELM device, each particle's evolution through the apparatus is independent of the other particle, and local measurements are made. LELM devices have been used to implement quantum teleportation and dense coding protocols in qubits [26–29], teleportation in qutrits [30], and dense coding in ququarts [31]. Given the relevance of qudit platforms and the use of LELM to enable quantum communication protocols experimentally, it is worthwhile to explore the structure of general Bell bases whose properties can be leveraged for Bell-state measurement.

In this Letter, we present a general complete basis of fully entangled bipartite qudit states, i.e., a qudit Bell basis, with definite symmetry under particle internal state exchange for any even single-particle dimension $d$. We show that no complete symmetrized basis exists for odd $d$. Previously, a complete symmetrized basis had been presented only for $d=2^n$ [32,33]. Using this basis, we determine the maximum number of general Bell states that can be unambiguously distinguished by an LELM device when $d$ is any even number: $2d-1$. This result joins previous upper limits for $d=2^n$ [32] and $d=3$ [34].

We note that the particle exchange we consider here is not the fundamental exchange symmetry of indistinguishable particles, but symmetry with respect to exchange of internal states between the two entangled particles or subsystems, which are already distinct from each other in an external, typically spatial, degree of freedom. Thus, our qudit Bell basis consists of eigenstates of the swap operator with eigenvalues $\pm 1$.

Beyond applications to qudit Bell-state measurement without conditional logic, our basis may also have relevance to extending other qubit-based protocols that rely on exchange symmetry of the qubit Bell states. For example, such exchange symmetry is important in numerous quantum communication protocols [35–37] and quantum algorithms [38–45].

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## II. BACKGROUND

An arbitrary pure state of a qudit pair has the general form [33]

$$|\Psi\rangle = \sum_{i,j=0}^{d-1} \chi_{ij} |i\rangle_L |j\rangle_R, \tag{1}$$

where $L$ (left) and $R$ (right) denote the individual particles distinguished by a spatial degree of freedom: by being stored in distinct memory locations or transmitted along two different communication channels.

Such a pure state is fully entangled if its reduced density matrix (for either particle) is proportional to the identity [46]. A complete basis of fully entangled states is known as a generalized Bell basis, after the $d = 2$ basis of four fully entangled, mutually orthogonal qubit states. The most commonly used qudit Bell basis [47] defines a Bell state in terms of two parameters, $n, m$:

$$|\Psi_{nm}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{2\pi i n k/d} |k\rangle_L |(k+m) \bmod d\rangle_R. \tag{2}$$

Almost all generalized Bell-state literature uses this definition, up to a difference of using $(k-m) \bmod d$ for the right ket state value or swapping the internal states of the left and right particles [34,48–56]. The only exception [57] introduced a slight variation by involving both parameters in the phases:

$$|\Psi_{nm}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{-2\pi i (m-k) n/d} |k\rangle_L |(m-k) \bmod d\rangle_R. \tag{3}$$

For $d = 2$, Eq. (2) yields the canonical Bell states that are explicitly symmetric or antisymmetric (hence referred to as symmetrized) under exchange of the left and right particle states. However, for general $d$, the resultant states are not eigenstates of exchange [33]. Previous work [32,33] on $d = 2^n$ constructed a basis of fully entangled mutually orthogonal states by taking explicit tensor products of the $d = 2$ Bell states, thus constructing the higher-dimensional basis as hyperentanglement in $n$ qubit degrees of freedom. We extend this work by demonstrating a symmetrized Bell basis for any even $d$, independent of a hyperentangled framework.

## III. SYMMETRIC BASIS FOR EVEN DIMENSION

To aid in construction of this basis, we establish a notation that builds naturally on the canonical $d = 2$ Bell states:

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}} (|0\rangle_L |0\rangle_R \pm |1\rangle_L |1\rangle_R),$$
$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}} (|0\rangle_L |1\rangle_R \pm |1\rangle_L |0\rangle_R). \tag{4}$$

The labels $|\Phi\rangle$ versus $|\Psi\rangle$ for these states fill the role of $m$ in Eq. (2); we say these specify the *correlation class* of a Bell state, since they identify a particular set of pairings between single-particle basis states for the left and right particles. The $\pm$ in Eq. (4) fills the role of $n$ in Eq. (2); we say these specify the *phase class* of a Bell state since they determine the relative phase between terms. Based on this correspondence, we rename $n, m$, respectively, in Eq. (2) as $p, c$ to designate sets of Bell states that share correlations $c$ and phases $p$ [34].

