# Stochastically perturbed billiards: Fingerprints of chaos and universality classes

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Billiards tables—a minimal model for particles moving in a confined region—are known to present different classical (and quantum) features according to their shape, ranging from strongly chaotic to integrable dynamics. Here, we consider the role of a stochastic perturbation of the elastic reflection law and show that while chaotic billiards maintain their key statistical feature, the behavior for integrable billiard tables is completely different: It can be linked, for tiny perturbations, to the Evans stochastic billiard, where at each collision the reflected angle is a uniformly distributed stochastic variable on $(-\pi/2, \pi/2)$. The resulting spatial stationary measure has peculiar aspects, like being typically nonuniform along the boundary, differently from any chaotic billiard table.

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## I. INTRODUCTION

Billiard tables, where a particle moves with constant velocity in a bounded domain, being elastically reflected upon colliding with the boundary, are known to display a rich variety of dynamical behavior [1–3], dictated by the geometric shape of the boundary. In particular, dispersing and defocusing billiards [4,5] are prototypical examples of ergodic and chaotic systems, while, on the opposite side, circular or elliptic billiards provide instances of regular, integrable motion [6]. We recall that these classical properties also reflect at the quantum and mesoscopic levels [7–9].

Elastic reflections are defined by the velocity change rule at the point of impact $\mathbf{q}$:

$$\mathbf{v}_{\text{out}} = \mathbf{v}_{\text{in}} - 2\mathbf{n}_q(\mathbf{v}_{\text{in}} \cdot \mathbf{n}_q), \tag{1}$$

where $\mathbf{n_q}$ is the (inward) normal at $\mathbf{q}$. Law (1) conserves the kinetic energy of the particle, and it guarantees that phase space volume is preserved [2] (so the dynamics is Hamiltonian). We mention that *deterministic* reflection laws different from specular reflections have been considered to study shear flows, in connection with fluctuation relations of phase space contraction [10,11].

On a different perspective, stochastic reflection laws have been considered since the foundation of kinetic theory of rarefied gases [12], in particular, by taking into account the Knudsen (or Lambert [13]) cosine law, where, upon collisions, the outgoing angles $\psi_i$ (measured with respect to $\mathbf{n}_q$) are i.i.d. random variables, independent of the incoming angle $\phi_i$ and distributed with a density

$$p_K(\psi) = \tfrac{1}{2}\cos(\psi), \quad \psi \in (-\pi/2, \pi/2). \tag{2}$$

Later, Knudsen prescription has been modified [14,15], considering a mixture of elastic and Knudsen collision laws, and this picture was supported by microscopic simulations [16].

We observe that the stochastic Knudsen law (2) has been mathematically supported as a model of collisions from rough surfaces [17,18], and it enjoys nice statistical properties [19]: Its rigorous derivation requires a *reciprocity* property [17], inspired by time reversibility of microscopic dynamics [20]. In the framework of stochastic billiards, other probability distributions, not necessarily satisfying reciprocity, have been analyzed: In Ref. [21], billiard tables with a uniform scattering angle (again independent of the incoming angle), corresponding to a density

$$p_E(\psi) = \frac{1}{\pi}, \quad \psi \in (-\pi/2, \pi/2) \tag{3}$$

were introduced. Evans proved that, both for continuous time dynamics and for the corresponding collision Markov chain, there exists a unique invariant probability measure. Such a measure displays peculiar properties: For instance, if we consider a circular table, the spatial invariant distribution is not uniform, being peaked at the boundary [22]. This closely reminds of what is observed in active matter confined in a bounded region [23–25] (see also Ref. [26] and references therein): Even if active matter interaction with the boundary, causing accumulation, is complex (see, for instance, Ref. [27]), the mechanism of active particles captured by the surface and released by random tumbling might be modeled exactly by Eq. (3) [28–30]. More explicitly, a number of studies on bacterial motility in confined geometries (see Ref. [29] and references therein) have pointed out how cells emerge from the boundary by repeated reorientations, until an escape direction is extracted: For a uniform tumbling probability, this mechanism generates exactly Evans billiard trajectories.

