

Anomalous diffusion in coupled viscoelastic media: A fractional Langevin equation approach

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Anomalous diffusion often arises in complex environments where viscoelastic or crowded conditions influence particle motion. In many biological and soft-matter systems, distinct components of the medium exhibit unique viscoelastic responses, resulting in time-dependent changes in the observed diffusion exponents. Here, we develop a theoretical model of two particles, each embedded in a distinct viscoelastic medium, and coupled via a harmonic potential. By formulating and solving a system of coupled fractional Langevin equations (FLEs) with memory exponents $0 < \alpha < \beta \leq 1$, we uncover rich transient anomalous diffusion phenomena arising from the interplay of memory kernels and bilinear coupling. Notably, we identify “recovery dynamics,” where a subdiffusive particle (α) transiently accelerates and eventually regains its intrinsic long-time mobility. This recovery emerges only when memory exponents differ ($\alpha < \beta$), whereas identical exponents ($\alpha = \beta$) suppress recovery. Our FLE models are further validated in comparison with Langevin dynamics simulations of polymer-based real systems. Our theoretical predictions offer insight into experimentally observed transient anomalous diffusions, such as polymer-protein complexes and cross-linked cytoskeletal networks, highlighting the critical role of memory heterogeneity and mechanical interactions in biological anomalous diffusion.

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I. INTRODUCTION

The motion of particles in biological and soft-matter systems often exhibits anomalous diffusion, in which the mean square displacement (MSD) of the particle follows a power law: $\langle \Delta x^2(t) \rangle \sim t^\nu$, where ν is anomalous exponent in the range of $0 < \nu \leq 2$ [1,2]. In particular, it is known that subdiffusive motions ($0 < \nu < 1$) are observed in many soft-matter systems, mainly due to the viscoelasticity and molecular crowding of the embedding media, which gives rise to memory effects in the diffusing dynamics of the embedded particles. A physics-based mathematical framework to describe such anomalous diffusion phenomena is the generalized Langevin equation (GLE) [3–5], which extends the Langevin equation by incorporating a time-dependent frictional kernel. The GLE takes the form

$$m\ddot{x}(t) = - \int_0^t K(t-t')\dot{x}(t')dt' + \xi(t), \quad (1)$$

where $K(t)$ is the memory kernel that governs the frictional response and $\xi(t)$ is the thermal noise satisfying the fluctuation-dissipation theorem (FDT) of the second kind [6]:

$$\langle \xi(t)\xi(t') \rangle = k_B T K(|t-t'|), \quad (2)$$

where k_B is the Boltzmann constant and T is the absolute temperature. When $K(t) = 2\gamma\delta(t)$, the GLE reduces to the ordinary Langevin equation described with the friction term $-\gamma\dot{x}$. In viscoelastic systems, friction has memory such that the velocities and positions in the past influence the diffusion dynamics at present.

An important class of the GLE is the so-called *fractional Langevin equation* (FLE) [7–9], where the memory kernel decays over time in a power-law manner, as follows:

$$K_\mu(t) = \frac{\gamma_\mu}{\Gamma(1-\mu)} t^{-\mu}, \quad 0 < \mu < 1. \quad (3)$$

Here, γ_μ is a generalized friction coefficient in the unit of $[\text{kg s}^{\mu-2}]$, μ is a memory exponent, and $\Gamma(z)$ is the Gamma function. Such power-law forms of the memory kernel are often observed in numerous biological and soft-matter systems, e.g., proteins [10,11], intracellular fluids [12–14], chromosomes [15,16], polymer networks [17–19], and complex fluids [20]. Plugging the kernel (3) into Eq. (1) leads to the FLE (neglecting the inertia term):

$$\int_0^t K_\mu(t-t')\dot{x}(t')dt' = \gamma_\mu \frac{d^\mu}{dt^\mu} x(t) = \xi_\mu(t), \quad (4)$$

where $\frac{d^\mu}{dt^\mu} x(t)$ is the Caputo fractional derivative of order μ [21,22]. The MSD of the FLE (4) exhibits the scaling law

$$\langle \Delta x^2(t) \rangle = \frac{2k_B T}{\gamma_\mu \Gamma(1+\mu)} t^\mu, \quad 0 < \mu < 1, \quad (5)$$

describing the subdiffusive motion observed in biological and soft-matter systems [23,24]. In the limit of $\mu \rightarrow 1$, the power-law memory kernel reduces to a delta function, and the

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FLE becomes the ordinary Langevin equation, recovering the Brownian motion.

The FLE framework has been widely applied to model various biocomplex systems. Theoretically, it can be derived from the general elastic models, including those incorporating hydrodynamic interactions [25]. A broad range of exponents μ emerges when considering the dynamics of a single monomer or a local segment as part of polymeric or membrane systems. For instance, polymer models predict specific values of μ : An entangled polymer has $\mu = 1/4$ [26]; a flexible polymer exhibits $\mu = 1/2$ [27–30], while a semiflexible polymer has $\mu = 3/4$ [31,32] and a self-avoiding (flexible) chain reveals $\mu \approx 0.54$ [26,33,34]. The Zimm model considering hydrodynamic interactions has $\mu = 2/3$ [25,35]. Interestingly, the transverse undulation dynamics in lipid membranes has $\mu = 2/3$ [25].

Apart from these model systems, the power-law memory kernel is revealed via the generalized Stokes-Einstein relation [23], using the microrheology experiments for viscoelastic gels and biological systems [12–14,17,36]. Furthermore, molecular dynamics simulations have reported viscoelastic subdiffusion in lipid bilayers [37,38] and protein condensates [39], demonstrating the relevance of FLE models in describing real-world systems.

On the other hand, anomalous diffusions in biological and soft-matter systems are, in many cases, *transient* such that the anomalous exponent $\nu(t)$ changes over time. Biological examples showing such transient anomalous diffusions include the chromatin motion in biological nuclei [40–46], cytoplasmic macromolecular transport [47–52], membrane-protein diffusion on biological membranes [53,54], tracer diffusion in synthetic gels [55–59] or cultured cells [60], and the migration of a cell in extracellular matrices [61,62].

As a possible mechanism, such transient anomalous diffusions originate from the presence of multiple viscoelastic components within complex biological systems and their coupled dynamics. When isolated in free space, each viscoelastic component would exhibit anomalous dynamics described by FLE (4) with a time-independent monoscaling exponent (μ). When several viscoelastic components comprise a complex system through mechanical couplings, the system can achieve temporally heterogeneous anomalous dynamics in which the anomalous exponent ν changes over time, depending on which viscoelastic component dominates or how strong the coupling between components is.

There are multiple biological examples that can be understood with our model of coupled FLE systems (or multicomponent viscoelastic systems). One such example is the chromosome dynamics where a chromatin polymer is associated with DNA-binding proteins, e.g., transcription factors [63], chromatin remodelers [64], and large protein condensates that function as membraneless organelles within the nucleus [65,66]. The schematic illustration for this example is depicted in Fig. 1(a). Here, the system consists of two components: the chromatin (i.e., a polymer chain) and a macromolecular complex (red). The chromatin is a viscoelastic material whose dynamics is modeled with FLE with the memory exponent $\alpha = 1/2$ (flexible chains) or $3/4$ (semiflexible chains). The macromolecular complex represents the other viscoelastic counterpart, governed by another FLE with the

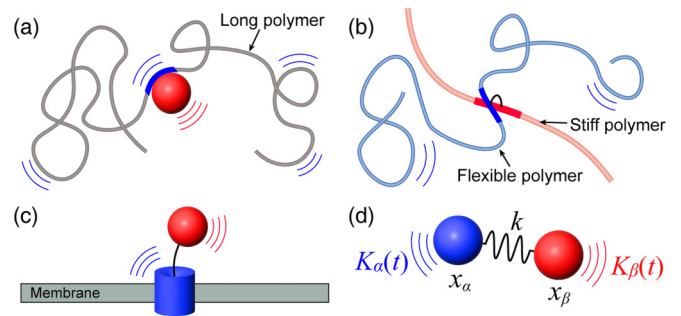


FIG. 1. Schematic illustrations of biological and soft-matter systems modeled by two coupled FLEs. (a) A polymer chain (e.g., chromatin) tethered to a macromolecular complex or protein condensate. (b) A composite cytoskeletal network comprising flexible polymers cross-linked with stiff polymers. (c) A membrane protein embedded within a lipid bilayer, interacting with both trans- and extracellular domains. (d) Our minimal bipartite model of these systems. Two particles, each governed by its own viscoelastic memory kernel characterized by distinct memory exponents (α and β), are mechanically coupled by a harmonic potential.

memory exponent β , which is unity if the embedding medium is a viscous fluid or <1 in a crowded (viscoelastic) medium. This macromolecular complex is bound to the chromatin, making the two FLE systems coupled.

Another interesting example is the cytoskeleton dynamics, as schematically illustrated in Fig. 1(b). For instance, a flexible polymer ($\alpha = 1/2$) cross-links with a stiff filament ($\beta = 3/4$). Then, the cross-linked polymer network exhibits the coupled monomer dynamics of these two distinct polymers. As an example, a microrheology experiment showed that when a composite gel is made with two distinct types of polymers with markedly different persistence lengths (l_p), the viscoelastic memory of the gel has two distinct power-law regimes at short and long timescales [55–57,67]. In the cell, the cytoskeleton is mainly composed of actin filaments ($l_p \approx 10 \mu\text{m}$) and microtubules ($l_p \approx 1 \text{ mm}$) [68–70]. These filaments are sterically entangled and frequently cross-linked by accessory proteins, facilitating essential cellular processes such as proliferation, differentiation, and migration.

An additional relevant example is the diffusion of membrane proteins on a plasma membrane. Membrane proteins, embedded in a lipid bilayer through its transmembrane domain (blue), typically possess a large cytoplasmic part (red) or form a complex (red) with macromolecules in the cytoplasm [Fig. 1(c)]. The transmembrane domain lies in a viscoelastic medium (lipid bilayer, α), which is connected to the extracellular part in contact with a cytoplasmic viscoelastic fluid (β). Such a mechanical coupling between the two distinct viscoelastic dynamics can result in a complex diffusion dynamics of membrane proteins. Furthermore, additional interactions with the cytoskeletal filaments can amplify the coupling effects, reinforcing the emergence of complex viscoelastic responses.

In this work, we investigate the dynamics of coupled viscoelastic systems using a minimal bipartite model based on two coupled FLEs [see Fig. 1(d)]. In this model, two particles are connected via a harmonic spring, and each undergoes

motion governed by its own FLE with a distinct memory kernel (with the index α or β). We solve the model analytically and show that the system exhibits rich transient anomalous diffusion characterized by multiple scaling regimes. A particularly surprising consequence is the appearance of recovery dynamics, where the slower particle (with the anomalous exponent α) temporarily accelerates and catches up with the faster one (with the exponent β), resulting in an effective scaling exponent $\nu(>\alpha)$ during an intermediate time window. The results highlight how coupling between components can lead to emergent anomalous behaviors that differ significantly from those of the individual subsystems.

The remainder of this paper is structured as follows. In Sec. II, we begin by introducing the coupled FLE model and defining relevant timescales based on system parameters. In Sec. III, we then solve the coupled equations and analyze the resulting effective memory kernel. In Sec. IV, we compute the MSDs of the two particles and their relative displacement. In Sec. V, we validate our theoretical FLE models with Langevin dynamics simulations of two physical examples. Finally, we summarize the key findings and discuss their physical implications.

II. COUPLED FRACTIONAL LANGEVIN EQUATIONS

Motivated by the coupled viscoelastic systems explained above, we consider a bipartite system shown in Fig. 1(d) where each particle performs a viscoelastic anomalous diffusion with a distinct (power-law decaying) memory kernel [Eq. (3)] and their motion is coupled via a harmonic spring. These coupled FLEs are described by the following equation of motion:

$$\int_0^t K_\alpha(t-t')\dot{x}_\alpha(t')dt' = -k(x_\alpha(t) - x_\beta(t)) + \xi_\alpha(t), \quad (6a)$$

$$\int_0^t K_\beta(t-t')\dot{x}_\beta(t')dt' = -k(x_\beta(t) - x_\alpha(t)) + \xi_\beta(t). \quad (6b)$$

In this expression, $x_\mu(t)$ is the position of the particle μ at time t , where $\mu = \alpha, \beta$, and $0 < \alpha < \beta \leq 1$ is assumed without loss of generality. Here, $K_\mu(t)$ denotes the power-law memory kernel given by Eq. (3), and $\xi_\mu(t)$ is the zero-mean fractional Gaussian noise satisfying FDT: $\langle \xi_\mu(t)\xi_\mu(t') \rangle = k_B T \delta_{\mu\mu'} K_\mu(|t-t'|)$. The two particles interact with each other through the harmonic potential of an interaction strength k . This setup can be considered as a generalization of two harmonically coupled Brownian particles in the context of viscoelastic media characterized with long-time memories [71]. In our model, the initial condition of the relative displacement $x_\alpha(t) - x_\beta(t)$ is assumed to follow a Gibbs-Boltzmann distribution, $x_\alpha(0) - x_\beta(0) \sim \mathcal{N}(0, k_B T/k)$.

Importantly, our FLE model (6) can be directly derived from coupled polymer Langevin dynamics, ensuring a clear physical foundation. Consider a bipartite polymer system composed of two subsystems, A and B , each represented by beads with coordinates $\mathbf{r}_i^Z(t) \in \mathbb{R}^d$ ($Z \in \{A, B\}$, $i = 1, \dots, M_Z$). The total potential energy of the system is

$$U_{\text{tot}} = U^A(\{\mathbf{r}_i^A\}) + U^B(\{\mathbf{r}_i^B\}) + \frac{1}{2}k(\mathbf{r}_a^A - \mathbf{r}_b^B)^2, \quad (7)$$

where the bead with index a in A and the bead with index b in B are coupled via the harmonic potential in U_{tot} . Microscopically, this bipartite polymer system is described by the overdamped Langevin equation

$$\gamma^Z \dot{\mathbf{r}}_i^Z(t) = -\frac{\partial U_{\text{tot}}}{\partial \mathbf{r}_i^Z} + \boldsymbol{\eta}_i^Z(t), \quad (8)$$

where γ^Z is the bead friction and $\boldsymbol{\eta}_i^Z$ are Gaussian white noises with zero mean and covariance $\langle \eta_{i,u}^Z(t) \eta_{j,v}^Z(t') \rangle = 2\gamma^Z k_B T \delta_{ZZ'} \delta_{ij} \delta_{uv} \delta(t-t')$, with u, v denoting Cartesian components. By integrating out all coordinate variables except for the tracers $\mathbf{r}_a^A(t)$ and $\mathbf{r}_b^B(t)$ and projecting along one Cartesian direction [25–33,35], we obtain the one-dimensional coupled FLEs in the form of Eq. (6). As an example, in Appendix A, we have derived the FLE for the macromolecule-polymer coupled system depicted in Fig. 1(a). The obtained FLE (A14) indeed follows the form in Eq. (6) with a power-law memory kernel of $\alpha = 1/2$ and a delta-correlated kernel (or $\beta = 1$). Refer to Appendix A for a complete description of the mathematical formalism.

Before solving the coupled FLEs (6), we define three important characteristic timescales through a comparative analysis of the memory kernels and the interaction part. In the Laplace domain (with the transformation $\tilde{f}(s) = \int_0^\infty e^{-st} f(t) dt$), the memory kernels become $\tilde{K}_\alpha(s) = \gamma_\alpha s^{-1+\alpha}$ and $\tilde{K}_\beta(s) = \gamma_\beta s^{-1+\beta}$, while the spring constant k is ks^{-1} . In the Laplace space, any two curves among them intersect each other, which allows us to define the following characteristic times:

$$\tau_\alpha = \left(\frac{\gamma_\alpha}{k}\right)^{1/\alpha}, \quad \tau_\beta = \left(\frac{\gamma_\beta}{k}\right)^{1/\beta}, \quad \tau_c = \left(\frac{\gamma_\beta}{\gamma_\alpha}\right)^{1/(\beta-\alpha)}. \quad (9)$$

The physical meaning of these timescales is as follows: τ_α and τ_β mark the times at which the particle α (or β) diffusing in free space ($k=0$) reaches the distance of $\sqrt{2k_B T/k}$ (i.e., the average confined distance in the harmonic potential). τ_c is the time when the free-space diffusion for particles α and β has the same diffusion distance, that is,

$$\begin{aligned} \tau_\alpha : \langle \Delta x_\alpha^2(\tau_\alpha) \rangle_{k=0} &\approx 2k_B T/k, \\ \tau_\beta : \langle \Delta x_\beta^2(\tau_\beta) \rangle_{k=0} &\approx 2k_B T/k, \\ \tau_c : \langle \Delta x_\alpha^2(\tau_c) \rangle_{k=0} &\approx \langle \Delta x_\beta^2(\tau_c) \rangle_{k=0}, \end{aligned} \quad (10)$$

where $\langle \Delta x_\mu^2(t) \rangle_{k=0} = \frac{2k_B T t^\mu}{\gamma_\mu \Gamma(1+\mu)}$.