By analogy with Eq. (2), we consider a basis of Bell states of the form

$$|\Psi_c^p\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{i\phi_{k,p}} |k\rangle_L |\pi_c(k)\rangle_R, \tag{5}$$

for $p, c \in [0, d-1]$, where $\pi_c$ is a permutation of the single-particle basis states. Individual states of this form are inherently fully entangled, so permutations $\pi_c$ and phase shifts $\phi_{k,p}$, as in Eq. (5), must be selected to construct $d^2$ states that are mutually orthogonal and symmetrized with respect to exchange of states between the left and right particles.

## IV. CORRELATION CLASSES FOR EXCHANGE SYMMETRY

To start construction of the exchange-symmetrized Bell basis, we can choose a set of permutations $\{\pi_c\}$ in Eq. (5) in a way that allows each Bell state to be symmetric or antisymmetric. Each permutation $\pi_c$ leads to a set of two-particle basis states $\{|k\rangle_L |\pi_c(k)\rangle_R\}$ that appear in $|\Psi_c^p\rangle$. These sets must satisfy the following conditions:

(1) If $|s_{jc}\rangle_L |t_{jc}\rangle_R$ is included in a set, so is $|t_{jc}\rangle_L |s_{jc}\rangle_R$, to allow symmetry with respect to particle exchange.

(2) No $|s_{jc}\rangle_L |t_{jc}\rangle_R$ may appear in multiple sets, so the Bell states in different correlation classes contain nonoverlapping two-particle basis states and are mutually orthogonal.

The correlation class arising from the trivial permutation, $\pi_0(k) = k$, certainly satisfies the first condition since it involves the two-particle basis states $\{|0\rangle_L |0\rangle_R, \ldots, |d-1\rangle_L |d-1\rangle_R\}$.

We can always find a set of $d-1$ additional correlation classes; the problem is equivalent to setting up a round-robin tournament of an even number of players. Consider a group of $d$ players representing the single-particle states from 0 to $d-1$. Each tournament "round" represents a distinct set of $d/2$ pairs of players. It is well known that a $(d-1)$-round complete set of pairings exists for any even $d$ [58]. Beginning from construction of a $d$-player round-robin tournament, if $(s_{jc}, t_{jc})$ play each other in round $c$, the two-particle basis states $|s_{jc}\rangle |t_{jc}\rangle$ and $|t_{jc}\rangle |s_{jc}\rangle$ are included in the Bell states of correlation class $c$. This prescription enforces our first condition above. Since each pairing can only occur once in the tournament, our second condition is satisfied as well.

This construction identifies one trivial correlation class ($c = 0$) and a further $d-1$ nontrivial correlation classes for use in Eq. (5).

## V. PHASE CLASSES FOR EXCHANGE SYMMETRY

Our $c \neq 0$ candidate Bell states now take the form

$$|\Psi_{c\neq 0}^p\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d/2-1} (e^{i\phi_{j,p}} |s_{jc}\rangle_L |t_{jc}\rangle_R + e^{i\eta_{j,p}} |t_{jc}\rangle_L |s_{jc}\rangle_R). \tag{6}$$

To enforce exchange symmetry, we require $e^{i\phi_{j,p}} = \pm e^{i\eta_{j,p}}$, reducing Eq. (6) to

$$|\Psi_{c\neq 0}^p\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d/2-1} e^{i\phi_{j,p}} (|s_{jc}\rangle_L |t_{jc}\rangle_R + (-1)^p |t_{jc}\rangle_L |s_{jc}\rangle_R). \tag{7}$$

This formulation also makes each Bell state pair $|\Psi_{c\neq 0}^{p=2\ell}\rangle$ and $|\Psi_{c\neq 0}^{p=2\ell+1}\rangle$ orthogonal. For mutual orthogonality of all Bell states within a correlation class, the coefficients $e^{i\phi_{j,p}}$ can now be chosen from the $(d/2)$th roots of unity as in a discrete Fourier transform (DFT):

$$\phi_{j,p} = \left\lfloor \frac{p}{2} \right\rfloor \frac{4\pi j}{d}. \tag{8}$$

To make the $c = 0$ Bell states orthogonal, there is more freedom in the coefficients of each $|k\rangle_L|k\rangle_R$, but for convenience, we choose these to match the $c \neq 0$ cases:

$$\left|\Psi_{c=0}^{p}\right\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d/2-1} e^{i\phi_{j,p}} (|2j\rangle_L|2j\rangle_R + (-1)^p|2j+1\rangle_L|2j+1\rangle_R), \tag{9}$$

with phases $\phi_{j,p}$ as in Eq. (8).

