

Combined framework for synchronization with distributed delays and noise in Kuramoto-Daido oscillators

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We derive the characteristic equation governing the linear stability of the incoherent state in delayed noisy Kuramoto-Daido oscillator ensembles with general periodic coupling. The calculation is formulated for arbitrary normalized delay kernels and natural-frequency distributions. For a Lorentzian frequency distribution and first-harmonic coupling, the characteristic equation reduces to a closed dispersion relation in which phase diffusion and frequency heterogeneity enter through the damping combination $D + \Delta$, while the delay kernel enters through its Laplace transform. The neutral conditions recover the established fixed- and distributed-delay limits and are evaluated explicitly for Gamma-distributed delays. We then derive the reduced phase dynamics of an overdamped Josephson-junction array coupled through a delayed common load. A finite-dc-load admittance produces a uniform drift that corresponds to the constant Fourier component of the reduced coupling function. When the ac coupling and dc-induced drift vary with the same load parameter, the resulting effective-frequency shift displaces the neutral stability boundary. Finite- N simulations of the reduced phase equations agree with the continuum prediction over the parameter range considered.

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I. INTRODUCTION

Synchronization in populations of interacting oscillators occurs in systems ranging from biological and neuronal networks to laser arrays, Josephson-junction arrays, and power-grid dynamics [1–8]. Kuramoto-type phase models describe weakly coupled self-sustained oscillators through the competition between frequency heterogeneity and interaction strength [2,3,9]. Finite propagation and processing times introduce delayed coupling, while microscopic fluctuations produce phase diffusion [10–12]. Both effects modify the linear stability of the incoherent state. Here, we study the continuum stability problem with distributed delays, additive phase diffusion, and a general periodic coupling function retained simultaneously.

The continuum stability problem for globally coupled phase oscillators has a well-developed history. For the standard Kuramoto model, Strogatz and Mirollo analyzed the stability of the incoherent state by linearizing the continuum kinetic equation [13], while noisy extensions were studied using Fokker-Planck descriptions [12,14,15]. Phase-lagged sinusoidal interactions lead to the Kuramoto-Sakaguchi model [16], whereas Daido and subsequent authors developed for-

mulations with general periodic coupling functions and higher Fourier harmonics [17–20]. In particular, Crawford and Davies considered arbitrary phase coupling together with stochastic frequency fluctuations in the continuum formulation [19]. These results establish the appropriate context for the general coupling and noise terms considered below.

Time-delayed Kuramoto models constitute a separate line of development. Yeung and Strogatz showed that a discrete interaction delay can generate multivalued neutral-stability curves and delay-induced resonance branches [21]. Lee, Ott, and Antonsen subsequently treated heterogeneous interaction delays in the continuum limit and showed that broadening the delay distribution suppresses the resonance structure associated with a sharply defined delay [22]. Their formulation applies to general delay distributions and was illustrated using Gamma-distributed kernels. Thus, neither distributed delays nor the smoothing of fixed-delay resonance branches is new to the present work. The extension considered here is the continuum linear stability problem in which distributed delays, additive phase diffusion, and a general Fourier coupling function are retained simultaneously. The linearization itself follows the established Fokker-Planck approach.

This structure also arises in reductions of physical oscillator arrays, where the effective phase interaction need not have zero mean. Load-coupled Josephson-junction arrays provide a concrete example. Averaging methods for overdamped series arrays coupled through linear loads produce tractable slow phase equations [23], and in the weak-coupling, weak-disorder regime Josephson arrays can be mapped onto Kuramoto-type phase models [7,24]. In the circuit considered here, retaining an explicit finite dc admittance of the common load generates, in addition to the resonant ac interaction, a

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uniform contribution to the reduced phase velocity. For fixed coupling, this contribution is only a rotating-frame frequency shift. Its relevance to delayed stability arises when the same load parameter controls both the ac coupling amplitude and the dc-induced shift, so that the effective rotation frequency changes as the stability boundary is traced.

In this work, we derive the characteristic equation governing the linear stability of the incoherent state for a delayed noisy Kuramoto-Daido ensemble with a general periodic coupling function. The derivation is formulated for arbitrary normalized delay kernels and natural frequency distributions. For a Lorentzian frequency distribution and first-harmonic coupling, the characteristic equation reduces to a closed dispersion relation in which phase diffusion and frequency broadening enter through the combination $D + \Delta$. The resulting neutral conditions recover the established fixed- and distributed-delay limits. We then apply the same stability relation to the phase-reduced overdamped Josephson-junction array and compare the continuum boundary with finite- N simulations of the reduced delayed phase equations.

The remainder of the paper is organized as follows. Section II introduces the delayed noisy Kuramoto-Daido model. Section III develops the continuum formulation and derives the characteristic equation for the incoherent state. Section IV evaluates the result for Lorentzian frequency distributions and fixed or Gamma-distributed delays. Section V presents the Josephson-junction reduction and its numerical comparison with the continuum prediction. Section VI discusses the interpretation and scope of the results, and Sec. VII summarizes the conclusions. Technical derivations are collected in Appendixes A–D.

II. DELAYED NOISY KURAMOTO-DAIDO MODEL

We consider a population of N globally coupled phase oscillators with distributed interaction delays and additive phase diffusion. The microscopic dynamics are governed by

$$\begin{aligned} \dot{\theta}_i(t) = & \omega_i + \frac{K}{N} \sum_{j=1}^N \int_0^\infty \rho(\tau) H[\theta_j(t - \tau) - \theta_i(t)] d\tau \\ & + \sqrt{2D} \xi_i(t), \quad i = 1, \dots, N, \end{aligned} \quad (1)$$

where ω_i is the natural frequency of the i th oscillator, K is the interaction strength, $\rho(\tau)$ is a normalized delay kernel satisfying

$$\rho(\tau) \geq 0, \quad \int_0^\infty \rho(\tau) d\tau = 1,$$

and $\xi_i(t)$ are independent Gaussian white noises with

$$\langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t) \xi_j(t') \rangle = \delta_{ij} \delta(t - t').$$

The natural frequencies are sampled from a normalized distribution $g(\omega)$. The kernel $\rho(\tau)$ represents the distribution of propagation delays between interacting oscillators and allows both fixed and distributed delays to be treated within the same formulation.

The interaction function $H(\phi)$ is assumed to be real and 2π periodic and is represented by the Fourier series

$$H(\phi) = \sum_{m=-\infty}^{\infty} h_m e^{im\phi}, \quad h_m = \frac{1}{2\pi} \int_0^{2\pi} H(\phi) e^{-im\phi} d\phi, \quad (2)$$

with $h_{-m} = h_m^*$. Throughout this work, we adopt the phase-difference convention used in Eq. (1), for which positive coupling ($K > 0$) corresponds to attractive interactions when $H(\phi) = \sin \phi$.