As a special case, there exists a critical interaction strength

$$k_c = (\gamma_\beta / \gamma_\alpha)^{1/(\beta-\alpha)} \quad (11)$$

at which the three terms ($\tilde{K}_\alpha(s)$, $\tilde{K}_\beta(s)$, and ks^{-1}) intersect at one point in the Laplace domain. It is noteworthy that the value of k/k_c determines the ordering of the three characteristic times: (1) in the weak-interaction regime ($k < k_c$), it is $\tau_\alpha > \tau_\beta > \tau_c$; and (2) in the strong-interaction regime ($k > k_c$), $\tau_\alpha < \tau_\beta < \tau_c$ is obtained.

In the next section, we explicitly solve the coupled FLEs (6) for x_α and x_β as well as for the relative displacement

$$r_{\alpha\beta}(t) \equiv x_\alpha(t) - x_\beta(t). \quad (12)$$

Because the two particles have distinct memory kernels, the normal mode decomposition is not applicable. Instead, we explicitly derive the generalized Langevin equation for each system, identifying the corresponding memory kernel and the effective noise term to find the analytic solution.

III. THE EFFECTIVE DECOUPLED LANGEVIN EQUATIONS

In this section, we introduce the GLE framework to derive effective equations for the relative displacement and individual particle dynamics, identifying the effective memory kernels and noise terms that incorporate the effects of the coupled dynamics.

A. GLE for the relative displacement $r_{\alpha\beta}$

Rewriting the coupled FLEs (6) in the Laplace domain, we obtain

$$\tilde{x}_\alpha(s) = -\frac{k}{\tilde{K}_\alpha(s)}(\tilde{x}_\alpha(s) - \tilde{x}_\beta(s)) + \frac{1}{\tilde{K}_\alpha(s)}\tilde{\xi}_\alpha(s), \quad (13a)$$

$$\tilde{x}_\beta(s) = -\frac{k}{\tilde{K}_\beta(s)}(\tilde{x}_\beta(s) - \tilde{x}_\alpha(s)) + \frac{1}{\tilde{K}_\beta(s)}\tilde{\xi}_\beta(s). \quad (13b)$$

Subtracting Eq. (13b) from Eq. (13a), we readily obtain the GLE for $r_{\alpha\beta}$ as

$$\tilde{r}_{\alpha\beta} = -k \frac{\tilde{K}_\alpha(s) + \tilde{K}_\beta(s)}{\tilde{K}_\alpha(s)\tilde{K}_\beta(s)}\tilde{r}_{\alpha\beta} + \frac{\tilde{\xi}_\alpha(s)}{\tilde{K}_\alpha(s)} - \frac{\tilde{\xi}_\beta(s)}{\tilde{K}_\beta(s)}. \quad (14)$$

From this expression, the corresponding GLE in the time domain reads

$$\int_0^t K_{\alpha\beta}^{(\text{eff})}(t-t')\dot{r}_{\alpha\beta}(t') = -kr_{\alpha\beta}(t) + \eta_{\alpha\beta}(t). \quad (15)$$

Thus, the relative motion can be understood such that a thermal particle governed by a fictional memory kernel

$$K_{\alpha\beta}^{(\text{eff})}(t) = \gamma_\alpha t^{-\alpha} E_{\alpha,\beta}(-(t/\tau_c)^{\beta-\alpha}) \quad (16)$$

diffuses in a confining harmonic potential of spring constant k . In this expression, $E_{a,b}(z)$ denotes the generalized Mittag-Leffler function defined through its Laplace transform [72,73]:

$$\mathcal{L}[t^{b-1} E_{a,b}(\pm ct^a)](s) = \frac{s^{-b}}{1 \mp cs^{-a}} \quad (17)$$

for $\Re a > 0$ and $\Re b > 0$. The $\eta_{\alpha\beta}(t)$ is the noise term written in the Laplace domain as

$$\tilde{\eta}_{\alpha\beta}(s) = \frac{\tilde{K}_\beta(s)\tilde{\xi}_\alpha(s) - \tilde{K}_\alpha(s)\tilde{\xi}_\beta(s)}{\tilde{K}_\alpha(s) + \tilde{K}_\beta(s)}, \quad (18)$$

which satisfies the FDT of the second kind:

$$\langle \eta_{\alpha\beta}(t)\eta_{\alpha\beta}(t') \rangle = k_B T K_{\alpha\beta}^{(\text{eff})}(|t-t'|) \quad (19)$$

(see Appendix B for details).

Using the asymptotic expressions for the Mittag-Leffler function around $z = 0$ and $z \rightarrow \infty$:

$$E_{a,b}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(b+an)} = \sum_{n=1}^{\infty} \frac{z^{-n}}{\Gamma(b-an)}, \quad (20)$$

we find that the memory kernel exhibits two distinct power-law characteristics:

$$K_{\alpha\beta}^{(\text{eff})}(t) \simeq \begin{cases} K_\alpha(t) = \frac{\gamma_\alpha}{\Gamma(1-\alpha)} t^{-\alpha}, & t \ll \tau_c, \\ K_\beta(t) = \frac{\gamma_\beta}{\Gamma(1-\beta)} t^{-\beta}, & t \gg \tau_c. \end{cases} \quad (21)$$

Here, τ_c is the crossover time introduced in Eq. (9). This indicates that the relative displacement follows the motion of particle α for times shorter than τ_c and then smoothly crosses over to that of particle β for times longer than τ_c .

B. GLE of particles α and β

We now solve the coupled FLE (6) and derive the equation of motion for x_α and x_β . Taking the Laplace transform of Eq. (6b) and using the identity $\tilde{x}(s) = (\tilde{x}(s) + x(0))/s$, we isolate $\tilde{K}_\beta(s)\tilde{x}_\beta(s)$ to obtain

$$\begin{aligned} \tilde{K}_\beta(s)\tilde{x}_\beta(s) = & -k(x_\beta(0) - x_\alpha(0)) \frac{\tilde{K}_\beta(s)}{s\tilde{K}_\beta(s) + k} \\ & + \frac{\tilde{K}_\beta(s)}{s\tilde{K}_\beta(s) + k} k\tilde{x}_\alpha(s) + \frac{s\tilde{K}_\beta(s)}{s\tilde{K}_\beta(s) + k} \tilde{\xi}_\beta(s). \end{aligned} \quad (22)$$

Next, the sum of Eqs. (6a) and (6b) yields

$$\tilde{K}_\alpha(s)\tilde{x}_\alpha(s) = -\tilde{K}_\beta(s)\tilde{x}_\beta(s) + \tilde{\xi}_\alpha(s) + \tilde{\xi}_\beta(s). \quad (23)$$

We plug the expression for $\tilde{K}_\beta(s)\tilde{x}_\beta(s)$ from Eq. (22) into Eq. (23), obtaining the effective GLE for x_α :

$$\begin{aligned} \int_0^t [K_\alpha(t') + \Phi_\beta(t')]\dot{x}_\alpha(t-t')dt' \\ = -kr_{\alpha\beta}(0)E_{\beta,1}\left(-\left(\frac{t}{\tau_\beta}\right)^\beta\right) + \xi_\alpha(t) + \eta_\beta(t). \end{aligned} \quad (24)$$

Here, $\Phi_\beta(t)$ and $\eta_\beta(t)$ are the memory and noise terms resulting from the motion of particle β , which read in the Laplace domain

$$\tilde{\Phi}_\beta(s) = \frac{k\tilde{K}_\beta(s)}{s\tilde{K}_\beta(s) + k}, \quad \tilde{\eta}_\beta(s) = \frac{s\tilde{\xi}_\beta(s)}{s\tilde{K}_\beta(s) + k}. \quad (25)$$

The memory kernel $\Phi_\beta(t)$ is identified to be

$$\Phi_\beta(t) = kE_{\beta,1}(-(t/\tau_\beta)^\beta), \quad (26)$$

which interpolates between two limiting behaviors:

$$\Phi_\beta(t) \simeq \begin{cases} k, & t \ll \tau_\beta, \\ K_\beta(t) = \frac{\gamma_\beta}{\Gamma(1-\beta)} t^{-\beta}, & t \gg \tau_\beta. \end{cases} \quad (27)$$

Hence, the additional memory term $\Phi_\beta(t)$ captures the confinement effect in the potential for times shorter than τ_β and becomes the memory kernel $K_\beta(t)$ for particle β . It is noted that Eq. (24) is seen as an effective GLE for x_α defined with the effective memory kernel

$$K_\alpha^{(\text{eff})}(t) = \frac{\gamma_\alpha}{\Gamma(1-\alpha)} t^{-\alpha} + kE_{\beta,1}(-(t/\tau_\beta)^\beta) \quad (28)$$

and the effective noise

$$\xi_\alpha(t) + \eta_\beta(t) = \xi_\alpha^{(\text{eff})}(t). \quad (29)$$

It can be shown that the effective GLE is a thermal equilibrium model such that at longer times ($t \gg \tau_\beta$) $\xi_\alpha^{(\text{eff})}$ satisfies the FDT of the second kind (Appendix B),

$$\langle \xi_\alpha^{(\text{eff})}(t) \xi_\alpha^{(\text{eff})}(t') \rangle \simeq k_B T K_\alpha^{(\text{eff})}(|t - t'|). \quad (30)$$

Interestingly, the effective GLE (24) for x_α has the extra term $kr_{\alpha\beta}(0)E_{\beta,1}(- (t/\tau_\beta)^\beta)$ attributed to the relaxation in the confining harmonic potential from the initial relative distance between the two particles. The effect of this initial condition decays out in a power-law manner and, in general, survives for long times. This term can be ignored if the system started from $t = -\infty$ or the system started at $t = 0$ with the equilibrium (stationary) condition: $\langle r_{\alpha\beta}^2(0) \rangle = \frac{k_B T}{k}$ and $\langle r_{\alpha\beta}(0)x_\alpha(0) \rangle = 0$ (see Appendix D for details). In this study, we focus on the coupled FLE dynamics under these equilibrium conditions, so the GLE (24) is rewritten in the following form:

$$\int_0^t K_\alpha^{(\text{eff})}(t - t') \dot{x}_\alpha(t') dt' = \xi_\alpha^{(\text{eff})}(t). \quad (31)$$

We repeat the derivation for the effective GLE for x_β . These expressions are obtained by symmetrically exchanging all α - and β - dependent terms in the steps above, resulting in

$$\tilde{K}_\beta^{(\text{eff})}(s) = \tilde{K}_\beta(s) + \tilde{\Phi}_\alpha(s), \quad (32)$$

$$\tilde{\xi}_\beta^{(\text{eff})}(s) = \tilde{\xi}_\beta(s) + \tilde{\eta}_\alpha(s), \quad (33)$$

where $\tilde{\Phi}_\alpha(s)$ and $\tilde{\eta}_\alpha(s)$ are given by

$$\tilde{\Phi}_\alpha(s) = \frac{k \tilde{K}_\alpha(s)}{s \tilde{K}_\alpha(s) + k}, \quad \tilde{\eta}_\alpha(s) = \frac{s \tilde{\xi}_\alpha(s)}{s \tilde{K}_\alpha(s) + k}. \quad (34)$$

Although the dynamics of particles α and β are separated using their respective GLEs, their motions remain correlated because the effective noise terms $\xi_\alpha^{(\text{eff})}$ and $\xi_\beta^{(\text{eff})}$ themselves are correlated. Consequently, their relative displacement $r_{\alpha\beta}$ exhibits the confined dynamics in the harmonic potential, as described by Eq. (15).

In Fig. 2, we analyze the asymptotic behavior of the memory kernels in the Laplace domain. In the strong-interaction regime ($k > k_c$) with the ordering of $\tau_\alpha < \tau_\beta < \tau_c$, we find that the kernel $\tilde{K}_\alpha^{(\text{eff})}(s)$ scales as

$$\tilde{K}_\alpha^{(\text{eff})}(s) \simeq \begin{cases} \gamma_\alpha s^{-1+\alpha}, & s \ll 1/\tau_c, \\ \gamma_\beta s^{-1+\beta}, & 1/\tau_c \ll s \ll 1/\tau_\beta, \\ ks^{-1}, & 1/\tau_\beta \ll s \ll 1/\tau_\alpha, \\ \gamma_\alpha s^{-1+\alpha}, & s \gg 1/\tau_\alpha. \end{cases} \quad (35a) \quad (35b) \quad (35c) \quad (35d)$$

See the theoretical curve of $\tilde{K}_\alpha^{(\text{eff})}(s)$ for $k/k_c = 10^4$ in Fig. 2. The two scaling relations (35b) and (35c) emerge only in the strong-interaction regime. In this case, the memory term $\tilde{\Phi}_\beta(s)$ dominates over $\tilde{K}_\alpha(s)$, producing the intermediate asymptotic regimes. For the opposite conditions ($k \lesssim k_c$), simply $\tilde{K}_\alpha^{(\text{eff})}(s) \simeq \tilde{K}_\alpha(s)$ across all timescales (see $k/k_c = 10^{-4}$ and $k/k_c = 1$).

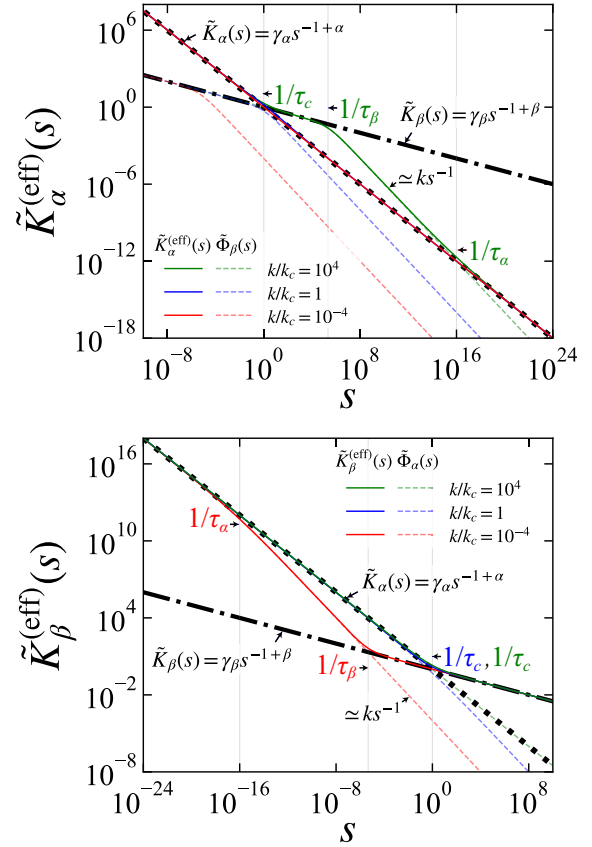


FIG. 2. Effective memory kernels in the Laplace domain: $\tilde{K}_\alpha^{(\text{eff})}(s)$ [Eq. (28), upper panel] and $\tilde{K}_\beta^{(\text{eff})}(s)$ [Eq. (32), lower panel]. In both panels, the dotted and dot-dashed curves depict $\tilde{K}_\alpha(s)$ and $\tilde{K}_\beta(s)$, respectively, while the dashed lines represent $\tilde{\Phi}_\alpha(s)$ and $\tilde{\Phi}_\beta(s)$ from Eqs. (25) and (34). Gray vertical lines indicate $1/\tau_\alpha$, $1/\tau_\beta$, and $1/\tau_c$ shown in Eq. (9). We set $\alpha = 0.25$, $\beta = 0.75$, $\gamma_\alpha = \gamma_\beta = 1$, and varied k , so τ_c remains the same for all cases.

