

# Reversible optical nonreciprocity and its scaling optimization in cold atoms via coherently induced four-wave mixing

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We present a further investigation on the optical nonreciprocity based on four-wave mixing (FWM) by leveraging coherently driven cold atoms, as a beneficial supplement to the usual cavity-waveguide solution in virtue of the on-demand tunability and broadband advantage. We consider two parameter regimes unexplored yet, where the probe and coupling fields are set with vanishing or large detunings while the dressing field is kept on resonance. Numerical results reveal that the forward (backward) probe exhibits a high transmittance (is strongly absorbed) in the resonant regime, but may be largely converted into an FWM field (propagate almost freely) in the far-detuned regime. The optimization of such a reversible unidirectional transmission featured by large isolation ratios and low insertion losses can be attained only when the coupling and dressing fields fulfill different scaling relations in amplitude, which has been well explained via analytical results. It is also viable to achieve reversible unidirectional transmission for a probe field of fixed large detunings by simply tuning the coupling field in a wide spectral range. Our findings provide additional insights into the FWM-based optical nonreciprocity and may enable more powerful unidirectional devices in quantum networks.

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## I. INTRODUCTION

As a crucial ingredient in optical information processing, optical nonreciprocity offers an efficient means to route light signals unidirectionally and also protect light sources from backscattering damage [1–3]. The most conventional approach for realizing optical nonreciprocity is to break the Lorentz reciprocity by generating circular birefringence via magneto-optical effects [4–6]. However, external magnetic fields and strong absorption in magnetic materials pose a great challenge in integrating nonreciprocal optical devices on chip with low insertion losses [7]. To overcome this challenge, various magnetic-free strategies have been proposed by adopting specially designed optical structures or modulation techniques, again in the regime of linear optical responses. Representative proposals include those based on spatiotemporal modulation of permittivity [8–11], optomechanical hybrid interactions [12–19], Bragg lattice in motion [20], synthetic angular momentum [21,22], modulated optical potential [23], non-Hermitian scattering [24–26], and atomic thermal motion [27–32]. Most of these proposals are limited in application scenarios due to certain complexity so that alternative solutions are always underpursued.

Meanwhile, much attention has been paid to breaking the Lorentz reciprocity via various optical nonlinearities like

Brillouin scattering, cross-Kerr effect, and four-wave mixing (FWM) [33–47], though these are typically very weak and just exist in specific solid materials. The basic physics of FWM-based solutions [42–47], for instance, lies in that FWM can occur for a signal propagating in one direction due to exact phase matching but is forbidden for a signal propagating in the opposite direction due to severe phase mismatch. A promising platform of such solutions is chip-scale hybrid systems that rely on high-Q ring cavities, in addition to nanofibers or waveguides facilitating light transport, to largely enhance light-matter interactions and thus involve complex and precision fabrications [42–44]. Resonances in these systems are sharp and once-made fixed such that relevant nonreciprocal devices exhibit limited adaptability and application scope due to narrow bandwidths and a missing tunability. This is why atomic systems in free space have been examined recently for inducing strong enough FWM nonlinearities around resonances to realize, e.g., broadband nonreciprocity and an on-demand tunability [45–47]. They may also promise the required miniaturization and integration of nonreciprocal devices since significant progress has been made in realizing on-chip cold or thermal atoms [48–51].

Resonantly enhanced nonlinearities are typically built on electromagnetically induced transparency (EIT) [52] in multilevel atoms. Various EIT effects have been proposed to achieve magnet- and cavity-free optical nonreciprocity or dissipation-induced transition between reciprocity and nonreciprocity [29–33] in combination with asymmetric Doppler shifts of thermal atoms for counterpropagating light signals. In contrast to Doppler-based schemes, FWM-based schemes can be realized in more compact cold atoms as well and are compatible with solid systems supporting EIT. To our knowledge, most FWM-based schemes are active and leverage parametric

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gain via continuous-wave laser excitation to compensate the loss of a signal propagating in one direction while leaving intrinsic absorption to a signal propagating in the opposite direction [42–45]. Passive FWM has also been explored in combination with a coexistent spontaneous parametric process for attaining extremely broadband optical nonreciprocity by virtue of pulsed laser excitation [46] or with a Mach-Zehnder interferometer for attaining both two-port isolators and four-port circulators [47].

In this work, we carry out a comprehensive investigation on the passive FWM-based optical nonreciprocity in four-level cold atoms in order to gain more physical insights. These atoms are in the double- $\Lambda$  configuration for a forward probe due to a converted FWM field, but in the  $N$ -type configuration for a backward probe in the absence of this FWM field. We consider two specific regimes that refer to (1) resonant probe, coupling, and dressing fields, or (2) resonant dressing but far-detuned probe and coupling fields. In case (1), the forward probe may exhibit a high transmittance due to negligible frequency conversion while the backward probe is severely attenuated due to strongly broken EIT. In case (2), the backward probe exhibits a high transmittance due to nearly decoupled interaction while the forward probe may be severely attenuated due to perfect frequency conversion.

It is of special interest that an optimized optical nonreciprocity requires the coupling and dressing fields to be scaled in amplitude by  $\sim 4$  in case (1) but  $\sim 1$  in case (2) as confirmed by approximate analytical expressions. Such scaling relations connecting simple microscopic laws with complex macroscopic phenomena allow us to achieve optimal control with less resources and are one of the goals that have been pursued all along in physics [53–55]. Moreover, we note that both cases deliver good nonreciprocal performance with isolation ratios being above 40 dB and insertion losses below 0.5 dB. Last but not least, the optical nonreciprocity in case (1) is reversed as compared to that in case (2) so that our atomic medium can be switched dynamically to achieve the unidirectional transmission of a forward resonant probe or a backward far-detuned probe. Further numerical calculations show that it is also viable to reverse the unidirectional transmission of probe fields of fixed detunings from forward to backward or vice versa in case (2) on demand. Clearly, our work differs essentially from all prior similar ones [42–47] by considering cold atoms driven into two specific parameter regimes of passive FWM frequency conversion in addition to revealing two nontrivial control capabilities of reversible isolation and scaling optimization.

## II. MODEL AND EQUATIONS

Two geometric arrangements of our model are shown in Figs. 1(a) and 1(b), both of which involve a coupling field of amplitude (wave vector)  $E_c(\mathbf{k}_c)$  and a dressing field of amplitude (wave vector)  $E_d(\mathbf{k}_d)$  propagating through a sample of cold atoms at angles  $\theta_c$  and  $\theta_d$ , respectively, relative to the  $z$  axis. We have also a forward probe field of amplitude (wave vector)  $E_p^f(\mathbf{k}_p^f)$  propagating along the  $z$  direction in Fig. 1(a) while a backward probe field of amplitude (wave vector)  $E_p^b(\mathbf{k}_p^b)$  propagating along the  $-z$  direction in Fig. 1(b). Moreover, a forward FWM field of amplitude (wave