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The main point of this present paper is to consider small stochastic perturbations of the deterministic elastic reflection law (1) [31] and to show that when such a perturbation is very small, the billiard tables exhibit either stationary properties associated with Knudsen law (2) or Evans law (3), depending on whether the corresponding deterministic table is chaotic or integrable.

## II. STOCHASTIC PERTURBATIONS AND UNIVERSAL PROPERTIES

First of all, let us fix our notation: We consider a particle of unit speed moving ballistically in a bounded region $\Omega \subset \mathbf{R}^2$ (our billiard table), whose boundary is denoted by $\partial\Omega$: The direction of the velocity is only changed upon collisions with $\partial\Omega$; moreover, we parametrize the boundary by the arclength $s \in [0, L)$. For $\mathbf{q} \in \partial\Omega$, $\mathbf{n_q}$ is the unit normal vector directed inward. Let $\theta_{\rm in} \in (-\pi/2, \pi/2)$ be the angle (with respect to $\mathbf{n_q}$) with which the incoming trajectory would be *elastically* reflected upon a collision in $\mathbf{q}$ (the sign is dictated by reference to the oriented tangent to $\partial\Omega$ in $\mathbf{q}$), while $\theta_{\rm out}$ is the actual angle the trajectory takes after collision (measured in the same way as above), so that elastic collisions (deterministic billiard tables) result in $\theta_{\rm in} = \theta_{\rm out}$. Stochastic (Markov) billiards are thus described by a stochastic transition kernel

$$\mathcal{W}(\theta_{\rm in} \to \theta_{\rm out}), \tag{4}$$

where we ignore a possible dependence upon the collision point $\mathbf{q}$. With our notation, we have

$$\mathcal{W}(\theta_{\rm in} \to \theta_{\rm out}) = \delta(\theta_{\rm in} - \theta_{\rm out}) \tag{5}$$

for elastic collisions and

$$\mathcal{W}(\theta_{\rm in} \to \theta_{\rm out}) = p_K(\theta_{\rm out}) \text{ or } p_E(\theta_{\rm out}) \tag{6}$$

for Knudsen or Evans stochastic billiards, respectively. We will be mainly concerned with the discrete stochastic process $\{\mathbf{q}_n\}$ of consecutive impact points with the boundary, where the particle proceeds with uniform unit velocity with direction induced by $\theta_{{\rm out}\,(n)}$ after collision at point $\mathbf{q}_n$ until it hits $\partial\Omega$ again at $\mathbf{q}_{n+1}$, from which it will exit with velocity directed according to $\theta_{{\rm out}\,(n+1)}$. If the distribution of outgoing angle is independent of the incoming angle, and given by a common law $p_\circ$, like Eq. (6), the relevant process is given by the sequence of collision points: In particular, the question is whether it admits an invariant density. We introduce a transition kernel [19] $\mathcal{K}(\xi \to \eta)$, where $\xi$ and $\eta$ are arclength coordinates corresponding to $\mathbf{x},\ \mathbf{y} \in \partial\Omega$,

$$\mathcal{K}(\xi \to \eta) = \mathcal{K}(\mathbf{x}\to\mathbf{y}) = p_\circ(\theta_\xi)\frac{\cos(\theta_\eta)}{\|\mathbf{x}-\mathbf{y}\|}, \quad \text{if } \mathbf{x} \rightleftharpoons \mathbf{y}, \tag{7}$$

where $\theta_\xi$ is the angle between $\mathbf{n_x}$ and $(\mathbf{y}-\mathbf{x})$, and $\theta_\eta$ is the angle between $\mathbf{n_y}$ and $(\mathbf{x}-\mathbf{y})$, while $\mathbf{x} \rightleftharpoons \mathbf{y}$ means that the open segment $(\mathbf{x}-\mathbf{y})$ is fully contained in $\Omega$ (this is always the case for convex billiard tables). Then, the invariant density $\mu(s)$ satisfies the equation

$$\mu(\xi) = \int_{\partial\Omega} d\eta\, \mu(\eta)\mathcal{K}(\eta \to \xi); \tag{8}$$

furthermore, if a density $\mu(s)$ satisfies the detailed balance condition

$$\mu(\xi)\mathcal{K}(\xi\to\eta) = \mu(\eta)\mathcal{K}(\eta \to \xi), \tag{9}$$

then Eq. (8) is ipso facto satisfied.

