

Effects of Berry curvature on ideal fractional Chern insulator many-body gaps

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We investigate the many-body ground states in a family of fractionally filled bands where the Berry curvature fluctuations can be tuned while maintaining ideal quantum geometry. We numerically find that the neutral gap of the fractional Chern insulator (FCI) ground state decreases as the Berry curvature becomes less homogeneous, ultimately driving an instability to a charge density wave. We further extend our analysis to bands perturbed away from the ideal limit and give examples where a less ideal band geometry results in a more stable FCI phase. To explain our findings, we apply the single-mode approximation to the ground-state wave functions of the ideal band, from which we obtain analytic expressions for the FCI many-body gap in the limit of weak Berry curvature variations. Finally, we make a connection between our results and experimentally relevant systems where FCIs have been observed.

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Introduction. Fractional Chern insulators (FCIs) are the analogs of fractional quantum Hall (FQH) states in the absence of magnetic field. They were proposed over a decade ago [1–6], but their realization remained elusive until recently, when observations in twisted transition metal dichalcogenide (TMD) homobilayers [7–10] and rhombohedral graphene multilayers [11] opened a new frontier in the field of topological order [12,13]. Follow-up experimental studies have observed rich new phenomenology in both the TMD [14–17] and graphene [18,19] platforms, giving rise to several puzzles regarding the nature of moiré FCIs. In this context, it is crucial to obtain a theoretical understanding of how electronic correlations in two-dimensional materials give rise to physics in the absence of a magnetic field that resembles—but at the same time reveals striking differences from—the FQH paradigm in Landau levels.

The observed FCIs are often studied theoretically by means of continuum models subject to periodic spatial modulations from the moiré superlattice [20]. The framework of quantum geometry [21–25] links those models to FQH physics through the notion of *ideal bands*, which are exactly flat bands whose wave functions can be mapped to those of the lowest Landau level (LLL) precisely via a spatial modulation. Hamiltonians with ideal bands are appealing because, for a particular form of the electron-electron interactions, analytic many-body ground-state wave functions can be obtained by generalizing the Laughlin construction [26,27]. Examples of ideal band

models include the chiral model of twisted bilayer graphene [28] and its multilayer generalizations [29,30], Dirac and spin-1/2 Schrödinger particles in a periodic magnetic field [31,32], the Kapit-Mueller model [33], and models with higher band touchings [34,35]. Although real materials are not expected to be described by fine-tuned ideal band models, the experimental discovery of FQH physics over wide ranges of twist angle and displacement field suggests that some moiré FCIs may be approximately described by perturbing away from an ideal band limit. Thus, it is important to understand how deviations from the limits of ideal geometry and constant Berry curvature impact the stability of putative FCI ground states. However, in generic models, finite bandwidth, violations of ideal band geometry, and variations in Berry curvature often appear simultaneously, making it difficult to disentangle their effects on the ground state. In this work, we gain insight by isolating these effects.

We first investigate the many-body physics of a family of ideal band models, which we refer to as Aharonov-Casher (AC) Hamiltonians. These models were introduced previously to approximate the continuum model for homobilayer TMDs [36,37], but their many-body physics still remains to be fully explored [38]. The AC models have the advantage that the Berry curvature distribution of the topmost band can be tuned while maintaining the ideal quantum geometry and vanishing bandwidth. This isolates the effect of nonhomogeneous Berry curvature on the FCI that results when the top band is partially filled. Our exact diagonalization (ED) calculations show that for a realistic Coulomb interaction, the neutral gap of the FCI ground state decreases when Berry curvature fluctuations are increased, despite the band remaining ideal. We also observe that the magnitude of the magnetoroton gap depends on the location of the Berry curvature extrema in the original band. We explain these observations by exploiting the exact solvability of the AC Hamiltonian. Specifically, we apply the

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single-mode approximation (SMA) in the limit of small Berry curvature inhomogeneity to obtain an analytic expression for the magnetoroton minimum, which agrees qualitatively with our numerics.

Next, we study the many-body physics of band models with small violations from the ideal condition, which is a more accurate approximation for moiré TMD homobilayers. Surprisingly, we find instances where sacrificing ideal geometry in favor of a more homogeneous Berry curvature increases the stability of the topologically ordered phase. We explain this effect using our SMA framework and relate it to realistic scenarios where FCIs have been experimentally observed. Our results show that the adiabatic approximation [36,37] provides a good description of the continuum model for twisted TMD homobilayers at both the single-particle and many-body levels, and justify the application of numerical and analytical many-body methods developed for the FQH problem to investigate FCI physics in moiré materials.

