

Thermalization of quantum many-body scars in kinetically constrained systems

Jia-wei Wang^{1,2,3}, Xiang-Fa Zhou^{1,2,3,4,*}, Guang-Can Guo^{1,2,3}, and Zheng-Wei Zhou^{1,2,3,4,†}

¹Anhui Province Key Laboratory of Quantum Network, *University of Science and Technology of China, Hefei 230026, China*

²Synergetic Innovation Center of Quantum Information and Quantum Physics, *University of Science and Technology of China, Hefei 230026, China*

³Hefei National Laboratory, *University of Science and Technology of China, Hefei 230088, China*

⁴Anhui Center for Fundamental Sciences in Theoretical Physics, *University of Science and Technology of China, Hefei 230026, China*



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The phenomenon of quantum many-body scars (QMBS) has been studied both theoretically and experimentally, due to its unusual violation of the eigenstate thermalization hypothesis (ETH). In this paper, we extend the ETH to a description based on the grand canonical ensemble to depict the thermal properties of QMBS models. For this purpose, we embed the dynamics of kinetically constrained systems within a Lindblad-like master equation, and demonstrate that the violation of the ETH by scar eigenstates is related to their slow decay in the corresponding dissipative process. Within this open system description, we reformulate the ETH to demonstrate that both scar eigenstates and thermal ones exhibit thermalization governed by grand canonical statistics. Consequently, our revised ETH unifies scars and thermal states under a cohesive thermodynamic rule. Our work resolves the fundamental tension between constraint-induced nonergodicity and thermalization paradigms, establishing a unified route to generalized thermalization in quantum many-body systems.

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I. INTRODUCTION

Eigenstate thermalization hypothesis (ETH) [1,2], first proposed in the 1990s, provides a foundational perspective on thermalization in quantum systems. ETH posits that for a many-body Hamiltonian H , all eigenstates exhibit thermal behavior. That is, the local characteristics of eigenstates should coincide with the microcanonical average at corresponding energies [3,4]. While ETH has been validated across numerous many-body systems [5–9], remarkable exceptions exist, including strong violations in integrable systems [10–12] and many-body localized phase [13–16], as well as weak violations in systems exhibiting Hilbert space fragmentation [17] or quantum many-body scars (QMBS) [18–22]. The last case is particularly intriguing—while most eigenstates thermalize, a nonthermal subspace of scar eigenstates persists, scaling extensively with system size. QMBS further separate into two distinct groups: (1) exactly solvable ones that constitute a Krylov subspace with well-defined algebraic structures [17,21,23–30] and (2) nonexactly-solvable scar models, which primarily include kinetically constrained systems, such as the well-known PXP model [18,31–36]. In class (2), significant efforts have been made to elucidate the

nonthermal characteristics of such scar eigenstates [37–41]. Notably, non-Hermitian dynamics have also been employed to depict the nonthermal behavior of QMBS in Refs. [42–44], which has served as an inspiration for this work.

Kinetically constrained QMBS models exhibit anomalous nonthermal characteristics despite their nonintegrability [31,32,35]. The absence of rigorous algebraic structures leaves the mechanism of ETH violation unresolved, posing fundamental challenges: To what extent can such a system thermalize, and what statistical principles govern the persistent nonergodicity of scar eigenstates?

This work investigates kinetically constrained systems that exhibit QMBS. We demonstrate that blockade-induced kinetic constraints can be realized by coupling unconstrained dynamics to two distinct dissipation channels, which represent information exchanges between the system and an auxiliary environment. Consequently, thermalization in such systems may encompass not only energy but also the implementation of kinetic constraints. Within the ETH framework, we refine the thermal description at the eigenstate level by introducing a set of variables—quasiparticle number and chemical potential—alongside energy and temperature. These variables serve as quantitative indicators of the information-exchange processes and enable an approximate grand canonical formalism of thermalization.

This paper is organized as follows: Section II develops a general description of kinetically constrained systems by combining positive and negative dissipation channels within a Lindblad-like framework. In the strong-dissipation regime, neglecting the negative channels yields a positive-definite approximation to the dissipative dynamics. In Sec. III, we analyze the behavior of eigenstates under the

*Contact author: xfzhou@ustc.edu.cn

†Contact author: zwzhou@ustc.edu.cn

constraint-dependent dissipations. Fixing a narrow energy window, we find that the scar states exhibit significantly lower decay rates than nearby thermal ones, reflecting their nonthermal characteristics. In Sec. IV, we revise the ETH based on the open system dynamics, utilizing a grand-canonical-like ensemble with constraint-dependent thermodynamic quantities. We numerically verify that this ensemble captures the local properties of both scar and thermal eigenstates, indicating that the constrained QMBS models studied here thermalize in accordance with the grand canonical framework. We conclude this work in the final section.

II. MASTER EQUATIONS GOVERNING GENERAL CONSTRAINED SYSTEMS

A general constrained system can be described by a series of blockade $\mathcal{B} = \{\pi_1, \pi_2, \dots, \pi_L\}$ acting on an unconstrained Hamiltonian $H_0 = \sum_{k=1}^D h_k$, where h_k is the k th local Hamiltonian and π_m 's are projectors satisfying $\pi_m^2 = \pi_m$, $[\pi_m, \pi_n] = 0$ for $\forall \pi_{m,n} \in \mathcal{B}$. Here, L and D denote the total number of the projectors and the local Hamiltonian h_k , respectively. Then, the projector onto the constrained Hilbert subspace $\mathcal{H}$ is $\hat{P} \equiv \prod_{k=1}^L (\mathbb{I} - \pi_k)$ and the constrained Hamiltonian writes

$$\begin{aligned} H &= \sum_{k=1}^D P_k h_k P_k = \sum_{k=1}^D (\mathbb{I} - \Pi_k) h_k (\mathbb{I} - \Pi_k) \\ &= \sum_{k=1}^D h_k - \Pi_k h_k - h_k \Pi_k + \Pi_k h_k \Pi_k, \end{aligned} \quad (1)$$

where $P_k = \prod_{[\pi_m, h_k] \neq 0} (\mathbb{I} - \pi_m)$ denotes the constraint comprising all the blockades that do not commute with h_k , and $\Pi_k \equiv \mathbb{I} - P_k$ marks the local blockade acting on h_k . H commutes with $\hat{P}$ and serves as the Hamiltonian governing the constrained dynamics within $\mathcal{H}$, where its eigenstates satisfy $H|E_i\rangle = E_i|E_i\rangle$, $\hat{P}|E_i\rangle = |E_i\rangle$. For clarity, we clarify the notation: $\hat{P}$ and P_k denote the projections onto the overall and locally constrained subspaces, respectively, while the Greek letters π_k and Π_k refer to the projectors targeting the prohibited states, as defined above.

In Eq. (1), $\sum_{k=1}^D h_k$ remains the unconstrained Hamiltonian, and the rest terms constitute the blockade interactions. In particular, the term $\Pi_k h_k$ cancels the mapping from the constrained subspace $\mathcal{H}$ to the blocked states, while $(\mathbb{I} - \Pi_k) h_k \Pi_k$ stops the latter mapping back to $\mathcal{H}$. The back mapping terms vanish when acting on any states in $\mathcal{H}$ from the left side, since we have $\Pi_k \hat{P} = (\mathbb{I} - P_k) \hat{P} = 0$. Hence, these terms can be altered without affecting the dynamics within $\mathcal{H}$. Specifically, we replace the term $(\mathbb{I} - \Pi_k) h_k \Pi_k$ with $\Pi_k h_k$'s anti-Hermitian conjugate and another imaginary part, and build up a non-Hermitian Hamiltonian as

$$\begin{aligned} H_N &= \sum_{k=1}^D h_k - \Pi_k h_k + h_k \Pi_k - ic \Pi_k \\ &= H_0 - \frac{i}{2} \sum_{\sigma, k=1}^D \gamma_\sigma L_{k, \sigma}^\dagger L_{k, \sigma}, \end{aligned} \quad (2)$$

where i stands for the imaginary unit and $c > 0$ is a tunable real parameter. The terms $-\Pi_k h_k + h_k \Pi_k - ic \Pi_k$ align with

the outcomes of the dissipative dynamics defined by two types of engineered dissipation channels $\sigma = \{1, 2\}$ as

$$\begin{aligned} \gamma_1 &= 2c, \quad L_{k,1} = \Pi_k - \frac{i}{c} \Pi_k h_k, \\ \gamma_2 &= -\frac{2}{c}, \quad L_{k,2} = \Pi_k h_k, \end{aligned} \quad (3)$$

where γ_σ 's denote the dissipation rates and $L_{k,\sigma}$'s are the dissipation operators. Then, we write out the Lindblad-like equation for the dissipative system as

$$\partial_t \rho = \mathcal{L}(\rho) \equiv -iH_N \rho + i\rho H_N^\dagger + \mathcal{J}(\rho), \quad (4)$$

where $\mathcal{L}$ is the Liouvillian superoperator acting on the system's density matrix (DM) ρ . The Hamiltonian H_N shares the same right eigenvector $|E_i\rangle$ with H , and the jumping term $\mathcal{J}(\rho) = \sum_{k,\sigma} \gamma_\sigma L_{k,\sigma} \rho L_{k,\sigma}^\dagger = \sum_{k=1}^D 2c \Pi_k \rho \Pi_k - 2i \Pi_k h_k \rho \Pi_k + 2i \Pi_k \rho h_k \Pi_k$ annihilates any ρ within the constrained subspace. In principle, the two types of engineered dissipations combine perfectly to form the effective constraints, while within $\mathcal{H}$, the corresponding jumping terms of both types adequately cancel each other. Hence, Eq. (4) exhibits the same constrained dynamics as H , with $\mathcal{H}$ serving as a jump-free subspace (JFS). This JFS differs from the conventional decoherence-free subspace, which is annihilated by all dissipators simultaneously [45,46].