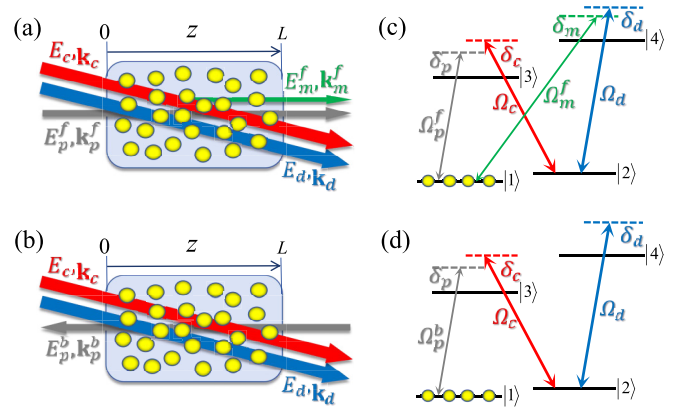


FIG. 1. Schematic diagrams for realizing optical nonreciprocity via directional phase matching of FWM occurring in cold atoms. (a) Geometric arrangement where a forward probe field  $\{E_p^f, \mathbf{k}_p^f\}$  facilitates the generation of a forward FWM field  $\{E_m^f, \mathbf{k}_m^f\}$  with the help of a coupling field  $\{E_c, \mathbf{k}_c\}$  and a dressing field  $\{E_d, \mathbf{k}_d\}$  due to phase matching. (b) Geometric arrangement where no FWM fields can be generated when a backward probe field  $\{E_p^b, \mathbf{k}_p^b\}$  is applied due to phase mismatch. (c) Double- $\Lambda$  configuration of four-level atoms interacting with a probe  $\{\Omega_p^f, \delta_p\}$ , a coupling  $\{\Omega_c, \delta_c\}$ , a dressing  $\{\Omega_d, \delta_d\}$ , and an FWM  $\{\Omega_m^f, \delta_m\}$  field. (d)  $N$ -type configuration of four-level atoms interacting with a probe  $\{\Omega_p^b, \delta_p\}$ , a coupling  $\{\Omega_c, \delta_c\}$ , and a dressing  $\{\Omega_d, \delta_d\}$  field.

vector)  $E_m^f(\mathbf{k}_m^f)$  can be generated due to exact phase matching  $|\mathbf{k}_p^f - \mathbf{k}_c + \mathbf{k}_d - \mathbf{k}_m^f| \rightarrow 0$  when the forward probe field is applied. It is however impossible to generate either a forward or a backward FWM field due to severe phase mismatch  $|\mathbf{k}_p^b - \mathbf{k}_c + \mathbf{k}_d - \mathbf{k}_m^{f,b}| \gg 0$  when the backward probe field is applied. The realization of above two assumptions depends on a reasonable choice of two ground states  $|1\rangle$  and  $|2\rangle$  and two excited states  $|3\rangle$  and  $|4\rangle$  for real atoms as shown in Figs. 1(c) and 1(d). In particular, transitions among these atomic states cannot exhibit identical wavelengths for achieving exact phase matching in the case of a forward probe field while severe phase mismatch in the case of a backward probe field, as discussed at the end of this section in detail.

For the geometric arrangement in Fig. 1(a), we can see from Fig. 1(c) that all atoms are in a double- $\Lambda$  configuration where the probe, coupling, dressing, and FWM fields of carrier frequencies  $\omega_p$ ,  $\omega_c$ ,  $\omega_d$ , and  $\omega_m$ , respectively, act upon transitions  $|1\rangle \leftrightarrow |3\rangle$ ,  $|2\rangle \leftrightarrow |3\rangle$ ,  $|2\rangle \leftrightarrow |4\rangle$ , and  $|1\rangle \leftrightarrow |4\rangle$  of resonant frequencies  $\omega_{31}$ ,  $\omega_{32}$ ,  $\omega_{42}$ , and  $\omega_{41}$ , respectively. These fields exhibit detunings  $\delta_p = \omega_p - \omega_{31}$ ,  $\delta_c = \omega_c - \omega_{32}$ ,  $\delta_d = \omega_d - \omega_{42}$ , and  $\delta_m = \omega_m - \omega_{41}$  as well as Rabi frequencies  $\Omega_p^f = \mu_{31}E_p^f/2\hbar$ ,  $\Omega_c = \mu_{32}E_c/2\hbar$ ,  $\Omega_d = \mu_{42}E_d/2\hbar$ , and  $\Omega_m^f = \mu_{41}E_m^f/2\hbar$ , with  $\mu_{mn}$  being the electric dipole moment on transition  $|n\rangle \leftrightarrow |m\rangle$ . For the geometric arrangement in Fig. 1(b), we have instead an  $N$ -type configuration for all atoms as can be seen from Fig. 1(d), where the FWM field is absent and the probe field exhibits a Rabi frequency  $\Omega_p^b = \mu_{31}E_p^b/2\hbar$ . The underlying physics is a nonlinear effect of directional phase matching, which is valid for the forward probe and FWM fields but impossible for the backward probe field and an FWM field propagating along any direction.

Based on all above considerations, under both electric-dipole and rotating-wave approximations, we can write down the following interaction Hamiltonian:

$$H_I^f = -\hbar[\Delta_1|3\rangle\langle 3| + \Delta_2|2\rangle\langle 2| + \Delta_3|4\rangle\langle 4| + (\Omega_p^f|3\rangle\langle 1| + \Omega_m^f|4\rangle\langle 1| + \Omega_c|3\rangle\langle 2| + \Omega_d|4\rangle\langle 2| + \text{H.c.})], \quad (1)$$

for the double- $\Lambda$  atomic system in Fig. 1(c) and that

$$H_I^b = -\hbar[\Delta_1|3\rangle\langle 3| + \Delta_2|2\rangle\langle 2| + \Delta_3|4\rangle\langle 4| + (\Omega_p^b|3\rangle\langle 1| + \Omega_c|3\rangle\langle 2| + \Omega_d|4\rangle\langle 2| + \text{H.c.})], \quad (2)$$

for the  $N$ -type atomic system in Fig. 1(d). Here, we have used H.c. to denote Hermitian conjugate while  $\Delta_1 = \delta_p$ ,  $\Delta_2 = \delta_p - \delta_c$ , and  $\Delta_3 = \delta_p - \delta_c + \delta_d$  to represent one-photon, two-photon, and three-photon detunings, in order. We have also considered  $\delta_m = \Delta_3$  in  $H_I^f$  because energy conservation is required in addition to phase matching to generate the FWM field  $\Omega_m^f$ . Taking  $H_I^{f,b}$  into the master equation of density operator  $\rho^{f,b}$  along with the Lindblad superoperator  $\mathcal{L}(\rho^{f,b})$  accounting for dissipation processes, we can obtain a set of dynamic equations for density matrix elements  $\rho_{mn}^{f,b}$  with  $m, n \in \{1, 2, 3, 4\}$ , governing optical responses of the atomic sample.

Below, we focus on a typical case where all atoms are initially prepared at state  $|1\rangle$ , and the probe and FWM fields keep much weaker than the coupling and dressing fields with  $|\Omega_{p,m}^f, \Omega_p^b| \ll |\Omega_c, \Omega_d|$ . Then, states  $|2\rangle$ ,  $|3\rangle$ , and  $|4\rangle$  will not be essentially populated at any time so that it is safe to assume  $\rho_{11}^{f,b} \simeq 1$ . In this case, density matrix equations for the double- $\Lambda$  atomic system become

$$\begin{aligned} \frac{\partial \rho_{21}^f}{\partial t} &= -(\gamma_{21} - i\Delta_2)\rho_{21}^f + i\Omega_c^*\rho_{31}^f + i\Omega_d^*\rho_{41}^f, \\ \frac{\partial \rho_{31}^f}{\partial t} &= -(\gamma_{31} - i\Delta_1)\rho_{31}^f + i\Omega_c\rho_{21}^f + i\Omega_p^f, \\ \frac{\partial \rho_{41}^f}{\partial t} &= -(\gamma_{41} - i\Delta_3)\rho_{41}^f + i\Omega_d\rho_{21}^f + i\Omega_m^f, \end{aligned} \quad (3)$$

restricted by  $\rho_{m1}^f = (\rho_{1m}^f)^*$  while density matrix equations for the  $N$ -type atomic system are reduced to

$$\begin{aligned} \frac{\partial \rho_{21}^b}{\partial t} &= -(\gamma_{21} - i\Delta_2)\rho_{21}^b + i\Omega_c^*\rho_{31}^b + i\Omega_d^*\rho_{41}^b, \\ \frac{\partial \rho_{31}^b}{\partial t} &= -(\gamma_{31} - i\Delta_1)\rho_{31}^b + i\Omega_c\rho_{21}^b + i\Omega_p^b, \\ \frac{\partial \rho_{41}^b}{\partial t} &= -(\gamma_{41} - i\Delta_3)\rho_{41}^b + i\Omega_d\rho_{21}^b, \end{aligned} \quad (4)$$

restricted by  $\rho_{m1}^b = (\rho_{1m}^b)^*$ . Above, we have introduced dephasing rates  $\gamma_{m1}$  for atomic coherences  $\rho_{m1}^{f,b}$  arising indeed from the Lindblad superoperator  $\mathcal{L}(\rho^{f,b})$ .