Adiabatic and Aharonov-Casher Hamiltonians. We study the Hamiltonian

$$H_{\text{ad}} = H_{\text{AC}} + U = -\frac{1}{2m} \Pi_- \Pi_+ + U, \quad (1)$$

which, in a certain limit, approximates the continuum model of TMD homobilayers such as tMoTe₂ [36–40]. In Eq. (1), $\Pi_{\pm} = (p_x + A_x) \pm i(p_y + A_y)$ are kinetic-momentum operators with $p_{\alpha} = -i\partial_{\alpha}$ for $\alpha = x, y$. $U = U(\mathbf{r})$ is a periodic potential (which we refer to as the residual potential) and $\mathbf{A}(\mathbf{r}) = (A_x(\mathbf{r}), A_y(\mathbf{r}))$ is a vector potential corresponding to a periodic magnetic field, $\nabla \times \mathbf{A}(\mathbf{r}) = B_0 + B(\mathbf{r})$, which we have split into a spatially uniform part with one flux quantum per unit cell, $B_0 = -1/\ell^2$, with ℓ the magnetic length, and a zero-average periodic piece $B(\mathbf{r})$. We assume that $B(\mathbf{r})$ and $U(\mathbf{r})$ have the same periodicity, with Fourier expansions

$$B(\mathbf{r}) = \sum_{\mathbf{G}} B_{\mathbf{G}} e^{i\mathbf{G}\cdot\mathbf{r}} \quad \text{and} \quad U(\mathbf{r}) = \sum_{\mathbf{G}} U_{\mathbf{G}} e^{i\mathbf{G}\cdot\mathbf{r}}, \quad (2)$$

where $\mathbf{G}$ are their shared reciprocal lattice vectors.

Following Ref. [37], we refer to Eq. (1) as the *adiabatic Hamiltonian* and to its first term, H_{AC} , as the *Aharonov-Casher Hamiltonian*, which is known to have a manifold of zero-energy states for arbitrary magnetic field [31,32]. See the Supplemental Material [41] for further details on the AC Hamiltonian. When the magnetic field is periodic, the resulting zero-energy manifold can be understood as a quasi-Bloch band, which we will refer to as an AC band [37,40]. These flat bands are known to satisfy the trace condition, namely, that their quantum metric g_k^{ab} and Berry curvature Ω_k satisfy $\text{tr} g_k^{ab} = |\Omega_k|$, guaranteeing an exact mapping to the LLL [21,22]. As a consequence, AC bands can support Laughlin-like ground states whose analytic wave functions are related to those of the FQH effect [24,42,43].

The wave functions of the AC band can be written as

$$\psi^{\text{AC}}(\mathbf{r}) = f(z) e^{-\frac{|z|^2}{4\ell^2} + \chi(\mathbf{r})} = \psi^{\text{LLL}}(\mathbf{r}) e^{\chi(\mathbf{r})}, \quad (3)$$

where $z = x + iy$, f is a holomorphic function, $\chi(\mathbf{r})$ is a real periodic function related to the magnetic field by $\nabla^2 \chi(\mathbf{r}) = B(\mathbf{r})$, and $\psi^{\text{LLL}}(\mathbf{r})$ is a lowest-Landau-level wave function. Equation (3) defines a mapping between LLL wave functions and AC wave functions, from which we can define a magnetic

quasi-Bloch basis for the AC band $|\psi_k^{\text{AC}}\rangle = e^{\chi(\mathbf{r})} |\psi_k^{\text{LLL}}\rangle$, in terms of the quasi-Bloch states of the LLL, $|\psi_k^{\text{LLL}}\rangle$ [44–47]. The Berry curvature of the AC band is given by [24,37]

$$\begin{aligned} \Omega_k &= \frac{2\pi}{A_{\text{BZ}}} + \frac{1}{2} \nabla_k^2 \ln \langle \psi_k^{\text{AC}} | \psi_k^{\text{AC}} \rangle \\ &= \Omega_0 + \frac{1}{2} \nabla_k^2 \ln \left(\sum_{\mathbf{G}} \Phi_{\mathbf{G}} \langle \psi_k^{\text{LLL}} | e^{i\mathbf{G}\cdot\hat{\mathbf{r}}} | \psi_k^{\text{LLL}} \rangle \right), \end{aligned} \quad (4)$$

