

Quantum thermodynamic uncertainty relation and macroscopic superconducting coherence

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Stability and efficiency are mutually exclusive in a thermodynamic process, e.g., in a thermal machine. Any effort to reduce the fluctuations of a certain output quantity is necessarily accompanied by an increase of entropy production, therefore lowering its efficiency. This interplay is beautifully captured by the so-called thermodynamic uncertainty relations (TURs), which set a lower bound on the relative uncertainty of a current for a given rate of entropy production. Their status in hybrid normal-superconducting (N-S) devices has remained unsettled. We show that, in the subgap regime, departures from the normal quantum TUR are governed by *macroscopic* superconducting coherence quantified by the pair amplitude, and that introducing a dephasing probe suppresses this coherence and restores the bound. We further derive a hybrid quantum TUR that is general for two-terminal N-S junctions in the Andreev regime: the inequality is never violated, is saturated only at vanishing current, and is related to the normal quantum bound under the replacement $e \rightarrow 2e$. For N-S quantum dot and Cooper-pair-splitter systems, we compute current and noise and show that deviations from the normal bound track the pair amplitude on the central region. The results establish a direct link between superconducting macroscopic coherence and nonequilibrium fluctuations and supply a general bound for the Andreev regime.

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I. INTRODUCTION

The thermodynamic uncertainty relation (TUR) for classical systems governed by Markovian dynamics can be expressed as the inequality $\sigma \Delta I / (k_B I^2) \geq 2$ [1–4], where I is the current associated with a thermodynamic process, ΔI is its variance, σ is the entropy-production rate, and k_B is the Boltzmann constant. The study of this and related bounds has attracted a lot of attention recently to understand under which circumstances they can be violated, and consequently, to search for inequalities that account for different dynamical regimes. An account of this activity can be found in Refs. [5,6]. More specifically, TURs in electronic-transport setups have been considered in several papers both in the static [2,6–33] and time-dependent case [34,35].

For charge currents, it is convenient to re-express the TUR employing the Fano factor $F = S / (e|J|)$, in the form

$$\frac{F}{\mathcal{B}[2k_B|J|/(e\sigma)]} - 1 \geq 0, \quad (1)$$

where the function $\mathcal{B}[x]$ depends only on the dimensionless quantity $x = 2k_B|J|/(e\sigma)$, with J being the charge current, S the noise, σ the entropy-production rate, k_B the Boltzmann constant, and $e > 0$ is the magnitude of the electron charge. By simply imposing the positivity of the entropy-production rate in the linear response regime, the function $\mathcal{B}$ is found to take the form [1,12,21,24] $\mathcal{B}_{cl}(x) = x$, which is equivalent to the TUR reported above and originally derived for classical Markovian processes. The inequality (1) with the function $\mathcal{B}_{cl}$, which we will refer to as the *classical TUR*, is generally found to hold, but fails in the presence of quantum coherence. This is the case of quantum dot systems at strong coupling, i.e., when (nonclassical) high-order tunneling processes contribute to transport [7–9,13,15,16,19,20,28]. Such violations of the classical TUR are a telltale for the quantum-coherent nature of transport. A step forward, to include quantum-coherent effects in determining a bound, has been recently achieved in Ref. [33]. A different inequality, the *quantum TUR*, has been derived making use of the Landauer-Büttiker scattering theory of quantum transport. In this case, the function $\mathcal{B}$ is given by $\mathcal{B}_{qu}(x) = \text{cosech}(1/x)$ and is never violated for noninteracting electrons in a quantum-coherent conductor.