Optical responses of our atomic sample to the forward probe and FWM fields are determined by  $\rho_{31}^f$  and  $\rho_{41}^f$ , respectively, for the double- $\Lambda$  configuration. Steady-state solutions of the two density matrix elements can be easily obtained by

setting  $\partial \rho_{m1}^f / \partial t = 0$  in Eq. (3) as

$$\begin{aligned} \rho_{31}^f &= \frac{i\Omega_p^f(|\Omega_d|^2 + \gamma_{21}'\gamma_{41}') - i\Omega_c\Omega_d^*\Omega_m^f}{\gamma_{21}'\gamma_{31}'\gamma_{41}' + |\Omega_d|^2\gamma_{31}' + |\Omega_c|^2\gamma_{41}'}, \\ \rho_{41}^f &= \frac{i\Omega_m^f(|\Omega_c|^2 + \gamma_{21}'\gamma_{31}') - i\Omega_c^*\Omega_d\Omega_p^f}{\gamma_{21}'\gamma_{31}'\gamma_{41}' + |\Omega_d|^2\gamma_{31}' + |\Omega_c|^2\gamma_{41}'}. \end{aligned} \quad (5)$$

Similarly, by setting  $\partial \rho_{m1}^b / \partial t = 0$  in Eq. (4), we have

$$\rho_{31}^b = \frac{i\Omega_p^b(|\Omega_d|^2 + \gamma_{21}'\gamma_{41}')}{\gamma_{21}'\gamma_{31}'\gamma_{41}' + |\Omega_d|^2\gamma_{31}' + |\Omega_c|^2\gamma_{41}'}, \quad (6)$$

determining the optical response of our atomic sample to the backward probe field for the  $N$ -type configuration. Here, we have defined  $\gamma_{21}' = \gamma_{21} - i\Delta_2$ ,  $\gamma_{31}' = \gamma_{31} - i\Delta_1$ , and  $\gamma_{41}' = \gamma_{41} - i\Delta_3$  as complex dephasings for  $\rho_{21}^{f,b}$ ,  $\rho_{31}^{f,b}$ , and  $\rho_{41}^{f,b}$ , respectively. Clearly,  $\rho_{31}^f$  comprises a direct-coupling linear term  $\propto \Omega_p^f$  and a cross-coupling nonlinear term  $\propto \Omega_m^f$ , yet  $\rho_{31}^b$  retains only a direct-coupling linear term  $\propto \Omega_p^b$ . The presence and absence of a cross-coupling nonlinear term in  $\rho_{31}^f$  and  $\rho_{31}^b$ , respectively, manifest the effect of directional phase matching in our model. There are also a direct-coupling linear term  $\propto \Omega_m^f$  and a cross-coupling nonlinear term  $\propto \Omega_p^f$  in  $\rho_{41}^f$ , the latter of which answers for the mutual conversion of forward probe and FWM fields together with the term  $\propto \Omega_m^f$  in  $\rho_{31}^f$ .

Under the slowly varying envelope approximation, the Maxwell equations describing propagation dynamics of the forward probe and FWM fields become

$$\begin{aligned} \frac{1}{c} \frac{\partial \Omega_p^f(z, t)}{\partial t} + \frac{\partial \Omega_p^f(z, t)}{\partial z} &= i \frac{\alpha_p \gamma_{31}}{2L} \rho_{31}^f, \\ \frac{1}{c} \frac{\partial \Omega_m^f(z, t)}{\partial t} + \frac{\partial \Omega_m^f(z, t)}{\partial z} &= i \frac{\alpha_m \gamma_{41}}{2L} \rho_{41}^f, \end{aligned} \quad (7)$$

while that for the backward probe field is

$$\frac{1}{c} \frac{\partial \Omega_p^b(z, t)}{\partial t} - \frac{\partial \Omega_p^b(z, t)}{\partial z} = i \frac{\alpha_p \gamma_{31}}{2L} \rho_{31}^b. \quad (8)$$

In these equations,  $\alpha_p = NL\sigma_{13}$  [ $\alpha_m = NL\sigma_{14}$ ] is defined as the optical depth for the probe (FWM) field with  $N$  being the atomic density,  $L$  the sample length, and  $\sigma_{13} = (2\pi\mu_{13}^2)/(\hbar\epsilon_0\lambda_p\gamma_{31})$  [ $\sigma_{14} = (2\pi\mu_{14}^2)/(\hbar\epsilon_0\lambda_m\gamma_{41})$ ] the absorption cross section on the  $|1\rangle \leftrightarrow |3\rangle$  [ $|1\rangle \leftrightarrow |4\rangle$ ] transition. Setting  $\partial \Omega_{p,m}^f / \partial t = 0$  in the steady state and taking  $\Omega_{p,m}^f(z, \infty) = \Omega_{p,m}^f(z)$  in Eq. (7) for simplicity, we can obtain the output probe and FWM fields

$$\begin{aligned} \Omega_p^f(L) &= \Omega_p^f(0)(A_+^f e^{iK_+^f} - A_-^f e^{iK_-^f}), \\ \Omega_m^f(L) &= \Omega_p^f(0)B_0^f(e^{iK_+^f} - e^{iK_-^f}), \end{aligned} \quad (9)$$

at  $z = L$ . With  $\partial \Omega_p^b / \partial t = 0$  and  $\Omega_p^b(z, \infty) = \Omega_p^b(z)$ , we can obtain from Eq. (8) the output probe field

$$\Omega_p^b(0) = \Omega_p^b(L)e^{iK_-^b}, \quad (10)$$

at  $z = 0$ . Taking  $\Omega_c$  and  $\Omega_d$  as real parameters without the loss of generality, above we have defined

$$\begin{aligned} K_{\pm}^f &= \frac{\pm\sqrt{G} - \alpha_m D_m - \alpha_p D_p}{4D}, \\ A_{\pm}^f &= \frac{\alpha_m D_m - \alpha_p D_p \pm \sqrt{G}}{2\sqrt{G}}, \\ B_0^f &= \frac{\alpha_m \gamma_{41} \Omega_c \Omega_d}{\sqrt{G}}, \\ K_0^b &= -\alpha_p D_p / 2D, \end{aligned} \quad (11)$$

with  $D_p = \gamma_{31}(\Omega_d^2 + \gamma_{21}'\gamma_{41}')$ ,  $D_m = \gamma_{41}(\Omega_c^2 + \gamma_{21}'\gamma_{31}')$ ,  $D = i\gamma_{21}'\gamma_{31}'\gamma_{41}' + i\Omega_d^2\gamma_{31}' + i\Omega_c^2\gamma_{41}'$ , and  $G = (\alpha_m D_m - \alpha_p D_p)^2 + 4\alpha_p\gamma_{31}\alpha_m\gamma_{41}\Omega_c^2\Omega_d^2$  under consideration.