where A_{BZ} is the area of the Brillouin zone and $\Phi_{\mathbf{G}}$ are the Fourier coefficients of the function $e^{2\chi(\mathbf{r})}$. Equation (4) shows that the Berry curvature distribution of the AC band can be controlled by tuning $B(\mathbf{r})$, while the geometry is known to remain ideal [24]. When $U(\mathbf{r}) \neq 0$ in Eq. (1), the topmost band is no longer ideal, and we will refer to it as an adiabatic band. However, we expect the FQH-like ground state of the AC Hamiltonian to survive for a range of strengths of the residual potential.

Many-body calculations. We fractionally fill the highest-energy band of the Hamiltonian in Eq. (1) with spinless holes and add band-projected unscreened Coulomb interactions, $V(\mathbf{r}) = e^2/\epsilon|\mathbf{r}|$, with ϵ the dielectric constant determining the interaction strength. The many-body Hamiltonian is

$$H = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}} + \frac{1}{2} \sum_{\substack{k_i, k_j \\ k_k, k_l}} V_{ijkl} c_{k_i}^{\dagger} c_{k_j}^{\dagger} c_{k_l} c_{k_k}, \quad (5)$$

where $\epsilon_{\mathbf{k}}$ are single-particle energies, $c_{\mathbf{k}}^{\dagger} (c_{\mathbf{k}})$ creates (destroys) a spinless hole with momentum $\mathbf{k}$, and V_{ijkl} is a two-particle matrix element. We then obtain the many-body spectrum by performing ED; see the Supplemental Material [41] for details on our numerics. Because we are projecting interactions to the lowest band, the energy scales determining the spectrum are the interaction strength $U_{\text{int}} = e^2/\epsilon\ell$ and, if present, the strength of the residual potential $U(\mathbf{r})$. For results presented in the main text, we fix the value of the dielectric constant to $\epsilon = 10$, which results in the interaction strength $U_{\text{int}} \approx 3.4\omega_c$. In the one-band-projected limit, tuning ϵ only renormalizes the many-body gap, as we show in the Supplemental Material [41]. When remote bands are considered, the value of the mixing parameter $\kappa = U_{\text{int}}/\omega_c$, where $\omega_c = 1/m\ell^2$ is the cyclotron frequency, may affect the nature of the ground state, but this is beyond the scope of the present work.

For simplicity, we assume both functions in Eq. (2) have sixfold symmetry and truncate them to the first shell of reciprocal lattice vectors, with magnitude $|\mathbf{G}| = G_0$. In this approximation, the first-shell harmonics, denoted by B_1 and U_1 , serve as tuning parameters that control the Berry curvature distribution and deviations from ideal geometry of the topmost band of Eq. (1). In the Supplemental Material [41], we justify this approximation by presenting a comparison between the continuum model, the first-shell adiabatic model, and an adiabatic model that includes 25 harmonics at both the single-particle and many-body levels. The results confirm that the first-shell adiabatic approximation accurately captures the behavior of the topmost continuum model band. In the following, B_1 has units of $1/\ell^2$ and U_1 has units of ω_c [48]. For calculations presented in the main text, we mainly focus on hole filling $\nu = 1/3$; in the Supplemental Material [41],