## VI. NUMBER OF SYMMETRIC VERSUS ANTISYMMETRIC STATES

Each $c \neq 0$ correlation class is evenly split between symmetric and antisymmetric states and the $c = 0$ correlation class contains only symmetric states, so in total there are $[(d-1)d]/2 + d = [(d+1)d]/2$ symmetric states and $[(d-1)d]/2$ antisymmetric states. These numbers match the case $d = 2^n$ [32,33]—for example, for $d = 2$, there are three symmetric states and one antisymmetric state: the triplet $(|\Phi^{\pm}\rangle, |\Psi^{+}\rangle)$ and singlet $(|\Psi^{-}\rangle)$ Bell states.

## VII. EXAMPLE SYMMETRIC BASIS FOR $d = 6$

We explicitly present an exchange-symmetric Bell basis for $d = 6$. First, we identify correlation classes. There are many different ways to construct these; choosing the pairings according to Ref. [58], one round-robin tournament configuration of six players 0–5 is

$$\begin{aligned}
&\text{Round 1}: [(0,1),(2,5),(3,4)],\\
&\text{Round 2}: [(0,2),(3,1),(4,5)],\\
&\text{Round 3}: [(0,3),(4,2),(5,1)],\\
&\text{Round 4}: [(0,4),(5,3),(1,2)],\\
&\text{Round 5}: [(0,5),(1,4),(2,3)].
\end{aligned}$$

To create the $c \neq 0$ correlation classes, we thus associate pairings $(s_{jc}, t_{jc})$ with basis states $|s_{jc}\rangle|t_{jc}\rangle$ and $|t_{jc}\rangle|s_{jc}\rangle$ to yield

$$\begin{aligned}
c=1:&\ \{|0\rangle|1\rangle, |1\rangle|0\rangle, |2\rangle|5\rangle, |5\rangle|2\rangle, |3\rangle|4\rangle, |4\rangle|3\rangle)\},\\
c=2:&\ \{|0\rangle|2\rangle, |2\rangle|0\rangle, |1\rangle|3\rangle, |3\rangle|1\rangle, |4\rangle|5\rangle, |5\rangle|4\rangle)\},\\
c=3:&\ \{|0\rangle|3\rangle, |3\rangle|0\rangle, |2\rangle|4\rangle, |4\rangle|2\rangle, |1\rangle|5\rangle, |5\rangle|1\rangle)\},\\
c=4:&\ \{|0\rangle|4\rangle, |4\rangle|0\rangle, |3\rangle|5\rangle, |5\rangle|3\rangle, |1\rangle|2\rangle, |2\rangle|1\rangle)\},\\
c=5:&\ \{|0\rangle|5\rangle, |5\rangle|0\rangle, |1\rangle|4\rangle, |4\rangle|1\rangle, |2\rangle|3\rangle, |3\rangle|2\rangle)\}.
\end{aligned} \tag{10}$$

Next, we define phase classes using the discrete Fourier transform of size $\frac{6}{2} \times \frac{6}{2}$:

$$\phi_{j,p} = \left\lfloor \frac{p}{2} \right\rfloor \frac{2\pi j}{3}. \tag{11}$$