Our main concern lies in the realization and optimization of nonreciprocal transmission behaviors for a forward and a backward probe field. Hence, below we further introduce two transmission coefficients defined as

$$\begin{aligned} t_p^f &= \zeta_p^f e^{i\phi_p^f} = A_+^f e^{iK_+^f} - A_-^f e^{iK_-^f}, \\ t_p^b &= \zeta_p^b e^{i\phi_p^b} = e^{iK_0^b}, \end{aligned} \quad (12)$$

with  $\zeta_p^{f,b}$  representing transmission amplitudes while  $\phi_p^{f,b}$  denoting phase shifts of the forward and backward probe fields. Distinct expressions of  $t_p^f$  and  $t_p^b$  clearly show that the directional phase matching in our model can break the Lorentz reciprocity to result in  $\zeta_p^f \neq \zeta_p^b$  and  $\phi_p^f \neq \phi_p^b$ . The degree of such a nonreciprocity will be evaluated in the next section by amplitude difference  $\Delta\zeta = \zeta_p^f - \zeta_p^b$ , relative phase  $\Delta\phi = \phi_p^f - \phi_p^b$ , contrast ratio  $C = (\zeta_p^f - \zeta_p^b)/(\zeta_p^f + \zeta_p^b)$ , isolation ratio  $IR = -10 \log_{10} |\zeta_p^f / \zeta_p^b|^2$ , and insertion loss  $IL = -10 \log_{10} |\zeta_p^{f,b}|^2$  [33].

To be more specific, now we consider cold  $^{87}\text{Rb}$  atoms as an example with states  $|1\rangle$ ,  $|2\rangle$ ,  $|3\rangle$ , and  $|4\rangle$  referring to  $|5^2S_{1/2}, F=2, m_F=0\rangle$ ,  $|5^2S_{1/2}, F=1, m_F=0\rangle$ ,  $|5^2P_{1/2}, F=2, m_F=-1\rangle$ , and  $|5^2P_{3/2}, F=2, m_F=1\rangle$ , respectively. Then, the probe and coupling transitions of a common  $\sigma^-$  polarization are on the D<sub>1</sub> line with wavelengths  $\lambda_{p,c} \simeq 795$  nm, dipole moments  $\mu_{13,14} \simeq 1.268 \times 10^{-29}$  C m, and dephasing rates  $\gamma_{31,32} \simeq 2\pi \times 5.75$  MHz, while the FWM and dressing transitions of a common  $\sigma^+$  polarization are on the D<sub>2</sub> line with wavelengths  $\lambda_{m,d} \simeq 780$  nm, dipole moments  $\mu_{23,24} \simeq 1.267 \times 10^{-29}$  C m, and dephasing rates  $\gamma_{41,42} \simeq 2\pi \times 6.06$  MHz [56]. We further assume  $\gamma_{21} = 2\pi \times 2$  kHz for a cold enough atomic sample and can find  $\alpha = \alpha_p \simeq \alpha_m$  by taking values of  $\mu_{13,14}$ ,  $\lambda_{p,m}$ , and  $\gamma_{31,41}$  into expressions of  $\sigma_{13,14}$ . Note, in particular, that  $\lambda_p \simeq \lambda_c \neq \lambda_m \simeq \lambda_d$  is crucial for achieving  $|\mathbf{k}_p^f - \mathbf{k}_c + \mathbf{k}_d - \mathbf{k}_m^f| \rightarrow 0$  in the case of a forward probe and  $|\mathbf{k}_p^b - \mathbf{k}_c + \mathbf{k}_d - \mathbf{k}_m^{b,f}| \gg 0$  in the case of a backward probe. Otherwise, it is possible to have  $|\mathbf{k}_p^b - \mathbf{k}_c + \mathbf{k}_d - \mathbf{k}_m^f| \rightarrow 0$  so that a backward probe may be converted into a backward FWM field if we choose magnetic sublevels only on the D<sub>1</sub> line. Finally, frequency difference  $\omega_{12} = 6.834$  GHz between states  $|1\rangle$  and  $|2\rangle$  requires  $\theta_c = 6.03$  mrad for getting  $|\mathbf{k}_p^f| = |\mathbf{k}_c| \cos \theta_c$  while  $\theta_d = 5.95$  mrad for getting  $|\mathbf{k}_m^f| = |\mathbf{k}_d| \cos \theta_d$ , which is an ideal case equivalent to phase matching for completely overlapping probe

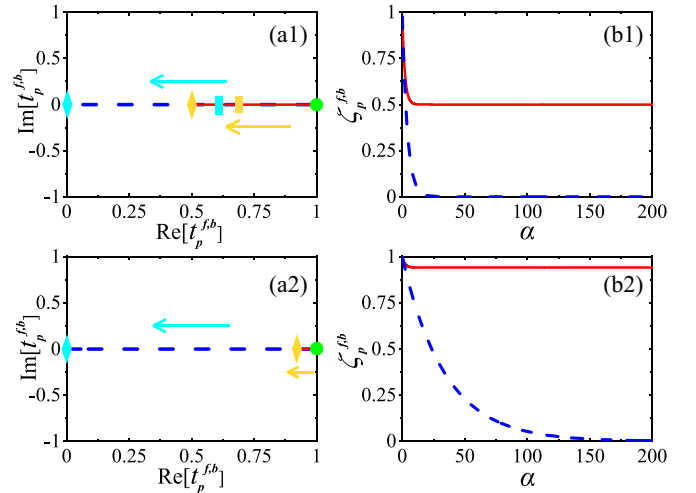


FIG. 2. (a1), (a2) Phase diagrams of transmission coefficients  $t_p^f$  (red solid) and  $t_p^b$  (blue dashed) for optical depth  $\alpha$  increased from 0 to 200. (b1), (b2) Transmission amplitudes  $\zeta_p^f$  (red solid) and  $\zeta_p^b$  (blue dashed) against  $\alpha$  in the range of  $\{0, 200\}$ . Parameters are  $\delta_p = \delta_c = \delta_d = 0$  and  $\Omega_c = \Omega_d = 20\gamma_{31}$  for (a1), (b1) while  $\Omega_c = 4\Omega_d = 20\gamma_{31}$  for (a2), (b2) except  $\gamma_{21}$ ,  $\gamma_{31}$ ,  $\gamma_{41}$ ,  $\theta_c$ , and  $\theta_d$  specified at the end of Sec. II.

and FWM fields. Phase matching occurs also for not strictly copropagating probe and FWM fields, e.g., when  $\theta_c$  and  $\theta_d$  are jointly misaligned up to 4.0 mrad. This small angular misalignment will result in a negligible transverse shift between the probe and FWM fields and is feasible for current techniques of stabilizing multiple intersecting light beams in relevant cold-atom experiments [57,58].

### III. RESULTS AND DISCUSSION

Here, we employ the formulas derived in the last section to examine the realization and optimization of nonreciprocal transmission with respect to a forward and a backward probe field. Numerical calculations and analytical discussion will be performed in two cases where (1) all three applied fields are kept on resonance with respective transitions, or (2) the dressing field is resonant but the probe and coupling fields are largely detuned. This aims at exploring different rules for optimizing nonreciprocal transmission in a broad range of the probe detuning so as to attain a more precise and efficient control.