An interesting question now arises if the presence of *macroscopic coherence* adds additional features to the picture, i.e., whether the quantum TUR should be obeyed in superconducting systems, where coherence has a macroscopic origin related to the superconducting condensate of Cooper pairs. TURs in hybrid superconducting systems have been

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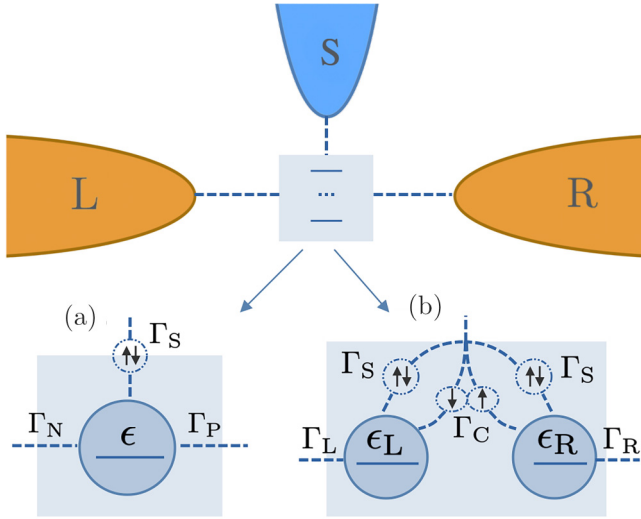


FIG. 1. A central region is coupled to a superconducting (S) lead with tunneling rate Γ_S , and to two normal leads $\eta = L, R$ with tunneling rates Γ_η , chemical potential μ_η , and kept at temperature T . (a) The central region corresponds to a quantum dot (with energy level ϵ) and one of the normal leads is considered as a probe coupled to the dot by a tunneling rate Γ_P and with a chemical potential μ_P that is adjusted so that the current flowing in P is zero. (b) The central region is a double quantum dot. The superconducting lead is also coupled to the dots through the nonlocal tunneling rate Γ_C .

considered in a few papers only [24,26–29]. Clear violations of the classical TUR have been demonstrated in such systems [26–28], where Andreev reflection plays a key role in exceeding this bound. What happens to the quantum inequality? Equivalently, how is the interplay between fluctuations and entropy production affected by the presence of macroscopic coherence?

The aim of this work is to answer this question by studying hybrid superconductor-normal systems and providing a general bound for this class of systems.

While single-particle coherence does not lead to any violation of Eq. (1) (in both cases), macroscopic coherence does. We will consider a system consisting of a superconducting lead and two normal leads coupled to a region containing localized levels, as shown in Fig. 1. By using a Green's function approach, we look for violations of both forms of TUR in the case where the central region is either a single [Fig. 1(a)] or two [Fig. 1(b)] quantum dots (QDs), to see also the effect of nonlocality introduced by the crossed Andreev reflection. Dephasing is introduced by using lead R as a voltage probe. In this case, the violation of the quantum TUR weakens by increasing the coupling to the probe, i.e., by suppressing the superconducting coherence, as measured by the superconducting pair amplitude of the QD. Our analysis is further supported by the derivation of an inequality that holds for two-terminal normal-superconducting junctions in the $|\Delta| \rightarrow \infty$ limit. This limit is commonly adopted since it effectively captures the essential physics of subgap transport dominated by coherent Andreev processes, and is equivalent to assuming temperatures and applied voltages much smaller than the superconducting gap [36].

II. MODEL AND FORMALISM

Let us consider a central region containing localized levels (QDs), coupled to one superconducting (S) lead and two normal leads $\eta = L, R$, as shown in Fig. 1. The normal leads are described by the Hamiltonians $H_\eta = \sum_{k,\sigma} \epsilon_k c_{\eta k \sigma}^\dagger c_{\eta k \sigma}$, where $c_{\eta k \sigma}$ is the annihilation operator for an electron with momentum k and spin σ in lead η . The single-particle energies in the leads are denoted by ϵ_k and we assume that are the same for both leads and independent of spin. The leads are coupled to the central region by means of the standard tunneling Hamiltonians $H_{\text{tunn},\eta} = \sum_{k,\sigma,i} V_{\eta,i} (c_{\eta k \sigma}^\dagger d_{i,\sigma} + \text{H.c.})$, where the label i denotes a localized state in the central region and will be omitted for the case of only one quantum dot, $d_{i\sigma}$ is the annihilation operator for an electron in level i , and without loss of generality, we take $V_{\eta,i}$ to be real. Both normal leads are kept at temperature T with chemical potentials μ_L and μ_R , respectively. The superconducting lead is grounded and we assume the gap Δ of the superconductor to be the largest energy scale ($|\Delta| \rightarrow \infty$ limit). In this limit, the quasiparticles in the superconductor do not contribute and the Andreev reflection processes between the superconductor and the central region can be included as an effective pairing term in the Hamiltonian of the central region [37–39].