When we consider the case of Knudsen law (2), the kernel (7) becomes *symmetric* [19], so detailed balance is satisfied for a constant density: Notice that, in the deterministic case [Eq. (1)], when one considers the discrete dynamics on the boundary (with the coordinates arclength $\xi$ and outgoing angle $\psi$), the invariant density

$$\mu(\xi)\rho(\psi) = \frac{1}{2|\partial\Omega|}\cos(\psi) \tag{10}$$

follows from the uniform invariant measure for the continuous time dynamics [2,32,33]. From a dynamical point of view, such a uniform measure coincides with the natural measure (associated with time averages) only when the billiard table is ergodic.

On the other side, when we consider a stochastic, uniformly distributed, outgoing angle [Eq. (3)], the analysis is not so simple: We do not even know whether (apart from few examples, as we will discuss further on) the corresponding Markov chain [Eq. (7)] satisfies detailed balance [21,34].

We are now ready to formulate our main results: When we introduce a very small stochastic perturbation on the collision dynamics, the billiard becomes ergodic, irrespective of the shape of the table [17,19,21,35], but what about the invariant measure of the discrete dynamics? We provide analytic arguments and numerical experiments showing that for chaotic billiards the invariant density is very close to the Knudsen case, while for integrable cases the density strongly resembles Evans billiards: In this sense, the stationary probability falls into different universality classes, depending on whether the classical deterministic table is chaotic or integrable. Billiards with mixed phase space require further analysis (see Fig. 1).

Before showing our results, we have to define which type of stochastic perturbation we will consider: Intuitively, we replace the elastic outgoing angle $\theta^{\rm el}_{\rm out}$ with a random variable uniformly distributed in $[\theta^{\rm el}_{\rm out} - \epsilon, \theta^{\rm el}_{\rm out} + \epsilon]$. We cannot use such a definition for any $\theta^{\rm el}_{\rm out}$, since close to $\pm\pi/2$ the trajectory might then escape the table: Close to tangency, we follow the choice of Markarian *et al.* [31]; when $\pi/2 - \theta^{\rm el}_{\rm out} < \epsilon$, the outgoing angle is uniformly chosen in $[\pi/2 - 2\epsilon, \pi/2]$, a symmetric choice being taken around $-\pi/2$. Actually, in the final part of the paper we will comment how the choice near tangency is not crucial, unless angular dynamics is absorbing at $\pm\pi/2$ (an example when this may happen is mentioned in Ref. [31]).

Let us first consider billiard tables with ergodic deterministic dynamics: In this case, we expect that the Markov chain invariant measure should be close to Eq. (10), due to stochastic stability of strongly chaotic systems [36–38]; this is indeed the case (see Figs. 1 and 2) when we consider a dispersive diamond billiard [39,40] (the concave region bounded by four arcs of circles of radius $R$ whose centers lie at the vertices of an $L \times L$ square).

It is interesting to check whether the same results hold for billiards with a mixed phase space. We consider here (generalized) lemon billiards [41–48]: The convex region is defined