In contrast to the Lindblad master equation, where the dissipation rates are positive according to the Born-Markov approximation (BMA), the coefficients γ_1 and γ_2 in Eq. (4) have opposite signs, indicating the presence of information feedback and non-Markovian characteristics [47]. However, due to its equivalence to the unitary dynamics governed by H , the Lindblad-like equation remains positive definite and trace-preserving within the JFS. Notably, the coefficient γ_2 tends to zero as the parameter $c \rightarrow +\infty$. In this limit, we can employ only the positive dissipations, yielding a master equation $\partial_t \rho = \mathcal{L}_+(\rho)$ comprised of

$$\begin{aligned} H_+ &\equiv H_0 - \frac{i}{2} \sum_{k=1}^D \gamma_1 L_{k,1}^\dagger L_{k,1} = H_N - \frac{i}{c} \sum_{k=1}^D h_k \Pi_k h_k, \\ \mathcal{J}_+(\rho) &\equiv \sum_{k=1}^D \gamma_1 L_{k,1} \rho L_{k,1}^\dagger = \mathcal{J}(\rho) + \frac{2}{c} \sum_{k=1}^D L_{k,2} \rho L_{k,2}^\dagger. \end{aligned} \quad (5)$$

The Liouvillian superoperator $\mathcal{L}_+$ is both trace preserving and positive definite within the entire Hilbert space, due to the positive dissipation rate γ_1 [48–51]. In Eq. (5), we express the jumping terms as the sum of two parts. Physically, when the parameter c is large, the quantum jumps $2c \Pi_k \rho \Pi_k$ from $\mathcal{J}(\rho)$ will confine the system to the constrained subspace $\mathcal{H}$ due to Zeno effect [52,53]. Meanwhile, the leakage out of $\mathcal{H}$ caused by the second term $2/c \sum_{k=1}^D L_{k,2} \rho L_{k,2}^\dagger = 2/c \sum_{k=1}^D \Pi_k h_k \rho h_k \Pi_k$ is also relatively small. Hence, the dynamics governed by $\mathcal{L}_+$ increasingly converge toward the constrained dynamics within $\mathcal{H}$. Experimentally, $\mathcal{L}_+$ can be approximated by compensating the total constrained Hamiltonian $\hat{P} H \hat{P}$ with dissipation channels $\{\gamma' = -\gamma_2, L'_k = L_{k,2}\}$. Provided that the dissipation rate $\gamma' \propto 1/c$ is sufficiently small, this results in a Lindblad master equation $\partial_t \rho = \mathcal{L}'_+(\rho)$ satisfying the BMA [54].

In order to demonstrate the feasibility of our framework, we take a 1D spin model hosting QMBS as an example. The unconstrained Hamiltonian comprises D spins of size $j \geq 1$: $H_0 = \sum_{k=1}^D s_k^x$. s_k^x is the k th spin operator in the x direction. The blockade is given by

$$\mathcal{B} = \{\pi_k = |x\rangle\langle x|_{k,k+1}; k = 1, 2, \dots, D\}, \quad (6)$$

where π_k denotes a projector of the product state $|x\rangle_{k,k+1} = |j\rangle_k \otimes |-j\rangle_{k+1}$ on k th and $k+1$ th sites, with $|m\rangle_k$ the z -direction spin eigenvector at the k th site: $s_k^z|m\rangle_k = m|m\rangle_k$, $m = -j, -j+1, \dots, j$. $\mathcal{B}$ prohibits the neighboring spins from occupying $|x\rangle_{k,k+1}$, and arranges the spins in a 1D chain with periodical boundary. Based on our framework, the non-Hermitian Hamiltonian writes $H_N = \sum_{k=1}^D s_k^x - \sqrt{j}M_k + \sqrt{j}M_k^\dagger - ic\sqrt{j/2}\pi_k$, where $M_k \equiv \frac{1}{\sqrt{j}}\pi_k(s_k^x + s_{k+1}^x) = |x\rangle\langle y|_{k,k+1}$ denotes the mapping from $\mathcal{H}$ to the blockaded region, with $|y\rangle_{k,k+1} = \frac{1}{\sqrt{2}}(|j\rangle_k \otimes |-j+1\rangle_{k+1} + |-j+1\rangle_k \otimes |j\rangle_{k+1})$. Hence, the dissipation channels of the corresponding Lindblad-like master equations can be defined as

$$\begin{aligned} \gamma_1 &= c\sqrt{2j}, \quad L_{k,1} = \pi_k - \frac{i\sqrt{2}}{c}M_k, \\ \gamma_2 &= -\sqrt{2j}\frac{2}{c}, \quad L_{k,2} = M_k. \end{aligned} \quad (7)$$

Notably, we have made minor model-dependent adjustments regarding the spin model above, resulting in slight differences between the dissipations in Eqs. (3) and (7); detailed analysis is available in Appendix A.

This constrained spin model is known to host QMBS within $\mathcal{H}$, and displays a quasiperiodical revival of the QMBS initial state: $|\psi(0)\rangle = |j, j, \dots, j\rangle$ [35]. Besides, when the spin size is $j = 1$, this model becomes equivalent to the PXP model [35,37]. In Fig. 1, we show the differences among H , $\mathcal{L}_+$, and $\mathcal{L}'_+$ via the evolution of the fidelity $\langle\psi(0)|\rho(t)|\psi(0)\rangle$. We confirm that as c increases, the evolutions of the two master equations nearly overlap, and they gradually approach the unitary dynamics given by H .

III. SLOW DECAY AND THE VIOLATION OF ETH

The dissipative mechanism provides a perspective on the constrained systems. In particular, the kinetic constraint can be viewed as arising from a two-step dissipative process: The first step, described by the positive dissipation, represents the outflow of information from the system to an auxiliary environment, while the second accounts for feedback from the environment to the system. In the large c regime, by neglecting the negative dissipation component of the constrained dynamics, allowing us to probe how the eigenstates evolve under this process. Specifically, when we focus on a specific energy and examine a narrow window around it, the scar eigenstates appear to show exceptionally lower decay rates than the thermal ones during the dissipative process defined by $\partial_t \rho = \mathcal{L}_+(\rho)$.

Starting from an eigenstate $|E_i\rangle$ of H , the system is approximately fixed during the jumpless process, since H_+ is dominated by the Hamiltonian H_N . Moreover, any quantum jump of $\mathcal{J}_+$ will project the system outside the constrained

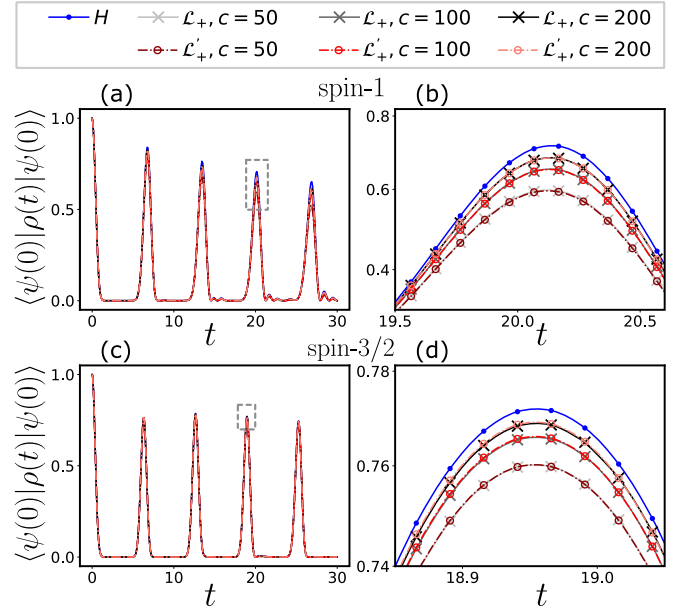


FIG. 1. The evolution of the initial states' fidelity in the constrained system with Hamiltonian H , compared with that under the master equations marked by $\mathcal{L}_+$ and $\mathcal{L}'_+$. In panels (a) and (b), we show the evolution of the model defined by Eqs. (6) and (7) in a spin-1 chain with nine sites, while panels (c) and (d) correspond to a spin-3/2 chain with seven sites. We have chosen the initial state as $|\psi(0)\rangle$. Panels (b) and (d) present detailed views of the regions enclosed within the gray boxes in panels (a) and (c), respectively.

subspace, ensuring that the resultant state is located in the blockaded area. Hence, the DM of the system approximately writes $\rho(t) \approx f_i(t)|E_i\rangle\langle E_i| + (\mathbb{I} - \hat{P})\rho(t)(\mathbb{I} - \hat{P})$, with the fidelity $f_i(t) = \langle E_i|\rho(t)|E_i\rangle$ satisfying

$$\begin{aligned} \frac{df_i(t)}{dt} &= \left\langle \frac{d\rho}{dt} \right\rangle_i = -i\langle H_+\rho - \rho H_+^\dagger \rangle_i \\ &\approx \gamma_2 \sum_{k=1}^D \langle L_{k,2}^\dagger L_{k,2} \rangle_i \cdot f_i(t) = -\alpha_i f_i(t), \end{aligned} \quad (8)$$

where $\langle \cdot \rangle_i = \langle E_i| \cdot |E_i\rangle$ denotes the expectation value under the eigenstate $|E_i\rangle$. We note that the jumping term $\mathcal{J}_+(\rho)$ vanishes when acting on the eigenstates. By defining $N_k = L_{k,2}^\dagger L_{k,2}$ and $\hat{N} = \sum_{k=1}^D N_k$, we obtain $f_i(t) \approx e^{-\alpha_i t}$ with the decay rate $\alpha_i \approx -\gamma_2 \langle \hat{N} \rangle_i$. For the spin model introduced above, $N_k = M_k^\dagger M_k$ is a projector on the neighboring state $|y\rangle_{k,k+1}$. $\hat{N} = \sum_{k=1}^D N_k$ counts the number of pattern $|y\rangle_{k,k+1}$ throughout the spin chain.

For eigenstates within a small energy interval ΔE , Fig. 2(c) illustrates the evolution of the fidelity $f_i(t)$, which is fitted with an exponential function $e^{-\alpha_i t}$. Then, we confirm the proportional relation $\alpha_i \approx -\gamma_{fit} \langle \hat{N} \rangle_i$ and verify that γ_{fit} approximates $\gamma_2 = -2\sqrt{2j}/c$ in Figs. 2(d) and 2(e). In Fig. 2(c), the scar eigenstate shows a lower decay rate than the thermal states close in energy. Correspondingly, in Fig. 2(b), the scar states demonstrate lower expectation values $\langle \hat{N} \rangle_i$, confirming their slower decay under the master equation $\partial_t \rho = \mathcal{L}_+(\rho)$.

