

Fractional Chern mosaic in supermoiré graphene

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We propose the realization of a fractional Chern mosaic: a state characterized by a spatially varying topological order. This arises in trilayer graphene with small sequential twist angles, where a large supermoiré lattice emerges alongside conventional moiré lattices. We show that electron correlations drive local fractionalization on the moiré scale, while its pattern varies across the supermoiré scale.

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Introduction. The importance of length scales in physics is exemplified in the rise of two-dimensional moiré materials. Twist engineered moiré lattices on the nanometer scale are an ideal platform for experimentally realizing a remarkable variety of quantum phenomena [1–4], from superconductivity [5–13] to fractional Chern insulators (FCIs) [14–20]. This rich physics is enabled by a separation of length scales between the atomic and moiré lattices. At experimentally relevant scales, the physics is governed by that of the moiré lattice, which behaves as a new artificial crystal.

The interplay of length scales is at the forefront in an emerging class of moiré materials with multiple incommensurate moiré lattices. Prominent examples are various multilayer graphene-hBN heterostructures [21–31] and twisted trilayer graphene at small, generic, twist angles [32–53]. The presence of two incommensurate moiré lattices introduces an additional large supermoiré (or “moiré-of-moiré”) length scale, which can be parametrically larger than the moiré lattices. This new supermoiré scale leads to another separation of length and energy scales, opening the door to especially intriguing correlated electronic states.

One striking example is the Chern mosaic [29,30,38,54–57], where electrons form Chern insulators at the moiré scale, but the Chern number varies across the supermoiré scale. The result is a mosaic of Chern insulators embedded in a network of topological domain walls. Here, we consider an even more exotic possibility: a “fractional Chern mosaic,” where electrons spontaneously fractionalize within local regions of FCI [58–64], but with the pattern of fractionalization varying at the supermoiré scale. Such a fractional Chern mosaic would represent a class of quantum states characterized by a spatially varying topological order.

We propose the realization of a fractional Chern mosaic. Specifically, we focus on helical trilayer graphene (HTG), a prototypical supermoiré material consisting of three graphene

sheets sequentially rotated by a small twist angle θ , i.e., in the configuration $(\theta_1, \theta_2, \theta_3) = (\theta, 0, -\theta)$. In HTG, adjacent layer pairs produce moiré lattices with periods $a_m \approx a_0/\theta$, which combine to form a large supermoiré lattice with period $a_{sm} \approx a_0/\theta^2$, where $a_0 = 0.246$ nm is the graphene lattice constant. This supermoiré lattice, on the scale of a few hundred nanometers, was recently imaged with a scanning single-electron transistor (SET) [65]. Lattice relaxation creates large triangular moiré-periodic domains, referred to as h-HTG and $\bar{h}$ -HTG, as illustrated in Fig. 1 [36,37]. Due to the separation of length scales $a_{sm} \gg a_m$, it suffices to model the system as locally periodic within each domain. Within these periodic domains, the local moiré band structure gives rise to topological flat bands with valley-contrasting Chern number and opposing topology in adjacent domains [36–38]. Flat band ferromagnetism then leads to spontaneous time-reversal symmetry breaking, as observed near a magic angle $\theta \approx 1.8^\circ$ [51]. The combined presence of broken time-reversal symmetry, flat Chern bands, and supermoiré periodicity sets the stage for the fractional Chern mosaic.

The task is to establish whether electrons within each domain can fractionalize. To achieve this, we perform a comprehensive analysis of the correlated physics at partial filling of the flat Chern bands. We begin by characterizing an additional MDT term in the Hamiltonian, which originates from the decay of the interlayer tunneling amplitude as a function of the in-plane distance between the atomic p_z orbitals [66]. We find that accounting for MDT not only explains the particle-hole asymmetry observed in transport [51] but also, crucially, stabilizes an FCI at filling factor $\nu = 3 + 1/3$. The predicted state, illustrated in Fig. 1(a), features electron fractionalization into charge $e/3$ quasiparticles in a spatially varying way due to the opposing band topology in the h-HTG and $\bar{h}$ -HTG domains, and can be experimentally unveiled by local scanning probes.