A frequently used special case is the first-harmonic coupling

$$H(\phi) = h_0 + \sin(\phi + \alpha), \quad (3)$$

whose zeroth Fourier coefficient is h_0 , while the nonzero oscillatory coefficients are

$$h_1 = -\frac{i}{2} e^{i\alpha}, \quad h_{-1} = \frac{i}{2} e^{-i\alpha}.$$

The constant Fourier component h_0 produces the uniform frequency shift Kh_0 . For fixed coupling strength, this contribution may be removed by transforming to a uniformly rotating frame. It is retained explicitly because, in the Josephson-junction application considered below, the delayed interaction and the corresponding frequency shift are controlled by the same physical load parameter.

Equation (1) defines the finite- N stochastic dynamics. Our objective is to determine the linear stability of the incoherent state in the thermodynamic limit, where the microscopic oscillator ensemble is replaced by a continuous phase density. The following continuum formulation yields the characteristic equation for the loss of linear stability of this state.

III. CONTINUUM FORMULATION AND LINEAR STABILITY

To determine the stability of the incoherent state, we consider the thermodynamic limit $N \rightarrow \infty$, in which the microscopic oscillator ensemble is described by the conditional probability density $f(\theta, \omega, t)$. Here, $f(\theta, \omega, t) d\theta$ denotes the fraction of oscillators with natural frequency ω whose phases lie in the interval $[\theta, \theta + d\theta]$ at time t , and satisfies the normalization

$$\int_0^{2\pi} f(\theta, \omega, t) d\theta = 1. \quad (4)$$

In the continuum limit, the discrete interaction term in Eq. (1) is replaced by its mean-field average over the phase density. The corresponding drift velocity is

$$\begin{aligned} v(\theta, \omega, t) = & \omega + K \int_0^\infty \rho(\tau) d\tau \int_{-\infty}^\infty g(\omega') d\omega' \\ & \times \int_0^{2\pi} H(\theta' - \theta) f(\theta', \omega', t - \tau) d\theta'. \end{aligned} \quad (5)$$

This velocity contains the delayed mean-field interaction generated by the oscillator population.

The evolution of the density is governed by the nonlinear Fokker-Planck equation

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \theta} (vf) + D \frac{\partial^2 f}{\partial \theta^2}, \quad (6)$$

where the first term represents deterministic phase advection under the mean-field drift $v(\theta, \omega, t)$, while the second describes phase diffusion arising from the additive white noise in Eq. (1).

The incoherent state is the uniform phase density

$$f_0(\theta, \omega) = \frac{1}{2\pi}. \quad (7)$$

For this state, the mean-field drift is $v_0(\omega) = \omega + Kh_0$, while the phase density remains uniform. To determine its stability, we perturb the density according to

$$f(\theta, \omega, t) = \frac{1}{2\pi} \left[1 + \sum_{n \neq 0} c_n(\omega, t) e^{in\theta} \right], \quad (8)$$

and retain only terms linear in the perturbation. The Fourier linearization is summarized in Appendix A. Seeking normal-mode solutions

$$c_n(\omega, t) = b_n(\omega) e^{\lambda t},$$

one obtains the characteristic equation

$$1 + inKh_{-n} \tilde{\rho}(\lambda) \int_{-\infty}^{\infty} \frac{g(\omega) d\omega}{\lambda + n^2 D + in(\omega + Kh_0)} = 0, \quad (9)$$

where

$$\tilde{\rho}(\lambda) = \int_0^{\infty} \rho(\tau) e^{-\lambda \tau} d\tau$$

is the Laplace transform of the delay kernel.

Equation (9) governs the linear stability of the incoherent state independently for each Fourier harmonic of the coupling function. The loss of stability of the incoherent state is determined by the harmonic whose eigenvalue first reaches $\text{Re } \lambda = 0$. In the following section, we specialize this general result to a Lorentzian frequency distribution and first-harmonic coupling, for which the characteristic equation reduces to a closed dispersion relation.

IV. STABILITY BOUNDARIES FOR LORENTZIAN FREQUENCY DISTRIBUTIONS

The characteristic equation derived in Sec. III is valid for arbitrary delay kernels and natural-frequency distributions. Explicit stability boundaries are obtained by specializing it to the Lorentzian distribution

$$g(\omega) = \frac{\Delta}{\pi[(\omega - \omega_0)^2 + \Delta^2]}, \quad (10)$$

and the first-harmonic coupling

$$H(\phi) = h_0 + \sin(\phi + \alpha),$$

for which $h_{-1} = (i/2)e^{-i\alpha}$. Because the Lorentzian distribution possesses a single simple pole in the lower half-plane, the frequency integral appearing in Eq. (9) is evaluated analytically by contour integration (Appendix B).

This yields the closed dispersion relation

$$\lambda + D + \Delta + i(\omega_0 + Kh_0) - \frac{K}{2} e^{-i\alpha} \tilde{\rho}(\lambda) = 0. \quad (11)$$

In Eq. (11), $D + \Delta$ sets the linear damping, $\tilde{\rho}(\lambda)$ gives the delay response, and Kh_0 shifts the mean rotation frequency.

The equivalent rotating-frame representation is summarized in Appendix B.

In the zero-delay limit, $\tilde{\rho}(\lambda) = 1$. For $\alpha = h_0 = 0$, Eq. (11) becomes

$$\lambda = \frac{K}{2} - (D + \Delta) - i\omega_0, \quad (12)$$

so that the incoherent state loses linear stability at

$$K_c(0) = 2(D + \Delta). \quad (13)$$

Equation (13) recovers the standard zero-delay threshold for this Lorentzian first-harmonic reduction. For the Lorentzian first-harmonic reduction, phase diffusion and frequency disorder enter the linear damping only through the combination $D + \Delta$. This simplification is specific to the Lorentzian reduction and should not be interpreted as a general equivalence between phase noise and frequency heterogeneity.

To examine the influence of delay dispersion, we employ the Gamma family

$$\rho_q(\tau) = \frac{q^q}{\Gamma(q)\bar{\tau}^q} \tau^{q-1} e^{-q\tau/\bar{\tau}}, \quad (14)$$

whose Laplace transform is

$$\tilde{\rho}_q(\lambda) = \left(1 + \frac{\lambda \bar{\tau}}{q} \right)^{-q}. \quad (15)$$

The parameter q interpolates between an exponential kernel ($q = 1$) and the fixed-delay limit ($q \rightarrow \infty$), for which

$$\tilde{\rho}(\lambda) = e^{-\lambda \bar{\tau}}.$$

For integer q , the Gamma kernel also admits the standard finite-dimensional linear-chain representation; here, it is used as an illustrative family of distributed-delay kernels.