#### A. Resonant probe and coupling fields

Focusing on case (1) with  $\delta_p = \delta_c = \delta_d = 0$ , we show in Figs. 2(a1) and 2(a2) the transmission phase diagrams of both forward and backward probe fields for two sets of parameters, as the optical depth  $\alpha$  varies in the range of  $\{0, 200\}$ . It is clear that the imaginary parts of  $t_p^f$  and  $t_p^b$  remain zero while their real parts decrease leftward, as indicated by the arrows, to different extents starting from an initial value 1.0 (green circle) with the gradual increase of  $\alpha$ . In particular, we find from Fig. 2(a1) that the real parts of  $t_p^f$  and  $t_p^b$  decrease, respectively, first to 0.69 (yellow square) and 0.61 (cyan square) for a smaller  $\alpha$  and then to 0.5 (yellow diamond) and 0 (cyan

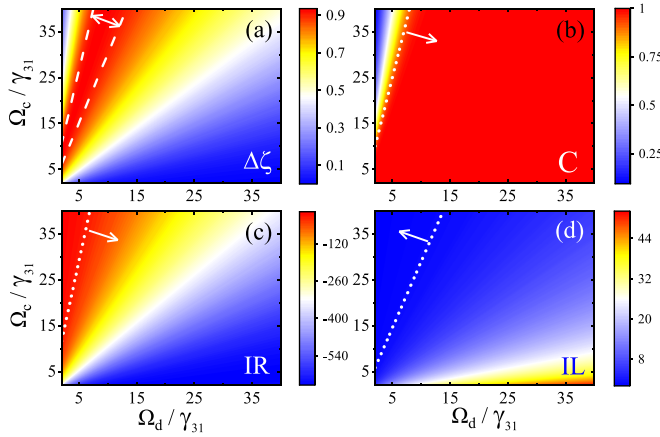


FIG. 3. Amplitude difference  $\Delta\zeta$  [panel (a)], contrast ratio  $C$  [panel (b)], isolation ratio  $IR$  [panel (c)], and insertion loss  $IL$  [panel (d)] of probe field against Rabi frequencies  $\Omega_c$  and  $\Omega_d$  of coupling and dressing fields, respectively. Two white dashed lines in panel (a) mark out the region in which  $\Delta\zeta \geq 0.9$ . White dotted lines refer to  $C \geq 90\%$  in panel (b) or  $IR \leq -20$  dB in panel (c) with a rightward arrow while to  $IL \leq 0.97$  dB in panel (d) with a leftward arrow. Other parameters are the same as in Fig. 2 except  $\alpha = 160$ .

diamond) for a larger  $\alpha$ , in the case of  $\Omega_c = \Omega_d$ . Hence, it is viable to realize a rather good transmission nonreciprocity for a sufficiently dense and long atomic sample. Keep in mind, however, that the optimal nonreciprocity should be featured by a unitary forward while a vanishing backward probe transmission or vice versa. We can get very close to such an ideal state in another case of  $\Omega_c = 4\Omega_d$  as shown by Fig. 2(a2) where the real parts of  $t_p^f$  and  $t_p^b$  approach 0.94 (yellow diamond) and 0 (cyan diamond), respectively, for a large enough  $\alpha$ . To have a more direct comparison, we plot  $\zeta_p^f$  and  $\zeta_p^b$  against  $\alpha$  in Figs. 2(b1) and 2(b2) corresponding to Figs. 2(a1) and 2(a2), respectively. It is not difficult to find that  $\zeta_p^f$  ( $\zeta_p^b$ ) finally stabilizes at 0.5 (0) for  $\alpha \gtrsim 12.5$  (25) in Fig. 2(b1) while at 0.94 (0) for  $\alpha \gtrsim 12.5$  (160) in Fig. 2(b2).

Above discussions show that the nonreciprocal performance depends critically on the ratio of Rabi frequencies  $\Omega_c$  and  $\Omega_d$  of the coupling and dressing fields. In this regard, we subsequently plot in Fig. 3 amplitude difference  $\Delta\zeta$ , contrast ratio  $C$ , isolation ratio  $IR$ , and insertion loss  $IL$  against both  $\Omega_c$  and  $\Omega_d$  in the range of  $\{2, 40\}\gamma_{31}$ . In Fig. 3(a), two white dashed lines are used to demarcate the region where  $\Delta\zeta$  exceeds the reference criterion 0.90 and takes the largest value 0.94 for  $\Omega_c/\Omega_d \simeq 4$ . This region locates clearly on the right side of the white dotted line in Fig. 3(b) and is accompanied by  $92\% < C < 98\%$  in fact. In terms of the isolation ratio, we expect to have  $IR < -20$  dB corresponding to  $|\zeta_p^f/\zeta_p^b|^2 > 100$  as indicated by the white dotted line with a rightward arrow in Fig. 3(c). As for the insertion loss, we expect to have  $IL < 0.97$  dB corresponding to  $|\zeta_p^f|^2 > 0.8$  as indicated by the white dotted line with a leftward arrow in Fig. 3(d). Taking these figures of merit together, we can say that it is viable to achieve a good transmission nonreciprocity in the range bounded by the two dashed lines in Fig. 3(a), where the optimal results refer to  $\Delta\zeta = 0.93$ ,  $C = 96\%$ ,  $IR = -40$  dB, and  $IL = 0.53$  dB for  $\Omega_c/\Omega_d \simeq 4$ , demonstrating a largely

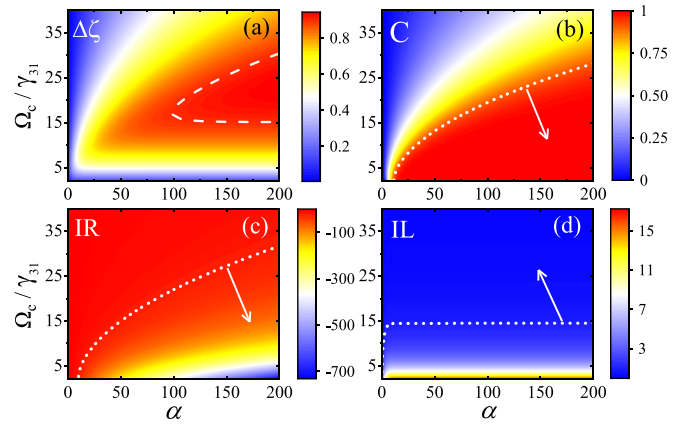


FIG. 4. Amplitude difference  $\Delta\zeta$  [panel (a)], contrast ratio  $C$  [panel (b)], isolation ratio  $IR$  [panel (c)], and insertion loss  $IL$  [panel (d)] of probe field against Rabi frequency  $\Omega_c$  and optical depth  $\alpha$ . The white dashed line in panel (a) marks out a parabolalike region in which  $\Delta\zeta \geq 0.9$ . White dotted lines refer to  $C \geq 90\%$  in panel (b) or  $IR \leq -20$  dB in panel (c) with a downward arrow while to  $IL \leq 0.97$  dB in panel (d) with an upward arrow. Other parameters are the same as in Fig. 2 except  $\Omega_d = 5\gamma_{31}$ .

enhanced nonreciprocal performance under refined parameter configurations.

We further present in Fig. 4 the amplitude difference  $\Delta\zeta$ , contrast ratio  $C$ , isolation ratio  $IR$ , and insertion loss  $IL$  as functions of  $\alpha$  and  $\Omega_c$  for a fixed  $\Omega_d$ . It is easy to find from Fig. 4(a) that, only in the case of  $\alpha > 100$ , we can obtain  $\Delta\zeta > 0.90$  in a parabolalike region as determined by the white dashed line, provided  $\Omega_c$  is neither too small nor too large as compared to  $4\Omega_d$ . Three white dotted lines are shown in Figs. 4(b)–4(d) to mark the boundaries beyond which we have  $C > 90\%$ ,  $IR < -20$  dB, and  $IL < 0.97$  dB, respectively, along relevant directions indicated by different arrows. An evident overlap between the parabolalike region and the regions bounded by white dotted lines confirms that the four criteria mentioned above can be simultaneously satisfied for proper control parameters. In other words, these numerical results characterize the interplay between atomic optical depth and coupling field strength for a sufficiently strong dressing field, allowing us to identify the parameters corresponding to optimized transmission nonreciprocity. All above findings can be well explained by reducing Eqs. (9)–(11) with reasonable assumptions to obtain analytical results, which will be given in the next subsection to facilitate a direct comparison between cases (1) and (2).