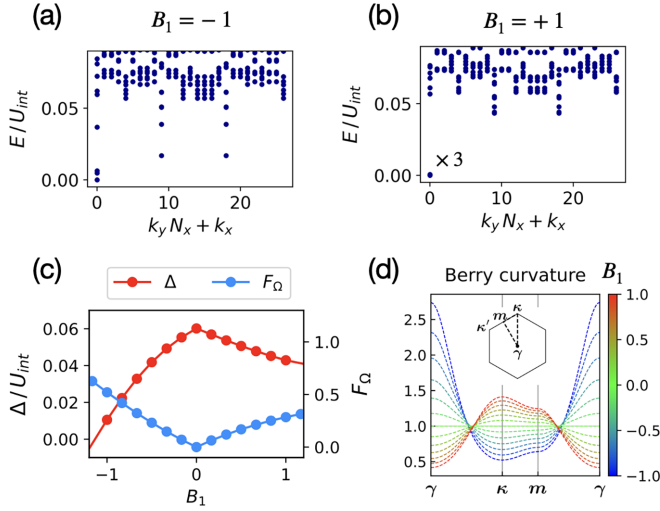


FIG. 1. (a), (b) Many-body spectra at filling $\nu = 1/3$ for an AC band with $B_1 = -1$ and $B_1 = +1$, respectively. (c) Dependence of the magnetoroton gap Δ at filling $\nu = 1/3$ (red) and Berry curvature standard deviation F_Ω [see Eq. (6)] of the AC band (blue) as a function of B_1 . (d) Berry curvature distribution of the AC band as a function of B_1 . Calculations were performed for systems with 27 unit cells.

we also consider hole filling $\nu = 2/3$. We observe a weak particle-hole asymmetry with respect to half-filling for both $B_1 \neq 0$ and $U_1 \neq 0$, which is expected due to the nontrivial quantum geometry of the topmost band [49,50].

Effects of inhomogeneous Berry curvature on ideal band FCIs. We first study the many-body spectrum of AC bands, which are obtained from the Hamiltonian in Eq. (1) with $U(\mathbf{r}) = 0$. Despite the AC band being ideal, an exact zero-energy Laughlin-like ground state at $\nu = 1/3$ is not required, as the interaction we chose contains terms beyond the first Haldane pseudopotential. In the Supplemental Material [41], we present results confirming that the FCI is stable to Berry curvature fluctuations for the Haldane pseudopotential interaction. In contrast, for the long-range Coulomb interaction, while we observe an FCI ground state for $B(\mathbf{r}) = 0$, strong Berry curvature inhomogeneity drives a transition out of that state. Previous studies of this transition simultaneously included deviations from the ideal band condition [51–55]. The unique aspect of our work is that we have isolated the effect of Berry curvature fluctuations from deviations from idealness on the nature of the ground state. We achieve this by tuning B_1 , which determines the Berry curvature of the ideal AC band via Eq. (4).

Figures 1(a) and 1(b) show the many-body spectrum at filling $\nu = 1/3$ for AC bands with $B_1 = -1$ and $B_1 = +1$, respectively. For both calculations, the ground state remains in the same universality class, with three quasidegenerate ground states in the same momentum sector, but the Berry curvature distribution significantly affects the low-energy excitation spectrum. Specifically, the gap to the lowest neutral excitation at κ/κ' , which we identify as a magnetoroton [56], is much smaller when B_1 is negative. Figure 1(c) shows this asymmetry explicitly by plotting the neutral gap, $\Delta = \varepsilon_4 - \varepsilon_3$, where ε_n is the n th many-body energy level.

We quantify Berry curvature inhomogeneity of the band through

$$F_\Omega = \sqrt{\int_{\text{BZ}} \frac{d^2\mathbf{k}}{A_{\text{BZ}}} \left(\frac{\Omega_{\mathbf{k}}}{\Omega_0} - C \right)^2}, \quad (6)$$

where C is the Chern number and Ω_0 the average Berry curvature. Figure 1(c) shows F_Ω as a function of B_1 , while Fig. 1(d) shows the Berry curvature distribution of the AC band across the Brillouin zone, as a function of B_1 . The location of the Berry curvature peaks in Fig. 1(d) follows from a perturbative expansion of Eq. (4) in B_1 , detailed in the Supplemental Material [41],

$$\Omega_{\mathbf{k}} \approx \Omega_0 - \frac{4\pi\ell^2 B_1 e^{-\frac{g_0^2 \ell^2}{4}}}{\sqrt{3}G_0^2} \sum_{j=1}^6 e^{i\mathbf{k} \cdot \mathbf{R}_j} + O(B_1^2), \quad (7)$$

where $\mathbf{R}_j$ are the six first-shell lattice vectors determining the periodicity of $B(\mathbf{r})$. We see that for $B_1 > 0$, Eq. (7) predicts that the Berry curvature has peaks at κ/κ' , while for $B_1 < 0$ it predicts a single peak at γ . However, Fig. 1(d) also reveals an asymmetry in the magnitude of peaks/troughs beyond the linear expansion in Eq. (7).