In contrast to $\tilde{K}_\alpha^{(\text{eff})}(s)$, the effective kernel $\tilde{K}_\beta^{(\text{eff})}(s)$ exhibits the full complicated scaling relations in the weak-interaction regime ($k < k_c$), which reads

$$\tilde{K}_\beta^{(\text{eff})}(s) \simeq \begin{cases} \gamma_\alpha s^{-1+\alpha}, & s \ll 1/\tau_\alpha, \\ ks^{-1}, & 1/\tau_\alpha \ll s \ll 1/\tau_\beta, \\ \gamma_\beta s^{-1+\beta}, & s \gg 1/\tau_\beta. \end{cases} \quad (36a) \quad (36b) \quad (36c)$$

The intermediate scaling relation (36b) appears only when $\tau_c < \tau_\beta < \tau_\alpha$ (i.e., the weak-interaction regime). Also note that the long-time (small $s \ll 1/\tau_c$, $s \ll 1/\tau_\alpha$) behavior of the kernel $\tilde{K}_\beta^{(\text{eff})}(s)$ follows that of $\tilde{K}_\alpha(s)$, manifesting the impact of the slower particle α . In the strong-interaction regime, the scaling of $\tilde{K}_\beta^{(\text{eff})}(s)$ is simpler such that $\tilde{K}_\beta^{(\text{eff})}(s)$ is $\tilde{K}_\alpha(s)$ for $s < 1/\tau_c$ and $\tilde{K}_\beta(s)$ for $s > 1/\tau_c$.

IV. MEAN SQUARE DISPLACEMENTS

In the previous section, we show that the dynamics of the relative displacement can be understood as that of a non-Markovian particle in a harmonic potential, described by the GLE (15), and that the individual particles diffuse as if in the free space governed by the effective memory kernel and

noise—described in GLE (31)—that incorporate the effect of the harmonic interaction between the two particles. MSDs of these particles can be evaluated using the expressions of the MSDs for a GLE with a given memory kernel $K(t)$ and noise $\xi(t)$ subject to a confining harmonic potential or in free space. For the GLE in the presence of a harmonic potential $\frac{1}{2}kx^2$, the corresponding MSD is given by

$$\begin{aligned}\langle \Delta x^2(t) \rangle &= \langle [x(t + t_0) - x(t_0)]^2 \rangle \\ &= 2k_B T \mathcal{L}^{-1} \left\{ \frac{1}{s(\tilde{K}(s) + k)} \right\}(t)\end{aligned}\quad (37)$$

at the stationary state (see the derivation in Appendix C). From the MSD, we define its (instantaneous) anomalous exponent $\nu(t)$ as the slope of $\langle \Delta x^2(t) \rangle$ in the log-log coordinate:

$$\nu(t) = \frac{d \log \langle \Delta x^2(t) \rangle}{d \log t} = \frac{t \mathcal{L}^{-1} \left\{ \frac{1}{s\tilde{K}(s) + k} \right\}(t)}{\mathcal{L}^{-1} \left\{ \frac{1}{s\tilde{K}(s) + k} \right\}(t)}.\quad (38)$$

For a GLE in free space, the MSD of a particle is obtained as

$$\langle \Delta x^2(t) \rangle = 2k_B T \mathcal{L}^{-1} \left\{ \frac{1}{s^2 \tilde{K}(s)} \right\}(t)\quad (39)$$

and the respective anomalous exponent is identified to be

$$\nu(t) = \frac{t \mathcal{L}^{-1} \left\{ \frac{1}{s\tilde{K}(s)} \right\}(t)}{\mathcal{L}^{-1} \left\{ \frac{1}{s^2 \tilde{K}(s)} \right\}(t)}.\quad (40)$$

The derivation of these results is explained in Appendix C. We numerically compute the inverse Laplace transform for these analytic expressions using the Gaver-Stehfest algorithm [74].

A. The MSD for the relative displacement

Employing Eq. (37) with plugging into $K(t) = K_{\alpha\beta}^{\text{eff}}(t)$, we obtain the expression for the MSD of the relative displacement $r_{\alpha\beta}(t)$. As a baseline, we first examine the limiting case that the confining force becomes zero (i.e., $k \rightarrow 0$). In this case, the MSD exhibits the time dependence as

$$\begin{aligned}\langle \Delta r_{\alpha\beta}^2(t) \rangle_{k=0} &= \frac{2k_B T}{\gamma_\alpha \Gamma(1+\alpha)} t^\alpha + \frac{2k_B T}{\gamma_\beta \Gamma(1+\beta)} t^\beta \\ &\simeq \begin{cases} \frac{2k_B T}{\gamma_\alpha \Gamma(1+\alpha)} t^\alpha, & t \ll \tau_c, \\ \frac{2k_B T}{\gamma_\beta \Gamma(1+\beta)} t^\beta, & t \gg \tau_c. \end{cases}\end{aligned}\quad (41)$$

As expected, the MSD is dominated by that of the slower (stronger subdiffusive) particle (α) for times shorter than the crossover time τ_c and that of the faster one (β) beyond τ_c . See the numerical plot for $k = 0$ in Fig. 3.

When a harmonic potential is present, the relative displacement becomes confined at long times. In the weak-interaction regime ($k < k_c$), the relative displacement displays the MSD of the following form (e.g., the case of $k/k_c = 10^{-3}$ in Fig. 3):

$$\langle \Delta r_{\alpha\beta}^2(t) \rangle \simeq \begin{cases} \frac{2k_B T}{\gamma_\alpha \Gamma(1+\alpha)} t^\alpha, & t \ll \tau_c, \\ \frac{2k_B T}{\gamma_\beta \Gamma(1+\beta)} t^\beta, & \tau_c \ll t \ll \tau_\beta, \\ \frac{2k_B T}{k}, & t \gg \tau_\beta. \end{cases}\quad (42)$$

For $t \ll \tau_\beta$, the confining harmonic potential effects negligibly. Thus, the MSD has the same forms as the above MSD

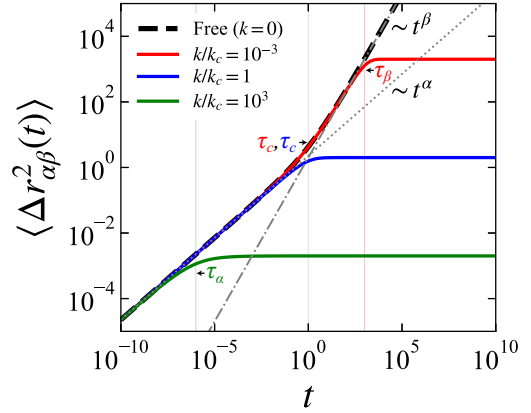


FIG. 3. The MSD of the relative displacement $r_{\alpha\beta}$. Solid lines depict the MSDs computed from Eqs. (16) and (37) for different values of k , while the dashed line represents the uncoupled case ($k = 0$) [Eq. (41)]. The dotted and dot-dashed lines correspond to the free subdiffusion [Eq. (5)] governed by $K_\alpha(t)$ and $K_\beta(t)$, respectively. Gray vertical lines indicate τ_α , τ_β , and τ_c shown in Eq. (9). We used the parameters $\alpha = 0.25$, $\beta = 0.75$, and $\gamma_\alpha = \gamma_\beta = 1$, with k varied such that τ_c remains unchanged for all cases.

(41) for $k = 0$. For $t \gg \tau_\beta$, however, the confining effect becomes significant, and the relative displacement ends up with thermal saturation in the harmonic potential.

In the critical and the strong-interaction regimes (where $\tau_\alpha \lesssim \tau_\beta \lesssim \tau_c$), the MSD is simpler. In this regime, the harmonic force is strong such that the MSD of the relative displacement grows as $\sim t^\alpha$ at short times and then becomes saturated for $t > \tau_\alpha$. Thus, the MSD displays a two-stage behavior as follows:

$$\langle \Delta r_{\alpha\beta}^2(t) \rangle \simeq \begin{cases} \frac{2k_B T}{\gamma_\alpha \Gamma(1+\alpha)} t^\alpha, & t \ll \tau_\alpha, \\ \frac{2k_B T}{k}, & t \gg \tau_\alpha. \end{cases}\quad (43)$$

The numerical plots for $k/k_c = 1$ and $k/k_c = 10^3$ confirm these relations (Fig. 3).

B. The MSD for particles α and β

Having analyzed the relative displacement, we now investigate the MSDs of the individual particles $x_\alpha(t)$ and $x_\beta(t)$. By substituting the effective memory kernels $K(t) = K_\mu^{\text{eff}}(t)$ from Eqs. (28) and (32) into Eq. (39), we can evaluate the MSDs for each particle.

The asymptotic form of each particle's MSD depends on whether it reaches the confinement-induced plateau $2k_B T/k$ before or after its counterpart. As mentioned in Eq. (10), $\langle \Delta x_\mu^2(t) \rangle$ reaches $2k_B T/k$ at $t \approx \tau_\mu$ ($\mu = \alpha, \beta$), and the order of timescales τ_α and τ_β [Eq. (9)] depends on k/k_c . We now present our results in three parts: the weak ($k < k_c$), critical ($k = k_c$), and strong ($k > k_c$) interactions.

In the weak-interaction regime ($k < k_c$ & $\tau_\beta < \tau_\alpha$), the MSD of particle β reaches the thermal plateau $2k_B T/k$ at $t \approx \tau_\beta$, while particle α is (sub)diffusing with negligible confinement effect. The particle β diffuses in the following way: For $t \ll \tau_\beta$, it moves as if in free space with the MSD of $\langle \Delta x_\beta^2(t) \rangle \approx \langle \Delta x_\beta^2(t) \rangle_{k=0}$. Beyond this timescale, the particle's movement is affected by the harmonic potential, and

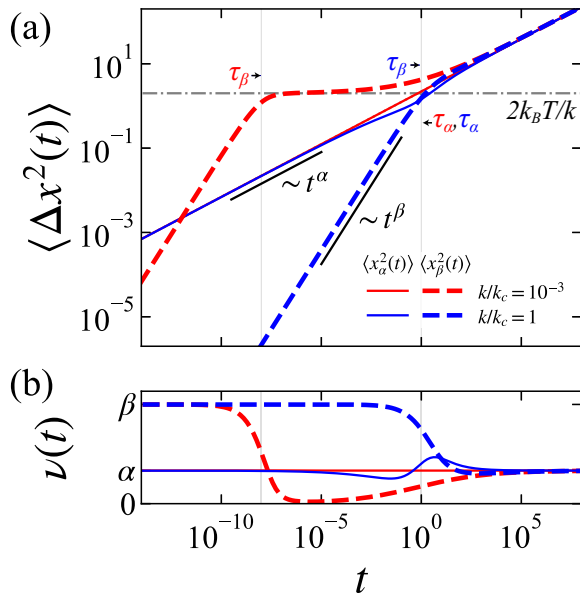


FIG. 4. The MSDs and the anomalous exponents in the weak-interaction regime ($k < k_c$). (a) MSDs (39) for $x_\alpha(t)$ (solid) and $x_\beta(t)$ (dashed) as a function of time t for two distinct values of k/k_c . (b) The time-dependent anomalous exponent (40) derived from the MSDs. Gray vertical lines indicate τ_α and τ_β . We set $\alpha = 0.25$, $\beta = 0.75$, $k = 1$, $\gamma_\alpha = 1$, and $k_B T = 1$ and varied γ_β while fixing τ_α for all cases.

eventually it reaches the confined state with $\langle \Delta x_\beta^2(t) \rangle \approx 2k_B T/k$ for $\tau_\beta \ll t \ll \tau_\alpha$. For $t \gg \tau_\alpha$, it follows the motion of the slower particle (α). We note that for $t \ll \tau_\alpha$, the MSD of particle β is identical with that of a single FLE particle confined to a harmonic potential [75]:

$$\langle \Delta x_\beta^2(t) \rangle \simeq 2k_B T \frac{t^\beta}{\gamma_\beta} E_{\beta, 1+\beta} \left(-\frac{k}{\gamma_\beta} t^\beta \right). \quad (44)$$

For particle α , the confining effect is negligible at short times ($t \ll \tau_\alpha$), and its MSD follows that of free subdiffusion, $\langle \Delta x_\alpha^2(t) \rangle \simeq \langle \Delta x_\alpha^2(t) \rangle_{k=0} \propto t^\alpha$. After then, the MSD of particle α approaches $2k_B T/k$ without confinement, and the two particles move together through the harmonic coupling. Hence, their MSDs share the same asymptotic form ($t \gg \tau_\alpha$):

$$\langle \Delta x_\alpha^2(t) \rangle \simeq \langle \Delta x_\beta^2(t) \rangle \simeq \frac{2k_B T}{\gamma_\alpha \Gamma(1+\alpha)} t^\alpha. \quad (45)$$

In this regime, both effective kernels $K_\alpha^{\text{eff}}(t)$ and $K_\beta^{\text{eff}}(t)$ are approximated by $K_\alpha(t)$ [see Eqs. (35a) and (36a)].

Summing up, particle α maintains the power-law scaling $\sim t^\alpha$ over the entire time domain, with a negligible intermediate crossover. In contrast, particle β undergoes a three-stage evolution ($k/k_c = 10^{-3}$ in Fig. 4):

$$\langle \Delta x_\beta^2(t) \rangle \simeq \begin{cases} \frac{2k_B T}{\gamma_\beta \Gamma(1+\beta)} t^\beta, & t \ll \tau_\beta, \\ \frac{2k_B T}{k}, & \tau_\beta \ll t \ll \tau_\alpha, \\ \frac{2k_B T}{\gamma_\alpha \Gamma(1+\alpha)} t^\alpha, & t \gg \tau_\alpha. \end{cases} \quad (46)$$

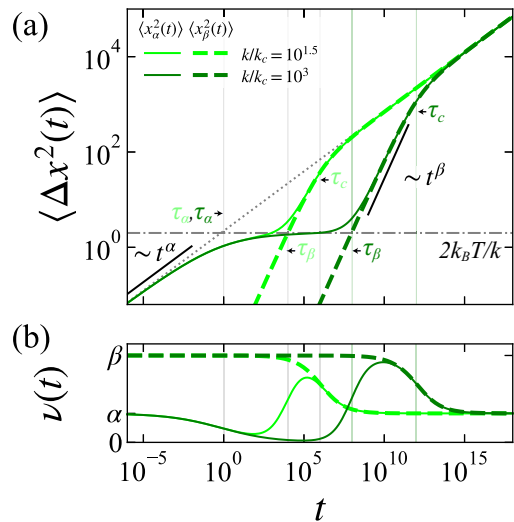


FIG. 5. The MSDs and the anomalous exponents in the strong-interaction regime ($k > k_c$). (a) MSDs (39) for $x_\alpha(t)$ (solid) and $x_\beta(t)$ (dashed) as a function of time t for different k/k_c . (b) The anomalous exponent (40). Gray vertical lines represent τ_α , τ_β , and τ_c shown in Eq. (9). We set $\alpha = 0.25$, $\beta = 0.75$, $k = 1$, $\gamma_\alpha = 1$, $k_B T = 1$, and varied γ_β while fixing τ_α for all cases.

The crossover from faster to slower subdiffusion highlights how harmonic interaction induces a delayed coherence in the coupled FLE system.

At the critical interaction strength ($k = k_c$), the three characteristic timescales become identical, i.e., $\tau_\alpha = \tau_\beta = \tau_c$. In this case, both particles terminate their free subdiffusion regime simultaneously at $t \approx \tau_c$, and immediately enter the long-time regime with $\langle \Delta x_\alpha^2(t) \rangle \simeq \langle \Delta x_\beta^2(t) \rangle \simeq \frac{2k_B T}{\gamma_\alpha \Gamma(1+\alpha)} t^\alpha$. There is no intermediate plateau in $\langle \Delta x_\beta^2(t) \rangle$, unlike in the weak-interaction case. The resulting asymptotic expressions are

$$\langle \Delta x_\alpha^2(t) \rangle \simeq \frac{2k_B T}{\gamma_\alpha \Gamma(1+\alpha)} t^\alpha \quad (47)$$

and

$$\langle \Delta x_\beta^2(t) \rangle \simeq \begin{cases} \frac{2k_B T}{\gamma_\beta \Gamma(1+\beta)} t^\beta, & t \ll \tau_c, \\ \frac{2k_B T}{\gamma_\alpha \Gamma(1+\alpha)} t^\alpha, & t \gg \tau_c. \end{cases} \quad (48)$$

These behaviors are illustrated in the numerical plot for $k/k_c = 1$ in Fig. 4.