## B. Detuned probe and coupling fields

Now, we transfer to case (2) where  $\delta_p$  and  $\delta_c$  deviate largely from zero while  $\delta_d$  keeps vanishing for achieving nonreciprocity not only in transmission amplitudes  $\zeta_p^{f,b}$  but also in phase shifts  $\phi_p^{f,b}$ . Taking  $\delta_p = \delta_c = 40\gamma_{31}$ , for instance, we first present in Fig. 5(a) the polar plots of both forward and backward probe fields as  $\alpha$  varies in the range of  $\{0, 350\}$ . Starting from a common point (1, 0) (green circle) but following different trajectories as indicated by relevant arrows, the forward and backward probe fields are found to exhibit

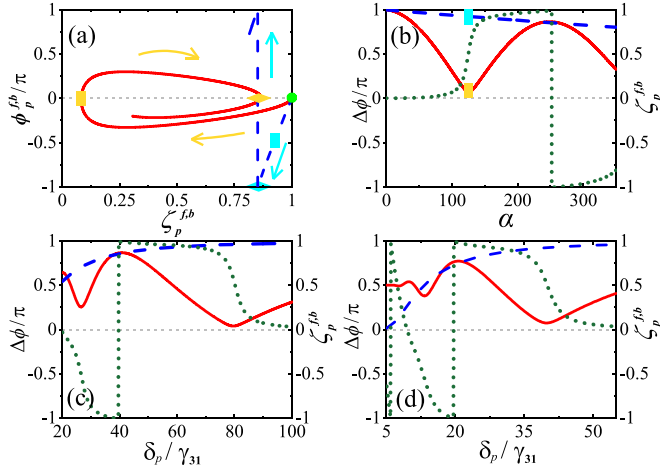


FIG. 5. (a) Phase diagrams of transmission coefficients  $t_p^f$  (red solid) and  $t_p^b$  (blue dashed) for  $\alpha$  increased from 0 to 350 with  $\delta_p = 40\gamma_{31}$ . Transmission amplitudes  $\zeta_p^f$  (red solid) and  $\zeta_p^b$  (blue dashed) as well as relative phase  $\Delta\phi$  (green dotted) against  $\alpha$  with  $\delta_p = 40\gamma_{31}$  in panel (b) but against  $\delta_p$  with  $\alpha = 250$  in panel (c) or  $\alpha = 125$  in panel (d). Other parameters are the same as in Fig. 2 except  $\delta_c = \delta_p$  and  $\Omega_c = \Omega_d = 20\gamma_{31}$ .

strong nonreciprocity in both transmission amplitudes and phase shifts. We have, in particular,  $\zeta_p^f = 0.07$  and  $\phi_p^f = 0$  (yellow square) for the forward probe while  $\zeta_p^b = 0.92$  and  $\phi_p^b = -0.5\pi$  (cyan square) for the backward probe when  $\alpha$  is tuned to 125, indicating a strong nonreciprocity in both transmission amplitude and phase shift. Moreover, we have  $\zeta_p^f = 0.86$  and  $\phi_p^f = 0$  (yellow diamond) for the forward probe while  $\zeta_p^b = 0.85$  and  $\phi_p^b = -\pi$  (cyan diamond) for the backward probe when  $\alpha$  is tuned to 250, indicating instead a pure nonreciprocal phase shift with high transmittance. The two instances are better substantialized in Fig. 5(b) where  $\zeta_p^f$ ,  $\zeta_p^b$ , and  $\Delta\phi$  are plotted against  $\alpha$ . We can see that  $\zeta_p^f$  exhibits a strong oscillation, while  $\zeta_p^b$  reduces slowly, with the increase of  $\alpha$ . As for  $\Delta\phi$ , it experiences an abrupt change when  $\zeta_p^f$  turns from decreasing to increasing and changes just slightly otherwise.

We then examine  $\zeta_p^f$ ,  $\zeta_p^b$ , and  $\Delta\phi$  against  $\delta_p = \delta_c$  in Figs. 5(c) and 5(d) for two specific values of  $\alpha$ . It is clear that, when the optical depth is halved from 250 to 125, the probe and coupling detunings corresponding to a pure nonreciprocal phase shift reduce also by half from  $40\gamma_{31}$  to  $20\gamma_{31}$ . That means, we have discovered an interesting pattern: The probe and coupling detunings for achieving a pure nonreciprocal phase shift scales linearly with the optical depth. Note also that  $\zeta_p^f$  ( $\zeta_p^b$ ) reduces evidently from 0.86 (0.85) in the case of  $\alpha = 250$  to 0.76 (0.72) in the case of  $\alpha = 125$ . Hence, too small optical depths should be avoided as they must lead to the degradation in transmission performance. Moreover, we note that a low forward transmittance with  $\zeta_p^f \rightarrow 0$  and a high backward transmittance with  $\zeta_p^b \rightarrow 1$  can be achieved only when  $\delta_p = \delta_c$  are large enough and take specific values because  $\zeta_p^f$  also oscillates against the two detunings.

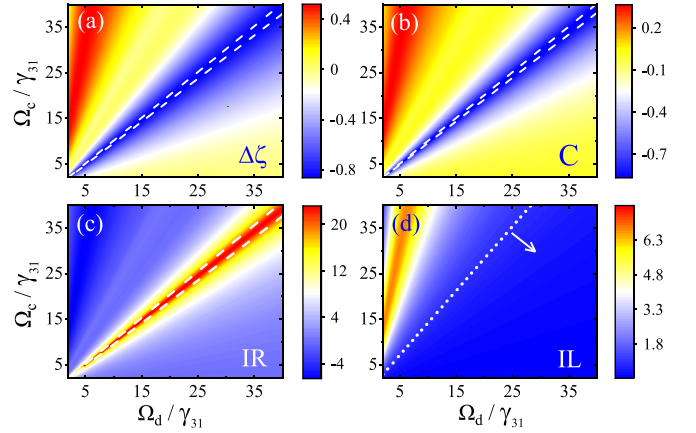


FIG. 6. Amplitude difference  $\Delta\zeta$  [panel (a)], contrast ratio  $C$  [panel (b)], isolation ratio IR [panel (c)], and insertion loss IL [panel (d)] of probe field against Rabi frequencies  $\Omega_c$  and  $\Omega_d$  of coupling and dressing fields, respectively. Two white dashed lines mark out a conical region with  $\Delta\zeta \leq -0.85$  in panel (a);  $C \leq -85\%$  in panel (b);  $IR \geq 20$  dB in panel (c). White dotted line in panel (d) refers to  $IL \leq 0.97$  dB with a downward arrow. Other parameters are the same as in Fig. 5 except  $\delta_p = \delta_c = 40\gamma_{31}$  and  $\alpha = 125$ .