Figure 1(c) shows that the maximum magnetoroton gap Δ is attained for homogeneous Berry curvature, i.e., $B_1 = 0$, and that fluctuations in Berry curvature lead to an approximately linear decrease in Δ where the slope depends on the sign of B_1 . To understand this behavior, we use the single-mode approximation to define low-energy neutral excitations—magnetorotons—on top of the AC ground state at $\nu = 1/3$. As discussed in the Supplemental Material [41], discrete translational symmetry imposed by the periodicity of $B(\mathbf{r})$ couples states that differ by a reciprocal lattice vector, causing the degenerate states at the κ/κ' points to split linearly with B_1 , while other states generically receive a correction of order B_1^2 . This leads to a generalized eigenvalue problem for the energies of excited states at the corners of the Brillouin zone, which in the limit $B_1 \ll 1$ has eigenvalues given by

$$E_j^{AC}(\kappa) = E^{\text{LLL}}(\kappa) + B_1 \text{Re}[e^{i\frac{2\pi j}{3}} \tilde{a}], \quad j = 0, \pm 1, \quad (8)$$

where $E^{\text{LLL}}(\kappa)$ is the energy of the neutral excitation when the field is uniform ($B_1 = 0$) and $\tilde{a}$ is a complex number defined in the Supplemental Material [41] that can be computed by evaluating expectation values of density operators on the LLL Laughlin state. The magnetoroton gap is obtained by selecting the lowest excitation and subtracting the ground-state energy, yielding

$$\Delta(\kappa) \approx \begin{cases} \Delta^{\text{LLL}}(\kappa) - \alpha |B_1|, & \text{for } B_1 < 0, \\ \Delta^{\text{LLL}}(\kappa) - \beta |B_1|, & \text{for } B_1 > 0. \end{cases} \quad (9)$$

Expressions for the coefficients α and β are provided in the Supplemental Material [41]; they are real and satisfy $\alpha > \beta > 0$, according to our numerics. Our SMA magnetoroton model captures the asymmetric behavior of Δ with respect to $B_1 = 0$ observed in Fig. 1(c). For completeness, in the Supplemental Material [41], we present momentum occupation plots $n_{\mathbf{k}}$, which show a smaller occupation in regions of the Brillouin zone where the Berry curvature is peaked, in agreement with Ref. [50]. Ultimately the large inhomogeneity in $\Omega_{\mathbf{k}}$, which translates into a large F_Ω , leads to a charge density wave

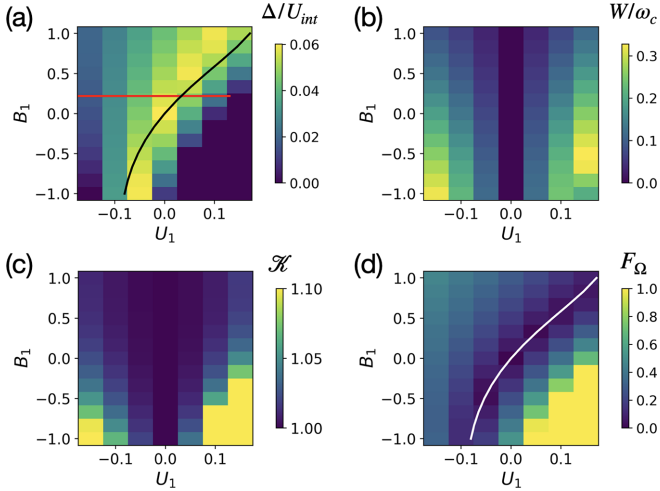


FIG. 2. (a) Magnetoroton gap Δ as a function of U_1 and B_1 ; $\Delta \leq 0$ indicates that the FCI is not the ground state. The red line indicates the parameters obtained from applying the first-shell adiabatic approximation to a continuum model of twisted MoTe₂ [57] for twist angles $\theta \sim 2.5^\circ$ – 5.0° . The black line traces the minimum value of F_Ω . (b) Bandwidth W , (c) quantum weight $\mathcal{K}$ (see main text), and (d) Berry curvature standard deviation for adiabatic bands, as a function of U_1 and B_1 . The white line in panel (d) indicates the minimum value of F_Ω [same as the black solid line in panel (a)]. The optimal lines in panels (b) and (c) correspond to the axis $U_1 = 0$.

(CDW) with $\Delta < 0$ for $B_1 < 0$, as can be seen in Fig. 1(c). In the Supplemental Material [41], we also present ED results for $\nu = 2/3$, which show a similar behavior of the many-body gap. However, at $\nu = 2/3$ the FCI might transition into other phases rather than a CDW and the SMA argument might not apply.