In the strong-interaction regime ($k > k_c$ & $\tau_\alpha < \tau_\beta < \tau_c$), Here, the confinement becomes dominant early in the dynamics. The MSD of particle α exhibits a four-regime behavior (Fig. 5):

$$\langle \Delta x_\alpha^2(t) \rangle \simeq \begin{cases} \frac{2k_B T}{\gamma_\alpha \Gamma(1+\alpha)} t^\alpha, & t \ll \tau_\alpha, \\ \frac{2k_B T}{k}, & \tau_\alpha \ll t \ll \tau_\beta, \\ \frac{2k_B T}{\gamma_\beta \Gamma(1+\beta)} t^\beta, & \tau_\beta \ll t \ll \tau_c, \\ \frac{2k_B T}{\gamma_\alpha \Gamma(1+\alpha)} t^\alpha, & t \gg \tau_c, \end{cases} \quad (49)$$

and particle β has

$$\langle \Delta x_\beta^2(t) \rangle \simeq \begin{cases} \frac{2k_B T}{\gamma_\beta \Gamma(1+\beta)} t^\beta, & t \ll \tau_c, \\ \frac{2k_B T}{\gamma_\alpha \Gamma(1+\alpha)} t^\alpha, & t \gg \tau_c. \end{cases} \quad (50)$$

Distinguished from the weak and critical regimes, the long-time MSD displays a two-step scaling relation for $t \gg \tau_\beta$. In this regime, the MSD of both particles can be obtained using a different approach, directly from the asymptotic in the Laplace domain:

$$\begin{aligned} \langle \Delta x_\alpha^2(t) \rangle &\simeq \langle \Delta x_\beta^2(t) \rangle \\ &\simeq 2k_B T \frac{t^\beta}{\gamma_\beta} E_{\beta-\alpha, 1+\beta}(-(t/\tau_c)^{\beta-\alpha}) \\ &\simeq \begin{cases} \frac{2k_B T}{\gamma_\beta \Gamma(1+\beta)} t^\beta, & t \ll \tau_c, \\ \frac{2k_B T}{\gamma_\alpha \Gamma(1+\alpha)} t^\alpha, & t \gg \tau_c. \end{cases} \end{aligned} \quad (51)$$

A striking consequence of Eq. (51) is that particle α , despite being intrinsically slower, undergoes transient acceleration. Specifically, it exhibits an effective power-law scaling $\sim t^\beta$ ($\beta > \alpha$) during the intermediate time window $\tau_\beta \ll \tau_c$, albeit its native subdiffusion is $\sim t^\alpha$. We refer to this postconfinement acceleration as *recovery dynamics* because it enables $\langle \Delta x_\alpha^2(t) \rangle$ to approach $\langle \Delta x_\alpha^2(t) \rangle_{k=0}$ in the long-time limit. Note that this recovery dynamics only occur for $k > k_c$.

As a result of this recovery dynamics, particle α ultimately regains the same generalized diffusivity it exhibits in the uncoupled limit ($k = 0$), regardless of the interaction strength k . Defining short- and long-time generalized diffusivities as

$$D_\alpha^{(0)} \equiv \lim_{t \rightarrow 0} \frac{\langle \Delta x_\alpha^2(t) \rangle}{2t^\alpha} \quad \text{and} \quad D_\alpha^{(\infty)} \equiv \lim_{t \rightarrow \infty} \frac{\langle \Delta x_\alpha^2(t) \rangle}{2t^\alpha}, \quad (52)$$

we find that they are equal to each other:

$$D_\alpha^{(0)} = D_\alpha^{(\infty)} = \frac{2k_B T}{\gamma_\alpha \Gamma(1+\alpha)}. \quad (53)$$

This indicates that particle α eventually recovers the same long-time behavior as in free space. (This long-time diffusivity is revisited in Sec. IV C.)

To consolidate our MSD findings, we present in Fig. 6 a heatmap of the anomalous exponent $\nu(t)$ as a function of dimensionless time t/τ_c (horizontal axis) and interaction strength k/k_c (vertical axis). Each horizontal slice of the heatmap represents $\nu(t)$ for a fixed value of k/k_c .

Panel (a) shows $\nu(t)$ for the relative displacement $r_{\alpha\beta}$. Since $\langle \Delta r_{\alpha\beta}^2(t) \rangle$ eventually saturates at $\sqrt{2k_B T/k}$, the corresponding $\nu(t)$ always decays to zero at long times for any condition of k/k_c . Notably, a clear t^β power-law region (highlighted in red) is present only when $k < k_c$. This fast dynamics emerges in this regime because $\langle \Delta r_{\alpha\beta}^2(t) \rangle \simeq \langle \Delta x_\alpha^2(t) \rangle + \langle \Delta x_\beta^2(t) \rangle$ is dominated by the MSD of particle β .

Panel (b) displays $\nu(t)$ for particle α , revealing the hallmark features of *recovery dynamics*. For $k > k_c$, particle α experiences a transient confinement in the intermediate regime, followed by the recovery regime where the particle moves with a larger anomalous exponent $\nu(t) \approx \beta > \alpha$. This transient acceleration reflects the postconfinement catch-up process, after which x_α smoothly regains its native subdiffusive dynamics (t^α) and the long-time generalized diffusivity.

Panel (c) shows $\nu(t)$ for particle β . In the strong-interaction regime, it exhibits subdiffusion of $\sim t^\beta$ at short times and later follows the diffusion ($\sim t^\alpha$) of the slower particle. When the interaction is weak ($k < k_c$), particle β has the intermediate

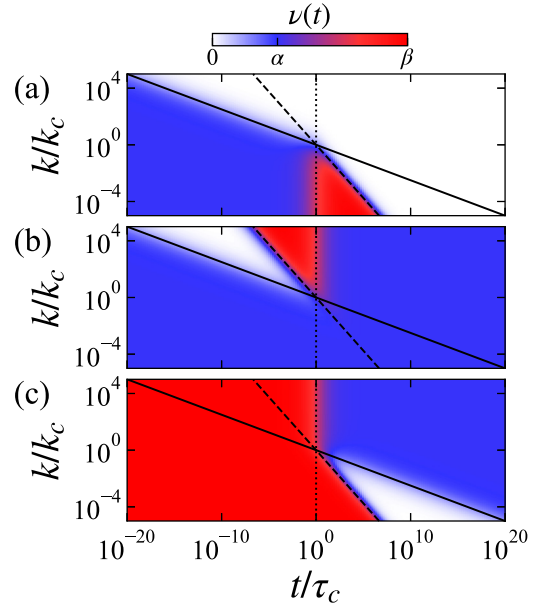


FIG. 6. Heatmap of the anomalous exponent $\nu(t)$ for the relative displacement [panel (a)] and the two individual particles [panels (b) and (c)], plotted as functions of the dimensionless time t/τ_c and the interaction strength k/k_c . The color scale encodes the anomalous exponents: 0 (white), α (blue), and β (red), while the solid and dashed diagonal lines mark $t = \tau_\alpha$ and $t = \tau_\beta$. The vertical dotted line indicates $t = \tau_c$. We used $\alpha = 0.25$ and $\beta = 0.75$ in the plot.

regime of transient confinement between the short-time ($\sim t^\beta$) and long-time ($\sim t^\alpha$) subdiffusion.

Altogether, the heatmaps illustrate a rich landscape of *transient anomalous diffusion* dynamics for the coupled FLE system. The intricate effects from the viscoelastic memories (α, β) and their coupling through the harmonic interaction lead to such complex dynamic behaviors, including the absence/emergence of transient confinement and an apparent speedup within a certain time window via the recovery process.

C. Coupled dynamics with identical memory exponents

Up to now, we have analyzed the coupled system with distinct viscoelastic memories ($\alpha \neq \beta$), where the intricate effects from the distinct viscoelastic memories and the mechanical coupling result in a hierarchy of the system's characteristic timescales and rich transient anomalous dynamics. Here, we consider the special case that the two memory exponents are equal to each other ($\alpha = \beta$).

Now, let us consider two particles (particles 1 and 2) whose positions are $x_{\alpha,1}(t)$ and $x_{\alpha,2}(t)$, respectively. We assume that their memory kernels share the same power-law exponent α but differ in amplitude, i.e.,

$$K_{\alpha,1}(t) = \frac{\gamma_\alpha}{\Gamma(1-\alpha)} t^{-\alpha}, \quad K_{\alpha,2}(t) = \lambda K_{\alpha,1}(t), \quad \lambda \geq 1. \quad (54)$$

Because both particles have the same MSD scaling law, among the three timescales in Eq. (9), τ_c diverges in the limit of $\beta \rightarrow \alpha$. However, the other two characteristic timescales

remain,

$$\tau_{\alpha,1} = \left(\frac{\gamma_\alpha}{k}\right)^{1/\alpha}, \quad \tau_{\alpha,2} = \left(\lambda \frac{\gamma_\alpha}{k}\right)^{1/\alpha}, \quad (55)$$

which satisfy the fixed ordering in magnitude and the ratio $\tau_{\alpha,2}/\tau_{\alpha,1} = \lambda^{1/\alpha} > 1$ remains the same independently of k . Hence, when $\alpha = \beta$, there is no critical strength k_c , and the qualitative shape of MSD is identical regardless of k .

For the relative displacement $r_{\alpha,12}(t) \equiv x_{\alpha,1}(t) - x_{\alpha,2}(t)$, the effective memory kernel for GLE (15) is given as a single power law,

$$K_{\alpha,12}^{\text{(eff)}}(t) = \frac{\lambda}{1+\lambda} \frac{\gamma_\alpha}{\Gamma(1-\alpha)} t^{-\alpha}, \quad (56)$$

leading to

$$\langle \Delta r_{\alpha,12}^2(t) \rangle \simeq \begin{cases} \frac{2k_B T}{\gamma_\alpha \Gamma(1+\alpha)} \left(1 + \frac{1}{\lambda}\right) t^\alpha, & t \ll \tau_{\alpha,12}, \\ \frac{2k_B T}{k}, & t \gg \tau_{\alpha,12}, \end{cases} \quad (57)$$

where $\tau_{\alpha,12} = [\lambda \gamma_\alpha / (1+\lambda) k]^{1/\alpha}$. The MSD crosses directly from the subdiffusion $\sim t^\alpha$ to the plateau $2k_B T/k$. No intermediate, k -dependent transient dynamics appears, which is in sharp contrast with the unequal-exponent case ($\alpha \neq \beta$).

For $x_{\alpha,1}(t)$ and $x_{\alpha,2}(t)$, their effective memory kernels for GLE (31) take the form

$$\begin{aligned} K_{\alpha,1}^{\text{(eff)}}(t) &= K_{\alpha,1}(t) + k E_{\alpha,1} \left(-\frac{k}{\lambda \gamma_\alpha} t^\alpha \right) \\ &\simeq \begin{cases} K_{\alpha,1}(t), & t \ll \tau_{\alpha,1}, \\ k, & \tau_{\alpha,1} \ll t \ll \tau_{\alpha,2}, \\ (1+\lambda)K_{\alpha,1}(t), & t \gg \tau_{\alpha,2} \end{cases} \end{aligned} \quad (58)$$

and

$$\begin{aligned} K_{\alpha,2}^{\text{(eff)}}(t) &= K_{\alpha,2}(t) + k E_{\alpha,1} \left(-\frac{k}{\gamma_\alpha} t^\alpha \right) \\ &\simeq \begin{cases} K_{\alpha,2}(t), & t \ll \tau_{\alpha,2}, \\ (1+\lambda)K_{\alpha,1}(t), & t \gg \tau_{\alpha,2}. \end{cases} \end{aligned} \quad (59)$$

Because both the kernel $K_{\alpha,1}(t)$ and the Mittag-Leffler term $k E_{\alpha,1}(-kt^\alpha/\lambda\gamma_\alpha)$ share the same power-law exponent α , their long-time contributions are simply added to be $(1+\lambda)K_{\alpha,1}(t)$ for $t \gg \tau_{\alpha,2}$. In the unequal-exponent system ($\alpha \neq \beta$), this does not happen: The slower-decaying kernel $K_\alpha(t) \sim t^{-\alpha}$ always dominates the long-time asymptotics, so $K_\beta(t) \sim t^{-\beta}$ becomes subleading and no factor like $(1+\lambda)$ appears.

Inserting these kernels to the general expression of MSD (39), we obtain

$$\langle \Delta x_{\alpha,1}^2(t) \rangle \simeq \begin{cases} \frac{2k_B T}{\gamma_\alpha \Gamma(1+\alpha)} t^\alpha, & t \ll \tau_{\alpha,1}, \\ \frac{2k_B T}{k}, & \tau_{\alpha,1} \ll t \ll \tau_{\alpha,2}, \\ \frac{2k_B T}{(1+\lambda)\gamma_\alpha \Gamma(1+\alpha)} t^\alpha, & t \gg \tau_{\alpha,2} \end{cases} \quad (60)$$

for particle 1 and

$$\langle \Delta x_{\alpha,2}^2(t) \rangle \simeq \begin{cases} \frac{2k_B T}{\lambda \gamma_\alpha \Gamma(1+\alpha)} t^\alpha, & t \ll \tau_{\alpha,2}, \\ \frac{2k_B T}{(1+\lambda)\gamma_\alpha \Gamma(1+\alpha)} t^\alpha, & t \gg \tau_{\alpha,2} \end{cases} \quad (61)$$

for particle 2. We note that the recovery dynamics and the regaining of the long-time diffusivity are absent in the condition of $\alpha = \beta$. Instead, at longer times, the individual particle

eventually attains the same reduced generalized diffusivity:

$$D_{\alpha,1}^{(\infty)} = D_{\alpha,2}^{(\infty)} = \frac{1}{1+\lambda} D_{\alpha,1}^{(0)}, \quad (62)$$

demonstrating that the mechanical coupling slows down the subdiffusive motion of both particles. This behavior contrasts sharply with the unequal-exponent case, where particle α eventually recovers its free-space generalized diffusivity [Eq. (53)].

V. COMPARISON WITH POLYMER SIMULATIONS

In the preceding sections, we have analytically studied the dynamics of coupled FLEs (6) in terms of MSDs. To assess how well our FLE model captures coupled viscoelastic dynamics of real systems, in this section, we perform Langevin dynamics simulations of two polymer-based physical systems schematically introduced in Figs. 1(a) and 1(b) and examine whether the simulated polymer dynamics indeed agree with our FLE models. Namely, we simulate the following two scenarios: (1) a flexible polymer whose central bead is bound to a Brownian tracer and (2) two polymers—one flexible and one stiff—whose central beads are cross-linked. Below, we summarize the comparison of the simulation results with our FLE model. Further information about the simulation model and parameters is described in Appendix E.

A. Flexible polymer bound to a Brownian particle

We simulate a Rouse polymer with its central monomer tethered to a macromolecule [see Fig. 1(a)], motivated by a chromatin locus tethered to a macromolecular complex or protein condensate [76–78]. The central bead of the Rouse chain is labeled as particle α and the tracer as particle β . In our framework, this system is explained by our coupled FLE: particle α with the memory exponent $\alpha = 1/2$ and particle β of $\beta = 1$. The corresponding MSDs are numerically obtained from Eq. (39) with the insertion of the corresponding memory kernels, Eqs. (28) and (32).

Figure 7(a) shows the comparison of MSDs obtained from the Langevin simulation (symbols) with the analytic counterpart (solid lines). After the kernel parameters are properly identified (Appendix E), our FLE model ($\alpha = 1/2$, $\beta = 1$) shows excellent agreement with the simulated dynamics of this macromolecule-polymer bipartite system over the entire time window of our interest. Under the given simulation condition, the system lies in the strong-coupling regime ($k/k_c \simeq 7.5 > 1$), in which the MSD of x_α displays the four distinct scaling regimes described in Eq. (49), which is clearly visible in the figure.