Experimentally, this dissipative mechanism can be achieved by building a Markovian open system governed by

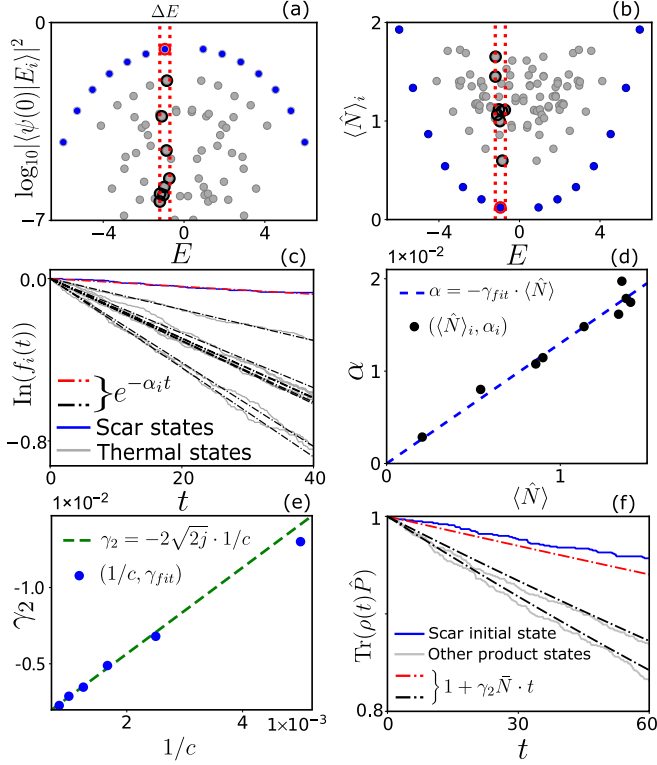


FIG. 2. The exponential decay of the eigenstates' fidelity. Panels (a) and (b) show the eigenstates' overlaps with $|\psi(0)\rangle$, and their expectation values of $\hat{N}$. We pick nine eigenstates from a small energy interval ΔE and illustrate the evolutions of $f_i(t)$ in panel (c). The blue dots and the blue curve denote scar states, while the gray ones denote thermal states. The dashed lines are the fitted exponential functions, yielding a parameter α_i for each $|E_i\rangle$. In panel (d), for nine randomly picked eigenstates, we perform a linear fit of α_i against $\langle \hat{N} \rangle_i$, obtaining a near-linear relation: $\alpha_i \approx -\gamma_{fit} \cdot \langle \hat{N} \rangle_i$. Panels (c) and (d) are computed with the parameter $c = 200$. In panel (e), by increasing c to 4, 6, 8, 10, 12 $\times 10^2$, γ_{fit} shows a near-linear relation with $1/c$: $\gamma_{fit} \approx \gamma_2 = -2\sqrt{2j} \cdot 1/c$. Panel (f) shows the time evolution of $\text{Tr}(\rho(t)\hat{P})$ under $\partial_t \rho = \mathcal{L}'_+(\rho)$, with $c = 1200$. The blue line indicates the initial state $|\psi(0)\rangle$, and the gray ones mark two other product states. Dashed lines indicate the estimated leakage $1 + \gamma_2 \bar{N}t$. The calculations are executed on a spin-1 chain with seven sites, employing 500 quantum trajectories [55–58].

$\partial_t \rho = \mathcal{L}'_+(\rho)$. For any initial state $|\phi\rangle = \sum_{i=1}^{\mathcal{D}} a_i |E_i\rangle$ located in $\mathcal{H}$, the system will leak out of $\mathcal{H}$ as $\text{Tr}(\rho(t)\hat{P}) \propto 1 + \gamma_2 \bar{N} \cdot t$. $\mathcal{D}$ denotes the dimension of the constrained subspace $\mathcal{H}$. $\bar{N} = \sum_{i=1}^{\mathcal{D}} |a_i|^2 \langle \hat{N} \rangle_i$ is the long-time average of $\hat{N}$ during the unitary evolution. In Fig. 2(f), we set the initial states to be product states with the same energy $\langle \phi | H | \phi \rangle = 0$ and confirm this by calculating the evolution of expectation value $\text{Tr}(\rho(t)\hat{P})$.

IV. REVISED THERMALIZATION THEORY BASED ON GRAND CANONICAL FRAMEWORK

ETH posits that for a nonintegrable many-body Hamiltonian H , the matrix elements of any local observable $\hat{O}$ in the

energy eigenbasis should satisfy [59–61]

$$\langle E_i | \hat{O} | E_{i'} \rangle = \mathcal{O}(\mathcal{E}) \delta_{i,i'} + \Omega(\mathcal{E})^{-1/2} f(\mathcal{E}, \omega) R_{i,i'}, \quad (9)$$

where $\mathcal{E} \equiv (E_i + E_{i'})/2$ and $\omega \equiv E_i - E_{i'}$ denote the energies average and difference of the two eigenstates, and $\Omega(\mathcal{E})$ is the density of states (DOS) at the energy $\mathcal{E}$. $\mathcal{O}(\mathcal{E})$ and $f(\mathcal{E}, \omega)$ are smooth functions of $\mathcal{E}$ and ω . $\delta_{i,i'}$ is the Kronecker delta function, while $R_{i,i'}$'s are erratically varying $\mathcal{O}(1)$ numbers. $\mathcal{O}(\mathcal{E})$ approaches the canonical average in the thermodynamic limit:

$$\mathcal{O}(\mathcal{E}) \sim \langle \hat{O} \rangle_{\beta_c} = \text{Tr}(e^{-\beta_c H} \hat{O}) / \text{Tr}(e^{-\beta_c H}), \quad (10)$$

where $\rho_c(\beta_c) = e^{-\beta_c H} / \text{Tr}(e^{-\beta_c H})$ is the canonical DM and β_c is the inverse-temperature calculated by fixing the energy: $\mathcal{E} = \text{Tr}(\rho_c(\beta_c) H)$. If the above conditions are met, ETH asserts that for any physically preparable initial state $|\phi\rangle$, the long-time average of $\hat{O}$ converges to $\mathcal{O}(\mathcal{E})$, with $\mathcal{E} = \langle \phi | H | \phi \rangle$.

Based on the above discussions, we realize that the eigenstate $|E_i\rangle$ of the constrained systems can be interpreted as a dynamic equilibrium state in open systems at a certain energy $\mathcal{E} = E_i$. Besides, the dissipative mechanism contributes to the state occupancy of the dynamic equilibrium. Specifically, since the expectation value of the operator $\hat{N} = \sum_{k=1}^{\mathcal{D}} N_k = \sum_{k=1}^{\mathcal{D}} L_{k,2}^\dagger L_{k,2}$ indicates the leaking rate from the system to the auxiliary environment, hence reflecting how each eigenstates would react to the effective kinetic constraints, we can regard N_k as a local “quasiparticle” operator, and $\hat{N} = \sum_{k=1}^{\mathcal{D}} N_k$ as the quasiparticle counter. Evidently, the energy E and the quasiparticle number jointly permit an effective description of the equilibrium state. Thus, for any eigenstate $|E_i\rangle$ within $\mathcal{H}$, we propose an adjustment to the diagonal terms from Eq. (9) by including another variable $\mathcal{N}$:

$$\langle E_i | \hat{O} | E_i \rangle \sim \mathcal{O}(\mathcal{E}, \mathcal{N}), \quad (11)$$

where $\mathcal{N} = \langle \hat{N} \rangle_i$ is the expectation value of the quasiparticle counter $\hat{N}$ under $|E_i\rangle$, and $\mathcal{O}(\mathcal{E}, \mathcal{N})$ is a smooth function of $\mathcal{E}$ and $\mathcal{N}$. For a general constrained model, Eq. (11) states that the energy eigenstates' expectation value of any local observable should approximate the value of a smooth function uniquely determined by the eigenstate's energy and its quasiparticle number $\langle \hat{N} \rangle_i$. In Fig. 3, we demonstrate that the expectation values of the scar states cannot be unified with the majority of other eigenstates in a smooth function $\mathcal{O}(\mathcal{E})$. However, all the expectation values can be unified in a smooth two-variable function $\mathcal{O}(\mathcal{E}, \mathcal{N})$, certifying the above discussions. Thus, all the eigenstates can be viewed as thermalized, as their local properties can be effectively described by Eq. (11).

Formally, the value of $\mathcal{O}(\mathcal{E}, \mathcal{N})$ should coincide with the ensemble average of the following grand-canonical-like DM:

$$\rho(\beta, \mu) \equiv \frac{\sum_{i=1}^{\mathcal{D}} e^{-\beta(E_i - \mu \langle \hat{N} \rangle_i)} |E_i\rangle \langle E_i|}{\sum_{i=1}^{\mathcal{D}} e^{-\beta(E_i - \mu \langle \hat{N} \rangle_i)}}, \quad (12)$$

where β and μ are two thermodynamic variables describing the equilibrium. Specifically, β is the inverse temperature and μ can be regarded as the chemical potential describing the dynamic equilibrium of quasiparticle exchange. This ensemble results from the dissipative mechanism employed

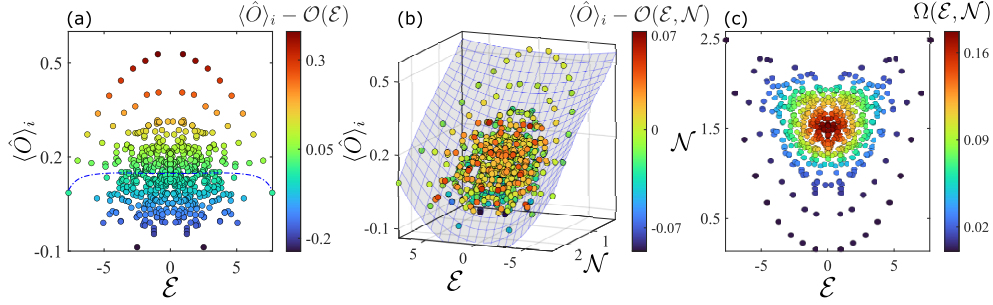


FIG. 3. The expectation values $\langle \hat{O} \rangle_i$ for local observable $\hat{O} = s_k^z \otimes s_{k+1}^z$ in a spin-1 chain with nine spins. Panel (a) shows $\langle \hat{O} \rangle_i$'s relation with the energy $\mathcal{E}$, with the blue curve being the canonical average $\mathcal{O}(\mathcal{E})$. In panel (b), $\langle \hat{O} \rangle_i$'s are fitted by a bivariate cubic function to form a surface $\mathcal{O}(\mathcal{E}, \mathcal{N})$ defined by $\mathcal{E}$ and quasiparticle number $\mathcal{N}$. The color bars show the deviations of $\langle \hat{O} \rangle_i$ from the curve (or surface): $\langle \hat{O} \rangle_i - \mathcal{O}(\mathcal{E})$ (or $\langle \hat{O} \rangle_i - \mathcal{O}(\mathcal{E}, \mathcal{N})$) through a range of changing colors. In panel (c), we show the distribution of eigenstates on the $\mathcal{E} - \mathcal{N}$ plane, with a color bar demonstrating the density of states.

to describe the constrained dynamics. Concerning an eigenstate $|E_i\rangle$, β and μ can be determined by fixing the energy and the quasiparticle number: $E_i = \text{Tr}(\rho(\beta, \mu)H)$, $\langle \hat{N} \rangle_i = \text{Tr}(\rho(\beta, \mu)\hat{N})$. Then, we can estimate $\langle \hat{O} \rangle_i$ by calculating $\mathcal{O}(\mathcal{E}, \mathcal{N}) \sim \text{Tr}(\rho(\beta, \mu)\hat{O})$.