The neutral stability boundary is obtained by setting

$$\lambda = i\Omega,$$

where Ω denotes the collective frequency of the marginal mode. Introducing

$$A = \text{Re}[e^{-i\alpha} \tilde{\rho}(i\Omega)], \quad B = \text{Im}[e^{-i\alpha} \tilde{\rho}(i\Omega)],$$

the real and imaginary parts of Eq. (11) yield

$$K_c = \frac{2(D + \Delta)}{A}, \quad (16)$$

$$\Omega + \omega_0 + K_c h_0 = \frac{K_c}{2} B. \quad (17)$$

Equations (16) and (17) determine the neutral stability boundary for arbitrary normalized delay kernels through the complex quantity $\tilde{\rho}(i\Omega)$.

Figure 1 illustrates the effects of delay dispersion and phase diffusion. Because $|\tilde{\rho}(i\Omega)| = 1$, a fixed delay changes the feedback phase without attenuating the marginal-frequency response, producing the resonance branches associated with discrete propagation delays. For the Gamma family,

$$|\tilde{\rho}_q(i\Omega)| = \left[1 + \left(\frac{\Omega \bar{\tau}}{q} \right)^2 \right]^{-q/2},$$

so increasing the width of the delay distribution progressively attenuates the frequency-dependent delayed feedback. Consequently, the higher-order resonance branches are suppressed

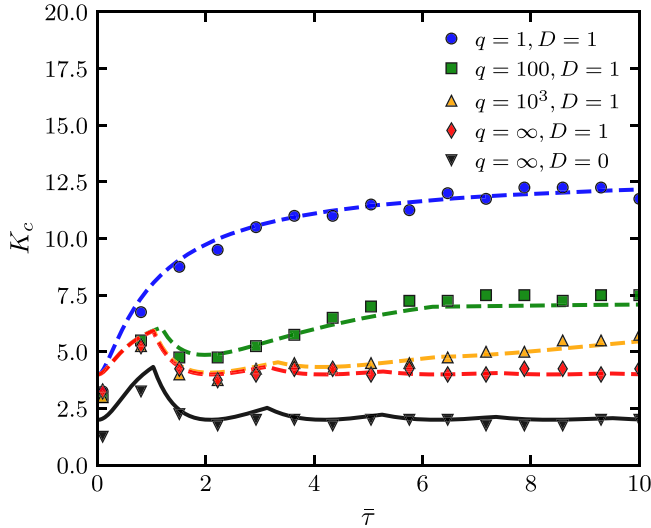


FIG. 1. Neutral stability boundaries of the incoherent state obtained from Eqs. (16) and (17) for fixed and Gamma-distributed delay kernels. The calculations use $\omega_0 = 3$, $\Delta = 1$, $\alpha = h_0 = 0$, with solid and dashed curves corresponding to $D = 0$ and $D = 1$, respectively. The Gamma shape parameter q controls the width of the delay distribution, recovering the fixed-delay limit as $q \rightarrow \infty$. Narrow delay kernels generate delay-induced resonance tongues, whereas broader kernels progressively suppress the higher-order resonances. For the Lorentzian first-harmonic reduction, phase diffusion enters the threshold through the damping combination $D + \Delta$, raising the critical coupling while preserving the qualitative dependence on the delay distribution.

and the neutral boundary evolves continuously toward the smooth monotonic profile reported previously for distributed-delay oscillator ensembles [22]. The attenuation of the distributed-delay transfer function accounts for the suppression of the higher-order branches.

Figure 1 also shows the effect of additive phase diffusion. Equation (11) shows that, for the Lorentzian first-harmonic reduction, diffusion and frequency disorder contribute to the linear damping through the combination $D + \Delta$. At zero delay, Eq. (13) therefore gives $K_c = 2$ for $D = 0$ and $K_c = 4$ for $D = 1$ using the parameters of Fig. 1. At finite delay, $D + \Delta$ continues to determine the damping, whereas the delay kernel modifies the threshold through the frequency-dependent phase and attenuation contained in $\tilde{\rho}(i\Omega)$. These boundaries recover the established fixed- and distributed-delay limits and provide the continuum reference used in Sec. V.

V. APPLICATION TO OVERDAMPED JOSEPHSON-JUNCTION ARRAYS

Overdamped Josephson-junction arrays provide a physical realization of the delayed phase model introduced in Sec. II. Common-load averaging and Kuramoto reductions of Josephson series arrays are established [7,23]. Here, we retain distributed propagation delays, effective phase diffusion, and the finite-dc-load drift in the reduced model and analyze the linear stability of its incoherent continuum state.

We consider the current-biased series array shown in Fig. 2. Neglecting junction capacitance, the k th junction obeys the

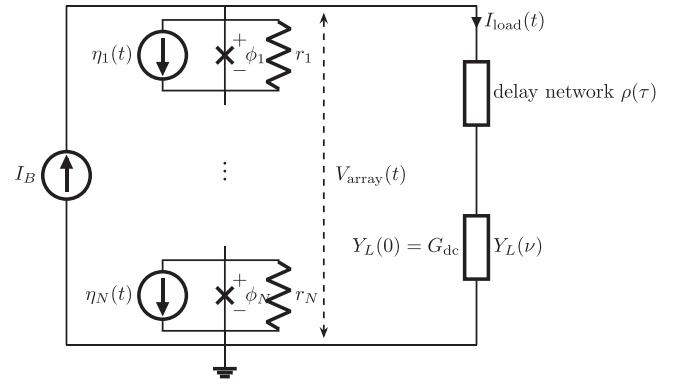


FIG. 2. Schematic of the delayed Josephson-junction array considered in Sec. V. Each overdamped junction is represented by a Josephson element shunted by the normal resistance r_k , with thermal fluctuations represented by the Langevin current source $\eta_k(t)$. The series array is biased by the dc current I_B . The total array voltage drives a common linear load through a propagation network described by the delay kernel $\rho(\tau)$. The ac load response generates the phase-dependent interaction, whereas a finite dc admittance $Y_L(0) = G_{dc}$ converts the nonzero mean array voltage into a dc load current that appears as a uniform drift in the reduced phase dynamics.

resistively shunted junction (RSJ) equation

$$\frac{\hbar}{2er_k} \dot{\phi}_k + I_k \sin \phi_k + I_{\text{load}}(t) = I_B + \eta_k(t), \quad (18)$$

where I_k and r_k are the critical current and normal-state resistance, respectively. The physical voltage across the series array is

$$V_{\text{array}}(t) = \frac{\hbar}{2e} \sum_{j=1}^N \dot{\phi}_j(t). \quad (19)$$

Propagation-time heterogeneity is represented by the normalized delay kernel $\rho(\tau)$, so that the voltage driving the load is

$$V_d(t) = \frac{\hbar}{2e} \sum_{j=1}^N \int_0^\infty \rho(\tau) \dot{\phi}_j(t - \tau) d\tau. \quad (20)$$

To obtain an $O(1)$ load current in the mean-field limit, we use the weak-load scaling

$$Y_L(\nu) = \frac{\mathcal{Y}(\nu)}{N}.$$

Equivalently, $\mathcal{Y}(\nu)$ acts on the mean array voltage V_d/N . The common load is taken to be linear,

$$\hat{I}_{\text{load}}(\nu) = Y_L(\nu) \hat{V}_d(\nu), \quad (21)$$

with $Y_L(\nu)$ its physical admittance.