Noninteracting model. We start with a generalized Bistritzer-MacDonald [67] continuum model describing the local band structure within each h-HTG domain [36]. The K valley of h-HTG is described by

$$H_K = \begin{bmatrix} -i\nu\sigma \cdot \nabla & T(r - d_t) & 0 \\ T^\dagger(r - d_t) & -i\nu\sigma \cdot \nabla & T(r - d_b) \\ 0 & T^\dagger(r - d_b) & -i\nu\sigma \cdot \nabla \end{bmatrix}, \quad (1)$$

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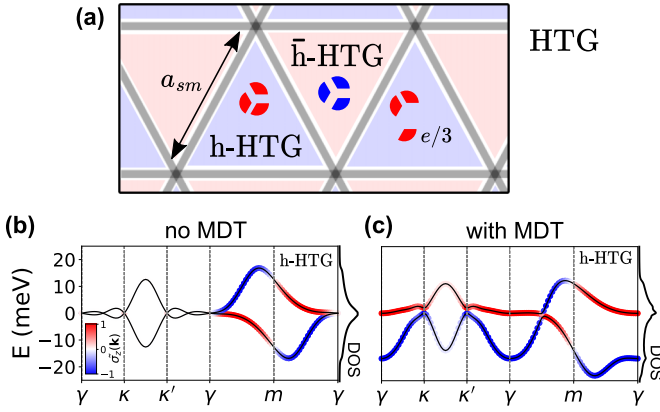


FIG. 1. (a) Helical trilayer (HTG) relaxes on the supermoiré scale a_{sm} into h -HTG and $\bar{h}$ -HTG domains that are locally moiré-periodic and separated by gapless domain walls (gray). Fractionalization within the domains generates a fractional Chern mosaic. (b) Band structure in valley K of the h -HTG domain in the absence of momentum-dependent tunneling (MDT). The central two bands are colored according to Chern-sublattice polarization $\tilde{\sigma}_z(k)$, where red (blue) indicates polarization onto the A (B) sublattice. Right line trace plots the density of states. (c) Same as panel (b) except in the presence of MDT.

where $\sigma = (\sigma_x, \sigma_y)$ acts on the microscopic sublattice, and $v = 8.8 \times 10^5 \text{ m s}^{-1}$.

In the absence of MDT, the interlayer tunneling is

$$T(\mathbf{r}) = \begin{bmatrix} w_{AA}t_0(\mathbf{r}) & w_{AB}t_{-1}(\mathbf{r}) \\ w_{AB}t_1(\mathbf{r}) & w_{AA}t_0(\mathbf{r}) \end{bmatrix},$$

$$t_\alpha(\mathbf{r}) = \sum_{n=0}^2 e^{\frac{2\pi i}{3}n\alpha} e^{iq_n \cdot \mathbf{r}}, \quad (2)$$

where $\mathbf{q}_n = k_\theta [\sin(\frac{2\pi}{3}n), \cos(\frac{2\pi}{3}n)]$, $k_\theta = \frac{8\pi}{3a_0} \sin \frac{\theta}{2}$, and we fix $(w_{AA}, w_{AB}) = (75 \text{ meV}, 110 \text{ meV})$. Within the h -HTG domain, the relative interlayer moiré shift is $d_t - d_b = \frac{1}{3}(a_2 - a_1)$, where $a_{1,2} = \frac{4\pi}{3k_\theta}(\pm \frac{\sqrt{3}}{2}, \frac{1}{2})$ are the basis moiré vectors. We focus only on the physics within an h -HTG domain, as the $\bar{h}$ -HTG domains are related by $\hat{C}_{2z}$ symmetry.

Equations (1) and (2) obey an exact particle-hole-inversion symmetry (PHS) that is absent in experiment [51], where correlated states appear only for electron doping (small twists in the Pauli term only weakly break PHS [36]). PHS is strongly broken by considering the finite range of interlayer hopping, which generates MDT. This is incorporated to leading order by making the replacement

$$t_\alpha(\mathbf{r}) \rightarrow \sum_{n=0}^2 e^{\frac{2\pi i}{3}n\alpha} e^{iq_n \cdot \mathbf{r}} (1 + \lambda_{\text{MDT}} \hat{\mathbf{q}}_{n,\perp} \cdot \vec{\nabla}), \quad (3)$$

where $\hat{\mathbf{q}}_{n,\perp} = [\cos(\frac{2\pi}{3}n), \sin(\frac{2\pi}{3}n)]$ is a unit vector perpendicular to $\mathbf{q}_n$. We take a realistic $\lambda_{\text{MDT}} \approx -2.3 \text{ \AA}$ (see the Supplemental Material [68]).