Effects of inhomogeneous Berry curvature on adiabatic band FCIs. In contrast to the AC bands studied in the previous section, bands that describe the low-energy physics of two-dimensional materials generically have a nonvanishing dispersion and do not satisfy the trace condition. To recover that scenario, we restore the residual potential $U(\mathbf{r})$ in the adiabatic Hamiltonian, Eq. (1), yielding a nonideal dispersive topmost band with nonuniform Berry curvature. Tuning the parameter U_1 will simultaneously change the Berry curvature distribution, destroy the trace condition, and introduce a finite bandwidth. These effects together ultimately determine the interacting ground state.

Figure 2(a) shows the evolution of Δ as a function of B_1 and U_1 . The region of FCI stability is determined by $\Delta > 0$. Figures 2(b)–2(d) show the bandwidth W , quantum weight $\mathcal{K}$, and Berry curvature inhomogeneity F_Ω , respectively, for adiabatic bands as a function of B_1 and U_1 . The quantum weight is defined as the Brillouin zone average of $\text{tr}g_k$ divided by 2π [58–60], which equals the Chern number C for ideal bands and thus measures the violation of trace condition. Both W and $\mathcal{K}$ are optimized at $U_1 = 0$. Deviations from the optimal value of these quantities contribute to the reduced stability of the FCI. The data reveal a surprising observation: For fixed $B_1 > 0$, the largest many-body gap occurs not for the ideal band ($U_1 = 0$), but for a nonideal band with finite $U_1 > 0$. Similarly, for fixed $B_1 < 0$, the largest many-body gap occurs

for a nonideal band with $U_1 < 0$. This results in a larger region of FCI stability for bands with $B_1 > 0$ than for the LLL, $B_1 = 0$. Comparing Figs. 2(a)–2(d) reveals that the maximum many-body gap approximately coincides with the minimum in F_Ω , suggesting that the Berry curvature variation may be more important than satisfying the ideal band condition in this nearly ideal parameter regime. Our numerical results are consistent with a perturbative calculation of the many-body gap using the SMA, as detailed in the Supplemental Material [41]. The inclusion of the residual potential shifts the value of B_1 at which the many-body gap in Eq. (9) is maximized from $B_1 = 0$ to $B_1 \neq 0$.

The behavior of F_Ω for adiabatic bands can be understood as follows: The Berry curvature induced by $U_1 > 0$ ($U_1 < 0$) and $B_1 > 0$ ($B_1 < 0$) is peaked in opposite locations in the Brillouin zone, so that the adiabatic Hamiltonian with finite U_1, B_1 has smaller Berry curvature fluctuations than when either term separately vanishes. In the Supplemental Material [41], we present plots for the Berry curvatures of the adiabatic bands to illustrate this behavior. We note that in our analysis it is crucial that U_1 is small (namely, $|U_1| \ll \omega_c$ and smaller than the many-body gap at $U_1 = 0$), which constrains us to the regime of weak trace condition violation. This is in contrast to previously studied topological band models where flattening the Berry curvature can destabilize the FCI phase [61] but the trace condition is strongly violated.

Application to moiré materials. The adiabatic Hamiltonian studied here gives insight into the stability of FCIs in twisted homobilayer TMDs. Specifically, the continuum model for the latter can be mapped to Eq. (1) by replacing the layer degree of freedom by a periodic magnetic field and a periodic potential, and discarding all but the lowest harmonics of $B(\mathbf{r})$ and $U(\mathbf{r})$. We applied the adiabatic approximation described in Refs. [36,37] to a particular model of twisted MoTe₂ (tMoTe₂) [57]. For the first-shell truncation, the harmonic of the magnetic field is independent of twist angle, $B_1 = 0.22 [1/\ell^2]$, and the harmonic of the residual potential in units of meV is given by

$$U_1 = \hbar\omega_c \Omega_1 + \Delta_1 = \left(\frac{4\pi\hbar^2\Omega_1}{\sqrt{3}m^*a_0^2} \right) \theta^2 + \Delta_1, \quad (10)$$