We separate the ac and dc responses of the load in the phase reduction. In the voltage state, each rotating junction has a nonzero mean voltage, and hence the complete series array has $V_{dc} \neq 0$. Whenever the common load has finite dc conductance,

$$G_{dc} = Y_L(0) \neq 0, \quad (22)$$

the mean array voltage necessarily drives the dc current

$$I_{dc} = G_{dc} V_{dc}. \quad (23)$$

This contribution vanishes for load topologies that block dc. The resulting uniform drift is therefore fixed by the dc response of the same load that generates the ac coupling.

The slow phase dynamics follow from the standard natural-angle transformation and first-order averaging over the Josephson period (Appendix D). The fundamental ac component of the load current produces the resonant delayed sine coupling, whereas the dc load current survives the averaging as a uniform drift. With the conventional mean-field normalization, the reduced equation is

$$\dot{\theta}_k = \omega_k + \Gamma_0 + \frac{K}{N} \sum_{j=1}^N \int_0^\infty \rho(\tau) \sin \times [\theta_j(t - \tau) - \theta_k(t) + \alpha] d\tau + \sqrt{2D} \xi_k(t). \quad (24)$$

Here,

$$\Gamma_0 = -\frac{\bar{\omega} I_B}{I_B^2 - \bar{I}^2} I_{dc}, \quad (25)$$

while K and α are determined by the magnitude and phase of the fundamental-frequency load response; their dependence on the load parameters is summarized in Appendix D. Under weak thermal noise, stochastic averaging over the running orbit replaces the multiplicative current-noise term by an effective additive phase noise. We denote the resulting phase-diffusion coefficient by D , consistent with the notation of Sec. II.

Equation (24) is therefore a specific realization of the first-harmonic Kuramoto-Daido model of Sec. II,

$$H(\phi) = h_0 + \sin(\phi + \alpha), \quad \Gamma_0 = K h_0. \quad (26)$$

For the comparison below, we consider a one-parameter variation of the load response for which the ac coupling and dc-induced drift scale together. The ratio $h_0 = \Gamma_0/K$ is therefore held fixed. For fixed K , the uniform contribution Γ_0 may be absorbed into a rotating frame. Its relevance to the delayed stability problem arises when the same physical load parameter changes both the ac interaction and the dc-induced drift. The effective frequency entering the neutral condition is then

$$\omega_{\text{eff}}(K) = \omega_0 + K h_0, \quad (27)$$

so that the phase accumulated during the delay changes as the stability boundary is traced.

Figure 3 compares this neutral boundary with finite- N simulations of the reduced phase model. The dashed curve is the analytical zero-dc-response reference $h_0 = 0$, whereas the solid curve corresponds to the finite-dc-load case $h_0 = -0.4$. At zero delay, the two curves coincide at

$$K_c(0) = 2(D + \Delta),$$

because h_0 enters the neutral-frequency condition but not the real threshold equation. At finite delay, Eq. (17) contains the combination $\omega_0 + K_c h_0$, and the two boundaries separate because the accumulated delay phase becomes coupling dependent.

The symbols in Fig. 3 are critical couplings obtained from finite- N simulations of Eq. (24). Within the statistical uncertainty of the simulations, the numerical thresholds follow the

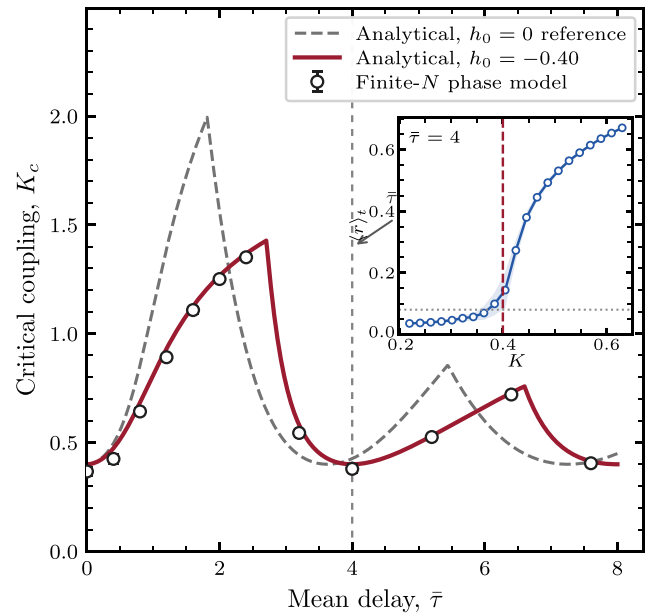


FIG. 3. Neutral stability boundary for the reduced Josephson-phase model. The dashed gray curve is the analytical $h_0 = 0$ reference, while the solid curve is the analytical boundary for the finite-dc-load case $h_0 = -0.4$. Open circles denote critical couplings obtained from finite- N simulations of Eq. (24); error bars show the standard deviation over five independent realizations. Parameters are $I_B = 2$, $\bar{I} = \bar{\tau} = \hbar/(2e) = 1$, $\omega_0 = \sqrt{3}$, $\Delta = 0.1$, $D = 0.1$, and $\alpha = 0$. The inset shows the time-averaged order parameter $\langle r \rangle_t$ as a function of K at $\bar{\tau} = \tau^* = 4$; the vertical guide in the main panel marks the same delay. The agreement of the numerical thresholds with the $h_0 = -0.4$ analytical boundary tests the continuum prediction for the reduced Josephson phase model.

analytical $h_0 = -0.4$ boundary over the range shown. The inset shows the time-averaged Kuramoto order parameter at $\bar{\tau} = \tau^* = 4$ and the crossing used to determine the numerical threshold. The $h_0 = 0$ reference isolates the displacement associated with the finite-dc-load response.