Figures 1(b) and 1(c) show the noninteracting band structure of $\theta = 1.8^\circ$ h -HTG. Regardless of MDT, we obtain two narrow bands separated by large gaps $\gtrsim 80 \text{ meV}$ to higher remote bands (not shown). The nontrivial topology is unveiled by diagonalizing the sublattice operator σ_z projected to the

central bands [69–71]. This yields the so-called Chern basis indexed by a “Chern-sublattice” label $\tilde{\sigma} = A, B$ according to the dominant sublattice polarization. Within valley K , these bands have Chern numbers

$$C_{K,A} = 1, \quad C_{K,B} = -2, \quad (4)$$

and the corresponding values for valley $\bar{K}$ and the $\bar{h}$ -HTG domain are obtained by time reversal and $\hat{C}_{2z}$. The blue and red colors indicate the expectation value of $\langle \tilde{\sigma}_z \rangle \equiv \langle P_A \rangle - \langle P_B \rangle$, where $P_{\tilde{\sigma}}$ is the projector onto the $\tilde{\sigma}$ band.

In the absence of MDT [Fig. 1(b)], the central bands obey PHS as reflected in the density of states. Band eigenstates with predominantly B character ($\langle \tilde{\sigma}_z \rangle \approx -1$) are more dispersive than those with A character, but both have the same energy when averaged over the mBZ. To make this concrete, we can rotate to the Chern basis and switch off the off-diagonal hybridization to define the dispersions $E_{\tilde{\sigma}_z}(k)$ (see the Supplemental Material [68]). At $\theta = 1.8^\circ$, we find that $E_A(k)$ and $E_B(k)$ have zero average, but bandwidths 7.2 and 30.4 meV, respectively.

In contrast, introduction of MDT [Fig. 1(c)] lowers the energy of the B basis relative to the A basis. At $\theta = 1.8^\circ$, $E_A(k)$ has a mean energy of 1.0 meV and is bounded by -2.9 and 4.2 meV , while $E_B(k)$ has a mean energy of -6.4 meV and is bounded by -22.4 and 9.4 meV . This strongly breaks the PHS, as evidenced by the substantial asymmetry of the density of states. These details will become important soon when considering correlated states.

Interactions. We consider gate-screened Coulomb interactions $V(q) = \frac{e^2}{2\epsilon_0\epsilon_r q} \tanh qd_{sc}$, where the gate distance is $d_{sc} = 25 \text{ nm}$ and the relative permittivity $\epsilon_r \simeq 8\text{--}15$ is left as a tuning parameter. The large single-particle remote gaps permit projection onto the two central bands per spin and valley:

$$H_{\text{int}} = \frac{1}{2A} \sum_q V(q) \delta \hat{\rho}_q \delta \hat{\rho}_{-q}, \quad (5)$$

where A is the system area and $\delta \hat{\rho}_q = \hat{\rho}_q - 4\bar{\rho}_q$ is the projected density operator measured relative to the mean density $\bar{\rho}_q$ of the central bands (this is the so-called “average” interaction scheme) [52]. H_{int} satisfies a $U(2)_K \times U(2)_{\bar{K}}$ flavor symmetry, which includes charge- $U(1)_c$ and valley- $U(1)_v$ conservation, and valley-dependent $SU(2)_s$ spin rotations.

Hartree-Fock phase diagram. We first consider the physics at integer fillings within Hartree-Fock (HF) theory. In the absence of MDT, Ref. [52] identified a hierarchy of so-called “strong-coupling” insulators at integer fillings ν . These phases can be understood by fully polarizing into $\nu + 4$ of the eight total Chern bands indexed by spin $s \in \{\uparrow, \downarrow\}$, valley $\tau \in \{K, \bar{K}\}$, and Chern sublattice $\tilde{\sigma}_z \in \{A, B\}$, with the associated Chern number following from Eq. (4). The phase diagram in Ref. [52] was identical under $\nu \leftrightarrow -\nu$ due to PHS.