We stress that our approach of defining quasiparticles deviates from the traditional route, where the grand canonical ensembles are introduced by regarding the conserved charges as the quasiparticles [6,10,62–64]. In contrast, the ETH normally does not incorporate conserved quantities as parameters for describing the local thermal properties of eigenstates. The emergence of such a conserved quantity would usually block-diagonalize the Hamiltonian and isolate the corresponding symmetry sectors, thereby precluding any ensemblelike mixing between eigenstates belonging to different sectors. In our framework, the quasiparticle number operator is explicitly nonconserved, as $[H, \hat{N}] \neq 0$. However, we numerically demonstrate that $\hat{N}$ is approximately diagonal in the eigenbasis of the Hamiltonian, as shown in Appendix C. This property justifies replacing $\hat{N}$ in the grand canonical ensemble by its diagonal terms.

Figures 4(a) and 4(b) illustrate the deviations between the ensemble average of local observables $\text{Tr}(\rho\hat{O})$ and their actual values $\langle \hat{O} \rangle_i$ for both the canonical DM $\rho_c(\beta_c)$ and the grand canonical DM $\rho(\beta, \mu)$ provided above; this is expressed as $\text{Tr}(\hat{O}\rho) - \langle \hat{O} \rangle_i$. In Figs. 4(e) and 4(f), we also compute the Schatten distance d_p between the local reduced DMs of the ensembles $\rho^A = \text{Tr}_{\bar{A}}(\rho(\beta, \mu))$ (or $\text{Tr}_{\bar{A}}(\rho_c(\beta_c))$) and that of the eigenstates $\sigma_i^A = \text{Tr}_{\bar{A}}(|E_i\rangle\langle E_i|)$, which is defined as [65,66] $d_p(\rho, \sigma) \equiv \left\| \frac{\rho}{\|\rho\|_p} - \frac{\sigma}{\|\sigma\|_p} \right\|_p$, with the Schatten p -norm $\|A\|_p \equiv \text{Tr}(|A|^p)^{1/p}$. These results indicate that the grand canonical DM $\rho(\beta, \mu)$ provides a valid characterization for the local properties of all the eigenstates. Moreover, the grand canonical ensemble gives extremely accurate descriptions of the scar eigenstates, whereas the canonical ensemble shows large deviations. In Figs. 4(g) and 4(h), we also show the time evolution of relevant operators $\hat{O}_{2,3}$ starting from the initial state $|\psi(0)\rangle$. It can be seen that $\langle \hat{O} \rangle(t)$ deviates significantly from the canonical average, while the grand canonical average gives a valid estimation for the long-time average of $\hat{O}$. Here, the thermal coefficient β and μ can be determined from the total energy and the long-time average of the quasiparticle

number $\bar{N}$, which can be obtained experimentally according to the previous section.

In Figs. 4(c) and 4(d), we computed the average deviation $|\text{Tr}(\hat{O}\rho) - \langle \hat{O} \rangle_i|$ over all eigenstates while gradually increasing the system size [60,67–69]. We observed that, regardless of which ensembles we employed, this deviation approximately scales as $\mathcal{D}^{-1/2}$, with $\mathcal{D}$ the dimension of the constrained subspace. However, the average deviation yielded by the grand canonical ensemble is significantly lower than that provided by the canonical ensemble. According to the ETH in Eq. (9), the discrepancy between the expectation value of local observables $\langle \hat{O} \rangle_i$ and the function $\mathcal{O}(\mathcal{E})$ is primarily governed by $\Omega^{-1/2}(\mathcal{E})$, with $\Omega(\mathcal{E})$ the DOS at the corresponding energy $\mathcal{E}$. In our revised ETH, this term should correspondingly be modified to the joint DOS in energy and quasiparticle number $\Omega(\mathcal{E}, \mathcal{N})$. Nevertheless, both definitions of DOS generally show proportional relations with $\mathcal{D}$, thus justifying the observed scaling behavior. Additionally, regarding the local properties of scar states, predictions from the grand canonical ensemble are extremely accurate across all system sizes, whereas the predictions from the canonical ensemble exhibit significant discrepancies, which do not diminish with increasing system size. Finally, our strategy will fundamentally influence the definition of the DOS and the assessment of the magnitudes of off-diagonal elements from Eq. (9). This will be addressed in our future investigations. We speculate that the diminished DOS $\Omega(\mathcal{E}, \mathcal{N})$ in proximity to scar states, as can be seen in Fig. 3(c), is likely the underlying reason for the quasiperiodic dynamical behavior observed in QMBS.

V. CONCLUSION

In summary, we establish a mapping between general constrained models and engineered dissipative dynamics. Our strategy describes the constrained subspace as a jump-free region, where the jumping terms induced by the two types of dissipations cancel each other out. In this context, we find that the scar eigenstates display markedly lower decay rates than the thermal states, which distinctly reflects their nonthermal characteristics. This inspires us to reformulate the ETH by introducing the quasiparticle number $\mathcal{N}$ to the hypothesis, leading to an equilibrium analogous to the grand canonical ensemble. The revised ETH successfully

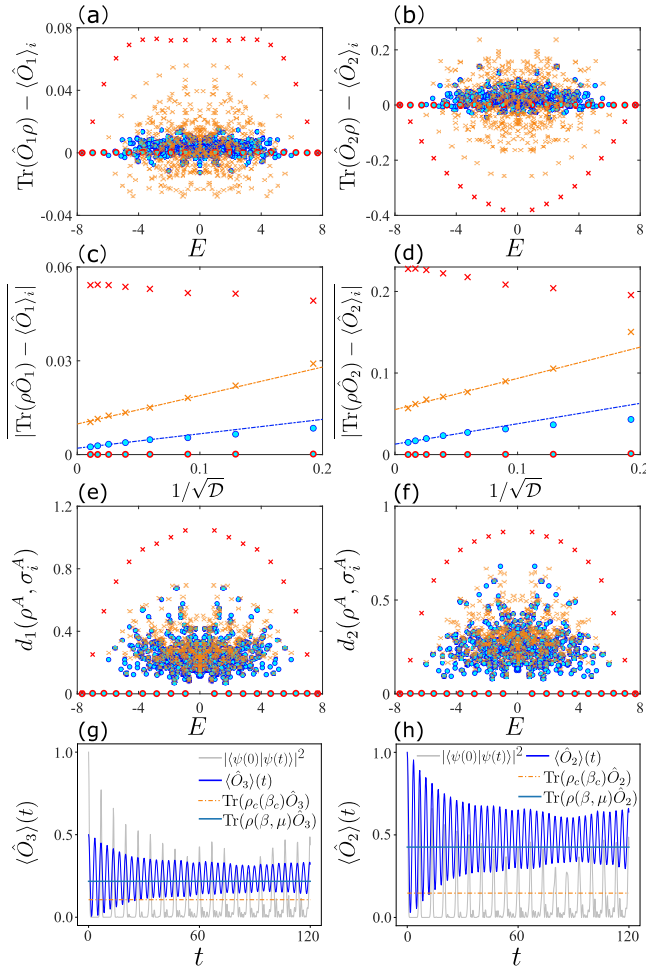


FIG. 4. Panels (a) and (b) compare the ensemble averages of local observables with their eigenstate expectation values, quantified by $\text{Tr}(\hat{O}\rho) - \langle \hat{O} \rangle_i$. Panels (c) and (d) show the average absolute value of these differences, calculated over all eigenstates and over scar states only, demonstrating how the averages scale with the Hilbert space dimension $\mathcal{D}$. Panels (e) and (f) present the Schatten distances $d_p(\rho^A, \sigma_i^A)$ for $p = 1, 2$, where $\rho^A = \text{Tr}_{\bar{A}}(\rho)$ is the reduced DM of the ensembles and $\sigma_i^A = \text{Tr}_{\bar{A}}(|E_i\rangle\langle E_i|)$ is that of the eigenstates, with both defined on the same two-site subsystem A . In panels (a)–(f), crosses denote results for the canonical DM $\rho = \rho_c(\beta_c)$, while solid points correspond to the grand canonical DM $\rho = \rho(\beta, \mu)$. All scar state data are highlighted in red. In panels (a), (b), (e), and (f), calculations are performed for a spin-1 chain with nine sites; analogous results for the spin-3/2 case are provided in Appendix D. Panels (c) and (d) report the mean absolute deviation $|\text{Tr}(\hat{O}\rho) - \langle \hat{O} \rangle_i|$ as the system size varies from $D = 5$ –12. For the subset with $D \geq 8$, the averages over all eigenstates are fitted to a linear function of $1/\sqrt{D}$, as indicated by the dotted lines. Panels (g) and (h) display the time evolution of local observable expectation values, setting the initial state as $|1, 1, \dots, 1\rangle$. Grey lines illustrate quasiperiodic revivals of the fidelity of the initial state, while blue curves depict the dynamics of $\hat{O}_3$ and $\hat{O}_2$, given by $\langle \hat{O}_{3,2}(t) \rangle = \langle \psi(t) | \hat{O}_{3,2} | \psi(t) \rangle$. The blue straight lines indicate the grand canonical averages, whereas red dashed lines show ETH-based canonical averages. We have employed the local observables $\hat{O}_1 = |1\rangle\langle 1|_k + |0\rangle\langle 0|_k$ for panels (a) and (c), $\hat{O}_2 = s_k^z \otimes s_{k+1}^z$ for panels (b), (d), and (h), $\hat{O}_3$ as the projector on the state $(|1\rangle_1 \otimes |1\rangle_2 - |-1\rangle_1 \otimes |-1\rangle_2)/\sqrt{2}$ for panel (e).

predicts local observables for all eigenstates and renders the QMBS model thermalized under the framework of the grand canonical ensemble. However, whether such a thermalization mechanism is applicable to exactly solvable QMBS and many-body localized phases remains open. In addition, although the nonintegrable constrained models discussed here show evidence of grand canonical thermalization, integrability is typically associated with the suppression of thermalization processes. A fundamental question is whether integrability can also suppress this grand canonical-like thermalization. Our work provides a unified pathway for exploring thermalization phenomena in a broader context.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: DETAILS IN THE DERIVATION OF THE MASTER EQUATION

We provide further details supporting the proof of the equivalence between the master equation and the original constrained Hamiltonian. First, we show that the Hermitian Hamiltonian H commutes with the projector $\hat{P}$ onto the constrained subspace $\mathcal{H}$ and shares the same right eigenvector with the non-Hermitian one H_N within $\mathcal{H}$. We directly write

$$\begin{aligned} [H, \hat{P}] &= \sum_{k=1}^N (P_k h_k P_k \hat{P} - \hat{P} P_k h_k P_k) \\ &= \sum_{k=1}^N (P_k h_k P_k \hat{P} - P_k h_k P_k \bar{P}_k) = 0. \end{aligned} \quad (\text{A1})$$

Here, $\bar{P}_k = \prod [\pi_m, h_k] = 0(\mathbb{I} - \pi_m)$ commutes with h_k and, together with the local constraint P_k , forms the projector $\hat{P}$. Hence, H can be diagonalized within the constrained subspace $\mathcal{H}$, where the eigen-equations of H and H_N are equivalent:

$$\begin{aligned} H|E_i\rangle &= \sum_{k=1}^D (h_k - \Pi_k h_k - h_k \Pi_k + \Pi_k h_k \Pi_k) \hat{P}|E_i\rangle \\ &= \sum_{k=1}^D (h_k - \Pi_k h_k + h_k \Pi_k - ic \Pi_k) \hat{P}|E_i\rangle \\ &= H_N|E_i\rangle. \end{aligned} \quad (\text{A2})$$

Here, we have considered the fact that the local projector Π_k 's are orthogonal with $\hat{P}$: $\Pi_k \hat{P} = (\mathbb{I} - P_k) \hat{P} = 0$.