Equations (7) and (9) specify a complete orthonormal basis of $d = 6$ Bell states. We display a few examples below, omitting the *L*/*R* subscripts for readability:

$$\begin{aligned}
\left|\Psi_0^4\right\rangle &= \frac{1}{\sqrt{6}}(|0\rangle|0\rangle + |1\rangle|1\rangle + e^{4\pi i/3}|2\rangle|2\rangle + e^{4\pi i/3}|3\rangle|3\rangle\\
&\quad + e^{8\pi i/3}|4\rangle|4\rangle + e^{8\pi i/3}|5\rangle|5\rangle),\\
\left|\Psi_0^5\right\rangle &= \frac{1}{\sqrt{6}}(|0\rangle|0\rangle - |1\rangle|1\rangle + e^{4\pi i/3}|2\rangle|2\rangle - e^{4\pi i/3}|3\rangle|3\rangle\\
&\quad + e^{8\pi i/3}|4\rangle|4\rangle - e^{8\pi i/3}|5\rangle|5\rangle),\\
\left|\Psi_1^1\right\rangle &= \frac{1}{\sqrt{6}}(|0\rangle|1\rangle - |1\rangle|0\rangle + |2\rangle|5\rangle - |5\rangle|2\rangle\\
&\quad + |3\rangle|4\rangle - |4\rangle|3\rangle),\\
\left|\Psi_2^5\right\rangle &= \frac{1}{\sqrt{6}}(|0\rangle|2\rangle - |2\rangle|0\rangle + e^{4\pi i/3}|1\rangle|3\rangle - e^{4\pi i/3}|3\rangle|1\rangle\\
&\quad + e^{8\pi i/3}|4\rangle|5\rangle - e^{8\pi i/3}|5\rangle|4\rangle).
\end{aligned} \tag{12}$$

We can compare these states to those given by Eq. (2) for the same correlation ($m = c$) and phase ($n = p$) classes:

$$\begin{aligned}
\left|\Psi_0^4\right\rangle &= \frac{1}{\sqrt{6}}(|0\rangle|0\rangle + e^{4\pi i/3}|1\rangle|1\rangle + e^{8\pi i/3}|2\rangle|2\rangle\\
&\quad + e^{12\pi i/3}|3\rangle|3\rangle + e^{16\pi i/3}|4\rangle|4\rangle + e^{20\pi i/3}|5\rangle|5\rangle),\\
\left|\Psi_0^5\right\rangle &= \frac{1}{\sqrt{6}}(|0\rangle|0\rangle + e^{5\pi i/3}|1\rangle|1\rangle + e^{10\pi i/3}|2\rangle|2\rangle + e^{5\pi i}|3\rangle|3\rangle\\
&\quad + e^{20\pi i/3}|4\rangle|4\rangle + e^{25\pi i/3}|5\rangle|5\rangle),\\
\left|\Psi_1^1\right\rangle &= \frac{1}{\sqrt{6}}(|0\rangle|1\rangle + e^{i\pi/3}|1\rangle|2\rangle + e^{2\pi i/3}|2\rangle|3\rangle + e^{\pi i}|3\rangle|4\rangle\\
&\quad + e^{4\pi i/3}|4\rangle|5\rangle + e^{5\pi i/3}|5\rangle|0\rangle),\\
\left|\Psi_2^5\right\rangle &= \frac{1}{\sqrt{6}}(|0\rangle|2\rangle + e^{5\pi i/3}|1\rangle|3\rangle + e^{10\pi i/3}|2\rangle|4\rangle\\
&\quad + e^{5\pi i}|3\rangle|5\rangle + e^{20\pi i/3}|4\rangle|0\rangle + e^{25\pi i/3}|5\rangle|1\rangle).
\end{aligned} \tag{13}$$

The entangled basis states of Eq. (12) are clearly eigenstates of particle state exchange, whereas those in Eq. (13) generated from the most common definition are not.

## VIII. CONNECTION TO HYPERENTANGLED BASIS

In the hyperentangled construction for $d = 2^n$ [32,33], the general form of particle correlations resembles ours while the phase factors take only values $\pm 1$. This can be understood as a slight variation on our framework. For $d = 2^n$, phase factors $e^{i\phi_{j,p}}$ can be restricted to the values $\pm 1$ by replacing the DFT with generalized Hadamard matrices, or Walsh matrices, defined recursively by [59]

$$H(2^n) = H(2) \otimes H(2^{n-1}), \tag{14}$$

where $H(2)$ is the $2 \times 2$ Hadamard matrix (normalized, so entries have magnitude 1). Then, phase classes for $d = 2^n$ Bell states have

$$e^{i\phi_{j,p}} = [H(2^{n-1})]_{j,p}. \tag{15}$$

Consider $d = 8$. We could by default use the $4 \times 4$ DFT to construct the phases:

$$\mathrm{DFT}(4) = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}, \tag{16}$$

or the phases could come from the $4 \times 4$ Walsh matrix:

$$H(4) = H(2) \otimes H(2) \tag{17}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \tag{18}$$

$H(4)$, like DFT(4), has mutually orthogonal columns. Thus, the Walsh matrices could be inserted seamlessly into the phase portion of the state construction procedure and would produce entangled qudit Bell states corresponding to the hyperentangled constructions [32,33].