In contrast, MDT leads to qualitatively different behaviors at negative and positive integer fillings in our HF calculations [Fig. 2(a)]. We diagnose the strong-coupling nature by the deviation from full $(\tau, s, \tilde{\sigma})$ -polarization:

$$\Lambda \equiv \max_{\tau,s,\tilde{\sigma}} [\min(\langle n_{\tau,s,\tilde{\sigma}} \rangle, 1 - \langle n_{\tau,s,\tilde{\sigma}} \rangle)], \quad (6)$$

where $\langle n_{\tau,s,\tilde{\sigma}} \rangle$ is the mBZ-averaged occupation number for the Chern band $\tilde{\sigma}$ in valley τ and spin s . Λ vanishes for an

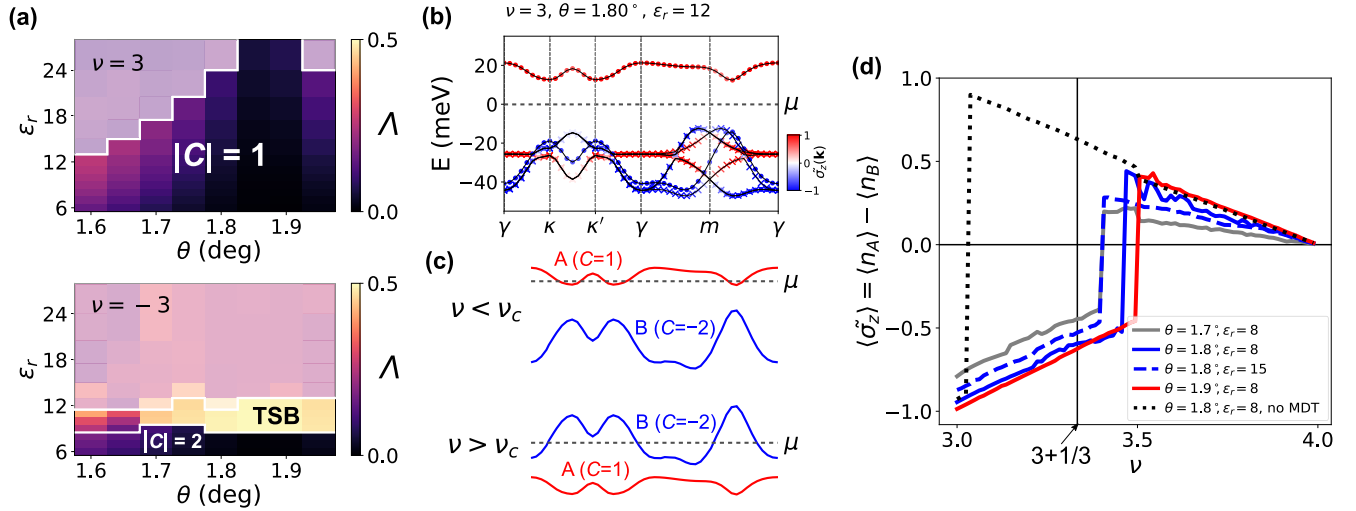


FIG. 2. (a) HF phase diagram as a function of twist angle θ and relative permittivity ϵ_r at $\nu = +3$ (top) and $\nu = -3$ (bottom). White shaded regions indicate gapless states. Λ , defined in the main text, is a measure of how far the HF solution is from an idealized strong-coupling state. TSB indicates various states that break moiré translation symmetry. System size is 12×12 , and we allow for spontaneous doubling or tripling of the moiré unit cell along both directions. (b) HF band structure of the $\nu = +3$ Chern insulator. The bands are colored according to their Chern-sublattice polarization. Dots (crosses) denote spin up (down). System size is 24×24 . (c) Schematic of Chern-sublattice transitions induced by the sublattice asymmetry in the effective dispersion. (d) HF calculations of Chern-sublattice polarization $\langle \tilde{\sigma}_z \rangle$ for fillings $3 \leq \nu \leq 4$. $\langle \tilde{\sigma}_z \rangle$ is defined as the difference of the occupation numbers of the A and B Chern-sublattice bands per moiré unit cell. We enforce moiré translation symmetry and valley $U_v(1)$ symmetry. Calculations are performed with MDT, unless otherwise indicated. System size is 18×18 .

idealized strong-coupling insulator, while it can take values as large as 0.5 for generic states. For $\nu = +3$, we find a strong-coupling $|C| = 1$ insulator for a range of θ , including for very weak interactions $\epsilon_r \gtrsim 25$. On the other hand, for $\nu = -3$, the strong-coupling phase only exists for relatively strong interactions $\epsilon_r \lesssim 9$. Lowering the interaction strength leads to gapless states or various gapped phases with intervalley coherence and translation symmetry breaking [52,72]. Analogous results hold for other integer fillings [68], demonstrating that MDT can explain the pronounced particle-hole asymmetry observed in transport [51].