Second, we provide a detailed deduction of Eq. (4). We equip the master equation with Hamiltonian H_0 and the dissipation channels described in Eq. (3), then we get the non-Hermitian effective Hamiltonian of the system as

$$\begin{aligned} H_0 - i \sum_{k,\sigma} \frac{\gamma_\sigma}{2} L_{k,\sigma}^\dagger L_{k,\sigma} \\ = H_0 + \frac{i}{c} \sum_{k=1}^D h_k \Pi_k h_k \\ - ic \sum_{k=1}^D \left(\Pi_k + \frac{i}{c} h_k \Pi_k \right) \left(\Pi_k - \frac{i}{c} \Pi_k h_k \right) \\ = \sum_{k=1}^D h_k - ic \Pi_k - \Pi_k h_k + h_k \Pi_k = H_N, \end{aligned} \quad (\text{A3})$$

with a tunnable parameter c . The jumping term $\mathcal{J}(\rho)$ can be obtained in the same manner:

$$\begin{aligned} \mathcal{J}(\rho) &= \sum_{k,\sigma} \gamma_\sigma L_{k,\sigma} \rho L_{k,\sigma}^\dagger \\ &= 2c \sum_{k=1}^D \left(\Pi_k - \frac{i}{c} \Pi_k h_k \right) \rho \left(\Pi_k + \frac{i}{c} h_k \Pi_k \right) \\ &\quad - \frac{2}{c} \sum_{k=1}^D \Pi_k h_k \rho h_k \Pi_k \\ &= \sum_{k=1}^D 2c \Pi_k \rho \Pi_k - 2i \Pi_k h_k \rho \Pi_k + 2i \Pi_k \rho h_k \Pi_k. \end{aligned} \quad (\text{A4})$$

Having demonstrated the deduction of the master equation, we turn to the master equation with only the positive dissipation rates. Formally, when $c \rightarrow \infty$, we can ignore the negative dissipation channels and obtain a Liouvillian superoperator $\mathcal{L}_+$, in which the non-Hermitian Hamiltonian writes $H_+ \equiv H_0 - \frac{i}{2} \sum_{k=1}^D \gamma_1 L_{k,1}^\dagger L_{k,1} = H_N - \frac{i}{c} \sum_{k=1}^D h_k \Pi_k h_k$ and the jumping term $\mathcal{J}_+(\rho)$ reads $\mathcal{J}_+(\rho) \equiv \sum_{k=1}^D \gamma_1 L_{k,1} \rho L_{k,1}^\dagger = \mathcal{J}(\rho) + \frac{2}{c} \sum_{k=1}^D \Pi_k h_k \rho h_k \Pi_k$. In the dynamical process generated by $\mathcal{L}_+$, the states within $\mathcal{H}$ leak out through the term $\Pi_k h_k \rho h_k \Pi_k$ to the blockade region, and then suffer a fast decay caused by the quantum jumps $2c \Pi_k \rho \Pi_k$ in $\mathcal{J}(\rho)$. Now, if we keep the leakage caused by the term $\Pi_k h_k \rho h_k \Pi_k$, and replace the confinement caused by the Zeno effect with the original blockade effect defined by $\hat{P}$, we will arrive at another Lindblad master equation $\partial_t \rho = \mathcal{L}'_+(\rho)$, with the constrained Hamiltonian $H_B = \hat{P} H \hat{P} = \hat{P} H_0 \hat{P}$ governing the unitary dynamics and dissipation channels with sufficiently low dissipation rates: $\{\gamma' = -\gamma_2 = \frac{2}{c}, L'_k = L_{k,2} = \Pi_k h_k\}$. Then, the non-Hermitian Hamiltonian of $\mathcal{L}'_+$

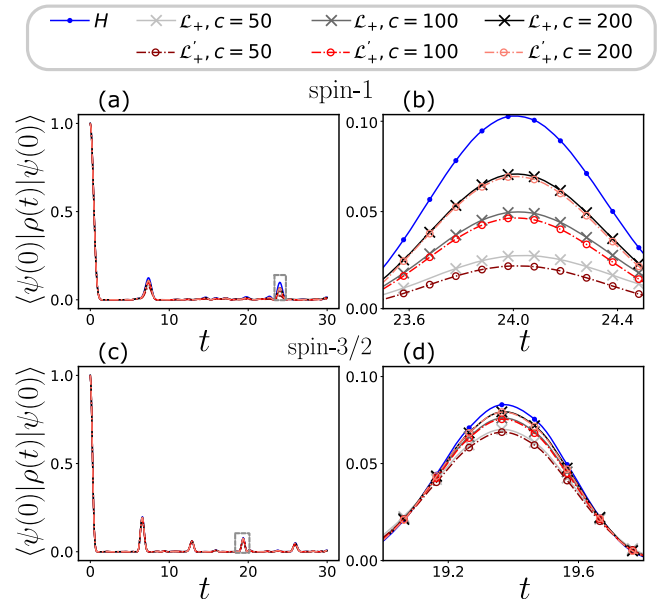


FIG. 5. The evolution of the initial states' fidelity in the constrained system with Hamiltonian H , compared with that under the master equations marked by $\mathcal{L}_+$ and $\mathcal{L}'_+$. In panels (a) and (b), we show the evolution in spin-1 chain with nine sites, while panels (c) and (d) correspond to a spin-3/2 chain with seven spins. In panels (a) and (c), we choose the initial states as random product states other than $|j, j, \dots, j\rangle$. Panels (b) and (d) present detailed views of the regions enclosed within the gray boxes in the corresponding figure on the right.

becomes

$$H'_+ \equiv H_B - \frac{i}{2} \sum_{k=1}^D \gamma' L_k'^\dagger L'_k = H_B - \frac{i}{c} \sum_{k=1}^D h_k \Pi_k h_k, \quad (\text{A5})$$

and the jumping term $\mathcal{J}'_+$ can be rewritten as

$$\mathcal{J}'_+(\rho) \equiv \sum_{k=1}^D \gamma' L'_k \rho L_k'^\dagger = \frac{2}{c} \sum_{k=1}^D \Pi_k h_k \rho h_k \Pi_k. \quad (\text{A6})$$

This dissipation rate is inversely small with large c . Hence, the evolution under $\mathcal{L}'_+$ approaches the unitary dynamics defined by H_B in the limiting case of $c \rightarrow \infty$.

In Fig. 1, we have illustrated the differences among the dynamics of H , $\mathcal{L}_+$, and $\mathcal{L}'_+$, employing the spin model with the initial state $|\psi(0)\rangle = |j, j, \dots, j\rangle$. Our calculations show that when c is large, both dynamical descriptions based on $\mathcal{L}_+$ and $\mathcal{L}'_+$ can approximate the ideal Hermitian dynamical evolution very well. Here, we also confirm this result by providing the evolution of these dynamical equations under non-QMBS initial states, as shown in Fig. 5.

Next, we provide some calculation details regarding the spin model discussed in the main text. Based on the specific form of the Hamiltonian $H_0 = \sum_{k=1}^D s_k^x$ and blockades $\mathcal{B} = \{\pi_k = |x\rangle\langle x|_{k,k+1}; k = 1, 2, \dots, D\}$, we follow the framework defined in the main text and write the constrained

Hamiltonian as

$$\begin{aligned} H &= \sum_{k=1}^D (\mathbb{I} - \Pi_k) s_k (\mathbb{I} - \Pi_k) = \sum_{k=1}^D s_k - \Pi_k s_k - s_k \Pi_k \\ &= \sum_{k=1}^D s_k^x - \sqrt{\frac{j}{2}} (|j\rangle\langle j|_{k-1} \otimes |-j\rangle\langle -j+1|_k + \text{H.c.}) \\ &\quad - \sqrt{\frac{j}{2}} (|j\rangle\langle j-1|_k \otimes |-j\rangle\langle -j|_{k+1} + \text{H.c.}), \quad (\text{A7}) \end{aligned}$$

where $\Pi_k = \pi_{k-1} + \pi_k$ stands for the constraints felt by the local Hamiltonian s_k^x . We note that the terms $\Pi_k h_k \Pi_k$ vanish for this model. Subsequently, we can obtain the explicit form of the dissipative channel by rigorously following Eq. (4), thereby arriving at the associated master equation. However, the obtained dissipation terms act on three adjacent lattice sites, akin to Π_k . To simplify computations and obtain more localized dissipations, we reformulate the Hamiltonian as

$$\begin{aligned} H &= \sum_{k=1}^D s_k^x - \sqrt{j} (|x\rangle_{k,k+1} \langle y|_{k,k+1} + \text{H.c.}) \\ &= \sum_{k=1}^D s_k^x - \sqrt{j} M_k - \sqrt{j} M_k^\dagger, \quad (\text{A8}) \end{aligned}$$

where $M_k \equiv \frac{1}{\sqrt{j}} \pi_k (s_k^x + s_{k+1}^x) = |x\rangle\langle y|_{k,k+1}$ is the mapping from $\mathcal{H}$ to the blockaded region, with $|y\rangle_{k,k+1} = \frac{1}{\sqrt{2}} (|j\rangle_k \otimes |-j+1\rangle_{k+1} + |j-1\rangle_k \otimes |-j\rangle_{k+1})$ the state mapped from the prohibited state, as defined in the manuscript. Following the same logic as established in the above framework, we then write out the non-Hermitian Hamiltonian as

$$H_N = \sum_{k=1}^D s_k^x - \sqrt{j} M_k + \sqrt{j} M_k^\dagger - ic \sqrt{\frac{j}{2}} \pi_k. \quad (\text{A9})$$

From which we can build up a set of dissipation channels acting on only nearest-neighboring sites, as given in Eq. (7).