An interesting note is that, despite the overall agreement between our FLE model and the simulation, the apparent MSD scalings can slightly differ from the asymptotic predictions, Eq. (49), obtained in the strong-coupling limit ($k/k_c \gg 1$). In particular, the deviation is noticeable in the two intermediate scalings; instead of the plateau ($\sim t^0$) the system exhibits subdiffusion of $\sim t^{0.3}$, and the Fickian recovery regime ($\sim t$) is replaced by an anomalous diffusion regime of $\sim t^{0.76}$. These results suggest that real coupled viscoelastic systems may display richer dynamical behavior than predicted by

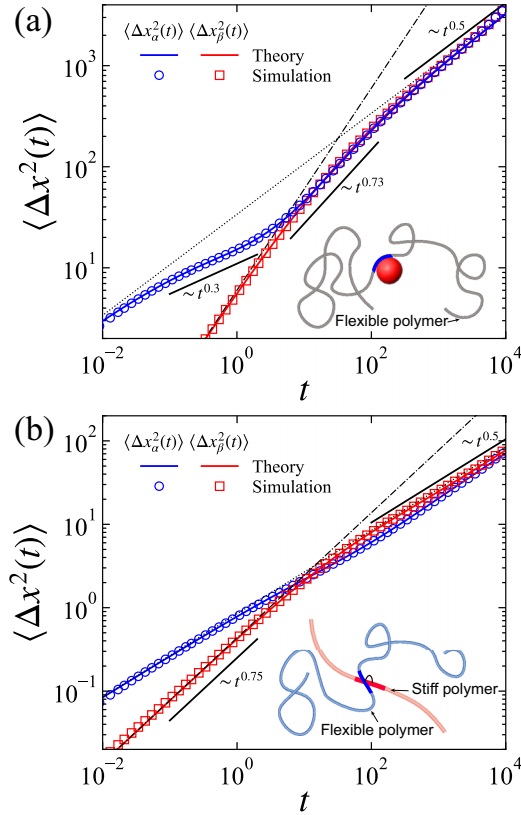


FIG. 7. Comparison of our FLE models [Eq. (6)] with the Langevin dynamics simulations of the two polymer-based physical examples. Simulation parameters are provided in Appendix E. (a) A tracer macromolecule is bound to the central bead of a flexible polymer in a viscous medium. The blue symbol is the simulated MSD of the central bead, while the red symbol represents the (simulated) MSD of the bound tracer. Solid lines are the theoretical MSD from the FLE model, with $\alpha = 1/2$ and $\beta = 1$. (b) Flexible (blue symbol) and stiff (red symbol) polymers whose center beads are cross-linked. Solid lines are the theoretical MSDs from the FLE model, with $\alpha = 1/2$ and $\beta = 3/4$. For both panels (a) and (b), dotted and dot-dashed lines represent the MSDs of $\langle \Delta x_\mu^2(t) \rangle_{k=0} = \frac{2k_B T}{\gamma_\mu \Gamma(1+\mu)} t^\mu$ ($k = 0$) for particles α and β , respectively.

asymptotic theory. The apparent exponents may be explained by the numerical solution of the full coupled FLE (39) at the chosen parameters.

B. Cross-linked flexible and stiff polymers

To test the generality of our framework, we further considered a second system where two polymers of differing stiffness are cross-linked. We simulate a flexible and a stiff polymers cross-linked at their central beads [Fig. 1(b)], labeled as particles α and β , respectively. In our theoretical framework, this system should be modeled by the FLE with the memory exponents $\alpha = 1/2$ and $\beta = 3/4$, respectively. See Appendix E for further details on the simulated polymer system and its connection to the corresponding FLE model. In Fig. 7(b), we present the simulated MSDs of the two cross-linked beads in comparison with the FLE prediction. Here again, it is confirmed that the coupled FLE (6) with the given

memory exponents captures the dynamics of this cross-linked polymer system with very good agreement. These two examples provide strong support that the coupled FLE framework presented in our study can serve as a quantitative dynamic model that accurately describes the complex viscoelastic dynamics in soft-matter and biological systems via couplings between components with distinct memory exponents.

VI. CONCLUDING REMARKS

In this work, we have introduced and analytically studied a minimal model of two viscoelastic systems coupled by a harmonic potential, each governed by an FLE with distinct memory exponents (α and β). Our analytic and numerical studies revealed that the coupling between distinct viscoelastic environments can generate rich, time-dependent behaviors not present in uncoupled systems. As a generic outcome, the system exhibits multiple transient anomalous diffusion behaviors in a wide range of time domain.

First, we identified and characterized distinct scaling regimes resulting from the interplay between the viscoelastic memories and the harmonic interaction. It turns out that there are three relevant timescales [Eq. (9)], τ_α , τ_β , and τ_c , characterizing the diffusion dynamics of individual particles (x_α and x_β) and their relative distance ($r_{\alpha\beta}$), and their dynamic behaviors differ whether the coupled FLE system is in the weak-, critical-, or strong-interaction regime, depending on the strength of the harmonic interaction.

Notably, we discovered the so-called *recovery dynamics* phenomenon: When a slower particle (or system) with exponent α is coupled with a faster one with exponent β (with $\alpha < \beta$), the slower particle suffers transient acceleration and eventually recovers its intrinsic long-time generalized diffusivity. This remarkable behavior disappears when both particles (or systems) share the same memory exponent, instead resulting in a persistent reduction of diffusivity.

A potentially important application of our model is to elucidate transient anomalous diffusion phenomena widely observed in various biological or soft-matter systems. For example, numerous *in vivo* chromatin-tracking experiments have reported transient anomalous diffusions and time-dependent anomalous exponents [40–46]. In human nuclei, Germier *et al.* tracked a single gene locus and found a crossover from a slow subdiffusion, $\nu \approx 0.3$ (0.4–4 s), to a faster subdiffusion, $\nu \approx 0.8$ (4–20 s) [40]. Using the displacement-correlation spectroscopy technique over the time window of 0.2–20 s, Zidovska *et al.* observed a similar two-step pattern: an initial slow subdiffusion with $\nu \approx 0.2$ (0.2–2 s) that crosses over to a faster subdiffusion with $\nu \approx 0.7$ in interphase or nearly diffusive motion with $\nu \approx 1$ in mitosis for 2–20 s [41]. Meanwhile, at a much longer time window (2–2000 s), Khanna *et al.* examined the relative genomic motion in B-cell nuclei and reported that its MSD has $\nu \approx 0.7$ for 2–20 s, then decreasing to $\nu \approx 0.5$ for 40–2000 s in B-cell nuclei [45]. These three eukaryotic-cell studies reveal a consistent picture: Chromatin motion has a common subdiffusive exponent ($\nu \approx 0.7$ –0.8) in the time window of 1–10 s, yet evolves into distinct regimes at shorter and longer times. Because a simple polymer model yields only a single power law (e.g., $\nu = 0.5$ for a flexible chain), this progression of multiple scaling regimes implies

that additional processes—condensate formation, chromatin looping, and chromatin remodeling, confinement, and so on—must be at work.

Our theory provides a minimal quantitative model for the complex diffusion of a locus via the interaction with DNA-protein assemblies. It is viewed that the tracked locus in a nucleus is mechanically tethered to macromolecular (protein) complexes or phase-separated condensates (e.g., transcription factories [76], DNA-repair hubs [77], and HP1 droplets [78]). In this picture [e.g., see Fig. 1(a)], we treat the chromatin fiber as a viscoelastic medium with memory exponent $\alpha = 0.5$ (a flexible polymer), while the bound focus, if it diffused freely, would follow a larger anomalous exponent $\beta \approx 0.7-1.0$, consistent with inert tracer measurements in biological nuclei [79–81]. Under the strong-interaction regime in our theory, the MSD of a chromatin locus has four distinct scaling regimes: (1) the short-time chromatin-dominated subdiffusion ($\nu = 0.5$); (2) the confined plateau ($\nu \sim 0$); (3) the recovery dynamics ($\nu = 0.7-1$); and (4) the late-time return to the chromatin-dominated subdiffusion ($\nu = 0.5$). For a certain finite value of k/k_c , the plateau in (2) can tilt into a shallow subdiffusive slope (e.g., $\nu \approx 0.2$), as observed in Fig. 5(b) with $k/k_c = 10^{1.5}$. We confirm that the four-stage dynamic pattern is indeed observed in our explicit bead-spring polymer simulation [Fig. 7(a)]: with $k/k_c \simeq 7.5$, the tilted plateau shows $\nu \approx 0.3$ and the recovery $\nu \approx 0.76$. Within this framework, the increase of the anomalous exponent from $\nu \approx 0.2-0.3$ to $\nu \approx 0.7-0.8$ reported by Germier *et al.* [40] and Zidovska *et al.* [41] seems to be consistent with the transition from regime (2) to (3). Meanwhile, the decrease from $\nu \approx 0.7$ to $\nu \approx 0.5$ observed by Khanna *et al.* [45] likely reflects the transition from regime (3) to (4).

A second illustrative example is the thermal transport of particles in cell-like composite gels comprising semiflexible actin filaments and stiff microtubules cross-linked by specialized proteins. Without cross-linkers (that mediate the interaction between filaments), tracer particles exhibit a single subdiffusive regime with $\nu \approx 0.75$, i.e., the thermal undulation dynamics of a semiflexible filament [57]. However, with cross-linkers present, the MSD reveals two distinct scaling regimes, depending on the tracer and the type of cross-linkers: For microspheres, they reveal the short-time subdiffusion with $\nu \approx 0.5-0.75$ and the long-time subdiffusion with $\nu \approx 0.25-0.5$ [57]. For a fluorescent-labeled DNA, it exhibits the short-time subdiffusion with $\nu \approx 0.75$ and the long-time subdiffusion with $\nu \approx 0.5$ [55,56]. Within our framework, the microtubules can be modeled as a stiff filament with a thermal undulation of $\beta = 0.75$, while the actin behaves as a flexible chain having a thermal undulation of $\alpha = 0.5$ at the timescales of interest [32]. In this scenario, the transition from faster to slower subdiffusion ($t^\beta \rightarrow t^\alpha$) can arise only when the components are mechanically coupled through the cross-linkers. Although these experiments do not directly measure fluctuations of the network itself, our model nonetheless provides a qualitative explanation for the observed two-regime behavior in the experiment that emerges only in the presence of the cross-linkers.

Our work bridges the gap between theoretical models and real biological systems, highlighting the critical role of memory heterogeneity and mechanical interactions in

understanding transient, heterogeneous anomalous dynamics. Indeed, numerous biological and soft-matter systems operate far from equilibrium, thereby violating FDT. Active processes—e.g., transcription, loop extrusion, or motor protein activity—can produce effective “hotter” noise or persistent driving forces. Extending our coupled FLE framework to incorporate such FDT-violating contributions would yield a more realistic description of chromatin and other complex materials. Additionally, our theoretical framework opens promising avenues to explore entropy production, energy dissipation, and thermodynamic efficiency in stochastic, nonequilibrium viscoelastic systems. Furthermore, extending our framework to include pinning potentials and periodic driving would expand the scope of the model to understand the stability and synchronism phenomena of coupled many viscoelastic systems [82]. A comprehensive analysis of this many-body, heterogeneous case is technically challenging but represents an exciting direction for future work.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [83].

APPENDIX A: DERIVATION OF THE COUPLED FLE (6) FROM POLYMER EXAMPLES

In this Appendix, we present the derivation of the coupled FLE (6) that is conceptually explained in Sec. II. We work out the system of a flexible polymer bound to a Brownian macromolecule [Fig. 1(a)], and subsequently discuss the generalization to other physical models. As described in the main text, the equation of motion for individual beads comprising the given polymer system is governed by the Langevin equation (8), along with the total potential energy equation (7). We derive the effective one-particle coupled FLE for the tracers $\mathbf{r}_a^A$ and $\mathbf{r}_b^B$ in subsystems *A* and *B* by integrating out all bead coordinates except for the two tracer variables. Our coarse-graining integration is based on the techniques employed in Refs. [25–33,35].

1. Derivation for a flexible polymer bound to a Brownian particle

As an illustrative example, let system *A* be a flexible Rouse chain with $M_A = 2N_A + 1$ beads, whose coordinate is represented by $\{\mathbf{r}_n^A\}$ ($n = 1, \dots, M_A$) and the central bead of index $a = N_A + 1$ is the tracer. The polymer bead is connected by harmonic springs of stiffness k^A , and each bead experiences an isotropic Stokes friction γ^A . System *B* is a single Brownian particle with a friction γ^B , and its coordinate is represented by $\{\mathbf{r}_n^B\}$, where the number of beads is $M_B = 1$ and the bead index is $b = 1$. The potential energy of this Rouse (Gaussian)

chain is

$$U^A = \frac{k^A}{2} \sum_{n=1}^{2N_A} (\mathbf{r}_{n+1}^A - \mathbf{r}_n^A)^2,$$

with the free-end (phantom) boundary conditions $\mathbf{r}_0^A = \mathbf{r}_1^A$ and $\mathbf{r}_{2N_A+2}^A = \mathbf{r}_{2N_A+1}^A$. The Langevin equation of a bead in A reads (where $u = 1, \dots, d$ is the Cartesian component)

$$\gamma^A \dot{r}_{n,u}^A(t) = -k^A (2r_{n,u}^A - r_{n-1,u}^A - r_{n+1,u}^A) - k(r_{a,u}^A - r_{b,u}^B) \delta_{n,a} + \eta_{n,u}^A(t), \quad (\text{A1})$$

while the Brownian particle in B obeys

$$\gamma^B \dot{r}_{b,u}^B(t) = -k(r_{b,u}^B - r_{a,u}^A) + \eta_{b,u}^B(t). \quad (\text{A2})$$

The thermal noises satisfy $\langle \eta_{n,u}^Z(t) \eta_{m,v}^Z(t') \rangle = 2\gamma^Z k_B T \delta_{nm} \delta_{uv} \delta(t - t')$ for $Z \in \{A, B\}$.

2. Mode decomposition

We split the polymer A into the left arm $n = 1, \dots, N_A$ and the right arm $n = N_A + 2, \dots, 2N_A + 1$, both attached to the center tracer $\mathbf{r}_a^A$. Introduce the arm's normal modes (for $p = 1, \dots, N_A$) [28,32]

$$\begin{aligned} X_{p,u}^L(t) &= \frac{2}{\sqrt{2N_A+1}} \sum_{n=1}^{N_A} c_n^{(p)} r_{n,u}^A(t), \\ X_{p,u}^R(t) &= \frac{2}{\sqrt{2N_A+1}} \sum_{n=N_A+2}^{2N_A+1} c_n^{(p)} r_{n,u}^A(t), \end{aligned} \quad (\text{A3})$$

with the orthogonal basis

$$c_n^{(p)} = \cos\left(\frac{2n-1}{2} \theta_p\right), \quad \theta_p = \frac{2p-1}{2N_A+1} \pi.$$

Projecting Eq. (A1) onto $X_{p,u}^\sigma$ with $\sigma \in \{L, R\}$ and using $2c_n^{(p)} - c_{n-1}^{(p)} - c_{n+1}^{(p)} = 4 \sin^2(\frac{\theta_p}{2}) c_n^{(p)}$, $c_{N_A+1}^{(p)} = 0$, $c_0^{(p)} = c_1^{(p)}$, we obtain

$$\gamma^A \dot{X}_{p,u}^\sigma(t) = -\lambda_p k^A X_{p,u}^\sigma(t) + k^A s_p r_{a,u}^A(t) + \xi_{p,u}^\sigma(t), \quad (\text{A4})$$

where $s_p \equiv \frac{2}{\sqrt{2N_A+1}} c_{N_A}^{(p)}$ and

$$\begin{aligned} \xi_{p,u}^L(t) &\equiv \frac{2}{\sqrt{2N_A+1}} \sum_{n=1}^{N_A} c_n^{(p)} \eta_{n,u}^A(t), \\ \xi_{p,u}^R(t) &\equiv \frac{2}{\sqrt{2N_A+1}} \sum_{n=N_A+2}^{2N_A+1} c_n^{(p)} \eta_{n,u}^A(t), \end{aligned} \quad (\text{A5})$$

for which $\langle \xi_{p,u}^\sigma(t) \xi_{p',v}^{\sigma'}(t') \rangle = 2\gamma^A k_B T \delta_{\sigma\sigma'} \delta_{pp'} \delta_{uv} \delta(t - t')$. The mode factor is

$$\lambda_p = 4 \sin^2\left(\frac{\theta_p}{2}\right), \quad (\text{A6})$$

and the relaxation time associated with the mode p is

$$\tau_p = \frac{\gamma^A}{k^A \lambda_p}. \quad (\text{A7})$$

Note that the normal-mode equation (A4) contains no explicit dependence on system B or on the coupling constant k . The

mode decomposition diagonalizes the internal elastic interactions of polymer A , while the external coupling to system B acts only on bead a (the central bead). Because the basis functions satisfy $c_a^{(p)} = 0$, the external force $-k(r_{a,u}^A - r_{b,u}^B)$ does not project onto any polymer mode in Eq. (A3). Hence, the relaxation spectrum $\{\lambda_p, \tau_p\}$ reflects only the intrinsic viscoelastic response of polymer A and is unaffected by the coupling.