A more comprehensive examination of the transmission nonreciprocity for largely detuned probe and coupling fields is given in Fig. 6 by plotting amplitude difference  $\Delta\zeta$ , contrast ratio  $C$ , isolation ratio IR, and insertion loss IL against Rabi frequencies  $\Omega_c$  and  $\Omega_d$ . Again, we find that the ratio of coupling and dressing Rabi frequencies for achieving optimal nonreciprocal transmission remains linear or constant, but it evidently changes from  $\Omega_c/\Omega_d \simeq 4$  in the case of  $\delta_p = \delta_c = 0$  to  $\Omega_c/\Omega_d \simeq 1$  in the case of  $\delta_p = \delta_c = 40\gamma_{31}$ . Along the line determined by  $\Omega_c/\Omega_d = 1$ , relevant plots show that the optimal value of amplitude difference is  $\Delta\zeta = -0.86$  while other three figures of merit seem also quite promising with contrast ratio, isolation ratio, and insertion loss being  $C = -86\%$ ,  $IR = 22$  dB, and  $IL = 0.66$  dB, respectively. Of course, these figures of merit become more sensitive to the ratio of  $\Omega_c/\Omega_d$  manifested as a narrower strip restricted by two white dashed lines, hence requiring the coupling and dressing fields to keep identical and stable strengths.

### C. Approximate analytical discussion

Next, we take  $\gamma_{21} = 0$ ,  $\gamma_{31} = \gamma_{41} = \gamma$ ,  $\alpha_p = \alpha_m = \alpha$ , and  $\eta = \Omega_c^2/(\Omega_c^2 + \Omega_d^2)$  for simplicity to better understand the results in Figs. 2 and 3. In the case of  $\delta_p = \delta_c = 0$ , it is easy to obtain  $D_p = \gamma\Omega_d^2$ ,  $D_m = \gamma\Omega_c^2$ ,  $D = i\gamma(\Omega_c^2 + \Omega_d^2)$ , and  $\sqrt{G} = i\alpha D$ , which further yields  $K_+^f = 0$ ,  $K_-^f = i\alpha/2$ ,  $A_+^f = \eta$ ,  $A_-^f = \eta - 1$ , and  $K_0^b = i(1 - \eta)\alpha/2$ . Taking these into Eq. (12), we have  $\phi_p^f = \phi_p^b = 0$  and

$$\begin{aligned}\zeta_p^f &= \eta + (1 - \eta)e^{-\alpha/2}, \\ \zeta_p^b &= e^{-(1-\eta)\alpha/2},\end{aligned}\quad (13)$$

which explains why  $\zeta_p^b$  decays a little (much) slower than  $\zeta_p^f$  as  $\alpha$  increases with the former and the latter finally stabilized at 0 and 0.5 (0.94), respectively, for  $\Omega_c/\Omega_d = 1$  ( $\Omega_c/\Omega_d = 4$ )

in Fig. 2. Asking a vanishing derivative of  $\Delta\zeta$  relative to  $\Omega_c^2/\Omega_d^2$ , we further learn that  $\Delta\zeta$  achieves its maximal value for  $\Omega_c/\Omega_d = \sqrt{\alpha/2/\ln(\alpha/2) - 1}$ , which turns out to be 4.15 for  $\alpha = 160$  and in good agreement with what are observed in Fig. 3(a). Here,  $\Omega_c/\Omega_d \simeq 4$  corresponding to  $\eta \simeq 0.94$  indicates a negligible nonlinear conversion from the forward probe to the FWM field when they satisfy the EIT conditions of  $\delta_p = \delta_c = 0$  and  $\delta_m = \delta_d = 0$ , respectively, with both coupling and dressing fields being resonant. Meanwhile, this scaling relation results in a rather strong absorption for the backward probe since the resonant EIT ( $\delta_p = \delta_c = 0$ ) interaction between it and the coupling field is largely broken by the resonant ( $\delta_d = 0$ ) dressing field. This remarkable difference is clear from the expressions of  $\zeta_p^f$  and  $\zeta_p^b$  in Eq. (13) and constitutes the physical basis of resonant optical nonreciprocity with respect to probe transmission.

In the case of  $\delta_p = \delta_c \gg \gamma$ , however, we have instead  $D = i\gamma\Omega_c^2 + \delta_p\Omega_d^2$ ,  $K_-^f = i\alpha\gamma(\Omega_c^2 + \Omega_d^2)/[2(\gamma\Omega_c^2 - i\delta_p\Omega_d^2)]$ , and  $K_0^b = i\alpha\gamma\Omega_d^2/[2(\gamma\Omega_c^2 - i\delta_p\Omega_d^2)]$  while  $D_p$ ,  $D_m$ ,  $\sqrt{G}$ ,  $K_+^f$ ,  $A_+^f$ , and  $A_-^f$  remain the same as in the case of  $\delta_p = \delta_c = 0$ . Taking these into Eq. (12), we obtain

$$\begin{aligned}\zeta_p^f &= \sqrt{\eta^2 + (1-\eta)^2 e^{\frac{-2\epsilon\eta}{(1-\eta)^2}} + 2\eta(1-\eta)e^{\frac{-\epsilon\eta}{(1-\eta)^2}} \cos\theta}, \\ \zeta_p^b &= e^{-\epsilon\eta/(1-\eta)},\end{aligned}\quad (14)$$

with  $\epsilon = \alpha\gamma^2/2\delta_p^2$  and  $\theta = \alpha\gamma/2\delta_p(1-\eta)$ . It is thus clear that  $\zeta_p^b$  decays slowly while  $\zeta_p^f$  oscillates with a period of  $4\pi\delta_p(1-\eta)/\gamma \simeq 251$  as  $\alpha$  increases in Fig. 5(b). Taking  $\theta = \pi$  to ensure a minimal  $\zeta_p^f$ , we further have

$$\Delta\zeta = |\eta - (1-\eta)e^{-\epsilon\eta/(1-\eta)^2}| - e^{-\epsilon\eta/(1-\eta)}, \quad (15)$$

which approaches the extreme value of  $-1$  for  $\Omega_c/\Omega_d = 1$  when  $\epsilon\eta/(1-\eta)^2 \rightarrow 0$  as considered in Fig. 6(a). Here,  $\Omega_c/\Omega_d \simeq 1$  corresponding to  $\eta \simeq 0.5$  indicates a perfect nonlinear conversion from the forward probe to the FWM field when they are on EIT resonances of  $\delta_p = \delta_c \gg \gamma$  and  $\delta_m = \delta_d = 0$  with the far-detuned coupling and resonant dressing fields, respectively. Meanwhile, this scaling relation results in a negligible absorption for the backward probe since it is almost decoupled from all atomic resonances as a joint action of the far-detuned coupling and resonant dressing fields. This remarkable difference constitutes the physical basis of far-detuned optical nonreciprocity with respect to probe transmission and is clear from the former  $\zeta_p^f$  and latter  $\zeta_p^b$  terms in Eq. (15).

#### D. Nonreciprocal reversal on demand

In practical applications, it would be very useful provided that one can reverse the direction of nonreciprocal transmission on demand. Such a nontrivial possibility is clearly shown in Fig. 7, where  $\Delta\zeta$  can be switched from a value approaching  $-1$  to a value approaching 1 or vice versa. This is attained, in fact, by adjusting the probe and coupling detunings as well as the coupling or dressing Rabi frequency to switch between case (1) and case (2) following a certain white arrow. To be more specific, a high forward transmittance corresponding to  $\Delta\zeta \geq 0.9$  requires  $\delta_p = \delta_c \simeq 0$  and  $\Omega_c \simeq 4\Omega_d$  while a