Next, we provide the deduction and the specific form of the master equation. The non-Hermitian Hamiltonian writes

$$\begin{aligned} H_N &= H_0 - i \sum_{k,\sigma} \frac{\gamma_\sigma}{2} L_{k,\sigma}^\dagger L_{k,\sigma} \\ &= H_0 - ic \sqrt{\frac{j}{2}} \sum_{k=1}^D \left(\pi_k + \frac{i\sqrt{2}}{c} M_k^\dagger \right) \left(\pi_k - \frac{i\sqrt{2}}{c} M_k \right) \\ &\quad + \frac{i\sqrt{2j}}{c} \sum_{k=1}^D M_k^\dagger M_k \\ &= \sum_{k=1}^D s_k^x - ic \sqrt{\frac{j}{2}} \pi_k - \sqrt{j} M_k + \sqrt{j} M_k^\dagger, \quad (\text{A10}) \end{aligned}$$

which shares the same right eigenvectors as the Hamiltonian from Eq. (A8) in the constrained subspace. We have used the relation $\pi_k M_k = M_k$ in the above deduction. Finally, we have

the jumping terms:

$$\begin{aligned} &\sum_{k,\sigma} \gamma_\sigma L_{k,\sigma} \rho L_{k,\sigma}^\dagger \\ &= c \sqrt{2j} \sum_{k=1}^D \left(\pi_k - \frac{i\sqrt{2}}{c} M_k \right) \rho \left(\pi_k + \frac{i\sqrt{2}}{c} M_k^\dagger \right) \\ &\quad - \sqrt{2j} \frac{2}{c} \sum_{k=1}^D M_k \rho M_k^\dagger \\ &= \sqrt{2j} \sum_{k=1}^D c \pi_k \rho \pi_k - i \sqrt{2} M_k \rho \pi_k + i \sqrt{2} \pi_k \rho M_k^\dagger. \quad (\text{A11}) \end{aligned}$$

We can see that any state within the constrained subspace remains unaffected by the jumping term since it is annihilated by the blockades π_k 's from either side: $\pi_k \hat{P} = \hat{P} \pi_k = 0$. Therefore, the above master equation equivalently describes the dynamics of the Hamiltonian from Eq. (A8).

We note that in the framework provided in the main text, we build up the dissipation channels based on each single local Hamiltonian part h_k and the blockades π_k 's that do not commute with h_k . On the contrary, here we build the dissipation channels in the spin model based on each single blockade π_k and local Hamiltonian terms that effectively act on it. For example, we have utilized the operator $M_k \propto \pi_k (s_k^x + s_{k+1}^x)$ instead of $\Pi_k s_k^x = (\pi_{k-1} + \pi_k) s_k^x$ to describe the connection between the constrained subspace and the blockaded states. Although the construction method utilized in the framework is universal, the approach applied to spin chains may provide significant convenience for subsequent discussions. Nevertheless, the latter necessitates a case-by-case examination when extending to other models.

Next, we provide a detailed deduction of Eq. (8), demonstrating the decay rate of H 's eigenstates under the dynamical equation $\partial_t \rho = \mathcal{L}_+(\rho)$:

$$\begin{aligned} \frac{df_i(t)}{dt} &= \frac{d}{dt} \langle E_i | \rho(t) | E_i \rangle = \langle E_i | \frac{d}{dt} \rho(t) | E_i \rangle \\ &= \langle E_i | \mathcal{L}_+(\rho(t)) | E_i \rangle = -i \langle E_i | H_+ \rho(t) | E_i \rangle + \text{H.c.} \\ &= \sum_{k=1}^D -i \langle E_i | \left(2h_k \Pi_k - \frac{i}{c} h_k \Pi_k h_k \right) \rho(t) | E_i \rangle + \text{H.c.} \quad (\text{A12}) \end{aligned}$$

We note that the jumping term $\mathcal{J}_+(\rho)$ vanishes due to the action of $\langle E_i | \cdot | E_i \rangle$. Then we make the following approximation to simplify the calculation. As c approaches infinity, the eigenstates of the constrained Hamiltonian H can be regarded as approximately equivalent to the right-eigenvectors of the non-Hermitian Hamiltonian H_+ :

$$\begin{aligned} H | E_i \rangle &= E_i | E_i \rangle = H_N | E_i \rangle \\ &= \left(H_+ + \frac{i}{c} \sum_{k=1}^D h_k \Pi_k h_k \right) | E_i \rangle \sim H_+ | E_i \rangle. \quad (\text{A13}) \end{aligned}$$

Consequently, for the trajectory without jumps, the system remains in its initial state $|E_i\rangle$. Moreover, any quantum jump of $\mathcal{L}_+$ projects the system outside the constrained subspace, thereby ensuring that the resultant state is orthogonal to the

initial one. Therefore, we have $\rho(t) \approx f_i(t) \cdot |E_i\rangle\langle E_i| + (\mathbb{I} - \hat{P})\rho(t)(\mathbb{I} - \hat{P})$, where $\hat{P}\rho\hat{P}$ denotes the part of the DM that has been mapped outside the subspace $\mathcal{H}$ by quantum jumps. Then, we arrive at the following equation:

$$\frac{df_i(t)}{dt} \approx -\frac{2}{c} \sum_{k=1}^D \langle h_k \Pi_k h_k \rangle_i \cdot f_i(t) = \gamma_2 \sum_{k=1}^D \langle L_{k,2}^\dagger L_{k,2} \rangle_i \cdot f_i(t). \quad (\text{A14})$$

Finally, we obtain $f_i(t) \approx e^{-\alpha_i t}$ with the decay rate $\alpha_i = -\gamma_2 \langle \hat{N} \rangle_i$, where $\hat{N} = \sum_{k=1}^D N_k$ and $N_k = L_{k,2}^\dagger L_{k,2}$ as defined in Sec. III.

Besides, for any physically preparable initial state $|\phi(0)\rangle = \sum_{i=1}^D a_i |E_i\rangle$, we define the possibility of the system remaining in the subspace $\mathcal{H}$ during the dissipative process governed by $\partial_t \rho = \mathcal{L}'_+(\rho)$: $h(t) = \text{Tr}(\rho(t)\hat{P})$. Similarly, we have

$$\begin{aligned} \frac{dh(t)}{dt} &= \text{Tr}\left(\hat{P} \frac{d}{dt} \rho(t)\right) \\ &= -\frac{\gamma'}{2} \sum_{k=1}^D \text{Tr}(\hat{P}(L_k^\dagger L'_k \rho + \rho L_k^\dagger L'_k)) \\ &= \frac{\gamma_2}{2} \text{Tr}(\hat{P}(\hat{N} \rho + \rho \hat{N})), \end{aligned} \quad (\text{A15})$$

then we utilize the former assumption that $\rho(t)$ is approximately governed by H for the process without jumps, and will be mapped out of the subspace $\mathcal{H}$ for any possible jump: $\rho(t) \approx h(t) \cdot \rho_{\mathcal{H}}(t) + (\mathbb{I} - \hat{P})\rho(t)(\mathbb{I} - \hat{P})$. $\rho_{\mathcal{H}}(t)$ denotes the DM within $\mathcal{H}$, and can be approximated by $e^{-iHt} \rho(0) e^{iHt}$. Following this, we have

$$\begin{aligned} \frac{dh(t)}{dt} &\approx \gamma_2 \text{Tr}(\hat{N} e^{-iHt} \rho(0) e^{iHt}) h(t), \\ h(t) &\approx 1 + \gamma_2 \int_0^t dt' h(t') \text{Tr}(\hat{N} e^{-iHt'} \rho(0) e^{iHt'}), \end{aligned} \quad (\text{A16})$$

then we can keep the terms up to the first order of $\gamma_2 \propto 1/c$ and get $h(t) \approx 1 + \gamma_2 \int_0^t \text{Tr}(\hat{N} e^{-iHt'} \rho(0) e^{iHt'}) dt'$. Hence, we write the integral above as

$$\sum_{k,l} \int_0^t a_l^* a_k e^{it'(E_l - E_k)} \langle E_l | \hat{N} | E_k \rangle dt' \xrightarrow{t \gg 1/\delta E} t \sum_{k=1}^D |a_k|^2 \langle \hat{N} \rangle_k, \quad (\text{A17})$$

where δE marks the minimal energy level spacing. Then, we approximately arrive at a leaking process denoted by $h(t) \approx 1 + \alpha_L t$, with $\alpha_L = \gamma_2 \bar{N}$ and $\bar{N} = \sum_{i=1}^D |a_i|^2 \langle \hat{N} \rangle_i$. The parameter range involved in the above calculations is given by $t \gg 1/\delta E$, $t \cdot \gamma_2 \ll 1$. Since $\gamma_2 \propto 1/c$, this leads to $c \gg t \gg 1/\delta E$.

APPENDIX B: MAPPING THE SPIN-1 SCENARIO TO PXP MODEL

The spin-1 scenario of the 1D spin chain model can be mapped to the PXP model. Here, we outline the derivation of the mapping from the Hamiltonian in Eq. (A8) to the standard PXP Hamiltonian. First, we rewrite Eq. (A8) in the matrix

form for spin-1 scenario as

$$\begin{aligned} H &= \sum_{k=1}^D s_k^x - M_k - M_k^\dagger \\ &= \frac{1}{\sqrt{2}} \sum_{k=1}^D \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}_k \\ &\quad - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_k \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{k+1} \\ &\quad - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_k \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}_{k+1} \\ &= \frac{1}{\sqrt{2}} \sum_{k=1}^D \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_k \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{k+1} \\ &\quad + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_k \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}_{k+1}, \end{aligned} \quad (\text{B1})$$

where the matrix expansion is written under the local spin basis of s_k^z : $\{|m=1\rangle, |m=0\rangle, |m=-1\rangle\}$. Then, we map these bases to two composite spin-1/2 sites as $|m=1\rangle_k = |\downarrow, \uparrow\rangle_{2k-1, 2k}$, $|m=0\rangle_k = |\downarrow, \downarrow\rangle_{2k-1, 2k}$, and $|m=-1\rangle_k = |\uparrow, \downarrow\rangle_{2k-1, 2k}$. Following this, we rewrite the Hamiltonian from Eq. (B1) in the spin-1/2 basis:

$$\begin{aligned} H &= \frac{1}{\sqrt{2}} \sum_{k=1}^D P_{2k-1}^\downarrow \otimes \sigma_{2k}^x \otimes P_{2k+1}^\downarrow + P_{2k}^\downarrow \otimes \sigma_{2k+1}^x \otimes P_{2k+2}^\downarrow \\ &= \frac{1}{\sqrt{2}} \sum_{l=1}^{2D} P_{l-1}^\downarrow \otimes \sigma_l^x \otimes P_{l+1}^\downarrow, \end{aligned} \quad (\text{B2})$$

where $P_l^\downarrow = |\downarrow\rangle\langle\downarrow|_l$ is the projector of spin-down state $|\downarrow\rangle_l$ on the l th site, σ_l^x is the Pauli operator on the x direction. The final expression in Eq. (B2) is exactly the Hamiltonian of a PXP model with $2D$ sites.