The current J_η flowing from lead η into the central region can be expressed in terms of the full Green's functions of the central region in Nambu representation and the Fermi function of lead η [40]. The other ingredient needed for the TUR is the zero-frequency noise [41–43] in lead S, which is defined as $\mathcal{S} = \frac{1}{2} \int dt \langle J_S(t) J_S(0) + J_S(0) J_S(t) \rangle - \langle J_S \rangle \langle J_S \rangle$. If the interactions in the central region can be treated at the level of mean field, also the zero-frequency noise can be written in terms of Green's function of the central region in Nambu representation and the Fermi function of lead η . The expressions for the current, the noise, and the Green's functions of the central region can be found in the Supplemental Material [47]. Since the normal currents J_η are the only dissipative currents in the system, the entropy-production rate is simply given by $\sigma = -\sum_\eta \mu_\eta J_\eta / (eT)$, where $-\mu_\eta/e$ is the voltage applied to each normal lead.

III. EFFECT OF MACROSCOPIC COHERENCE ON TUR

The effects of macroscopic quantum coherence can be explored by modeling the central region in its minimal form as a single QD. The right normal lead will be used as a dephasing probe ($R = P$) by setting its chemical potential $\mu_R = \mu_P$ in such a way to nullify the current flowing through it. The left normal lead ($L = N$) is kept at a chemical potential $\mu_L = \mu_N$. We denote the tunneling rates between the leads P and N and the QD, as Γ_P and Γ_N , respectively. These rates are calculated in second order in $H_{\text{tunn},\eta}$ in the wideband limit.

In the $|\Delta| \rightarrow \infty$ limit, one can write an effective Hamiltonian for the QD that takes into account the coupling to the superconductor [37–39]:

$$H_{\text{QD}} = \epsilon \sum_\sigma d_\sigma^\dagger d_\sigma - \frac{\Gamma_S}{2} (d_\uparrow^\dagger d_\downarrow^\dagger + d_\downarrow d_\uparrow).$$

Here, ϵ represents the energy level of the QD. We start with the ideal system, where decoherence is absent, by setting $\Gamma_P = 0$.

The exact expressions for the current J flowing from the lead N into the QD and its zero-frequency noise are provided in the Supplemental Material [47].

Here, we examine the limit where $\Gamma_N \ll k_B T$, but making no assumptions on ϵ , Γ_S , and μ_N . This is in contrast to standard perturbative expansions in Γ_N as, e.g., in Ref. [23], where it is also assumed that Γ_N is much smaller than the splitting of the Andreev bound states, $\epsilon_A = \sqrt{\epsilon^2 + \frac{\Gamma_S^2}{4}}$, and therefore of Γ_S . The standard approach leads to expressions that are not valid in the region of parameter space where TUR violations occur. In the case considered here, the current and noise can be written as

$$J = -\frac{e}{h} \pi \frac{\Gamma_N \frac{\Gamma_S^2}{2}}{\bar{\Omega}^2} \text{Tr}[\tau_3 F_N^+(\epsilon_A)], \quad (2)$$

$$S = \frac{e^2}{h} \pi \frac{\Gamma_N \frac{\Gamma_S^2}{2}}{\bar{\Omega}^3} \left\{ (\bar{\Omega}^2 + \Gamma_N^2) \frac{\Gamma_S^2}{4} \text{Tr}[F_N^+(\epsilon_A) F_N^-(\epsilon_A)] \right. \\ \left. + \left[2\bar{\Omega}^4 - (\bar{\Omega}^2 + \Gamma_N^2) \frac{\Gamma_S^2}{4} \right] \text{Tr}[F_N^+(\epsilon_A) F_N^+(\epsilon_A)] \right\}, \quad (3)$$