The finite- N thresholds in Fig. 3 were obtained by integrating Eq. (24) with the Euler-Maruyama method using $N = 1000$, time step $dt = 0.01$, and total integration time $T = 600$. The initial interval $0 < t < 300$ was discarded as a transient, and the time-averaged order parameter $\langle r \rangle_t$ was computed over the remaining interval. Natural frequencies were generated from the Lorentzian distribution in Eq. (10) by stratified inverse-transform sampling, with one frequency drawn from each equal-probability interval of the cumulative distribution and without truncating the Lorentzian tails. For each delay $\bar{\tau}$, the numerical threshold was defined by $\langle r \rangle_t > r_c$, with $r_c = 0.08$. The crossing was first bracketed in K and then refined by nine bisection steps. Within each threshold search, the same realization of the natural frequencies and the same initial phase configuration were retained as K was varied. For delays not commensurate with the integration step, the delayed complex order parameter was obtained by linear interpolation of the stored history. The reported K_c is the mean over five independent realizations, and the error bars denote the corresponding sample standard deviation.

VI. DISCUSSION

The characteristic equation derived in Sec. III separates the contributions to the continuum stability problem. The natural-frequency distribution enters through the spectral integral over $g(\omega)$, phase diffusion damps the n th Fourier perturbation at the rate n^2D , and the delay distribution enters through $\tilde{\rho}(\lambda)$. Each Fourier coefficient of the coupling function acts on its corresponding perturbation mode. For the Lorentzian first-harmonic reduction, the damping depends on $D + \Delta$, while $\tilde{\rho}(i\Omega)$ supplies the attenuation and phase shift of the delayed feedback. Figure 1 recovers the established fixed- and distributed-delay limits: Fixed delay changes the feedback phase without attenuation, whereas a broad delay distribution suppresses higher resonance branches through $|\tilde{\rho}(i\Omega)| < 1$.

The Josephson-junction reduction shows how the constant Fourier component enters a physical phase model. A finite dc response of the common load produces the uniform drift $\Gamma_0 = Kh_0$, while the ac response produces the delayed first-harmonic interaction. For fixed K , the drift can be removed by a rotating-frame transformation. Along the one-parameter load variation used in Sec. V, $h_0 = \Gamma_0/K$ is fixed, so varying K also changes the effective frequency $\omega_{\text{eff}}(K) = \omega_0 + Kh_0$. The resulting change in delay phase accounts for the displacement of the $h_0 = -0.4$ neutral boundary relative to the $h_0 = 0$ reference. The finite- N simulations in Fig. 3 agree with the continuum boundary within the statistical uncertainty of the extracted thresholds.

The same stability calculation can be adapted to other oscillator networks that admit weak-coupling phase reductions. Electrically coupled spin-torque nano-oscillator arrays provide one example of collective phase dynamics under common electrical coupling [25,26]. Coupled-oscillator phased arrays provide a related setting in which the phase distribution determines beam steering and propagation delays modify the phase dynamics [27,28]. The beam-steering models cited here use spatially structured or feedforward coupling rather than the global mean-field interaction assumed in Eq. (1); applying the present calculation to those systems therefore requires the corresponding network coupling operator.

The analytical results are subject to the assumptions used in their derivation. Equation (9) determines the linear stability of the incoherent state in the continuum limit and does not determine the nonlinear state reached beyond threshold. The explicit dispersion relation in Sec. IV assumes a Lorentzian frequency distribution and first-harmonic coupling. The Josephson-junction reduction assumes overdamped junctions, weak loading, and weak disorder, and the numerical comparison tests the reduced phase equations rather than the full circuit dynamics. Higher coupling harmonics, non-Lorentzian frequency distributions, stronger loading, and nonlinear dynamics beyond the instability threshold require separate analysis.

VII. CONCLUSIONS

We derived the characteristic equation for the linear stability of the incoherent state in a delayed noisy Kuramoto-Daido continuum model with general periodic coupling. The delay kernel enters through its Laplace transform, phase diffusion

contributes the harmonic-dependent damping n^2D , and the Fourier coefficients of the coupling function define the independent linear stability channels. For a Lorentzian frequency distribution and first-harmonic coupling, the result reduces to a closed dispersion relation that recovers the zero-delay threshold $K_c(0) = 2(D + \Delta)$ and the established fixed- and distributed-delay limits.

For the overdamped Josephson-junction array considered here, natural-angle averaging gives the same delayed phase model together with the load-induced drift $\Gamma_0 = Kh_0$ when the common load has finite dc admittance. The drift is removable at fixed coupling, but along the load variation considered in Sec. V it changes the effective frequency entering the delay condition. Finite- N simulations of the reduced phase equations reproduce the resulting displacement of the neutral boundary. These results identify how delay dispersion, phase diffusion, coupling harmonics, and a load-induced frequency shift enter the loss of linear stability of the incoherent state.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: FOURIER LINEARIZATION OF THE CONTINUUM EQUATION

This Appendix gives the Fourier linearization leading to the characteristic equation used in Sec. III. The calculation follows the standard stability analysis of the incoherent state, but is included to fix the Fourier and sign conventions used throughout the paper.

For the coupling expansion

$$H(\phi) = \sum_{m=-\infty}^{\infty} h_m e^{im\phi}, \quad (\text{A1})$$

we perturb the uniform density according to

$$f(\theta, \omega, t) = \frac{1}{2\pi} \left[1 + \sum_{n \neq 0} c_n(\omega, t) e^{in\theta} \right]. \quad (\text{A2})$$

The uniform part of the mean-field drift is

$$v_0(\omega) = \omega + Kh_0.$$

Using Eq. (A2) in the continuum velocity field, the perturbation of the drift is

$$\delta v(\theta, t) = K \sum_{n \neq 0} h_{-n} e^{in\theta} \int_0^\infty \rho(\tau) C_n(t - \tau) d\tau, \quad (\text{A3})$$

where

$$C_n(t) = \int_{-\infty}^{\infty} g(\omega) c_n(\omega, t) d\omega. \quad (\text{A4})$$

Substituting $f = f_0 + \delta f$ and $v = v_0 + \delta v$ into the nonlinear Fokker-Planck equation and retaining only terms linear in the perturbation gives

$$\partial_t \delta f = -v_0 \partial_\theta \delta f - f_0 \partial_\theta \delta v + D \partial_\theta^2 \delta f.$$

Projection onto the Fourier mode $e^{in\theta}$ then yields

$$\begin{aligned} \dot{c}_n(\omega, t) = & -[n^2 D + in(\omega + Kh_0)]c_n(\omega, t) \\ & - inKh_{-n} \int_0^\infty \rho(\tau) C_n(t - \tau) d\tau. \end{aligned} \quad (\text{A5})$$