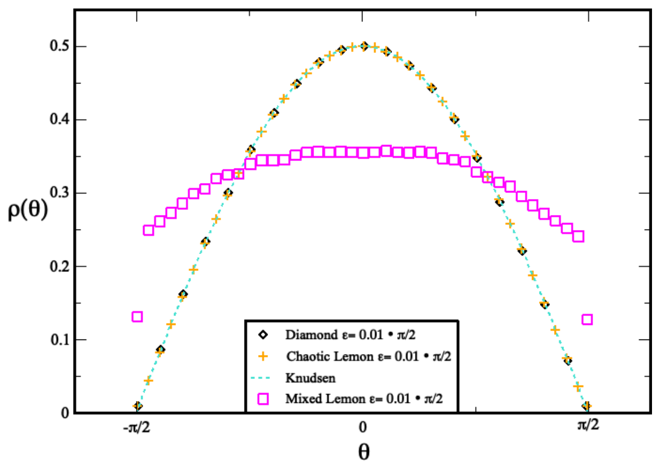


FIG. 1. Probability density of outgoing angles under a weak stochastic perturbation: Knudsen distribution is well reproduced by a fully chaotic diamond billiard ($R = \sqrt{2}$ and $L = 1 + \sqrt{3}$) and an ergodic lemon billiard ($R_1 = 1$, $R_2 = 1.5$, $d = 1.15$), while significant deviations appear for $R_1 = R_2 = 1$, $d = 0.01$, close to the integrable case (circular billiard). All numerical data are obtained by $10^9$ collisions.

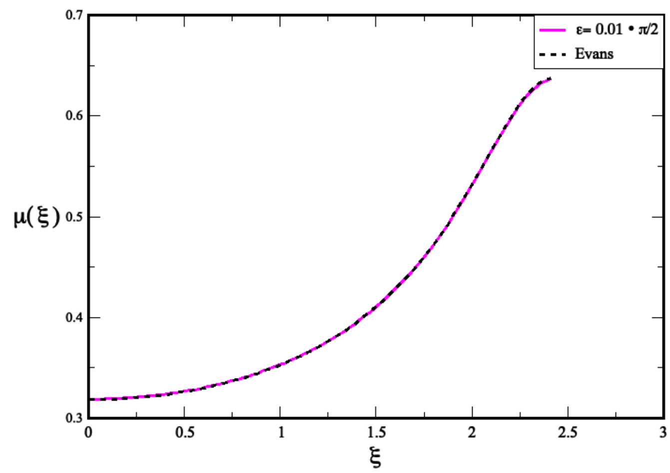


FIG. 3. Probability density along the boundary for one quarter of an elliptical billiard ($a = 2$, $b = 1$) under a weak stochastic perturbation: Numerical data (with $10^9$ collisions) are compared to the invariant density of the Evans billiard [Eq. (14)].

by the intersection of two circles of radii $R_1$ and $R_2$, whose centers are separated by a distance $d$; when $R_1 = R_2 = R$ and $d/R$ is very small, the phase space has only a small chaotic component, while for appropriate choices of $R_1 \neq R_2$ and $d$, the billiard is ergodic [43]. We see from Fig. 1 that indeed in the ergodic case again we recover Knudsen distribution, while strong deviations (manifesting through a flattened distribution) appear close to the integrable case. We remark that our numerical experiments for chaotic billiards encompass examples of both mechanisms of chaoticity: dispersion and defocusing [4,49,50].

When we turn to integrable billiard tables, the situation is completely different: Here, we present results concerning elliptic billiards, the simplest example beyond circular billiards (where rotation symmetry plays an important role). Their purely deterministic dynamics is integrable: Any trajectory is tangent to a caustic (confocal ellipse or hyperbola [51,52]; see also Ref. [53], where the existence of caustics is proven for more general convex tables). What happens if we apply a small stochastic perturbation? We know from Refs. [19,31] that the system is now ergodic, but as we can see (Figs. 3 and 4) the spatial and angular densities are completely different. They are very well reproduced by Evans results: The outgoing angle distribution [Eq. (4)] is flat (apart from a tiny region around the grazing angles $\pm\pi/2$), while the spatial density is not uniform along the boundary, and it is nearly indistinguishable from the invariant spatial distribution of an elliptic Evans billiard. As a matter of fact, the invariant density for the Evans elliptic billiard may be explicitly found [54]: Let $a$ and $b$ be the semiaxes ($a^{-2}x^2 + b^{-2}y^2 = 1$); then, given two