The formal solution of Eq. (A4) is

$$\begin{aligned} X_{p,u}^\sigma(t) &= X_{p,u}^\sigma(0) e^{-t/\tau_p} \\ &+ \frac{1}{\gamma^A} \int_0^t dt' e^{-(t-t')/\tau_p} [k^A s_p r_{a,u}^A(t') + \xi_{p,u}^\sigma(t')]. \end{aligned} \quad (\text{A8})$$

3. Tracer FLEs

The equation for the center tracer (the bead index a) in A follows from Eq. (A1):

$$\begin{aligned} \gamma^A \dot{r}_{a,u}^A(t) &= -k(r_{a,u}^A(t) - r_{b,u}^B(t)) \\ &+ k^A \sum_{p=1}^{N_A} (-1)^{p-1} s_p [X_{p,u}^L(t) + X_{p,u}^R(t)] \\ &+ \eta_{a,u}^A(t) - 2k^A r_{a,u}^A(t). \end{aligned} \quad (\text{A9})$$

Substituting Eq. (A8) into Eq. (A9) yields the generalized Langevin equation

$$\int_0^t K_A(t-t') \dot{r}_{a,u}^A(t') dt' = -k(r_{a,u}^A(t) - r_{b,u}^B(t)) + \Xi_\mu^A(t), \quad (\text{A10})$$

with the memory kernel (exact for finite N_A)

$$K_A(t) = \gamma^A \delta(t) + \frac{2(k^A)^2}{\gamma^A} \sum_{p=1}^{N_A} s_p^2 \tau_p e^{-t/\tau_p}, \quad (\text{A11})$$

and the effective Gaussian noise obeying the FDT

$$\langle \Xi_\mu^A(t) \Xi_\nu^A(t') \rangle \simeq k_B T K_A(|t-t'|) \delta_{\mu\nu} \quad (\text{A12})$$

at a long time.

For $N_A \gg 1$, the spectrum $\{\tau_p\}$ becomes quasicontinuous between $\tau_0^A \sim \tau_{N_A} \sim \gamma^A/(\pi^2 k^A)$ and $\tau_R^A \sim \tau_1 \sim \gamma^A(2N_A+1)^2/(\pi^2 k^A)$. In the broad intermediate window $\tau_0^A \ll t \ll \tau_R^A$, the sum in Eq. (A11) is well approximated by an integral, leading to the Rouse long-time tail

$$K_A^{\text{flex}}(t) \simeq \sqrt{\frac{4k^A \gamma^A}{\pi (k_B T)^2}} t^{-1/2}. \quad (\text{A13})$$

Hence, the generalized Langevin equation (A10) reduces to an FLE with the exponent $\alpha = \frac{1}{2}$ for system A .

Collecting the tracer equations for $\mathbf{r}_a^A$ and $\mathbf{r}_b^B$ [Eqs. (A10) and (A2)], finally, results in the coupled FLE:

$$\begin{aligned} \int_0^t K_A^{\text{flex}}(t-t') \dot{r}_{a,u}^A(t') dt' &= -k(r_{a,u}^A - r_{b,u}^B) + \Xi_u^A(t), \\ \gamma^B \dot{r}_{b,u}^B(t) &= -k(r_{b,u}^B - r_{a,u}^A) + \eta_{b,u}^B(t). \end{aligned} \quad (\text{A14})$$

Identifying $x_\alpha \equiv r_{a,u}^A$ and $x_\beta \equiv r_{b,u}^B$, we obtain the coupled FLE (6) for a flexible polymer bound to a Brownian particle [Fig. 1(a)].

4. Generalization

The above coarse graining for the Rouse chain extends to other polymer models or systems by modifying the mode spectrum $\{\lambda_p, \tau_p\}$ according to the underlying elastic and hydrodynamic physics [33,35]. Examples include self-avoiding or fractally packed chains (altered long-wavelength dispersion and density of modes) [33,35,84] and hydrodynamic interactions (Zimm-type relaxation spectrum) [33,35]. In each case, the kernel retains the form (A11) with a model-specific set of $\{\tau_p, s_p\}$, leading to a power-law tail $K_A(t) \propto t^{-\alpha}$ with an exponent α characteristic of the chosen polymer model.

In addition, the coarse-graining procedure for the polymer (A) bound to a Brownian particle (B) can be readily extended to polymer-polymer junction. Notably, the derivation of Eq. (A10) for system A is independent of the detailed structure of system B. Hence, Eq. (A10) remains valid even when system B itself represents another polymer, as illustrated in Fig. 1(b). Applying the same procedure to system B, characterized by its own parameters (k^B, γ^B, M_B) and the coupling site b , yields an analogous generalized Langevin equation:

$$\int_0^t K_A(t-t') \dot{r}_{a,u}^A(t') dt' = -k(r_{a,u}^A - r_{b,u}^B) + \Xi_u^A(t), \quad (\text{A15})$$

$$\int_0^t K_B(t-t') \dot{r}_{b,u}^B(t') dt' = -k(r_{b,u}^B - r_{a,u}^A) + \Xi_u^B(t), \quad (\text{A16})$$

where the effective noises satisfy the fluctuation-dissipation relation

$$\langle \Xi_u^Z(t) \Xi_v^Z(t') \rangle = k_B T K^Z(|t-t'|) \delta_{uv}, \quad Z \in \{A, B\}. \quad (\text{A17})$$

With the identifications $x_\alpha \equiv r_{a,u}^A$ and $x_\beta \equiv r_{b,u}^B$, this reproduces the coupled FLEs (6) in full generality.

APPENDIX B: THE CORRELATION FUNCTIONS OF THE EFFECTIVE NOISES

In this section, we derive the correlation functions of $\eta_{\alpha\beta}(t)$ [Eq. (18)] and $\xi_\alpha^{\text{eff}}(t)$ [Eq. (29)]. For a stationary process $\phi(t)$ with the correlation function $\langle \phi(t)\phi(t') \rangle = \psi(|t-t'|)$, its Laplace transform satisfies

$$\langle \tilde{\phi}(s)\tilde{\phi}(s') \rangle = \frac{\tilde{\psi}(s) + \tilde{\psi}(s')}{s + s'}. \quad (\text{B1})$$

Employing this result to fractional Gaussian noise ξ_p , which obeys the FDT

$$\langle \tilde{\xi}_\mu(s)\tilde{\xi}_\mu(s') \rangle = k_B T \frac{\tilde{K}_\mu(s) + \tilde{K}_\mu(s')}{s + s'}, \quad (\text{B2})$$

where $\tilde{K}_\mu(s) = \gamma_\mu s^{-1+\mu}$, we proceed to compute the correlations.

1. The correlation function of $\eta_{\alpha\beta}(t)$

Using Eq. (B2), the correlation of $\tilde{\eta}_{\alpha\beta}(s)$ yields

$$\begin{aligned} \langle \tilde{\eta}_{\alpha\beta}(s)\tilde{\eta}_{\alpha\beta}(s') \rangle &= \frac{\tilde{K}_\beta(s)}{\tilde{K}_\alpha(s) + \tilde{K}_\beta(s)} \frac{\tilde{K}_\beta(s')}{\tilde{K}_\alpha(s') + \tilde{K}_\beta(s')} \langle \tilde{\xi}_\alpha(s)\tilde{\xi}_\alpha(s') \rangle \\ &+ \frac{\tilde{K}_\alpha(s)}{\tilde{K}_\alpha(s) + \tilde{K}_\beta(s)} \frac{\tilde{K}_\alpha(s')}{\tilde{K}_\alpha(s') + \tilde{K}_\beta(s')} \langle \tilde{\xi}_\beta(s)\tilde{\xi}_\beta(s') \rangle \\ &= \frac{k_B T}{s + s'} \frac{K_\beta(s)K_\beta(s')(\tilde{K}_\alpha(s) + \tilde{K}_\alpha(s'))}{(\tilde{K}_\alpha(s) + \tilde{K}_\beta(s))(\tilde{K}_\alpha(s') + \tilde{K}_\beta(s'))} \\ &+ \frac{k_B T}{s + s'} \frac{K_\alpha(s)K_\alpha(s')(\tilde{K}_\beta(s) + \tilde{K}_\beta(s'))}{(\tilde{K}_\alpha(s) + \tilde{K}_\beta(s))(\tilde{K}_\alpha(s') + \tilde{K}_\beta(s'))}. \end{aligned} \quad (\text{B3})$$

After rearranging terms, we arrive at the expression

$$\langle \tilde{\eta}_{\alpha\beta}(s)\tilde{\eta}_{\alpha\beta}(s') \rangle = k_B T \frac{\frac{\tilde{K}_\alpha(s)\tilde{K}_\beta(s)}{\tilde{K}_\alpha(s) + \tilde{K}_\beta(s)} + \frac{\tilde{K}_\alpha(s')\tilde{K}_\beta(s')}{\tilde{K}_\alpha(s') + \tilde{K}_\beta(s')}}{s + s'}. \quad (\text{B4})$$

Since the obtained expression follows the form defined in Eq. (B1), we can identify the correlation function in the time domain, which satisfies the FDT in the following:

$$\langle \eta_{\alpha\beta}(t)\eta_{\alpha\beta}(t') \rangle = k_B T K_{\alpha\beta}^{\text{eff}}(|t-t'|). \quad (\text{B5})$$

This is Eq. (19). Because no specific assumption on $\tilde{K}_\mu(s)$ was made, this result holds for any memory kernels other than the power law.

2. The correlation function of $\xi_\alpha^{\text{eff}}(t)$

Similarly, using Eq. (B2), we calculate the correlation function of $\tilde{\eta}_\beta(s)$ as

$$\begin{aligned} \frac{\langle \tilde{\eta}_\beta(s)\tilde{\eta}_\beta(s') \rangle}{k_B T} &= \frac{k}{s\tilde{K}_\beta(s) + k} \frac{k}{s'\tilde{K}_\beta(s') + k} \frac{\tilde{K}_\beta(s) + \tilde{K}_\beta(s')}{s + s'} \\ &= \frac{k\tilde{K}_\beta(s)(s'\tilde{K}_\beta(s') + k) + k\tilde{K}_\beta(s')(s\tilde{K}_\beta(s) + k)}{(s\tilde{K}_\beta(s) + k)(s'\tilde{K}_\beta(s') + k)(s + s')} \\ &\quad - \frac{k\tilde{K}_\beta(s)\tilde{K}_\beta(s')(s + s')}{(s\tilde{K}_\beta(s) + k)(s'\tilde{K}_\beta(s') + k)(s + s')} \\ &= \frac{\frac{k\tilde{K}_\beta(s)}{s\tilde{K}_\beta(s) + k} + \frac{k\tilde{K}_\beta(s')}{s'\tilde{K}_\beta(s') + k}}{s + s'} \\ &\quad - k \frac{\tilde{K}_\beta(s)}{s\tilde{K}_\beta(s) + k} \frac{\tilde{K}_\beta(s')}{s'\tilde{K}_\beta(s') + k}. \end{aligned} \quad (\text{B6})$$

Through the inverse Laplace transform, we obtain

$$\langle \eta_\beta(t)\eta_\beta(t') \rangle = k_B T \Phi_\beta(|t-t'|) - k_B T k E_{\beta,1}(-(t/\tau_\beta)^\beta) E_{\beta,1}(-(t'/\tau_\beta)^\beta). \quad (\text{B7})$$

Here, the Mittag-Leffler function asymptotically decays as $E_{\beta,1}(-z^\beta) \simeq z^{-\beta}$ for $z \gg 1$, so the nonstationary terms $E_{\beta,1}(-(t/\tau_\beta)^\beta)$ and $E_{\beta,1}(-(t'/\tau_\beta)^\beta)$ become negligible as $t/\tau_\beta \rightarrow \infty$ and $t'/\tau_\beta \rightarrow \infty$. Finally, neglecting nonstationary contributions and using the independency between the noises ($\langle \xi_\alpha(t)\eta_\beta(t') \rangle = 0$), we obtain the FDT for the effective noise $\xi_\alpha^{\text{eff}}(t) = \xi_\alpha(t) + \eta_\beta(t)$ as follows:

$$\langle \xi_\alpha^{\text{eff}}(t)\xi_\alpha^{\text{eff}}(t') \rangle \simeq k_B T [K_\alpha(|t-t'|) + \Phi_\beta(|t-t'|)]. \quad (\text{B8})$$

APPENDIX C: DERIVATION OF MSDs (37) AND (39)

In this Appendix, we derive the MSDs for the generalized Langevin equations quoted in Eqs. (37) and (39).

1. The GLE subject to a harmonic potential

We begin with a GLE described with a memory kernel $K(t)$ and subject to a harmonic potential of stiffness k :

$$\int_0^t K(t-t')\dot{x}(t')dt' = -kx(t) + \xi(t). \quad (\text{C1})$$

The free-space GLE is recovered in the limit of $k \rightarrow 0$.

In the Laplace domain, the solution of the GLE above can be written as

$$\tilde{x}(s) = \frac{\tilde{K}(s)}{s\tilde{K}(s) + k}x(0) + \frac{\tilde{\xi}(s)}{s\tilde{K}(s) + k}. \quad (\text{C2})$$

Using this, we obtain the expression for the two-point correlation function as follows:

$$\langle \tilde{x}(s)\tilde{x}(s') \rangle = \langle x^2(0) \rangle \tilde{\Psi}(s)\tilde{\Psi}(s') + \frac{\langle \tilde{\xi}_\alpha(s)\tilde{\xi}_\alpha(s') \rangle}{[s\tilde{K}(s) + k][s'\tilde{K}(s') + k]}, \quad (\text{C3})$$

where $\tilde{\Psi}(s)$ is defined by

$$\tilde{\Psi}(s) \equiv \frac{\tilde{K}(s)}{s\tilde{K}(s) + k}, \quad (\text{C4})$$

and $\langle x(0)\tilde{\xi}(s) \rangle = 0$ is used.

Employing the Laplace-transformed FDT (B2), we have

$$\langle \tilde{x}(s)\tilde{x}(s') \rangle = \langle x^2(0) \rangle \tilde{\Psi}(s)\tilde{\Psi}(s') + \frac{k_B T}{k} \left[\frac{\tilde{\Psi}(s) + \tilde{\Psi}(s')}{s + s'} - \tilde{\Psi}(s)\tilde{\Psi}(s') \right]. \quad (\text{C5})$$

The inverse Laplace transformation of the above results is performed to obtain the autocovariance of $x(t)$:

$$\langle x(t)x(t') \rangle = \left[\langle x^2(0) \rangle - \frac{k_B T}{k} \right] \Psi(t)\Psi(t') + \frac{k_B T}{k} \Psi(|t - t'|). \quad (\text{C6})$$

If we set the equilibrium initial condition, $\langle x^2(0) \rangle = \frac{k_B T}{k}$ and the first term in the right-hand side vanishes. Assuming $\Psi(0) = 1$ (i.e., $\lim_{s \rightarrow \infty} s\tilde{K}(s) = \infty$), the MSD

$$\langle [x(t + t_0) - x(t_0)]^2 \rangle = \langle x^2(t + t_0) \rangle + \langle x^2(t_0) \rangle - 2\langle x(t + t_0)x(t_0) \rangle \quad (\text{C7})$$

is simplified to

$$\langle [x(t + t_0) - x(t_0)]^2 \rangle = \frac{2k_B T}{k} [1 - \Psi(t)]. \quad (\text{C8})$$

Equivalently, in the Laplace domain, this can be written as

$$\langle [x(t + t_0) - x(t_0)]^2 \rangle = 2k_B T \mathcal{L}^{-1} \left\{ \frac{1}{s(s\tilde{K}(s) + k)} \right\} (t), \quad (\text{C9})$$

which recovers Eq. (37).