For a normal mode,

$$c_n(\omega, t) = b_n(\omega) e^{\lambda t}, \quad (\text{A6})$$

the delayed collective amplitude becomes

$$C_n(t - \tau) = B_n e^{\lambda t} e^{-\lambda \tau}, \quad B_n = \int_{-\infty}^\infty g(\omega) b_n(\omega) d\omega.$$

Hence,

$$[\lambda + n^2 D + in(\omega + Kh_0)]b_n(\omega) = -inKh_{-n} \tilde{\rho}(\lambda) B_n, \quad (\text{A7})$$

with

$$\tilde{\rho}(\lambda) = \int_0^\infty \rho(\tau) e^{-\lambda \tau} d\tau.$$

Solving Eq. (A7) for $b_n(\omega)$, multiplying by $g(\omega)$, and integrating over ω gives

$$\begin{aligned} 0 = B_n \left[1 + inKh_{-n} \tilde{\rho}(\lambda) \right. \\ \left. \times \int_{-\infty}^\infty \frac{g(\omega) d\omega}{\lambda + n^2 D + in(\omega + Kh_0)} \right]. \end{aligned} \quad (\text{A8})$$

A nontrivial collective perturbation requires $B_n \neq 0$, which gives Eq. (9).

APPENDIX B: LORENTZIAN EVALUATION AND FRAME CONVENTION

This Appendix evaluates the frequency integral appearing in the first-harmonic characteristic equation and clarifies the relation between the laboratory- and rotating-frame representations. We retain the convention of Eq. (1), for which $K > 0$ is attractive when $H(\phi) = \sin \phi$.

1. Lorentzian frequency integral

For the Lorentzian distribution

$$g(\omega) = \frac{\Delta}{\pi[(\omega - \omega_0)^2 + \Delta^2]}, \quad \Delta > 0, \quad (\text{B1})$$

the $n = 1$ characteristic equation contains

$$I(\lambda) = \int_{-\infty}^\infty \frac{g(\omega) d\omega}{\lambda + D + i(\omega + Kh_0)}. \quad (\text{B2})$$

For $\text{Re } \lambda > -D$, the pole generated by the denominator lies in the upper half of the ω plane. Closing the contour in the lower half-plane therefore encloses only the Lorentzian pole

$$\omega = \omega_0 - i\Delta.$$

Evaluation of the residue gives

$$I(\lambda) = \frac{1}{\lambda + D + \Delta + i(\omega_0 + Kh_0)}. \quad (\text{B3})$$

Substituting Eq. (B3) into Eq. (9), together with

$$h_{-1} = \frac{i}{2} e^{-i\alpha},$$

gives

$$\lambda + D + \Delta + i(\omega_0 + Kh_0) - \frac{K}{2} e^{-i\alpha} \tilde{\rho}(\lambda) = 0, \quad (\text{B4})$$

which is Eq. (11).

2. Laboratory and rotating frames

Equation (B4) is written in the laboratory frame. For a collective perturbation proportional to $e^{\lambda t}$,

$$C(t - \tau) = C_0 e^{\lambda(t - \tau)} = C_0 e^{\lambda t} e^{-\lambda \tau},$$

and averaging over the delay distribution therefore produces

$$\tilde{\rho}(\lambda) = \int_0^\infty \rho(\tau) e^{-\lambda \tau} d\tau. \quad (\text{B5})$$

No additional factor involving ω_0 appears in the laboratory-frame delay transform.

To compare with a rotating-frame formulation, define

$$\vartheta = \theta - \omega_0 t. \quad (\text{B6})$$

Then,

$$\theta_j(t - \tau) - \theta_i(t) = \vartheta_j(t - \tau) - \vartheta_i(t) - \omega_0 \tau. \quad (\text{B7})$$

The n th Fourier harmonic of the interaction therefore acquires the factor $e^{-in\omega_0 \tau}$, and the corresponding delay transform becomes

$$\tilde{\rho}(\lambda + in\omega_0). \quad (\text{B8})$$

The two forms are equivalent when the entire calculation is performed in the same frame. In this work, we use the laboratory representation throughout: $\tilde{\rho}(\lambda)$ appears in the delay term, while $\omega_0 + Kh_0$ remains explicit in the dispersion relation. This convention is used in all analytical stability boundaries presented in Secs. IV and V.

APPENDIX C: NEUTRAL STABILITY CONDITIONS

This Appendix gives the explicit neutral-stability equations used to evaluate the analytical boundaries in Sec. IV.

At marginal stability, the leading eigenvalue satisfies

$$\lambda = i\Omega, \quad (\text{C1})$$

where Ω is the collective frequency of the neutral mode. Substitution into Eq. (11) gives

$$D + \Delta + i(\Omega + \omega_0 + Kh_0) - \frac{K}{2} e^{-i\alpha} \tilde{\rho}(i\Omega) = 0. \quad (\text{C2})$$

Define

$$A(\Omega) = \text{Re}[e^{-i\alpha} \tilde{\rho}(i\Omega)], \quad B(\Omega) = \text{Im}[e^{-i\alpha} \tilde{\rho}(i\Omega)]. \quad (\text{C3})$$

Separating the real and imaginary parts of Eq. (C2) gives

$$D + \Delta = \frac{K_c}{2} A(\Omega), \quad (\text{C4})$$

$$\Omega + \omega_0 + K_c h_0 = \frac{K_c}{2} B(\Omega). \quad (\text{C5})$$

Thus,

$$K_c = \frac{2(D + \Delta)}{A(\Omega)}, \quad A(\Omega) > 0, \quad (\text{C6})$$

while Eq. (C5) determines the corresponding neutral frequency. These are the equations quoted as Eqs. (16) and (17).

For the Gamma family,

$$\tilde{\rho}_q(i\Omega) = \left(1 + i \frac{\Omega \bar{\tau}}{q}\right)^{-q}, \quad (\text{C7})$$

so the neutral boundary is obtained directly from Eqs. (C4) and (C5). The fixed-delay limit follows from

$$\lim_{q \rightarrow \infty} \tilde{\rho}_q(i\Omega) = e^{-i\Omega \bar{\tau}}. \quad (\text{C8})$$

For a fixed delay, it is useful to introduce

$$y = \Omega \bar{\tau}. \quad (\text{C9})$$

Since

$$e^{-i\alpha} \tilde{\rho}(i\Omega) = e^{-i(\alpha+y)},$$

one has

$$A = \cos(\alpha + y), \quad B = -\sin(\alpha + y).$$

The neutral coupling can therefore be written parametrically as

$$K_c(y) = \frac{2(D + \Delta)}{\cos(\alpha + y)}, \quad \cos(\alpha + y) > 0. \quad (\text{C10})$$

Using $\Omega = y/\bar{\tau}$ in Eq. (C5) then gives

$$\bar{\tau}(y) = \frac{y}{-\omega_0 - K_c(y)h_0 - \frac{K_c(y)}{2} \sin(\alpha + y)}. \quad (\text{C11})$$

Equations (C10) and (C11) provide the fixed-delay neutral branches used in Secs. IV and V. For Gamma-distributed delays the corresponding boundaries are obtained by solving Eqs. (C4) and (C5) with Eq. (C7).