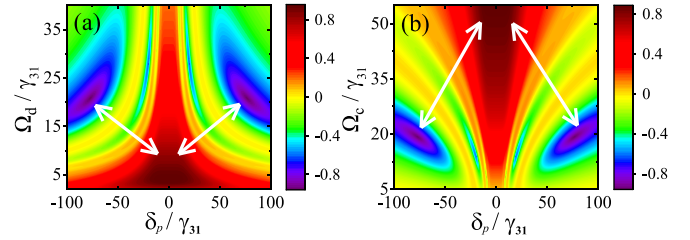


FIG. 7. Amplitude difference  $\Delta\zeta$  against (a) probe detuning  $\delta_p$  and dressing Rabi frequency  $\Omega_d$  with coupling Rabi frequency  $\Omega_c = 20\gamma_{31}$  or (b) probe detuning  $\delta_p$  and coupling Rabi frequency  $\Omega_c$  with dressing Rabi frequency  $\Omega_d = 20\gamma_{31}$ . White double-arrow lines show a possible reversal of optical nonreciprocity from  $\Delta\zeta \rightarrow 1$  to  $\Delta\zeta \rightarrow -1$  or vice versa. Other parameters are the same as in Fig. 5 except  $\alpha = 250$ .

high backward transmittance corresponding to  $\Delta\zeta \leq -0.85$  requires  $|\delta_p = \delta_c| \simeq 2\Omega_c \simeq 2\Omega_d$ . Note that simultaneously adjusting two or more field parameters inevitably introduces the operational complexity and should be avoided as much as possible in experiment. Consequently, it is of great interest provided that one can reverse the direction of nonreciprocal transmission by tuning a single parameter, which promotes us to consider a more general case by going beyond the previous restriction of  $\delta_p = \delta_c$ .

With above considerations, we finally plot in Fig. 8(a) the amplitude difference  $\Delta\zeta$  against both probe detuning  $\delta_p$  and coupling detuning  $\delta_c$ . It is clear that  $\Delta\zeta$  can be switched between a value close to  $-1$  and a value close to 1 by just adjusting  $\delta_p$  or  $\delta_c$  when they have very large absolute values and identical signs, as indicated by two black-dashed blocks. A better insight into such a single-parameter adjustment can be gained from a blow up of the upper-right block shown in Fig. 8(b). According to this blow up, we further plot in Fig. 8(c)  $\zeta_p^f$  and  $\zeta_p^b$  against  $\delta_p$  with  $\delta_c = 80\gamma_{31}$  while

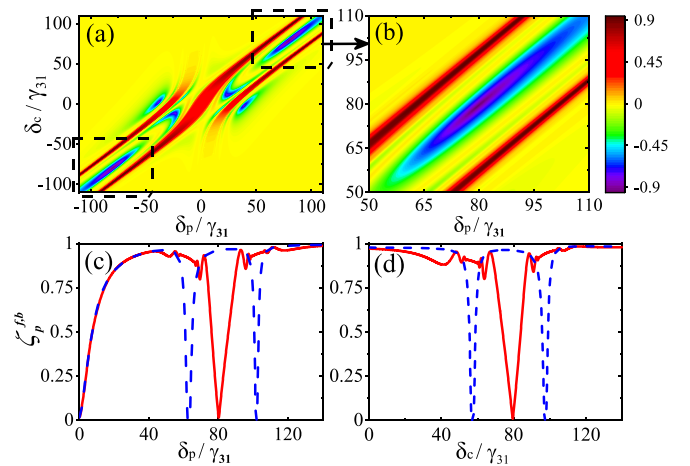


FIG. 8. (a) Amplitude difference  $\Delta\zeta$  against detunings  $\delta_c$  and  $\delta_p$  of coupling and probe fields, respectively. (b) A blow up of the upper-right region marked out by a black dashed square in panel (a). Transmission amplitudes  $\zeta_p^f$  (red solid) and  $\zeta_p^b$  (blue dashed) against (c)  $\delta_p$  for  $\delta_c = 80\gamma_{31}$  or (d)  $\delta_c$  for  $\delta_p = 80\gamma_{31}$ . Other parameters are the same as in Fig. 5 except  $\alpha = 250$ .

in Fig. 8(d)  $\zeta_p^f$  and  $\zeta_p^b$  against  $\delta_c$  with  $\delta_p = 80\gamma_{31}$ . These plots show that  $\zeta_p^b \rightarrow 1$  and  $\zeta_p^f \rightarrow 0$  occur at  $\delta_p \simeq 80\gamma_{31}$  while  $\zeta_p^b \rightarrow 0$  and  $\zeta_p^f \rightarrow 1$  can be found at  $\delta_p \simeq 63\gamma_{31}$  or  $102\gamma_{31}$  in Fig. 8(c). The corresponding isolation ratio and insertion loss are IR = 38 dB and IL = 0.34 dB, respectively, for a backward transmittance while IR = -45 dB and IL = 0.53 dB, respectively, for a forward transmittance. Similarly, we have  $\zeta_p^b \rightarrow 1$  and  $\zeta_p^f \rightarrow 0$  at  $\delta_c \simeq 80\gamma_{31}$  while  $\zeta_p^b \rightarrow 0$  and  $\zeta_p^f \rightarrow 1$  at  $\delta_c \simeq 57\gamma_{31}$  or  $98\gamma_{31}$  in Fig. 8(d). The corresponding isolation ratio and insertion loss are IR = 50 dB and IL = 0.34 dB, respectively, for a backward transmittance while IR = -47 dB and IL = 0.49 dB, respectively, for a forward transmittance. These results indicate that it is viable to reverse the quite favorable transmission nonreciprocity under a single-parameter adjustment.

#### IV. CONCLUSIONS

In summary, a magnet- and cavity-free scheme based on resonantly enhanced FWM nonlinearity in cold atoms has been proposed for achieving, manipulating, and optimizing the transmission nonreciprocity of a probe field. These atoms exhibit the double- $\Lambda$  configuration allowing for FWM for a forward probe while the  $N$ -type configuration forbidding FWM for a backward probe so that the two probe fields must undergo distinct optical responses. We find via numerical calculations that the forward and backward probe are highly transmitted and strongly absorbed, respectively, when they are applied on resonance together with the coupling and dressing fields. This nonreciprocal behavior becomes reversed to yield a strongly absorbed forward probe and a highly transmitted backward probe when they are far detuned and on two-photon resonance with the coupling field. Such a reversible unidirectional transmission can be optimized only if the coupling and dressing Rabi frequencies fulfill specific scaling ratios:  $\Omega_c/\Omega_d \simeq 4$  for a resonant probe while  $\Omega_c/\Omega_d \simeq 1$  for a far-detuned probe, supported also by analytical results. The unidirectional transmission, featured by high isolation

ratios and low insertion losses, can be reversed even for two counterpropagating probe fields of identical detunings. All above findings constitute an essential supplement to prior understandings of FWM-based optical nonreciprocity and are expected to facilitate the development of more powerful non-reciprocal devices needed to route complex photon flows in quantum networks.

The validity of our proposed model relies on the weak-probe approximation featured by  $\rho_{11}^{f,b} \simeq 1$  and has been examined for an ideal situation with vanishing stray magnetic fields. Full calculations based on 16 mutually coupled density-matrix equations confirm that  $\rho_{11}^{f,b} \simeq 1$  holds even for  $\Omega_p^{f,b}/\gamma_{31} \lesssim 1.0$ , and optical depth  $\alpha$  should be scaled up by several tens of percent to compensate for a few times stronger probe beyond this upper limit. Moreover, in realistic experimental environments, the presence of stray magnetic fields inevitably results in different Zeeman splittings of our four states [59,60], which need to be incorporated into detunings of the probe, coupling, dressing, and FWM field but can be easily compensated by tuning these light fields in frequency. Alternatively, one may adopt active magnetic compensation via Helmholtz coils and  $\mu$ -metal shielding, or consider magnetically insensitive  $|1\rangle \leftrightarrow |2\rangle$  transition to preserve the desired quantum coherence  $\rho_{21}^{f,b}$ .

#### ACKNOWLEDGMENTS

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#### DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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