2. GLE in free space

As explained above, the GLE (C1) converges to that in free space in the limit of $k \rightarrow 0$. Using this property, we obtain the expression for MSD in free space by taking $k \rightarrow 0$ in Eq. (C5). In this limit, the autocovariance is

$$\langle \tilde{x}(s)\tilde{x}(s') \rangle = \frac{\langle x^2(0) \rangle}{ss'} + \frac{k_B T [\tilde{K}(s) + \tilde{K}(s')]}{ss'(s + s')\tilde{K}(s)\tilde{K}(s')}. \quad (\text{C10})$$

Now, we define $\tilde{G}(s) = \frac{1}{s^2\tilde{K}(s)}$ and rewrite the autocovariance in terms of G as such:

$$\langle \tilde{x}(s)\tilde{x}(s') \rangle = \frac{\langle x^2(0) \rangle}{ss'} + k_B T \left[\frac{\tilde{G}(s)}{s'} + \frac{\tilde{G}(s')}{s} - \frac{\tilde{G}(s) + \tilde{G}(s')}{s + s'} \right]. \quad (\text{C11})$$

Transforming this expression back to the time domain yields

$$\langle x(t)x(t') \rangle = \langle x_0^2 \rangle + k_B T [G(t) + G(t') - G(|t - t'|)]. \quad (\text{C12})$$

Inserting Eq. (C12) into the definition of MSD, Eq. (C7), we finally obtain the expression for the MSD:

$$\langle [x(t + t_0) - x(t_0)]^2 \rangle = 2k_B T G(t), \quad (\text{C13})$$

which recovers Eq. (39). We can also confirm that taking $k \rightarrow 0$ in Eq. (C9) yields Eq. (C13), which ensures a smooth connection between the MSDs at the conditions of confined and free space.

APPENDIX D: THE EXACT SOLUTION OF EQ. (24)

In Sec. III B, we neglected the initial-position term in Eq. (24) and assumed a stationary noise correlation. Under this condition, we obtained the effective GLE (31) with the MSD (39). Here, we retain every initial-condition contribution and demonstrate that the full solution still reproduces the same MSD and two-point correlation.

We take the Laplace transform of Eq. (24) to find the formal solution:

$$\tilde{x}_\alpha(s) = \frac{1}{s}x_\alpha(0) - \frac{\tilde{\Phi}_\beta(s)}{s\tilde{K}_\alpha^{(\text{eff})}(s)}r_{\alpha\beta}(0) + \frac{\tilde{\xi}_\alpha(s) + \tilde{\eta}_\beta(s)}{s\tilde{K}_\alpha^{(\text{eff})}(s)}. \quad (\text{D1})$$

Using this, we obtain the autocovariance as

$$\begin{aligned} \langle \tilde{x}_\alpha(s)\tilde{x}_\alpha(s') \rangle &= \frac{\langle x_\alpha^2(0) \rangle}{ss'} + \langle x_\alpha(0)r_{\alpha\beta}(0) \rangle \left[\frac{\tilde{\Phi}_\beta(s')}{ss'\tilde{K}_\alpha^{(\text{eff})}(s')} + \frac{\tilde{\Phi}_\beta(s)}{s'\tilde{K}_\alpha^{(\text{eff})}(s)} \right] \\ &\quad + \langle r_{\alpha\beta}^2(0) \rangle \frac{\tilde{\Phi}_\beta(s)}{s\tilde{K}_\alpha^{(\text{eff})}(s)} \frac{\tilde{\Phi}_\beta(s')}{s'\tilde{K}_\alpha^{(\text{eff})}(s')} \\ &\quad + \frac{1}{s\tilde{K}_\alpha^{(\text{eff})}(s)s'\tilde{K}_\alpha^{(\text{eff})}(s')} [\langle \tilde{\xi}_\alpha(s)\tilde{\xi}_\alpha(s') \rangle + \langle \tilde{\eta}_\beta(s)\tilde{\eta}_\beta(s') \rangle]. \end{aligned} \quad (\text{D2})$$

Using the relations of Eqs. (28) and (29), we can rearrange the above expression as follows:

$$\begin{aligned} \langle \tilde{x}_\alpha(s) \tilde{x}_\alpha(s') \rangle &= \frac{\langle x_\alpha^2(0) \rangle}{ss'} + \langle x_\alpha(0) r_{\alpha\beta}(0) \rangle \left[\frac{\tilde{\Phi}_\beta(s')}{ss' \tilde{K}_\alpha^{(\text{eff})}(s')} + \frac{\tilde{\Phi}_\beta(s)}{s' \tilde{K}_\alpha^{(\text{eff})}(s)} \right] + \left[\langle r_{\alpha\beta}^2(0) \rangle - \frac{k_B T}{k} \right] \frac{\tilde{\Phi}_\beta(s)}{s \tilde{K}_\alpha^{(\text{eff})}(s)} \frac{\tilde{\Phi}_\beta(s')}{s' \tilde{K}_\alpha^{(\text{eff})}(s')} \\ &\quad + \frac{k_B T}{k} \left[\frac{\tilde{G}^{(\text{eff})}(s)}{s'} + \frac{\tilde{G}^{(\text{eff})}(s')}{s} - \frac{\tilde{G}^{(\text{eff})}(s) + \tilde{G}^{(\text{eff})}(s')}{s + s'} \right], \end{aligned} \quad (\text{D3})$$

where $\tilde{G}^{(\text{eff})}(s) = \frac{1}{s^2 \tilde{K}_\alpha^{(\text{eff})}(s)}$. Performing the inverse Laplace transformation, we obtain the autocovariance in the time domain

$$\begin{aligned} \langle x_\alpha(t) x_\alpha(t') \rangle &= \langle x_\alpha^2(0) \rangle + \langle x_\alpha(0) r_{\alpha\beta}(0) \rangle [\Psi_{\alpha\beta}(t) + \Psi_{\alpha\beta}(t')] + \left[\langle r_{\alpha\beta}^2(0) \rangle - \frac{k_B T}{k} \right] \Psi_{\alpha\beta}(t) \Psi_{\alpha\beta}(t') \\ &\quad + k_B T [G^{(\text{eff})}(t) + G^{(\text{eff})}(t') - G^{(\text{eff})}(|t - t'|)], \end{aligned} \quad (\text{D4})$$

with

$$\begin{aligned} \Psi_{\alpha\beta}(t) &= \mathcal{L}^{-1} \left\{ \frac{\tilde{\Phi}_\beta(s)}{s \tilde{K}_\alpha^{(\text{eff})}(s)} \right\} (t) \\ &\simeq (t/\tau_\beta)^{-\beta}, \quad t \gg \tau_\alpha, \tau_\beta, \tau_c. \end{aligned} \quad (\text{D5})$$

Equation (D4) is the most general expression obtained under the nonequilibrium initial condition. Even in such a case, the second and third terms in the right-hand side decay out in the infinite-time limit as $\Psi_{\alpha\beta}(t) \sim (t/\tau_\beta)^{-\beta}$. These contributions are absent either in the long-time limit $t \gg \tau_\beta$ (or if the process starts at $t = -\infty$) or under the equilibrium initial conditions: $\langle r_{\alpha\beta}^2(0) \rangle = k_B T/k$ and $\langle r_{\alpha\beta}(0) x_\alpha(0) \rangle = 0$. In such cases, Eq. (D4) reduces to the form, Eq. (C12), presented in the main text. Finally, Eq. (C7) is used to estimate MSD, yielding

$$\langle [x(t + t_0) - x(t_0)]^2 \rangle = 2k_B T G^{(\text{eff})}(t). \quad (\text{D6})$$

Thus, the MSD is stationary and identical to the expression shown in Eq. (39).

APPENDIX E: POLYMER SIMULATIONS

In Sec. V and Fig. 7, we present the Langevin dynamics simulation results of the bipartite polymer systems. Here, we describe the simulation details.

We have performed the overdamped Langevin dynamics simulations of the coupled polymer systems introduced in Figs. 1(a) and 1(b). The setup is identical to the two-component model discussed in Sec. II, which consists of two polymeric components, systems A and B, each represented by a chain of beads. The total potential energy is given by Eq. (7), and the equation of motion follows the overdamped Langevin equation (8). The initial positions of all beads are sampled from the Boltzmann distribution, $P(\{\mathbf{r}_i^A\}, \{\mathbf{r}_i^B\}) \propto e^{-U_{\text{tot}}}$. We numerically integrate the equations of motion using a second-order Langevin integrator [85], with a time step of $dt = 0.001$. Each simulation runs for 10^7 time steps. The MSDs are computed as both time- and ensemble-averaged quantities over 10^3 independent trajectories.

Below, we specify the system parameters for the two cases studied in our simulation.

1. Flexible polymer bound to a Brownian particle

In system A [Fig. 1(a)], we model the polymer with a Rouse chain connected via the potential

$$U^A = \frac{k^A}{2} \sum_{i=1}^{M_A-1} |\mathbf{r}_{i+1}^A - \mathbf{r}_i^A|^2, \quad (\text{E1})$$

with $M_A = 2001$. We set $k^A = 10$ and $\gamma^A = 0.1$, which makes the characteristic time of a monomer and the Rouse relaxation time

$$\tau_0^A \simeq \frac{\gamma^A}{k^A} \simeq 10^{-2}, \quad \tau_R^A \simeq (M_A/2)^2 \tau_0^A \simeq 10^4, \quad (\text{E2})$$

respectively. As shown in Appendix A, a Rouse monomer in the intermediate timescale $\tau_0^A \ll t \ll \tau_R^A$ obeys an FLE with the memory kernel given by Eq. (A13). The central monomer ($a = 1001$) is connected to a single Brownian particle (system B): $U^B = 0$, $M_B = 1$, and $b = 1$. The central monomer interacts with the Brownian particle via a harmonic spring of stiffness $k = 0.3$. Combining $K_\alpha(t) \simeq K_A^{\text{flex}}(t)$ explained in Eq. (A13) and $K_\beta(t) = 2\gamma^B \delta(t)$, the critical coupling strength k_c is obtained from Eq. (11). The corresponding dimensionless interaction strength is

$$\frac{k}{k_c} \simeq 7.5 > 1, \quad (\text{E3})$$

placing the system in the strong-interaction regime.

2. Cross-linked flexible and semiflexible polymer

The system A is modeled as a Rouse chain with the same potential of Eq. (E1). We set $M_A = 2001$, $a = 1001$, $k^A = 40$, and $\gamma^A = 0.4$, which results in

$$\tau_0^A \simeq \frac{\gamma^A}{k^A} \simeq 10^{-2}, \quad \tau_R^A \simeq (M_A/2)^2 \tau_0^A \simeq 10^4. \quad (\text{E4})$$

The system B is a Gaussian semiflexible chain [32,86] with $M_B = 2N_B + 1$ beads. In this polymer model, the potential energy is given by

$$U^B = \frac{k^B}{2} \left[\sum_{i=1}^{M_B-1} \lambda_i^B |\mathbf{q}_i^B|^2 + \sum_{i=1}^{M_B-2} \mu_i^B \mathbf{q}_i^B \cdot \mathbf{q}_{i+1}^B \right], \quad (\text{E5})$$

where $\mathbf{q}_i^B = \mathbf{r}_{i+1}^B - \mathbf{r}_i^B$ is the bond vector. In the right-hand side, the first term represents bond stretching and the second term accounts for bending energy. The spring constant is given

by $k^B = d k_B T / \ell$, where ℓ is the bond length, yielding the contour length $L_c = 2N_B \ell$. The Lagrange multipliers λ_i^B and μ^B are

$$\lambda_i^B = \begin{cases} \frac{1}{2} \frac{1+\kappa^2}{1-\kappa^2}, & i = 2, 3, \dots, 2N_B - 1, \\ \frac{1}{2} \frac{1}{1-\kappa^2}, & i = 1, 2N_B, \end{cases}$$

$$\mu^B = \frac{\kappa}{1-\kappa^2}, \quad (\text{E6})$$

which enforce the constraints

$$\langle |\mathbf{q}_i|^2 \rangle = \ell^2, \quad (\text{E7})$$

$$\langle \mathbf{q}_{i+1} \cdot \mathbf{q}_i \rangle = \kappa \ell^2.$$

The parameter $0 \leq \kappa < 1$ determines the persistence length

$$L_p = \frac{\ell}{1-\kappa}. \quad (\text{E8})$$

For $L_c \lesssim L_p$, the chain behaves as a stiff polymer whose local fluctuations are described by an FLE with kernel [31]

$$K_B^{\text{stiff}}(t) = c_{\text{stiff}} \left((\gamma^B)^3 k^B \frac{L_p}{\ell} \right)^{1/4} t^{-3/4}, \quad (\text{E9})$$

where c_{stiff} is a numerical prefactor that originates from the limitation of the scaling theory.

We use $M_B = 101$, $b = 51$, $k^B = 1$, $d = 3$, $\gamma^B = 2$, and $\kappa = 0.995$, so that the persistence length exceeds the contour length ($L_p/L_c \simeq 2$). The characteristic timescales are

$$\tau_0^B \simeq \frac{\gamma^B}{k^B} \frac{\ell}{L_p} \simeq 10^{-2}, \quad \tau_R^B \simeq (M_B/2)^2 \tau_0^A \simeq 6.25 \times 10^4. \quad (\text{E10})$$

In between these two timescales, the dynamics of the central monomer (b) is governed by the FLE with the kernel (E9). We have simulated the free Gaussian semiflexible chain ($k = 0$) and obtained $c_{\text{stiff}} \approx 0.66$ by fitting the MSD.

The spring constant is set to $k = 0.5$. Combining $K_\alpha(t) \simeq K_A^{\text{flex}}(t)$ and $K_\beta(t) = K_B^{\text{stiff}}(t)$, we find from Eq. (11) that

$$\frac{k}{k_c} \simeq 0.22 \lesssim 1, \quad (\text{E11})$$

which places the system in the weak-interaction regime.

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- [1] R. Metzler, J.-H. Jeon, A. G. Cherstvy, and E. Barkai, Anomalous diffusion models and their properties: Non-stationarity, non-ergodicity, and ageing at the centenary of single particle tracking, *Phys. Chem. Chem. Phys.* **16**, 24128 (2014).
 - [2] M. Baggioli, G. La Nave, and P. W. Phillips, Anomalous diffusion and Noether's second theorem, *Phys. Rev. E* **103**, 032115 (2021).
 - [3] G. W. Ford, M. Kac, and P. Mazur, Statistical mechanics of assemblies of coupled oscillators, *J. Math. Phys.* **6**, 504 (1965).
 - [4] H. Mori, Transport, collective motion, and Brownian motion, *Prog. Theor. Phys.* **33**, 423 (1965).
 - [5] R. Zwanzig, Nonlinear generalized Langevin equations, *J. Stat. Phys.* **9**, 215 (1973).
 - [6] R. Kubo, The fluctuation-dissipation theorem, *Rep. Prog. Phys.* **29**, 255 (1966).
 - [7] E. Lutz, Fractional Langevin equation, *Phys. Rev. E* **64**, 051106 (2001).
 - [8] I. Goychuk, Viscoelastic subdiffusion: From anomalous to normal, *Phys. Rev. E* **80**, 046125 (2009).
 - [9] I. Goychuk, Viscoelastic subdiffusion: Generalized Langevin equation approach, in *Advances in Chemical Physics* (John Wiley & Sons, Ltd., New York, NY, 2012), pp. 187–253.
 - [10] S. C. Kou and X. S. Xie, Generalized Langevin equation with fractional Gaussian noise: Subdiffusion within a single protein molecule, *Phys. Rev. Lett.* **93**, 180603 (2004).
 - [11] W. Min, G. Luo, B. J. Cherayil, S. C. Kou, and X. S. Xie, Observation of a power-law memory kernel for fluctuations within a single protein molecule, *Phys. Rev. Lett.* **94**, 198302 (2005).
 - [12] B. Fabry, G. N. Maksym, J. P. Butler, M. Glogauer, D. Navajas, and J. J. Fredberg, Scaling the microrheology of living cells, *Phys. Rev. Lett.* **87**, 148102 (2001).
 - [13] C. Wilhelm, Out-of-equilibrium microrheology inside living cells, *Phys. Rev. Lett.* **101**, 028101 (2008).
 - [14] M. Guo, A. J. Ehrlicher, M. H. Jensen, M. Renz, J. R. Moore, R. D. Goldman, J. Lippincott-Schwartz, F. C. Mackintosh, and D. A. Weitz, Probing the stochastic, motor-driven properties of the cytoplasm using force spectrum microscopy, *Cell* **158**, 822 (2014).
 - [15] S. C. Weber, A. J. Spakowitz, and J. A. Theriot, Bacterial chromosomal loci move subdiffusively through a viscoelastic cytoplasm, *Phys. Rev. Lett.* **104**, 238102 (2010).
 - [16] T. J. Lampo, N. J. Kuwada, P. A. Wiggins, and A. J. Spakowitz, Physical modeling of chromosome segregation in *Escherichia coli* reveals impact of force and DNA relaxation, *Biophys. J.* **108**, 146 (2015).
 - [17] T. G. Mason, T. Gisler, K. Kroy, E. Frey, and D. A. Weitz, Rheology of F-actin solutions determined from thermally driven tracer motion, *J. Rheol.* **44**, 917 (2000).
 - [18] T. Gisler and D. A. Weitz, Scaling of the microrheology of semidilute F-actin solutions, *Phys. Rev. Lett.* **82**, 1606 (1999).
 - [19] B. Schnurr, F. Gittes, F. C. MacKintosh, and C. F. Schmidt, Determining microscopic viscoelasticity in flexible and semiflexible polymer networks from thermal fluctuations, *Macromolecules* **30**, 7781 (1997).
 - [20] B. R. Dasgupta, S.-Y. Tee, J. C. Crocker, B. J. Frisken, and D. A. Weitz, Microrheology of polyethylene oxide using diffusing wave spectroscopy and single scattering, *Phys. Rev. E* **65**, 051505 (2002).
 - [21] M. Caputo, Linear models of dissipation whose Q is almost frequency independent—II, *Geophys. J. Int.* **13**, 529 (1967).
 - [22] E. C. de Oliveira and J. A. Tenreiro Machado, A review of definitions for fractional derivatives and integral, *Math. Probl. Eng.* **2014**, 238459 (2014).
 - [23] T. G. Mason and D. A. Weitz, Optical measurements of frequency-dependent linear viscoelastic moduli of complex fluids, *Phys. Rev. Lett.* **74**, 1250 (1995).
 - [24] I. Goychuk and P. Hänggi, Anomalous escape governed by thermal $1/f$ noise, *Phys. Rev. Lett.* **99**, 200601 (2007).