## IX. NO SYMMETRIZED BELL BASIS FOR ODD *d*

The construction above relies on even single-particle Hilbert space dimension $d$; here, we establish that no Bell basis symmetrized with respect to particle exchange exists for odd $d$. It is well known that every fully entangled bipartite pure state can be expressed as $|\Psi\rangle = (1/\sqrt{d}) \sum_{j=0}^{d-1} |\phi_j\rangle_L |\psi_j\rangle_R$ for some orthonormal bases $\{|\phi_j\rangle_L\}$ and $\{|\psi_j\rangle_R\}$. Thus, when re-expressed in the form of Eq. (1), every fully entangled bipartite state has a unitary $d \times d$ coefficient matrix $\chi$. A complete basis of entangled states requires the existence of $d^2$ linearly independent coefficient matrices. Exchange (anti)symmetry corresponds to $\chi_{ij} = (-)\chi_{ji}$. However, every skew-symmetric matrix of odd dimension has determinant zero, and there are only $d(d+1)/2$ linearly independent symmetric matrices. Therefore, no qudit Bell basis of exchange-symmetrized states exists for odd $d$.

## X. MAXIMAL DISTINGUISHABILITY OF BELL STATES FOR EVEN *d*

We exploit the symmetry of our generalized Bell basis to yield a simple proof of the maximum number of qudit Bell states that can be distinguished by an LELM device. Bell-basis measurements are required in many applications, including quantum teleportation [47,60], quantum repeaters [61,62], and quantum dense coding [63]. Despite their in-principle limitations, LELM devices remain relevant to demonstrations of these quantum communication protocols because of their reliability in practice [28,64,65].

Let $k$ be the number of qudit Bell states unambiguously distinguishable under LELM. Previously, only the results for $d = 2$, $k = 3$ [24,25], $d = 3$, $k = 3$, and $d = 2^n$, $k = 2^{n+1} - 1$ [32,66] were known. Most relevant to our work is Ref. [32], which examined Bell states with single-particle dimension $d = 2^n$ and considered a hyperentangled basis composed of symmetrized states. That work demonstrated that, using an exchange-symmetrized Bell basis, the maximal distinguishability is

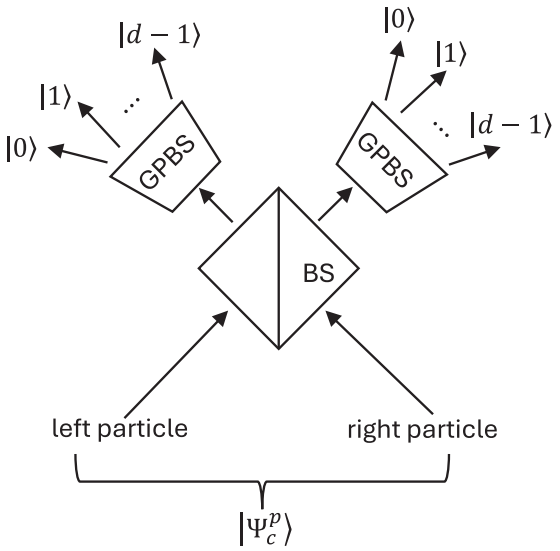



FIG. 1. Optimal LELM device to distinguish qudit Bell states ($|\Psi_c^p\rangle$ as in Eqs. (7) and (9) of even single-particle dimension $d$. BS is a 50/50 beam splitter, and trapezoids represent GPBSs that spatially separate single-particle basis states. The output channels shown couple to particle-number resolving detectors.