where $\bar{\Omega}^2 = \epsilon_A^2 + \Gamma_N^2/4$ and τ_3 is the z Pauli matrix. The symbol $F_\eta^\pm(\omega)$ denotes the 2×2 diagonal matrices $F_\eta^\pm(\omega) = \text{Diag}[f^\pm(\omega - \mu_\eta), f^\mp(-\omega - \mu_\eta)]$, where $f^+(\omega) \equiv f(\omega)$ and $f^-(\omega) \equiv 1 - f(\omega)$ with $f(\omega)$ being the Fermi function at temperature T . In this limit, the Fano factor is minimized for $\epsilon = 0$ and $\Gamma_S = \sqrt{5/3} \Gamma_N$, taking a very simple form

$$F = \frac{9}{16} [1 - 2f(-\mu_N)] - \frac{1}{1 - 2f(-\mu_N)}, \quad (4)$$

when $\Gamma_N \ll \mu_N$ is further assumed. This expression breaks the quantum and classical bounds when

$$|\mu_N|/(k_B T) < 2 \ln[1/7(9 + 4\sqrt{2})] \approx 1.478 \quad (F < \mathcal{B}_{\text{qu}}), \\ |\mu_N|/(k_B T) \lesssim 4.074 \quad (F < \mathcal{B}_{\text{cl}}).$$

This shows that both the classical and quantum-coherent TUR can be violated for small values of Γ_N as long as $\Gamma_S \approx \Gamma_N$. This contrasts with what happens in the corresponding normal system, where the S lead is in the normal phase. In this case, violations of the classical TUR occur only at strong coupling, i.e., when (nonclassical) high-order tunneling processes contribute to transport [7,16].

To go beyond the small coupling regime, in Fig. 2 we plot the Fano factor for $\Gamma_N = 0.6k_B T$, obtained by numerical integration of the full expressions for J and S provided in the Supplemental Material [47]. In panel (a), F is plotted as a function of the level position and the tunnel-coupling ratio Γ_S/Γ_N for fixed $\mu_N = k_B T$, while in panel (b), F is plotted as a function of chemical potential and the tunnel-coupling ratio for fixed $\epsilon = 0$. The region of parameter space where both bounds are violated is colored in red, while in the yellow region only the classical bound is violated. According to panel (a), also in this situation, we find that both bounds are violated in a region of parameter space around $\epsilon = 0$ and similar coupling to the leads S and N. More interestingly, there is also a smaller region (red part of the surface) where the