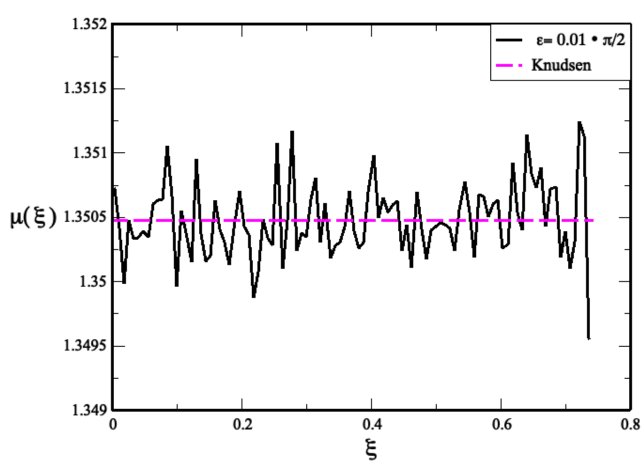


FIG. 2. Probability density along the boundary under a weak stochastic perturbation for the chaotic diamond billiard ($R = \sqrt{2}$ and $L = 1 + \sqrt{3}$): We observe small fluctuations around a uniform density (due to symmetry, only one out the four arcs is considered). Numerical data are obtained by $10^9$ collisions.

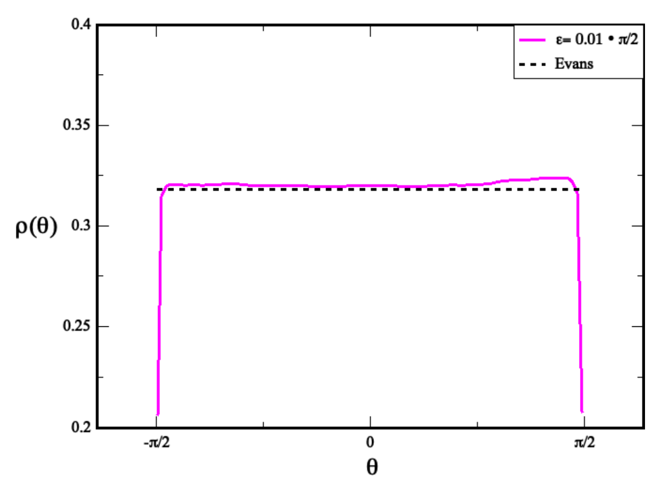


FIG. 4. Probability density of outgoing angle for one quarter of an elliptical billiard ($a = 2$, $b = 1$) under a weak stochastic perturbation: Numerical data (with $10^9$ collisions) are compared to a uniform distribution; deviations only appear close to the boundary (see text).

points $\mathbf{q}$, $\mathbf{p}$ on the ellipse, we may write the transition kernel (7) as

$$\mathcal{K}(\mathbf{q}\to\mathbf{p}) = \frac{1}{\pi}\frac{(\mathbf{q}-\mathbf{p})\cdot\mathbf{n_p}}{\|\mathbf{q}-\mathbf{p}\|^2}. \tag{11}$$

Then, once we write the inward normal at the boundary point $\mathbf{q}$ as

$$(n_x, n_y) = \frac{1}{\sqrt{\left(a^{-4}q_x^2 + b^{-4}q_y^2\right)}}(-a^{-2}x, -b^{-2}y), \tag{12}$$

we can get the identity

$$\begin{aligned}&\sqrt{\left(a^{-4}q_x^2 + b^{-4}q_y^2\right)}\mathbf{n_q}\cdot(\mathbf{p}-\mathbf{q})\\ &\quad = \sqrt{\left(a^{-4}p_x^2 + b^{-4}p_y^2\right)}\mathbf{n_p}\cdot(\mathbf{q}-\mathbf{p});\end{aligned} \tag{13}$$

then, some simple calculations show that the invariant density satisfying detailed balance is

$$\mu(\mathbf{q}) = \frac{C}{\sqrt{a^{-4}q_x^2 + b^{-4}q_y^2}}, \tag{14}$$

where $C$ is the normalization constant [55]: In Fig. 3, we show how Eq. (14) reproduces with high precision the numerical experiments. We remark that a curvature-dependent density on $\partial\Omega$, together with a nonuniform measure in $\Omega$, has been observed in active matter [23,25], and in this respect, the Evans billiard may be considered like a toy model reproducing some key issues of complex matter in confined geometry.