APPENDIX D: NATURAL-ANGLE REDUCTION OF THE JOSEPHSON-JUNCTION MODEL

This Appendix derives the reduced phase equation used in Sec. V. The calculation follows the standard weak-coupling averaging procedure for overdamped common-load Josephson arrays and is included to identify explicitly the ac coupling term and the dc-induced uniform drift.

1. Microscopic circuit equations

For an overdamped junction, the RSJ equation is

$$\frac{\hbar}{2er_k} \dot{\phi}_k(t) + I_k \sin \phi_k(t) + I_{\text{load}}(t) = I_B + \eta_k(t), \quad (\text{D1})$$

where I_k , r_k , and $\eta_k(t)$ denote the critical current, normal-state resistance, and thermal-noise current, respectively.

The physical voltage across the complete series array is

$$V_{\text{array}}(t) = \frac{\hbar}{2e} \sum_{j=1}^N \dot{\phi}_j(t). \quad (\text{D2})$$

Including propagation-time heterogeneity, the load is driven by

$$V_d(t) = \frac{\hbar}{2e} \sum_{j=1}^N \int_0^\infty \rho(\tau) \dot{\phi}_j(t - \tau) d\tau. \quad (\text{D3})$$

The physical load current satisfies

$$\hat{I}_{\text{load}}(v) = Y_L(v) \hat{V}_d(v), \quad (\text{D4})$$

with

$$Y_L(0) = G_{\text{dc}}.$$

2. Natural-angle transformation

For an uncoupled junction, $I_{\text{load}} = \eta_k = 0$, the running-state frequency is

$$\omega_k = \frac{2er_k}{\hbar} \sqrt{I_B^2 - I_k^2}. \quad (\text{D5})$$

Defining

$$A_k = I_B^2 - I_k^2, \quad (\text{D6})$$

the physical phase may be mapped to a uniformly rotating natural angle θ_k through

$$I_B - I_k \sin \phi_k = \frac{A_k}{I_B - I_k \cos \theta_k}. \quad (\text{D7})$$

Using this identity in Eq. (D1) gives the exact natural-angle equation

$$\dot{\theta}_k = \omega_k - \frac{\omega_k}{A_k} [I_{\text{load}}(t) - \eta_k(t)] (I_B - I_k \cos \theta_k). \quad (\text{D8})$$

The factor $I_B - I_k \cos \theta_k$ is the phase-response factor that couples the load current to the slow phase.

3. Fourier decomposition of the junction voltage

The instantaneous voltage of the k th junction is

$$V_k(\theta_k) = \frac{\hbar}{2e} \dot{\phi}_k = r_k \frac{A_k}{I_B - I_k \cos \theta_k} \quad (\text{D9})$$

in the uncoupled running state. Its Fourier expansion may be written as

$$V_k(\theta_k) = V_{0k} + \sum_{m=1}^{\infty} V_{mk} \cos(m\theta_k), \quad (\text{D10})$$

with

$$V_{0k} = r_k \sqrt{A_k}, \quad (\text{D11})$$

$$V_{1k} = 2r_k \sqrt{A_k} q_k, \quad q_k = \frac{I_B - \sqrt{A_k}}{I_k}. \quad (\text{D12})$$

The dc coefficient satisfies

$$V_{0k} = \frac{\hbar \omega_k}{2e},$$

as required by the Josephson voltage-frequency relation.

4. Separation of dc and ac load responses

The delayed array voltage contains both a mean component and oscillatory harmonics. Accordingly, the load current may be decomposed as

$$I_{\text{load}}(t) = I_{\text{dc}} + I_{\text{ac}}(t). \quad (\text{D13})$$

For weak disorder, the mean voltage per junction is

$$\bar{V}_0 \simeq \frac{\hbar \bar{\omega}}{2e},$$

so the total mean array voltage is $V_{\text{dc}} \simeq N \bar{V}_0$. With the weak-load scaling $Y_L(v) = \mathcal{Y}(v)/N$, the zero-frequency admittance satisfies $G_{\text{dc}} = \mathcal{G}_{\text{dc}}/N$, and the dc load current remains $O(1)$:

$$I_{\text{dc}} = G_{\text{dc}} V_{\text{dc}} \simeq \mathcal{G}_{\text{dc}} \bar{V}_0. \quad (\text{D14})$$

The oscillatory part is filtered by the complex load admittance at the harmonic frequencies. Retaining the fundamental component, write

$$\mathcal{Y}(\bar{\omega}) = |\mathcal{Y}_1| e^{i\alpha_1},$$

so that $Y_L(\bar{\omega}) = |\mathcal{Y}_1| e^{i\alpha_1}/N$. The ac current has the form

$$I_{\text{ac}}(t) \propto \sum_{j=1}^N \int_0^\infty \rho(\tau) \cos[\theta_j(t - \tau) - \alpha_1] d\tau \quad (\text{D15})$$

up to the overall amplitude determined by $|\mathcal{Y}_1|$ and V_{1j} .

The two contributions enter the averaging differently. The dc current multiplies the constant component I_B of the phase-response factor and therefore survives the fast average as a uniform drift. The fundamental ac current multiplies the $\cos \theta_k$ component and produces the resonant phase-difference interaction. Higher voltage harmonics do not contribute at this order because the phase-response factor of the overdamped junction contains only the fundamental harmonic.

5. First-order averaged phase dynamics

We now assume weak loading and weak disorder so that the load-induced phase evolution is slow compared with the Josephson period and the phase-response factor may be evaluated at the mean junction parameters $(\bar{I}, \bar{r}, \bar{\omega})$.

The dc contribution obtained from Eq. (D8) is

$$\Gamma_0 = -\frac{\bar{\omega} I_B}{I_B^2 - \bar{I}^2} I_{\text{dc}}. \quad (\text{D16})$$

For a load with $G_{\text{dc}} = 0$, one has $I_{\text{dc}} = 0$ and therefore $\Gamma_0 = 0$.