APPENDIX C: VALIDITY OF THE QUASIPARTICLE NUMBER $\hat{N}$

Regarding the grand canonical ensemble in classical statistical mechanics, the total energy and total particle number of the system and its environment are conserved. However, fluctuations in both energy and particle number within the system arise due to interactions between the system and its environment. Since these quantities are c -numbers in classical physics rather than operators, noncommutativity does not become an issue. In the quantum case, however, such straightforward commutation relations cannot be guaranteed. According to the ETH, an isolated system can be viewed as comprising a local subsystem with the remaining that act as the environment. In this framework, the Hamiltonian of the system assumes a dual significance: first, as a conserved quantity that governs the unitary dynamics of the isolated system; and second, as

the energy of the local subsystem, which experiences local fluctuations described by the canonical ensemble.

In our framework, we treat the kinetically constrained model equivalently as an open system coupled to an auxiliary environment, and introduce the definition of the system's quasiparticle number. Examining such a system from a thermodynamic perspective, we regard the steady state of the master equation—also the eigenstate of the isolated system—as a microstate contributing to the thermal description of a grand canonical ensemble. Here, the quasiparticle number $\hat{N} = \sum_{k=1}^D L_{k,2}^\dagger L_{k,2} = \sum_k h_k \Pi_k h_k$ quantifies the information exchange process between the system and its environment. Specifically, the expectation value of $\hat{N}$ reflects the decay rate in the information-outflow process governed by $\mathcal{L}_+$. Therefore, $\hat{N}$ cannot be considered as the total quasiparticle number of the system and its environment, implying that the operator $\hat{N}$ is not necessarily a conserved quantity throughout the entire dynamical process. However, since energy and quasiparticle number are viewed as two quantities that describe this microstate. It is necessary for the Hamiltonian H and the quasiparticle number $\hat{N}$ to have an approximately commuting relationship, in order to facilitate the calculation of the grand canonical ensemble. In Figs. 6(a) and 6(b), we numerically validate this assertion, demonstrating that $\hat{N}$ is approximately diagonal in the eigenbasis of the Hamiltonian H . Moreover, this quasideagonal structure becomes increasingly pronounced as the system size grows. Thus, we can retain only the diagonal elements of $\hat{N}$ and approximate the grand canonical ensemble with $\sum_{k=1}^D e^{-\beta(E_k - \mu \langle \hat{N} \rangle_k)} |E_k\rangle \langle E_k| / Z$.

On another note, our revised ETH posits that all the local observables should satisfy Eq. (11). Hence, we provide additional examples in Figs. 6(c)–6(f) to testify to our hypothesis. We randomly choose several local observables other than the one we employed in Fig. 3, and demonstrate that all the eigenstates' expectation values can be expressed as a coherent function of energy $\mathcal{E}$ and quasiparticle number $\mathcal{N}$.

APPENDIX D: APPLICATION IN OTHER MODELS HOSTING QMBS

The results from Fig. 4 can be generalized to the same spin model with larger spins. In Fig. 7, we provide the results calculated for the aforementioned model in the spin-3/2 scenario. The calculation shows that the grand canonical ensembles fit the local properties of the eigenstates better than the canonical ensemble, confirming our discussion in Sec. IV.

Besides, our framework can be applied to other constrained systems. Here, we consider another QMBS model based on a 1D spin chain, which is known as the high-dimensional PXP model. The Hamiltonian writes

$$\begin{aligned} H_{\text{hPXP}} &= \hat{P} \sum_{k=1}^D s_k^x \hat{P}, \\ \hat{P} &= \prod_{k=1}^D (\mathbb{I} - T_k \otimes T_{k+1}), \\ T_k &= \sum_{m=-j+1}^j |m\rangle \langle m|_k, \end{aligned} \quad (\text{D1})$$

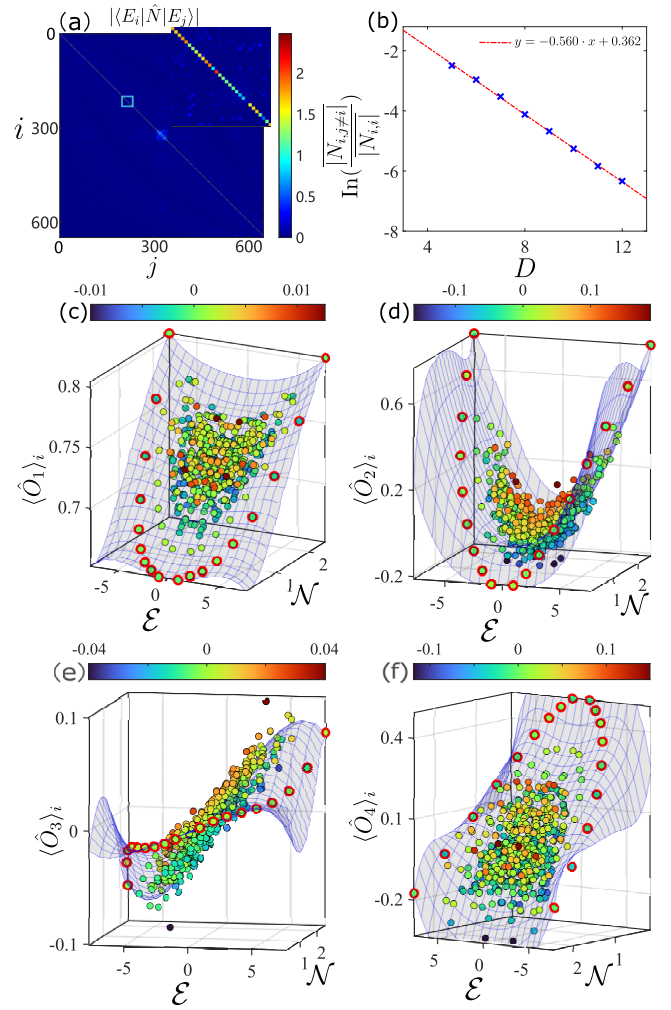


FIG. 6. In panel (a), we show the matrix form of quasiparticle number $\hat{N}$ under the eigenbasis of the system Hamiltonian. We use the color bar to demonstrate the absolute value of each matrix element $|\langle E_i | \hat{N} | E_j \rangle|$, and show the characteristics of the local diagonal portion of the matrix in the inset plot. We have chosen a spin-1 chain with nine sites. In panel (b), we calculate the ratio of the average absolute value of the off-diagonal matrix elements to that of the diagonal matrix elements, expressed as $\frac{\overline{|N_{i \neq j}|}}{\overline{|N_{i=i}|}}$, where $N_{i,j} = \langle E_i | \hat{N} | E_j \rangle$, $|\cdot|$ denotes the absolute value operation, and $\overline{\cdot}$ signifies the average taken over all relevant matrix elements. We take the same model, varying the system size from 5 to 12, and show that this ratio exhibits an exponential decrease as the system size increases. In panels (c)–(f), we fit the local observables' expectation values $\langle \hat{O} \rangle_i$ by polynomial functions of $\mathcal{E}$ and quasiparticle number $\mathcal{N}$. The color bars show the deviations of $\langle \hat{O} \rangle_i$ from the fitted surface: $\langle \hat{O} \rangle_i - \mathcal{O}(\mathcal{E}, \mathcal{N})$ through a range of changing colors. We have chosen the local observables as $\hat{O}_1 = |1\rangle\langle 1|_k + |0\rangle\langle 0|_k$, $\hat{O}_2 = s_k^x \otimes s_{k+1}^x$, $\hat{O}_3 = s_k^x \otimes s_{k+1}^z$, and $\hat{O}_4 = s_k^y \otimes s_{k+1}^y$.

where $\hat{P}$ stands for the projector of the constrained subspace. $\mathcal{B} = \{\pi_k = T_k \otimes T_{k+1}, k = 1, 2, \dots, D\}$ mark the blockades that prohibit nearest neighboring spins from spontaneously occupying states $\{|-j+1\rangle_k, |-j+2\rangle_k, \dots, |j\rangle_k\}$. $|m\rangle_k$ denotes the eigenstate of $s_k^z: s_k^z |m\rangle_k = m |m\rangle_k$. Therefore, a spin

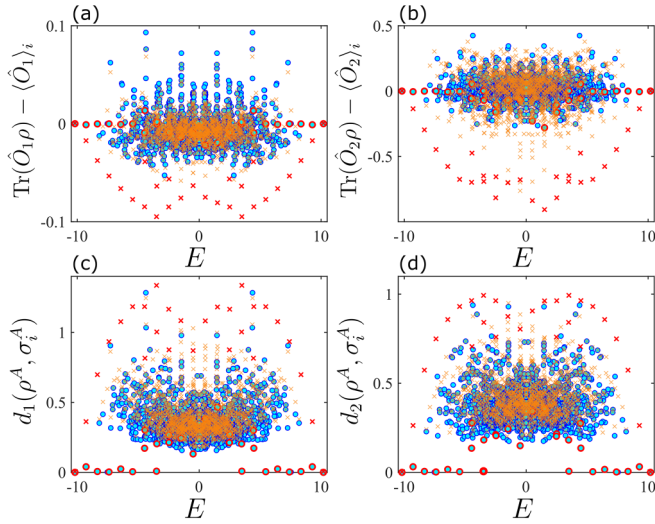


FIG. 7. Deviations between the eigenstates and the ensemble estimations for the spin-3/2 case. Panels (a) and (b) show the differences between the DM average of local observables and their expectation values in the eigenstates $|E_i\rangle$, expressed as $\text{Tr}(\hat{O}\rho) - \langle \hat{O} \rangle_i$. In panel (a), the local observable is chosen as $\hat{O}_1 = |3/2\rangle\langle 3/2|_k$. In panel (b), the local observable is also $\hat{O}_2 = s_k^z \otimes s_{k+1}^z$. Panels (c) and (d) present the Schatten distances $d_p(\rho^A, \sigma_i^A)$ between the reduced DM $\rho^A = \text{Tr}_{\bar{A}}(\rho)$ and the reduced DM of the eigenstates $\sigma_i^A = \text{Tr}_{\bar{A}}(|E_i\rangle\langle E_i|)$ in the same subsystem A , where we take $p = 1, 2$. In panels (a)–(d), the horizontal axis corresponds to energy E and the crosses represent results from the canonical DM $\rho = \rho_c(\beta_c)$, with red crosses indicating scar states and yellow ones denoting thermal states. Solid points represent results calculated from the grand canonical DM $\rho = \rho(\beta, \mu)$, with scar states highlighted in red circles. We have employed a spin-3/2 chain with seven sites.

remains dynamically constrained unless its two neighboring spins are both in the lowest spin- z eigenstates $|-j\rangle_k$.