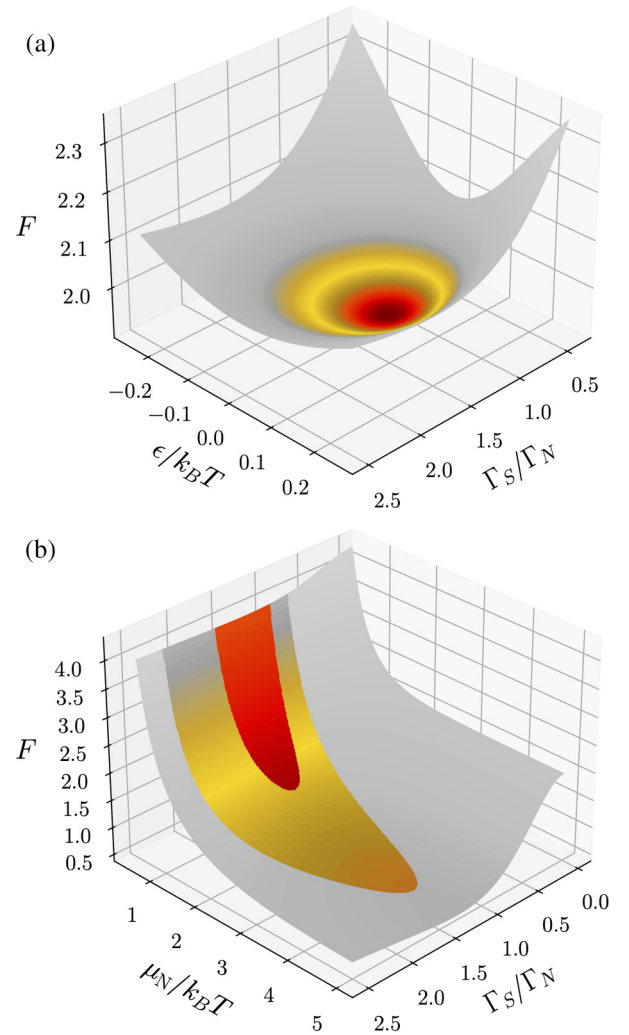


FIG. 2. Fano factor as a function of (a) ϵ and (b) μ_N and the ratio Γ_S/Γ_N . The red area shows the region in which the quantum bound $\mathcal{B}_{\text{qu}}$ is broken, and the yellow area corresponds to violations of the classical bound $\mathcal{B}_{\text{cl}}$. Parameters (when fixed): $\Gamma_N = 0.6k_B T$, $\mu_N = k_B T$, and $\epsilon = 0$.

quantum bound $\mathcal{B}_{\text{qu}}$ is also violated. We also find that there is a maximum value for the magnitude of the chemical potential to observe TUR violations [see panel (b) of Fig. 2]. This is trivially due to the fact that for large μ_N the entropy-production rate suppresses both bounds $\mathcal{B}_{\text{cl}}$ and $\mathcal{B}_{\text{qu}}$. Note that F is even in μ_N and diverges as μ_N goes to zero since the current J vanishes. We note that the Fano factor obtained from Eqs. (2) and (3) agrees quantitatively very well with the numerical results for $\Gamma_N \leq 0.1k_B T$. For larger values of Γ_N , as the one in Fig. 2, we still find a good qualitative agreement.

Physically, the violation of the quantum bound can be attributed to the presence of macroscopic quantum coherence due to the presence of the superconducting condensate. The superconducting correlations in the QD can be quantified by the pair amplitude $|\langle d_\downarrow d_\uparrow \rangle| = |\int \frac{d\omega}{2\pi} (\mathbf{G}^<(\omega))_{1,2}|$. From a mathematical point of view, the violation of the quantum bound in our system is not unexpected, as it was originally derived based on currents and noise expressions for a normal system. In contrast, in our case the expressions for current and

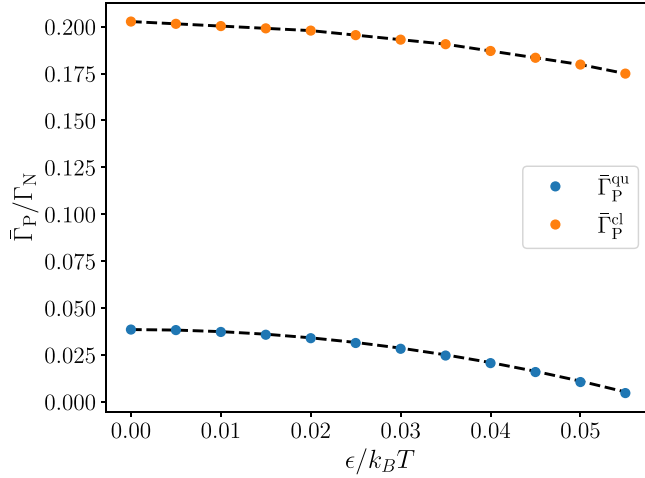


FIG. 3. The value of Γ_P for which the decoherence is strong enough that the quantum (blue) and classical (orange) TUR stop being violated as a function of ϵ . Parameters: $\Gamma_N = 0.6k_B T$, $\Gamma_S = \sqrt{5/3}\Gamma_N$, and $\mu_N = k_B T$.

noise have a different structure due to the coupling between electrons and holes induced by superconducting correlations.