The resonant part of the fundamental ac response gives the delayed first-harmonic interaction. After absorbing the physical load amplitude and the array-size scaling into the mean-field coupling coefficient K , the averaged equation takes the form

$$\begin{aligned} \dot{\theta}_k = & \omega_k + \Gamma_0 + \frac{K}{N} \sum_{j=1}^N \int_0^\infty \rho(\tau) \sin[\theta_j(t - \tau) - \theta_k(t) + \alpha] d\tau \\ & + \sqrt{2D} \xi_k(t). \end{aligned} \quad (\text{D17})$$

Here, $\alpha = \alpha_1$ is the phase of the fundamental load response. With $Y_L = \mathcal{Y}/N$, the factor $1/N$ in the physical admittance cancels the N scaling of the array voltage. The remaining first-harmonic load amplitude is absorbed into the mean-field coefficient K , so that Eq. (D17) has the normalization of Eq. (1).

Comparing Eq. (D17) with the first-harmonic Kuramoto-Daido interaction

$$H(\phi) = h_0 + \sin(\phi + \alpha)$$

identifies

$$Kh_0 = \Gamma_0. \quad (\text{D18})$$

Thus, a finite-dc-load conductance generates the constant Fourier component in the reduced coupling function, whereas a load with vanishing dc admittance reduces to the usual zero-mean first-harmonic interaction.

The reduction is valid when the load-induced phase evolution is slow compared with the Josephson frequency and the parameter disorder is sufficiently weak for averaging about the mean junction orbit. These are the assumptions under which the reduced phase dynamics are compared with the continuum stability boundary in Sec. V.

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- [1] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2001).
 - [2] J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort, and R. Spigler, The Kuramoto model: A simple paradigm for synchronization phenomena, *Rev. Mod. Phys.* **77**, 137 (2005).
 - [3] F. A. Rodrigues, T. K. DM Peron, P. Ji, and J. Kurths, The Kuramoto model in complex networks, *Phys. Rep.* **610**, 1 (2016).
 - [4] H. Sompolinsky, D. Golomb, and D. Kleinfeld, Global processing of visual stimuli in a neural network of coupled oscillators, *Proc. Natl. Acad. Sci. USA* **87**, 7200 (1990).
 - [5] E. M. Izhikevich, *Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting* (MIT Press, Cambridge, MA, 2007).
 - [6] S. Yanchuk and M. Wolfrum, A multiple time scale approach to the stability of external cavity modes in the Lang-Kobayashi system using the limit of large delay, *SIAM J. Appl. Dyn. Syst.* **9**, 519 (2010).
 - [7] K. Wiesenfeld, P. Colet, and S. H. Strogatz, Synchronization transitions in a disordered Josephson series array, *Phys. Rev. Lett.* **76**, 404 (1996).
 - [8] A. E. Motter, S. A. Myers, M. Anghel, and T. Nishikawa, Spontaneous synchrony in power-grid networks, *Nat. Phys.* **9**, 191 (2013).
 - [9] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1984).
 - [10] M. G. Earl and S. H. Strogatz, Synchronization in oscillator networks with delayed coupling: A stability criterion, *Phys. Rev. E* **67**, 036204 (2003).

- [11] S. Yanchuk and G. Giacomelli, Spatio-temporal phenomena in complex systems with time delays, *J. Phys. A: Math. Theor.* **50**, 103001 (2017).
- [12] H. Sakaguchi, Cooperative phenomena in coupled oscillator systems under external fields, *Prog. Theor. Phys.* **79**, 39 (1988).
- [13] S. H. Strogatz and R. E. Mirollo, Stability of incoherence in a population of coupled oscillators, *J. Stat. Phys.* **63**, 613 (1991).
- [14] L. L. Bonilla, J. C. Neu, and R. Spigler, Nonlinear stability of incoherence and collective synchronization in a population of coupled oscillators, *J. Stat. Phys.* **67**, 313 (1992).
- [15] J. D. Crawford, Amplitude expansions for instabilities in populations of globally-coupled oscillators, *J. Stat. Phys.* **74**, 1047 (1994).
- [16] H. Sakaguchi and Y. Kuramoto, A soluble active rotator model showing phase transitions via mutual entrainment, *Prog. Theor. Phys.* **76**, 576 (1986).
- [17] H. Daido, Order function and macroscopic mutual entrainment in uniformly coupled limit-cycle oscillators, *Prog. Theor. Phys.* **88**, 1213 (1992).
- [18] H. Daido, Onset of cooperative entrainment in limit-cycle oscillators with uniform all-to-all interactions: Bifurcation of the order function, *Physica D* **91**, 24 (1996).
- [19] J. D. Crawford and K. T. R. Davies, Synchronization of globally coupled phase oscillators: Singularities and scaling for general couplings, *Physica D* **125**, 1 (1999).
- [20] H. Chiba, Linear stability of the incoherent solution and the transition formula for the Kuramoto–Daido model, *RIMS Kôkyûroku Bessatsu B* **21**, 109 (2010).
- [21] M. K. S. Yeung and S. H. Strogatz, Time delay in the Kuramoto model of coupled oscillators, *Phys. Rev. Lett.* **82**, 648 (1999).
- [22] W. S. Lee, E. Ott, and T. M. Antonsen, Large coupled oscillator systems with heterogeneous interaction delays, *Phys. Rev. Lett.* **103**, 044101 (2009).
- [23] K. Wiesenfeld and J. W. Swift, Averaged equations for Josephson junction series arrays, *Phys. Rev. E* **51**, 1020 (1995).
- [24] B. C. Daniels, S. T. M. Dissanayake, and B. R. Trees, Synchronization of coupled rotators: Josephson junction ladders and the locally coupled Kuramoto model, *Phys. Rev. E* **67**, 026216 (2003).
- [25] J. Turtle, P.-L. Buono, A. Palacios, C. Dabrowski, V. In, and P. Longhini, Synchronization of spin torque nano-oscillators, *Phys. Rev. B* **95**, 144412 (2017).
- [26] B. Sturgis-Jensen, P.-L. Buono, A. Palacios, J. Turtle, V. In, and P. Longhini, On the synchronization phenomenon of a parallel array of spin torque nano-oscillators, *Physica D* **396**, 70 (2019).
- [27] T. Levasseur, A. Palacios, S. Sharan, and V. In, Beam steering and signal amplification through feedforward networks. Part I: Transmission, *Int. J. Bifurc. Chaos* **32**, 2230034 (2022).
- [28] T. Levasseur, A. Palacios, S. Sharan, and V. In, Beam steering and signal amplification through feedforward networks: Part II reception, *Int. J. Bifurc. Chaos* **35**, 2530004 (2025).