As regards the distribution of outgoing angles in the same case, we also get a picture very similar to the Evans case (see Fig. 4). This is a consequence of the strong correlations of successive outgoing angles in the unperturbed (deterministic) dynamics. Consider, for simplicity, a circular table: In the elastic case with no perturbation, if we start with an outgoing angle $\theta_0$, then the whole temporal sequence will be $\theta_n = \theta_0$; the introduction of the stochastic perturbation then amounts to turn the dynamics into a random walk (we consider only half of the domain $[0, \pi/2]$; the rule is symmetric in the left half):

$$\theta_{i+1} = \begin{cases}\theta_i + \delta, & \theta_i < \pi/2 - \epsilon,\\ \pi/2 - \epsilon + \delta, & \theta_i > \pi/2 - \epsilon,\end{cases} \tag{15}$$

where $\delta$ is a stochastic variable uniformly distributed in $[-\epsilon, \epsilon]$. For very small perturbations, we have a large region $[0, \pi/2 - \epsilon]$ where the random walk is homogeneous and symmetric, suggesting a uniform stationary probability distribution, as observed in Fig. 4, provided the end point $\pi/2$ is not absorbing, which, with our choice of the stochastic perturbation, is guaranteed by a net drift (if $\theta_n > \pi/2 - \epsilon$, then $\langle\theta_{n+1}\rangle = \pi/2 - \epsilon < \theta_n$). The same conclusion can be obtained by writing a Fokker-Planck equation [56] and considering its stationary solution. As a matter of fact, outside a region of size $\epsilon$ close to the boundary (grazing angles), we have a constant diffusion $D^{(2)}(\theta) = D = \epsilon^2/3$, and zero drift $D^{(1)}(\theta) = 0$, and the stationary solution $W(\theta) = N\exp(-\Phi(\theta))$, where $\Phi(\theta) = \ln D^{(2)} - \int^{\theta} d\phi\, D^{(1)}(\phi)/D^{(2)}(\phi)$, is constant. The actual form of the perturbation around grazing angles modifies the stationary solution only close to the boundary, unless divergences arise in the computation of $W(\theta)$ as $\theta$ tends to $\pi/2$ (which is not the case for the perturbation we employ here [57]).

## III. CONCLUSIONS AND PERSPECTIVES

In conclusion, we have considered the effects of small stochastic perturbations of the elastic reflection law of billiard tables: While stochastic stability leads to the preservation of the cosine law and spatial uniform distribution along the boundary in the case of chaotic billiards, the results for integrable billiard tables are quite different; the statistical behavior is quite close to Evans random billiards, where the reflection angle is a uniformly distributed random variable on $(-\pi/2, \pi/2)$. Such billiards present a number of peculiar statistical properties: The spatial measure along the boundary depends upon the curvature, the volume measure is not uniform (being concentrated along the boundary [22,25]), and the Cauchy invariance principle [58–61] is not satisfied [22].

Several perspectives and directions for future work remain open: On the theoretical side, a natural problem is to better characterize the invariant measure, under weak stochastic perturbations, of "mixed" billiards where a chaotic sea coexists with regular components.

As regards the relationship to active matter (in particular, "pulled"-type swimmers [62]), further work will scrutinize the simultaneous use of "Evans tumbling" [28,30] and the Cauchy invariance principle to estimate the path length between collisions [28,62]: As we mentioned, the mean chord in the Evans billiard does not satisfy the Cauchy estimate, as long as the motion is purely ballistic between collisions with the boundary. What happens if we replace ballistic trajectories with run-and-tumble motion, which seems more appropriate in describing microorganism dynamics? In Ref. [63], there are indications that the Cauchy invariance principle is restored when the characteristic length of random motion is much smaller than the linear size of the container; however, this remains an open problem.

Finally, if we consider "pushed"-type swimmers (see, for instance, Ref. [62]), where agents slide along the boundary before leaving it, investigating the recently proposed pensive billiards [64] might be of interest.

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## DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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