Figure 2(b) shows the HF band structure of the $|C| = 1$ insulator at $\nu = +3$. The state is fully spin-valley polarized within the flat-band manifold, and the strong-coupling nature is evidenced by the near-total $\tilde{\sigma}_z$ polarization of the conduction band. In the partially filled flavor, the system occupies the B band (blue) with lower mean kinetic energy, leaving a narrow unfilled A band (red) with $|C| = 1$.

Sublattice transitions. At $\nu = 3$, the unfilled conduction band is a narrow $|C| = 1$ band of predominantly $\tilde{\sigma} = A$ character, with favorable quantum geometry [68] for hosting FCIs when fractionally occupied. For this to occur, it is crucial that the notion of an isolated conduction A band remains valid at higher fractional fillings like $\nu = 3 + \frac{1}{3}$.

To this end, we discover a cascade of sublattice transitions [Figs. 2(c) and 2(d)], whose mechanism relies on two key ingredients: (1) sublattice ferromagnetism induced by the dominant Coulomb interactions and (2) the lower average but higher peak kinetic energy of the B band compared to the A band. In the following, we discuss the cascade for the filling range $3 \leq \nu \leq 4$ assuming full flavor polarization (i.e., flat bands in three of four spin-valley flavors are fully filled). Exactly at $\nu = 3$, occupying either Chern band of the re-

maining flavor yields a valid exchange-driven strong-coupling insulator, but the system chooses to form the $|C| = 1$ insulator and fill the B band due to its lower average kinetic energy, yielding a $\langle \tilde{\sigma}_z \rangle \approx -1$ state. As electrons are doped into the system, they naturally populate the empty A band [Fig. 2(c), top], which gradually degrades the initial negative sublattice polarization $-1 < \langle \tilde{\sigma}_z \rangle < 0$. Eventually we reach a critical filling ν_c where the system energetically prefers to instead fully occupy the A band and partially occupy the B band so that $\langle \tilde{\sigma}_z \rangle > 0$ [Fig. 2(c), bottom]. The reason is that doing so allows the system to empty the parts of the B band that are kinetically costly, while respecting the tendency toward sublattice polarization. ν_c is determined by the interplay between the relative energy gain from fully polarizing the B band over the A band, and the kinetic cost of occupying the high energy parts of the B band. The outcome is an abrupt sign change of the sublattice polarization at ν_c .

Figure 2(d) tracks the self-consistent HF sublattice polarization. For all parameters, $\langle \tilde{\sigma}_z \rangle$ starts off near -1 and increases linearly with a gradient close to 1 as the empty A band is progressively occupied. With MDT, we find a strongly first-order sublattice jump at which $\langle \tilde{\sigma}_z \rangle$ changes sign. Crucially, the transition filling $\nu_c \approx 3.5$ is greater than $3 + \frac{1}{3}$ for realistic parameters. The enhanced dispersion of the B band means that the Fermi surfaces for $\nu > \nu_c$ are often flavor-degenerate, but the phase for $\nu < \nu_c$ is always fully flavor polarized. These findings are consistent with the experimental observation of AHE persisting to $\nu \approx 3.5$ [51]. ν_c is lowered when increasing ϵ_r . If MDT is neglected (dotted lines), the sublattice transition occurs almost immediately upon electron doping $\nu = +3$. The reason is that without MDT, the average kinetic energies of the A and B bands are identical [Fig. 2(a)], and only the sublattice exchange anisotropy of $\lesssim 0.5$ meV