To corroborate the fact that the violation of the quantum bound is due to the presence of macroscopic quantum coherence, we introduce a source of decoherence by adding a fictitious voltage probe, which reduces the presence of superconducting correlations as measured by the pair amplitude. The probe consists of a normal lead (denoted by P in Fig. 1) whose voltage is adjusted to make the current flowing through the lead vanish. To do so, we calculate the current in the probe J_P numerically for different values of μ_P and find the one that nullifies the current. The coupling strength of the probe is set by its tunneling rate Γ_P . We find that for small values of the probe coupling $\Gamma_P \ll \Gamma_S$, the pairing parameter is reduced by the noise as

$$|\langle d_\downarrow d_\uparrow \rangle| = |\langle d_\downarrow d_\uparrow \rangle|_{\Gamma_P=0} - \alpha \frac{\Gamma_P}{\Gamma_S},$$

where $\alpha > 0$ is a parameter that depends on Γ_N , ϵ , and μ_N . For the parameter values used for the figures, we have found $\alpha \approx 0.05$. As the probe introduces noise in the system, we find that both the quantum and classical TUR stop being violated. We define $\bar{\Gamma}_P^{\text{qu}}$ ($\bar{\Gamma}_P^{\text{cl}}$), the value of the probe's coupling for which decoherence is strong enough that the quantum (classical) bound is no longer broken. In Fig. 3, we show the values of $\bar{\Gamma}_P$ for both bounds. As expected, the presence of the noise makes both TUR violations disappear. We can see that the quantum bound is more sensible to this noise since the values of $\bar{\Gamma}_P^{\text{qu}}$ are always smaller than those of $\bar{\Gamma}_P^{\text{cl}}$, which makes sense considering that the violation of the quantum bound is always smaller than that of the classical one.

IV. NONLOCAL CORRELATION EFFECTS ON THE TUR

In order to investigate the effect of nonlocal correlations on the TUR, we model the central region as a Cooper-pair splitter (CPS), as shown in Fig. 1(b). The system consists of a double quantum dot symmetrically coupled to two normal

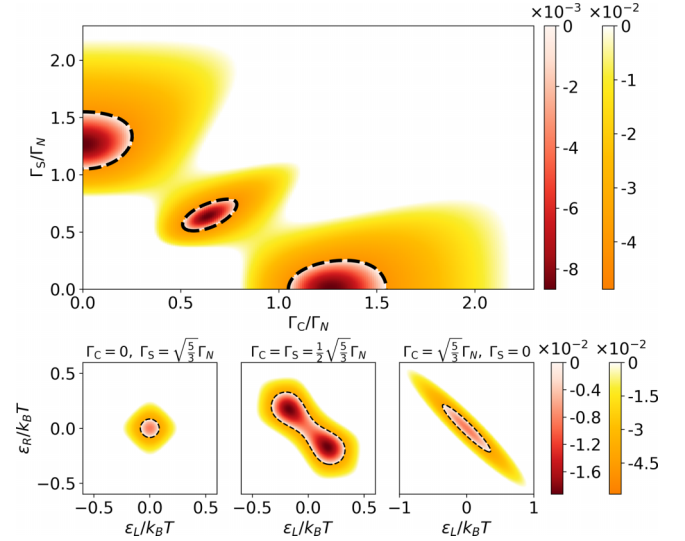


FIG. 4. Regions of violation of the classical (yellow) and quantum (red) TURs in the CPS system. The plotted quantities correspond to the left-hand side of Eq. (1), where the respective inequality is violated. The top panel displays the dependence on the tunneling rates Γ_S/Γ_N and Γ_C/Γ_N , while the bottom panels show the dependence on the energy levels ϵ_η for the three representative points in the top panel where the violations are maximal. Simultaneous LAR and CAR processes result in the maximal violation of the TURs. The black dashed line marks the saturation of the quantum bound. Parameters (when fixed): $\Gamma_N = 0.6k_B T$, $\epsilon_\eta = 0$, and $\mu_N = k_B T$.