Our framework can be applied to this model directly. However, solving the energy spectrum of this model may pose significant challenges. Considering a spin- j chain with D sites, the dimension of the corresponding Hilbert space is $(2j+1)^D$ with numerous unnecessary degrees of freedom. We can reduce these redundant degrees of freedom by re-encoding two adjacent spins $|m, m'\rangle_{2l-1, 2l}$ into a logic site $|\cdot\rangle_l$ with $4j+1$ degrees of freedom. Formally, we can divide the blockades $\mathcal{B}$ into two subsets of blockade projectors: $\mathcal{B}_o = \{\pi_{2l-1}, l = 1, 2, \dots, D/2\}$ and $\mathcal{B}_e = \{\pi_{2l}, l = 1, 2, \dots, D/2\}$, where $\hat{B}_o(\hat{B}_e)$ act only on the neighboring $2l-1$ th and $2l$ th sites ($2l$ th and $2l+1$ th sites). The projector of the constrained subspace can also be decomposed into two parts: $\hat{P} = \hat{P}_o \cdot \hat{P}_e$, where $\hat{P}_{o(e)} = \prod_{\pi_l \in \mathcal{B}_{o(e)}} (\mathbb{I} - \pi_l)$. Following this, we can rewrite the Hamiltonian as

$$\begin{aligned} H_{\text{hXP}} &= \hat{P}_e \hat{P}_o \sum_{k=1}^D s_k^x \hat{P}_o \hat{P}_e = \hat{P}_e H_{\text{odd}} \hat{P}_e, \\ H_{\text{odd}} &= \hat{P}_o \sum_{k=1}^D s_k^x \hat{P}_o = H_0 \cdot \hat{P}_o = \hat{P}_o \cdot H_0, \end{aligned} \quad (\text{D2})$$

where H_0 commutes with $\hat{P}_o$: $[H_0, \hat{P}_o] = 0$, and is equivalent to H_{odd} within the subspace denoted by $\hat{P}_o$. Considering the

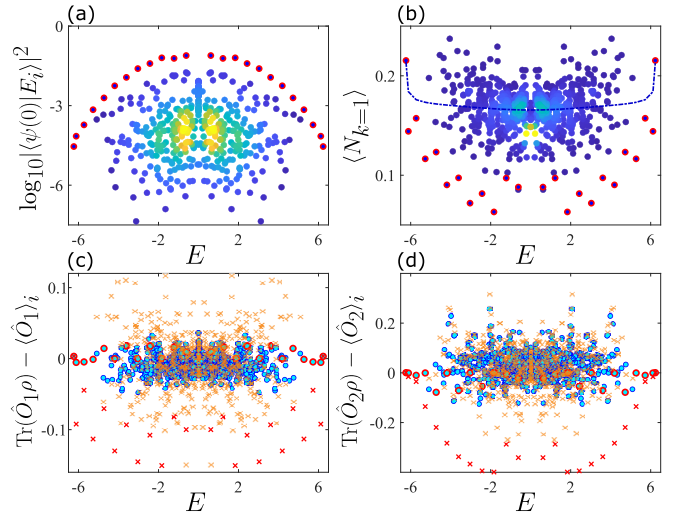


FIG. 8. Panel (a) shows the overlap between the eigenstates and the initial state $|\psi(0)\rangle = |j, -j, \dots, j, -j\rangle$. Panel (b) shows the expectation values of the local observable $N_{k=1}$, with the points indicating the results obtained from the eigenstates: $\langle N_{k=1} \rangle_i = \langle E_i | N_{k=1} | E_i \rangle$, and the dashed line stands for the expected values given by the prediction of canonical ensemble. In both panels (a) and (b), we mark the scar states in the red circles. Panels (c) and (d) demonstrate the deviation between the eigenstates expectations of local observables $\hat{O}_1, \hat{O}_2$ and that of the DMs: $\text{Tr}(\hat{O}_{1,2}\rho) - \langle \hat{O}_{1,2} \rangle_i$. We take ρ to be either the canonical DM or the adjusted version from our work. The results from the canonical DM $\rho_c(\beta_c)$ are represented by the yellow and red crosses, with the former marking the thermal states, and the latter marking the scar states. The results from the adjusted grand canonical DM $\rho(\beta, \nu)$ are represented by blue dots, where the scar states are highlighted with red circles. We choose $\hat{O}_1$ to be the single-site spin- z operator $\hat{O}_1 = s_k^z$, and $\hat{O}_2$ to be the product of spin- z operators on the next neighboring sites $\hat{O}_2 = s_k^z \otimes s_{k+2}^z$. We carried out our calculations in a spin-1 chain with 12 sites.

null space of $\hat{P}_o$, H_{odd} is entirely constrained while H_0 can still drive the system evolving in the unconstrained area. Moreover, $H_0 = \sum_{l=1}^{D/2} h_l$ is comprised of local terms h_l 's, which signify the blockade $\mathcal{B}_o$ with local interactions between neighboring spins:

$$\begin{aligned} h_l &= (\mathbb{I} - \pi_{2l-1})(s_{2l-1}^x + s_{2l}^x)(\mathbb{I} - \pi_{2l-1}) \\ &= (\mathbb{I} - T_{2l-1} \otimes T_{2l})(s_{2l-1}^x + s_{2l}^x)(\mathbb{I} - T_{2l-1} \otimes T_{2l}) \\ &= (\mathbb{I} - T_{2l-1}) \otimes s_{2l}^x + s_{2l-1}^x \otimes (\mathbb{I} - T_{2l}) \\ &= |-j\rangle\langle -j|_{2l-1} \otimes s_{2l}^x + s_{2l-1}^x \otimes |-j\rangle\langle -j|_{2l}. \end{aligned} \quad (\text{D3})$$

Then, we rewrite the constrained Hamiltonian as

$$\begin{aligned} H_{\text{hXP}} &= \hat{P}_e H_{\text{odd}} \hat{P}_e = \hat{P}_e H_0 \hat{P}_e \cdot \hat{P}_o \\ &= \sum_{l=1}^{D/2} \prod_{l_1, l_2=l-1}^l (\mathbb{I} - \pi_{2l_1}) h_l (\mathbb{I} - \pi_{2l_2}) \cdot \hat{P} \\ &= \sum_{l=1}^{D/2} \prod_{l_1, l_2=l-1}^l (\hat{P}_o - \pi_{2l_1} \cdot \hat{P}_o) h_l (\hat{P}_o - \pi_{2l_2} \cdot \hat{P}_o) \cdot \hat{P} \end{aligned}$$

$$\begin{aligned}
&= \sum_{l=1}^{D/2} \prod_{l_1, l_2=l-1}^l (\mathbb{I} - \pi'_{2l_1}) h_l (\mathbb{I} - \pi'_{2l_2}) \cdot \hat{P} \\
&= H \cdot \hat{P}, \\
H &= \sum_{l=1}^{D/2} h_l - \Pi_l h_l - h_l \Pi_l + \Pi_l h_l \Pi_l, \quad (D4)
\end{aligned}$$

where we have utilized the commutation relation between $\hat{P}_o$ and $\hat{P}_e$ in the first line. Besides, $\hat{P}_o^2 = \hat{P}_o$ is a projector, which enables us to project the blockades π_{2l} 's onto the subspace denoted by $\hat{P}_o$ and arrive at $\pi' = \pi \cdot \hat{P}_o$. This further leads us to define the local blockade projectors $\Pi_l = \pi'_{2l-2} + \pi'_{2l}$. Finally, H commutes with $\hat{P}$ and is the equivalent Hamiltonian for this model.

Now we take the critical step by viewing the subspace denoted by $\hat{P}_o$ as the entire Hilbert space. Consequently, we can regard h_l as the local Hamiltonian acting on the l th logic site, and consider this model as an outcome of applying the blockades $\mathcal{B}_e$ on H_0 . Correspondingly, we should eliminate the states $|m, m'\rangle_{2l-1, 2l}$ with m and m' both greater than $-j$, since these states are located in the null space of $\hat{P}_o$. Then we map the remaining $4j + 1$ states as

$$\begin{aligned}
|-j, m\rangle_{2l-1, 2l} &\rightarrow |a_m\rangle_l, \quad m = -j + 1, \dots, j - 1, j, \\
|m, -j\rangle_{2l-1, 2l} &\rightarrow |b_m\rangle_l, \quad m = -j + 1, \dots, j - 1, j, \\
|-j, -j\rangle_{2l-1, 2l} &\rightarrow |c\rangle_l, \quad (D5)
\end{aligned}$$

with $l = 1, 2, \dots, D/2$ denotes the logic site number. We note that all the terms in H can be expressed with these states. Therefore, we have effectively reduced the dimension of the Hilbert space to $(4j + 1)^{D/2}$.

Following the framework and Eq. (D4), we can build up a master equation equivalent to the unitary dynamics governed by H , and construct the quasiparticle counting operator $\hat{N} = \sum_{k=1}^{D/2} L_{k,1}^\dagger L_{k,1}$, which then leads to the alternation of the Gibbs ensemble as discussed in Sec. V. We spare the repetitive derivation process and provide the calculations done upon this model below.

First, we show the eigenstates' deviation from ETH by calculating the expectation of the local observable $N_k = L_{k,1}^\dagger L_{k,1}$ in Fig. 8(b). We also show the overlap between the eigenstates and the appointed initial state in Fig. 8(a) as a reference. Scar eigenstates appear to be the ones that show higher overlaps with the initial state, and at the same time, mostly diverge from the canonical average. Then, we display the deviation between the eigenstates' expectation values and the ensemble average for both the grand canonical ensemble and the canonical ensemble in Figs. 8(c) and 8(d). We have employed two local observables for the calculation. It can be seen that the grand canonical ensemble shows a significant advantage over the canonical ensemble, as the grand canonical DM can accurately predict the expectation values for both the scar states and the thermal eigenstates. Specifically, the grand canonical ensemble provided in Eq. (12) demonstrates better fits for the local properties of the scar eigenstates, while these properties show significant deviation from the canonical ensemble average originating from ETH.

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