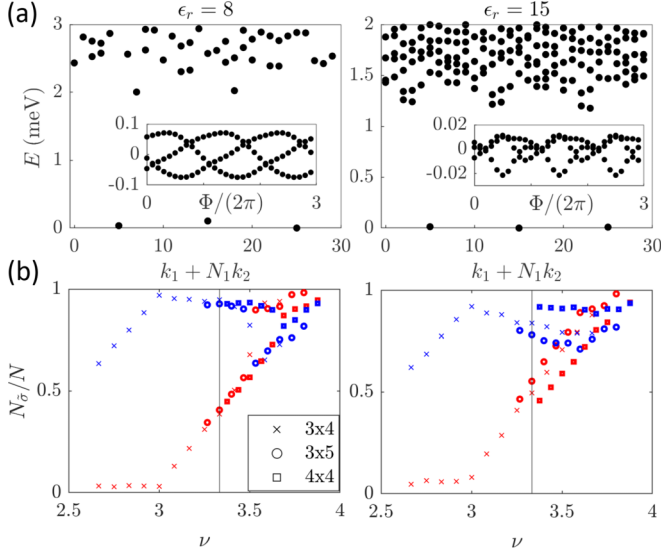


FIG. 3. (a) One-band ED many-body spectrum on the 5×6 cluster at $\nu = 3 + 1/3$ for $\epsilon_r = 8$ and 15 . Inset shows the corresponding evolutions of the three quasidegenerate ground states under flux threading Φ . (b) Two-band ED calculation of the Chern-sublattice polarization as a function of filling. Different symbols (see legend) refer to various lattice geometries with corresponding system sizes $N = 12, 15, 16$. $N_{\tilde{\sigma}}$ is the expectation value of the particle occupation in the Chern-sublattice basis. Red (blue) corresponds to the A (B) basis. The vertical line indicates $\nu = 3 + 1/3$. All calculations are performed at $\theta = 1.8^\circ$ and with MDT.

favors occupying the B band at $\nu = +3$ [52]. This tiny splitting is easily swamped at small ν_c by the kinetically driven mechanism explained above, underscoring the importance of MDT in faithfully capturing the sublattice physics. Similar sublattice transitions occur between every non-negative integer filling, but none occur for $\nu < 0$, where there is significantly less symmetry breaking [68].

Exact diagonalization. We use exact diagonalization (ED) to investigate the stability of FCIs at fractional fillings. Since full calculations in the eight central bands are expensive, truncating the “active” Hilbert space is necessary. In Fig. 3(a), we first present one-band ED results at $\nu = 3 + 1/3$, where the active subspace is restricted to the $|C| = 1$ conduction band of the $\nu = +3$ HF insulator. In other words, we constrain the other seven HF bands to be fully occupied, and work at $1/3$ filling of the remaining Chern band, which is spin-valley polarized and of predominantly $\tilde{\sigma} = A$ character [see Fig. 2(b)]. We obtain an FCI for $\epsilon_r = 8, 15$ on a system with $N = 30$ unit cells, as evident from the threefold quasidegenerate ground states with correct momenta that remain gapped under flux threading. In the Supplemental Material [68], we show $N = 27$ results that rule out charge density waves.

Next, we perform ED calculations where we only constrain that three flavors are fully occupied, leading to the restriction $2 \leq \nu \leq 4$. We emphasize that this effective two-band calculation is *exact* within the maximally spin-valley-polarized symmetry sector of the eight central bands. From the HF results, we anticipate that the ground state indeed lies in this flavor sector from $\nu = +3$ up to at least the sublattice

transition at ν_c [68]. Importantly, this two-band calculation faithfully captures the interplay between the $\tilde{\sigma} = A, B$ bands.

Figure 3(b) plots the Chern-sublattice densities $\langle n_A \rangle$ and $\langle n_B \rangle$ in the two-band calculation as a function of ν . Owing to the increased computational difficulty, we collate results for several system sizes. Consistent with the HF analysis, the $\nu = +3$ ground state is strongly polarized into the B band and is essentially a gapped single Slater determinant [68]. Upon electron doping for $\epsilon_r = 8$, the B band maintains a large occupancy $> 90\%$ such that the additional carriers primarily populate the A band. Crucially, this picture still holds at $\nu = 3 + 1/3$ across multiple lattice geometries, lending credence to the one-band ED. The theory of sublattice transitions is corroborated by the abrupt jump at $\nu_c \approx 3.5$. For $\epsilon_r = 15$, the B band depopulates more appreciably for certain system sizes when adding electrons to the $\nu = +3$ insulator, though we still find kinks in the sublattice occupations around $\nu_c \approx 3.5$. We note that direct signatures of FCIs in our multiband calculations are present but not fully robust [68], likely due to finite-size effects since the largest size for which we have two-band data at $\nu = 3 + 1/3$ is smaller than the threshold N beyond which one-band ED robustly yields FCIs.