leads ($\Gamma_L = \Gamma_R \equiv \Gamma_N$) and one superconducting lead. In the $|\Delta| \rightarrow \infty$ limit, the effective Hamiltonian takes the form

$$H_{\text{CPS}} = \sum_{\eta,\sigma} \epsilon_\eta d_{\eta,\sigma}^\dagger d_{\eta,\sigma} - \frac{\Gamma_S}{2} \sum_\eta (d_{\eta,\uparrow}^\dagger d_{\eta,\downarrow}^\dagger + \text{H.c.}) - \frac{\Gamma_C}{2} (d_{\text{R},\uparrow}^\dagger d_{\text{L},\downarrow}^\dagger - d_{\text{R},\downarrow}^\dagger d_{\text{L},\uparrow}^\dagger + \text{H.c.}),$$

where $\eta = \text{L, R}$ refers to the dots index. The last term describes crossed Andreev reflection, in which a Cooper pair is coherently split between the two dots, generating nonlocal superconducting correlations characterized by the rate Γ_C .

The combined influence of local (Γ_S) and nonlocal (Γ_C) tunneling processes on the quantum TUR for the CPS system is studied numerically for the current J_S flowing in lead S and presented in the top panel of Fig. 4. The violation of the classical and quantum TURs appears in three distinct regions around the points: $\Gamma_S = \sqrt{5/3}\Gamma_N$ and $\Gamma_C = 0$ (Region 1), $\Gamma_S = \Gamma_C = \frac{1}{2}\sqrt{5/3}\Gamma_N$ (Region 2), and $\Gamma_C = \sqrt{5/3}\Gamma_N$ and $\Gamma_S = 0$ (Region 3). For $\epsilon_L = \epsilon_R = 0$, the current J_S and the noise S_S are invariant under the exchange $\Gamma_S \leftrightarrow \Gamma_C$. This transformation interchanges the local (LAR) and crossed (CAR) Andreev reflection current contributions, leaving their sum unchanged.

In the bottom panels of Fig. 4, the violation of the quantum TUR as a function of the energy levels ϵ_η is presented for the three different regions. In Region 1 (Region 3), transport is mediated exclusively by LAR (CAR) process, and the Fano factor of the superconducting current J_S reaches its minimum in a finite range around the resonant condition $\epsilon_\eta = 0$

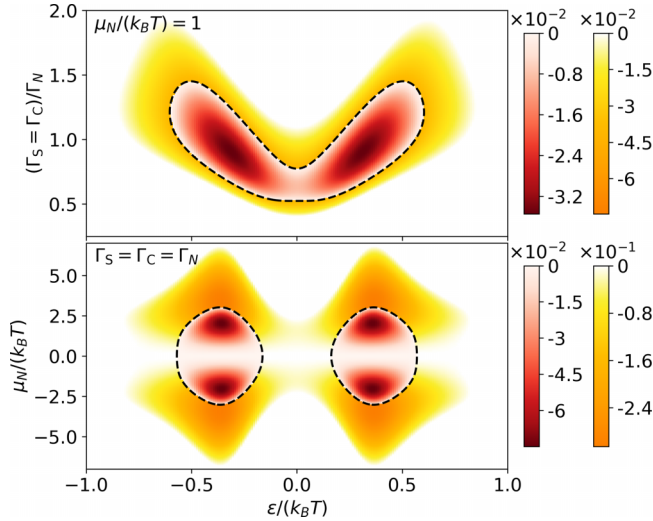


FIG. 5. Regions of violation of the classical (yellow) and quantum (red) TURs in the CPS system. The plotted quantities correspond to the left-hand side of Eq. (1), where the corresponding inequality is violated. The dependence on the energy level $\varepsilon \equiv \varepsilon_L = -\varepsilon_R$ and the tunneling rates $\Gamma_S/\Gamma_N = \Gamma_C/\Gamma_N$ is displayed in the top panel, while the dependence on the normalized chemical potential $\mu_N/(k_B T)$ is shown in the bottom panel. In the top panel, with $\mu_N = k_B T$ fixed, the largest departures from both bounds occur for nearly symmetric couplings, $\Gamma_S = \Gamma_C \approx \Gamma_N$, at $|\varepsilon| \simeq \sqrt{5/12} \Gamma_N$. In the bottom panel, fixing $\Gamma_S = \Gamma_C = \Gamma_N$, the classical TUR reaches a minimum of -0.3 around $|\mu_N|/(k_B T) = 3.4$, while the quantum TUR attains -0.075 near $|\mu_N|/(k_B T) = 2$ (both at $|\varepsilon| \simeq \sqrt{5/12} \Gamma_N$). Parameters (when fixed): $\Gamma_N = 0.6 k_B T$, $\mu_N = k_B T$, and $\Gamma_S = \Gamma_C = \Gamma_N$.