- [25] A. Taloni, A. Chechkin, and J. Klafter, Generalized elastic model yields a fractional Langevin equation description, *Phys. Rev. Lett.* **104**, 160602 (2010).
- [26] D. Panja, Anomalous polymer dynamics is non-Markovian: Memory effects and the generalized Langevin equation formulation, *J. Stat. Mech.* (2010) P06011.
- [27] L. Lizana, T. Ambjörnsson, A. Taloni, E. Barkai, and M. A. Lomholt, Foundation of fractional Langevin equation: Harmonization of a many-body problem, *Phys. Rev. E* **81**, 051118 (2010).
- [28] H. Vandebroek and C. Vanderzande, On the generalized Langevin equation for a Rouse bead in a nonequilibrium bath, *J. Stat. Phys.* **167**, 14 (2017).
- [29] C. Maes and S. R. Thomas, From Langevin to generalized Langevin equations for the nonequilibrium Rouse model, *Phys. Rev. E* **87**, 022145 (2013).
- [30] S. Shinkai, S. Onami, and T. Miyaguchi, Generalized Langevin dynamics for single beads in linear elastic networks, *Phys. Rev. E* **110**, 044136 (2024).
- [31] H.-T. Han, S. Joo, T. Sakaue, and J.-H. Jeon, Nonequilibrium diffusion of active particles bound to a semiflexible polymer network: Simulations and fractional Langevin equation, *J. Chem. Phys.* **159**, 024901 (2023).
- [32] X. Durang, C. Lim, and J.-H. Jeon, Generalized Langevin equation for a tagged monomer in a Gaussian semiflexible polymer, *J. Chem. Phys.* **161**, 244906 (2024).
- [33] T. Sakaue, Memory effect and fluctuating anomalous dynamics of a tagged monomer, *Phys. Rev. E* **87**, 040601 (2013).
- [34] J. Kappler, F. Noé, and R. R. Netz, Cyclization and relaxation dynamics of finite-length collapsed self-avoiding polymers, *Phys. Rev. Lett.* **122**, 067801 (2019).
- [35] D. Panja, Generalized Langevin equation formulation for anomalous polymer dynamics, *J. Stat. Mech.* (2010) L02001.
- [36] A. Rigato, A. Miyagi, S. Scheuring, and F. Rico, High-frequency microrheology reveals cytoskeleton dynamics in living cells, *Nat. Phys.* **13**, 771 (2017).
- [37] J.-H. Jeon, H. M.-S. Monne, M. Javanainen, and R. Metzler, Anomalous diffusion of phospholipids and cholesterol in a lipid bilayer and its origins, *Phys. Rev. Lett.* **109**, 188103 (2012).
- [38] E. Yamamoto, T. Akimoto, A. C. Kalli, K. Yasuoka, and M. S. P. Sansom, Dynamic interactions between a membrane binding protein and lipids induce fluctuating diffusivity, *Sci. Adv.* **3**, e1601871 (2017).
- [39] F. Watanabe, T. Akimoto, R. B. Best, K. Lindorff-Larsen, R. Metzler, and E. Yamamoto, Diffusion of intrinsically disordered proteins within viscoelastic membraneless droplets, *Phys. Rev. Res.* **7**, 043117 (2025).
- [40] T. Germier, S. Kocanova, N. Walther, A. Bancaud, H. A. Shaban, H. Sellou, A. Z. Politi, J. Ellenberg, F. Gallardo, and K. Bystricky, Real-time imaging of a single gene reveals transcription-initiated local confinement, *Biophys. J.* **113**, 1383 (2017).
- [41] A. Zidovska, D. A. Weitz, and T. J. Mitchison, Micron-scale coherence in interphase chromatin dynamics, *Proc. Natl. Acad. Sci. USA* **110**, 15555 (2013).
- [42] I. Bronstein, Y. Israel, E. Kepten, S. Mai, Y. Shav-Tal, E. Barkai, and Y. Garini, Transient anomalous diffusion of telomeres in the nucleus of mammalian cells, *Phys. Rev. Lett.* **103**, 018102 (2009).
- [43] I. Bronshtein, E. Kepten, I. Kanter, S. Berezin, M. Lindner, A. B. Redwood, S. Mai, S. Gonzalo, R. Foisner, Y. Shav-Tal, and Y. Garini, Loss of lamin A function increases chromatin dynamics in the nuclear interior, *Nat. Commun.* **6**, 8044 (2015).
- [44] S. S. Ashwin, T. Nozaki, K. Maeshima, and M. Sasai, Organization of fast and slow chromatin revealed by single-nucleosome dynamics, *Proc. Natl. Acad. Sci. USA* **116**, 19939 (2019).
- [45] N. Khanna, Y. Zhang, J. S. Lucas, O. K. Dudko, and C. Murre, Chromosome dynamics near the sol-gel phase transition dictate the timing of remote genomic interactions, *Nat. Commun.* **10**, 2771 (2019).
- [46] V. Levi, Q. Ruan, M. Plutz, A. S. Belmont, and E. Gratton, Chromatin dynamics in interphase cells revealed by tracking in a two-photon excitation microscope, *Biophys. J.* **89**, 4275 (2005).
- [47] B. D. Hoffman, G. Massiera, K. M. Van Citters, and J. C. Crocker, The consensus mechanics of cultured mammalian cells, *Proc. Natl. Acad. Sci. USA* **103**, 10259 (2006).
- [48] M. Otten, A. Nandi, D. Arcizet, M. Gorelshvili, B. Lindner, and D. Heinrich, Local motion analysis reveals impact of the dynamic cytoskeleton on intracellular subdiffusion, *Biophys. J.* **102**, 758 (2012).
- [49] P. Annibale and E. Gratton, Single cell visualization of transcription kinetics variance of highly mobile identical genes using 3D nanoimaging, *Sci. Rep.* **5**, 9258 (2015).
- [50] M. Mak, S. Anderson, M. C. McDonough, F. Spill, J. E. Kim, A. Boussommier-Calleja, M. H. Zaman, and R. D. Kamm, Integrated analysis of intracellular dynamics of menainv cancer cells in a 3D matrix, *Biophys. J.* **112**, 1874 (2017).
- [51] S. S. Rogers, T. A. Waigh, and J. R. Lu, Intracellular microrheology of motile *Amoeba proteus*, *Biophys. J.* **94**, 3313 (2008).
- [52] E. Lee, D. Kim, Y. H. Song, K. Shin, S. Song, M. Lee, Y. Goh, M. H. Lim, J.-H. Kim, J. Sung, and K. T. Lee, Real-time tracking of vesicles in living cells reveals that tau-hyperphosphorylation suppresses unidirectional transport by motor proteins, *Chem. Biomed. Imaging* **2**, 362 (2024).
- [53] K. Murase, T. Fujiwara, Y. Umemura, K. Suzuki, R. Iino, H. Yamashita, M. Saito, H. Murakoshi, K. Ritchie, and A. Kusumi, Ultrafine membrane compartments for molecular diffusion as revealed by single molecule techniques, *Biophys. J.* **86**, 4075 (2004).
- [54] K. Suzuki, K. Ritchie, E. Kajikawa, T. Fujiwara, and A. Kusumi, Rapid hop diffusion of a G-protein-coupled receptor in the plasma membrane as revealed by single-molecule techniques, *Biophys. J.* **88**, 3659 (2005).
- [55] D. M. Wulstein, K. E. Regan, J. Garamella, R. J. McGorty, and R. M. Robertson-Anderson, Topology-dependent anomalous dynamics of ring and linear DNA are sensitive to cytoskeleton crosslinking, *Sci. Adv.* **5**, eaay5912 (2019).
- [56] J. Garamella, K. Regan, G. Aguirre, R. J. McGorty, and R. M. Robertson-Anderson, Anomalous and heterogeneous DNA transport in biomimetic cytoskeleton networks, *Soft Matter* **16**, 6344 (2020).
- [57] S. J. Anderson, J. Garamella, S. Adalbert, R. J. McGorty, and R. M. Robertson-Anderson, Subtle changes in crosslinking drive diverse anomalous transport characteristics in actin-microtubule networks, *Soft Matter* **17**, 4375 (2021).
- [58] G. Lee, G. Leech, M. J. Rust, M. Das, R. J. McGorty, J. L. Ross, and R. M. Robertson-Anderson, Myosin-driven

- actin-microtubule networks exhibit self-organized contractile dynamics, *Sci. Adv.* **7**, eabe4334 (2021).
- [59] J. Y. Sheung, J. Garamella, S. K. Kahl, B. Y. Lee, R. J. McGorty, and R. M. Robertson-Anderson, Motor-driven advection competes with crowding to drive spatiotemporally heterogeneous transport in cytoskeleton composites, *Front. Phys.* **10**, 1055441 (2022).
- [60] B. A. Krajina, B. L. LeSavage, J. G. Roth, A. W. Zhu, P. C. Cai, A. J. Spakowitz, and S. C. Heilshorn, Microrheology reveals simultaneous cell-mediated matrix stiffening and fluidization that underlie breast cancer invasion, *Sci. Adv.* **7**, eabe1969 (2021).
- [61] E. L. Baker, R. T. Bonnecaze, and M. H. Zaman, Extracellular matrix stiffness and architecture govern intracellular rheology in cancer, *Biophys. J.* **97**, 1013 (2009).
- [62] G. Higgins, F. Higgins, J. Peres, D. M. Lang, T. Abdalrahman, M. H. Zaman, S. Prince, and T. Franz, Intracellular mechanics and TBX3 expression jointly dictate the spreading mode of melanoma cells in 3D environments, *Exp. Cell Res.* **428**, 113633 (2023).
- [63] F. Spitz and E. E. M. Furlong, Transcription factors: From enhancer binding to developmental control, *Nat. Rev. Genet.* **13**, 613 (2012).
- [64] C. R. Clapier, J. Iwasa, B. R. Cairns, and C. L. Peterson, Mechanisms of action and regulation of ATP-dependent chromatin-remodelling complexes, *Nat. Rev. Mol. Cell Biol.* **18**, 407 (2017).
- [65] Q. Su, S. Mehta, and J. Zhang, Liquid-liquid phase separation: Orchestrating cell signaling through time and space, *Mol. Cell* **81**, 4137 (2021).
- [66] T. Hirose, K. Ninomiya, S. Nakagawa, and T. Yamazaki, A guide to membraneless organelles and their various roles in gene regulation, *Nat. Rev. Mol. Cell Biol.* **24**, 288 (2023).
- [67] S. N. Ricketts, J. L. Ross, and R. M. Robertson-Anderson, Co-entangled actin-microtubule composites exhibit tunable stiffness and power-law stress relaxation, *Biophys. J.* **115**, 1055 (2018).
- [68] F. Gittes, B. Mickey, J. Nettleton, and J. Howard, Flexural rigidity of microtubules and actin filaments measured from thermal fluctuations in shape., *J. Cell Biol.* **120**, 923 (1993).
- [69] M. Kikumoto, M. Kurachi, V. Tosa, and H. Tashiro, Flexural rigidity of individual microtubules measured by a buckling force with optical traps, *Biophys. J.* **90**, 1687 (2006).
- [70] F. Huber, A. Boire, M. P. López, and G. H. Koenderink, Cytoskeletal crosstalk: When three different personalities team up, *Curr. Opin. Cell Biol.* **32**, 39 (2015).
- [71] M. Schulz, S. Stepanow, and S. Trimper, Two harmonically coupled Brownian particles in random media, *Europhys. Lett.* **54**, 424 (2001).
- [72] H. J. Haubold, A. M. Mathai, and R. K. Saxena, Mittag-Leffler functions and their applications, *J. Appl. Math.* **2011**, 298628 (2011).
- [73] F. Mainardi, Why the Mittag-Leffler function can be considered the queen function of the fractional calculus? *Entropy* **22**, 1359 (2020).
- [74] H. Stehfest, Algorithm 368: Numerical inversion of Laplace transforms [D5], *Commun. ACM* **13**, 47 (1970).
- [75] J.-H. Jeon and R. Metzler, Inequivalence of time and ensemble averages in ergodic systems: Exponential versus power-law relaxation in confinement, *Phys. Rev. E* **85**, 021147 (2012).
- [76] K. Rippe and A. Papantonis, RNA polymerase II transcription compartments—from factories to condensates, *Nat. Rev. Genet.* **26**, 775 (2025).
- [77] E. Alghoul, J. Basbous, and A. Constantinou, Compartmentalization of the DNA damage response: Mechanisms and functions, *DNA Repair* **128**, 103524 (2023).
- [78] A. G. Larson, D. Elnatan, M. M. Keenen, M. J. Trnka, J. B. Johnston, A. L. Burlingame, D. A. Agard, S. Redding, and G. J. Narlikar, Liquid droplet formation by HP1 α suggests a role for phase separation in heterochromatin, *Nature (London)* **547**, 236 (2017).
- [79] M. Wachsmuth, W. Waldeck, and J. Langowski, Anomalous diffusion of fluorescent probes inside living cell nuclei investigated by spatially-resolved fluorescence correlation spectroscopy, *J. Mol. Biol.* **298**, 677 (2000).
- [80] A. Bancaud, S. Huet, N. Daigle, J. Mozziconacci, J. Beaudouin, and J. Ellenberg, Molecular crowding affects diffusion and binding of nuclear proteins in heterochromatin and reveals the fractal organization of chromatin, *EMBO J.* **28**, 3785 (2009).
- [81] M. K. Daddysman and C. J. Fecko, Revisiting point FRAP to quantitatively characterize anomalous diffusion in live cells, *J. Phys. Chem. B* **117**, 1241 (2013).
- [82] L. Meng, R. Zhang, L. Yu, and H. Wang, Tempered anomalous dynamics of globally coupled harmonic oscillators in the fluctuating potential field: Stability, synchronism, and collective behaviors, *Eur. Phys. J. Plus* **139**, 63 (2024).
- [83] M. C. Lim and M. J.-H. Jeon, Simulation files are freely available at <https://github.com/chanlimresearch/coupled-FLE-polymer-sim>.
- [84] M. V. Tamm, L. I. Nazarov, A. A. Gavrilov, and A. V. Chertovich, Anomalous diffusion in fractal globules, *Phys. Rev. Lett.* **114**, 178102 (2015).
- [85] E. Vanden-Eijnden and G. Ciccotti, Second-order integrators for Langevin equations with holonomic constraints, *Chem. Phys. Lett.* **429**, 310 (2006).
- [86] R. G. Winkler, P. Reineker, and L. Harnau, Models and equilibrium properties of stiff molecular chains, *J. Chem. Phys.* **101**, 8119 (1994).