Fractional Chern mosaics. Having found evidence for FCIs within the h-HTG domains, we now zoom out and reexamine the structure of HTG at $\nu = 3 + 1/3$ as a whole. In the h-HTG domain, electrons within the A band of a particular spin-valley (τ, s) flavor fractionalizes. The resulting state has a local Hall conductivity $\sigma_{xy} = -\frac{2}{3}\tau_z\frac{e^2}{h}$, where $\tau_z = 1$ (for K) or -1 (for K'). The fractional Hall conductivity can be decomposed as $-\frac{2}{3} = -1 + \frac{1}{3}$, where -1 arises from the Chern insulator at $\nu = 3$, and $1/3$ from the FCI. Since the local physics in the h-HTG domain is related by $\hat{C}_{2z}$, the corresponding FCI there will arise from a partially filled B band with local $\sigma_{xy} = \frac{2}{3}\tau_z\frac{e^2}{h}$. Since the spin-valley polarization is determined spontaneously, there are multiple qualitatively distinct fractional Chern mosaics that may occur. Which of these scenarios is favored will likely depend on fine details of the domain wall energetics [52,55].

Consider the case where the valley polarizations in the h-HTG and h-HTG domains are identical. In this case, a mosaic of local $\sigma_{xy} = \pm\frac{2}{3}\frac{e^2}{h}$ domains is formed. The domain walls necessarily host fractionally charged topological chiral edge modes. In both types of domains, electrons fractionalize into charge $e/3$ quasiparticles as is known for the $1/3$ Laughlin state. However, the anyonic exchange phase $e^{\pm i\frac{\pi}{3}}$ will be opposite for quasiparticles in the two types of domains. The result is a state in which fractionally charged quasiparticles exhibit spatially varying anyonic statistics.

Another possibility is that the valley polarizations will be opposite in the two domains. This configuration will be favored by a small external magnetic field. For concreteness, suppose that the FCI arises in the K valley for h-HTG, and in the K' valley for h-HTG. In this case, while both domains have local $\sigma_{xy} = -\frac{2}{3}\frac{e^2}{h}$, we still consider this state a fractional Chern mosaic since the fractionally charged $e/3$ quasiparticles carry opposite valley quantum numbers in the two domains.

Discussion. Our results have important implications for recent and future experiments. The transport study of Ref. [51] observed correlated features only for positive fillings in

multiple magic-angle ($\theta \approx 1.7^\circ$ – 1.8°) HTG devices, consistent with our HF calculations in the presence of MDT. The anomalous Hall effect and density-tuned hysteresis observed near $3 \lesssim \nu \lesssim 3.5$ are consistent with the sublattice transition identified here via the kinetic mechanism at $\nu_c \approx 3.5$. Combined with the recent observation of supermoiré domains in (non-magic-angle) HTG by Ref. [65], our analysis implies that a fractional Chern mosaic can be realized at $\nu = 3 + 1/3$, but is not likely visible in transport [51] due to the network of conducting domain walls and sample inhomogeneity. Local probes are ideal for resolving the real-space mosaic and detecting the local topological state. Scanning SETs [15,65,73,74] can identify the state as an incompressible feature that disperses with magnetic field B according to the Streda formula. Scanning tunneling microscopy [75–82] would further be able to directly image the sublattice transitions predicted in this work. We also note that the global $\hat{C}_{2z}$ symmetry of pristine HTG implies that identical physics would be present in the h-HTG and $\bar{h}$ -HTG domains, but alignment to the hBN substrate could introduce a sublattice splitting that favors fractionalization in only one type of domain.

Finally, while we have mainly focused on $\nu = 3 + \frac{1}{3}$, FCIs may exist at other fillings, especially those that are compatible with the picture above. The simplest candidates are Jain states at $\nu = 3 + \frac{k}{2k+1}$ for which $\nu < \nu_c$. The periodic resets in flavor and sublattice physics suggest that fractionalization could also occur around other integer fillings, a possibility that we leave for future theoretical studies.

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Data availability. The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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