($\varepsilon_L = -\varepsilon_R$), where the corresponding Andreev reflection is maximal. In Region 2, the combined contribution of the LAR and CAR processes enhances the quantum-TUR violation by roughly a factor of 2.25 relative to Regions 1 and 3 at finite energy levels around $\varepsilon_{\eta} = -\varepsilon_{\bar{\eta}} \approx \Gamma_N/2\sqrt{3}$ with $\bar{\eta} \neq \eta$.

Figure 5 quantifies how the tunneling rates and the normal-lead chemical potential influence the degree of violation of the TURs along the CAR resonance line, $\varepsilon_L = -\varepsilon_R \equiv \varepsilon$. In the bottom panel, the shift between the classical and quantum TURs minima originates from the distinct chemical potential dependence entering the two bounds through the Fano factor: In the classical case, the violation grows approximately linearly with chemical potential, whereas the quantum bound, which incorporates the $\sinh(\mu_N/k_B T)$ dependence, saturates earlier.

V. HYBRID QUANTUM BOUND

For a two-terminal normal-superconducting junction in the $|\Delta| \rightarrow \infty$ limit with the superconducting terminal grounded, we use the scattering approach to derive the hybrid quantum bound $\mathcal{B}_{H,qu}(x) = 2 \operatorname{cosech}(2/x)$ for $x = 2k_B|J|/(e\sigma)$ (see the Supplemental Material [47] for the derivation). Consequently, the hybrid quantum TUR takes the form

$$\frac{S}{2e|J|} \sinh\left(\frac{e\sigma}{k_B|J|}\right) - 1 \geq 0, \quad (5)$$

which is never violated in the systems considered here and is saturated only in the limit of vanishing current. A

comparison of the hybrid quantum TUR of Eq. (5) with the normal quantum TUR shows that the former can be obtained from the normal quantum TUR with the replacement $e \rightarrow 2e$, reflecting the $2e$ charge transferred in Andreev processes. Equation (5) also holds for finite superconducting gap Δ , although it is not saturated in this case.

VI. CONCLUSIONS AND OUTLOOK

In this work, we demonstrate that the macroscopic coherence arising from superconducting correlations in hybrid systems causes a violation of the recently formulated quantum TUR [33]. The classical TUR is also violated, and to a greater extent, even in the weak coupling regime, in contrast to what happens in normal (nonsuperconducting) systems. We do this by calculating the Fano factor, using a Green's function method, for the minimal system consisting of a QD coupled to a normal and a superconducting lead. We find that the quantum TUR is violated in a smaller region of parameter space compared to the classical TUR, and it is more sensitive to decoherence. To understand the role of superconducting correlations, we introduce a decoherence probe. We find that this probe reduces the pairing amplitude in the QD, and for sufficiently strong decoherence, the TUR is no longer violated.

Macroscopic quantum coherence can also introduce non-local correlations. We find that in a Cooper-pair splitter, in which two QDs are coupled to the superconducting lead, these correlations can lead to a stronger violation of the quantum TUR.

To account for macroscopic superconducting coherence, we have derived a hybrid quantum TUR that captures subgap Andreev transport in two-terminal normal-superconducting junctions in the $|\Delta| \rightarrow \infty$ limit. In the setups studied here, the bound is saturated in the limit of vanishing current.

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DATA AVAILABILITY

The data supporting this study's findings are available within the article.

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