with θ in degrees, $a_0 = 0.352$ nm the monolayer lattice constant, and $m^* = 0.62 m_0$ the effective mass. $\Omega_1 = 0.23$ and $\Delta_1 = -7.36$ meV are parameters specific to the model in Ref. [57]. The red line in Fig. 2(a) traces the values of U_1 that correspond to twist angles within the experimentally relevant range $\theta \sim 2.5^\circ$ – 5.0° , justifying the parameter ranges that we have chosen for our study. Figure 3 shows the evolution of the many-body gaps at $\nu = 1/3$ and $\nu = 2/3$ as a function of twist angle for the continuum model [57] and two adiabatic models: one that keeps several harmonics in $B(\mathbf{r})$ and $U(\mathbf{r})$, and the first-shell truncation. We see that the adiabatic approximation reproduces the qualitative behavior of the continuum model many-body physics. For completeness, in the Supplemental Material [41] we show typical many-body spectra for the three models in Fig. 3, which display remarkable qualitative agreement. This result suggests that the stability of the FCI phase in semiconductor moiré systems over a large range of

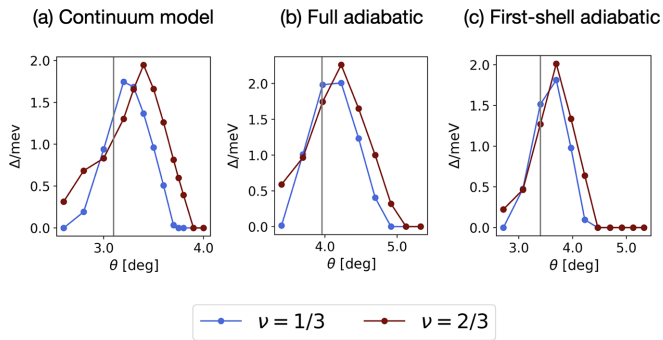


FIG. 3. Many-body gap as a function of twist angle for (a) the continuum model of tMoTe₂ from Ref. [57], (b) the adiabatic approximation to the continuum model including 25 harmonics in $B(\mathbf{r})$ and $U(\mathbf{r})$, and (c) the adiabatic approximation truncated to the first shell of harmonics. The vertical lines indicate the twist angle where the quantum weight K is minimized.

twist angles, as numerically [57,62–66] and experimentally [7–10] observed, can be explained in terms of a simplified model of Landau levels in the presence of periodic spatial modulations.

Discussion. We have numerically and analytically investigated the effect of Berry curvature fluctuations on the many-body gap of FCIs in ideal and near-ideal bands, in the presence of a realistic Coulomb interaction. We derived perturbative expressions for the FCI magnetoroton gaps in terms of the LLL magnetoroton gap. These expressions explain the asymmetry of the neutral FCI gap with respect to the momentum-space position of the Berry curvature maxima, for both AC and adiabatic bands at filling $\nu = 1/3$. This powerful analytical result sheds light on the properties of Berry curvature that favor the stability of FCI ground states in nearly ideal bands and is complementary to other approaches to address FCI excitations [56,67–76]. By exploiting the adiabatic approximation, we have used our results to explain the large stability region for topologically ordered phases in twisted TMDs, where the band geometry is no longer ideal, through a connection to FQH physics. This connection establishes a

framework that provides analytical insights into the role of band geometry on FCIs in real materials, like tMoTe₂.

We focused on spinless fermions, for which the FCI at $\nu = 1/3$ is destabilized by neutral magnetoroton modes. When spin or orbital degrees of freedom are taken into account, other candidate phases can become competitive [52,77]. In the Supplemental Material [41], we present results showing that the valley-flip excitations have higher energies than the lowest valley-polarized excitations, which justifies the restriction to a single valley in this work. Our band-projected results yield finite and comparable magnetoroton gaps for $\nu = 1/3$ and $\nu = 2/3$ (See the Supplemental Material [41]). This is in contrast to experiments in tMoTe₂, where the FCI state is robust at $\nu = 2/3$ and absent at $\nu = 1/3$, pointing to the importance of remote band mixing in realistic systems, as discussed by other authors [78,79]. We leave the analysis of remote band effects in our model for future work.

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N.M.-D. and J.S. designed the project and J.C. supervised it. J.S. calculated the noninteracting band structures and provided the basis wave functions for exact diagonalization. N.M.-D. and J.C. developed the magnetoroton theory. N.M.-D. performed the ED calculations and wrote the paper with input from all authors.

Data Availability. The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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