$$k = 2d - 1. \tag{19}$$

In the argument of Ref. [32], an upper bound of $2d - 1$ LELM-distinguishable Bell-state classes is independent of the specific dimension $d$ of the single-particle Hilbert space. The criteria of linear evolution and local measurement lead to detection signatures that consist of a pair of particle detections across $2d$ detectors. Maximal entanglement implies that any Bell state must be capable of triggering any detector at least once, meaning that for any "first" detection there are only $2d$ available "second" detections available to distinguish between Bell states. For indistinguishable particle pairs, the number of LELM-distinguishable Bell states is further decreased by 1 because of fundamental particle exchange symmetry: For fermions, no single detector may be triggered twice, and for bosons, one of the detection signatures involving each detector $i$ either does not occur or overlaps with the Bell states in another detection signature involving $i$. Reference [32] presented this reasoning for hyperentangled qubit Bell states with $d = 2^n$, but it applies equally for any $d$, provided the Bell basis is symmetrized under particle exchange.

For qudit Bell states in our symmetrized basis, the optimal distinguishability can be realized with a 50-50 beam splitter and two generalized polarizing beam splitters (GPBSs) of dimension $d$, as shown in Fig. 1. This schematic is analogous to previous optimal distinguishability schemes for qubit pairs [24,25,67–69]. As in Ref. [32], it exploits the symmetrized nature of the input states: By the generalized Hong-Ou-Mandel effect, for bipartite symmetric states both detections occur on the same output port of the beam splitter (bunching), whereas antisymmetric states give one detection on each side (antibunching). Thus, some phase information—the parity of $p$—becomes spatial information.

Our argument does not consider auxiliary modes. However, it has been shown that the use of auxiliary modes of definite particle number cannot increase the number of distinguishable Bell states in an LELM device with projective measurement [70,71].

## XI. APPLICATION: DENSE CODING

We outline a dense coding protocol for qudits using our Bell-state distinguishability result for LELM devices.

Suppose Alice and Bob use Bell states of an even $d$. They share the fully entangled state $|\Psi_0^0\rangle$ and define a set of $2d-1$ codewords consisting of one Bell state per distinguishability class. Alice's message to Bob is encoded in which of these states $|\Psi_c^p\rangle$ she creates via a local unitary operation on her qudit, i.e., $U_A \otimes I_B$. After her chosen local operation, Alice sends her qudit to Bob, who can reliably distinguish between all $2d-1$ codewords using the device in Fig. 1.

For instance, Alice and Bob may choose as their $2d-1$ codewords the $c=0$, $p=0$ and $c\neq 0$, $p=0,1$ states. Alice can transform an initially shared $c=0$, $p=0$ state to any of the others by pairwise swapping her single-qudit basis states to achieve the desired $c$ and, to make $p=1$, flipping the phases of an appropriate 50% of them. For example, for $d=6$, suppose Alice wishes to send the codewords $c=1$, $p=0$ or $c=1$, $p=1$. To transform $|\Psi_0^0\rangle \to |\Psi_1^0\rangle$, Alice performs

$$\begin{aligned} |0\rangle &\to |1\rangle, \quad |1\rangle \to |0\rangle, \\ |2\rangle &\to |5\rangle, \quad |5\rangle \to |2\rangle, \\ |3\rangle &\to |4\rangle, \quad |4\rangle \to |3\rangle. \end{aligned} \tag{20}$$

To create $|\Psi_1^1\rangle$, Alice performs

$$\begin{aligned} |0\rangle &\to -|1\rangle, \quad |1\rangle \to |0\rangle, \\ |2\rangle &\to -|5\rangle, \quad |5\rangle \to |2\rangle, \\ |3\rangle &\to -|4\rangle, \quad |4\rangle \to |3\rangle. \end{aligned} \tag{21}$$

## XII. CONCLUSION

In qudit pairs of even single-particle Hilbert space dimension $d$, we have introduced a Bell basis whose states have definite symmetry under exchange of the internal states of the particles in the two distinct spatial channels. We have shown that an LELM device can unambiguously distinguish $2d-1$ of these exchange-symmetrized qudit Bell states. Furthermore, we have sketched a qudit dense coding protocol with channel size $2d-1$ with these states.

It remains an open question whether the same maximal distinguishability bound is achievable for unsymmetrized Bell states, as symmetrized and unsymmetrized bases are not, in general, equivalent under local unitary transformations.

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## DATA AVAILABILITY

No data were created or analyzed in this study.

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