

Construction of hopfion crystals

Wen-Tao Hou^{1,*}, Zhuoxian Xiang¹, Yizhou Liu^{2,†} and Jiadong Zang^{3,‡}¹*School of Physical Science and Technology, Tiangong University, Tianjin 300387, China*²*Anhui Province Key Laboratory of Low-Energy Quantum Materials and Devices, High Magnetic Field Laboratory, HFIPS, Chinese Academy of Sciences, Hefei 230031, China*³*Department of Physics and Astronomy, University of New Hampshire, Durham, New Hampshire 03824, USA*

(Received 8 April 2025; accepted 17 October 2025; published 19 November 2025)

Hopfions, three-dimensional topological solitons characterized by nontrivial Hopf indices, represent a fundamental class of field configurations that emerge across diverse areas of physics. Despite extensive studies of isolated hopfions, a framework for constructing spatially ordered arrays of hopfions, i.e., hopfion crystals, has been lacking. Here, we present a systematic approach for generating hopfion crystals with cubic symmetry by combining the Hopf map with rational mapping techniques. By superposing helical waves in $\mathbb{R}^4$, we construct hopfion crystals with tunable Hopf indices and controllable topology. We demonstrate simple cubic, face-centered cubic, and body-centered cubic hopfion crystals and extend our framework to create crystals of more complex topological structures, including axially symmetric tori, torus links, and torus knots with higher Hopf indices. Our results provide a foundation for searching hopfions in real materials and studying their collective phenomena.

DOI: [10.1103/rr4k-x9m5](https://doi.org/10.1103/rr4k-x9m5)

In 1975, Faddeev introduced hopfions as a class of truly three-dimensional (3D) topological solitons characterized by integer Hopf indices [1–3]. Since then, hopfions have sparked intense research interest and have been studied in a variety of physical systems across scales, including optical fields [4–8], Bose-Einstein condensates [9–13], liquid crystals [14,15], magnetic materials [16–20], ferroelectric materials [21], superconductors [22], superfluid [23], etc. Hopfions, as 3D topological solitons, exhibit rich topological structures such as rings, links, and knots, which give rise to fascinating physical phenomena. For example, in magnetic systems, theoretical studies have revealed hopfions' unique nonlinear transport effects and 3D dynamic characteristics [24–29], highlighting their potential applications in 3D spintronics [30]. Recent advances in fabrication and three-dimensional imaging techniques from nano- to microscales have made it possible to access the details of hopfions in laboratories [15,30–32].

Despite growing interest, searching for hopfions in real physical systems and investigating their collective behaviors remain challenging due to their sophisticated configurations and complex models. Obtaining analytical expressions for hopfion configurations is important for studying their stability, dynamics, and physical properties, which in combination with computational methods could enable efficient explo-

ration of the parameter space and provide comprehensive understanding. While analytical ansatzes for single hopfions with various topologies have been developed based on methods like rational maps [33–35], the theoretical framework for constructing spatially ordered hopfion crystals, critical for unlocking real material models that host hopfions as ground states and studying emergent phenomena in hopfion ensembles, has been lacking.

In this Letter, we present a method for constructing hopfion crystals, which offers a robust framework for generating topological solitons in three dimensions. We start by first reviewing the construction process of a single hopfion. A hopfion realizes the map from $\mathbb{R}^3$ to $\mathbb{S}^2$. As long as the spins at infinity are polarized to the same direction as required by the finite energy, the base manifold $\mathbb{R}^3$ is compactified to $\mathbb{S}^3$. The isomorphism between $\mathbf{r} \in \mathbb{R}^3$ and $\mathbf{u} \in \mathbb{S}^3$ is enabled by the stereographic projection

$$\begin{aligned} u_i &= \frac{r_i}{r} \sin f, \\ u_4 &= \cos f, \end{aligned} \quad (1)$$

where $i = \{1, 2, 3\}$ are the Cartesian coordinates. A common choice of the angle f is $f = \arccos \frac{1-r^2}{1+r^2}$, but it can be any smooth function interpolating between 0 and π from origin to infinity. The Hopf map $\mathbb{S}^3 \rightarrow \mathbb{S}^2$ can be constructed by the rational map [34,36]. Introducing two complex coordinates on $\mathbb{S}^3$,

$$Z_0 = u_1 + iu_2, \quad Z_1 = u_3 + iu_4. \quad (2)$$

One can construct a variable $\xi \in \mathbb{C}$ in the Riemann complex plane by

$$\xi = \frac{P(Z_1, Z_0, \bar{Z}_1, \bar{Z}_0)}{Q(Z_1, Z_0, \bar{Z}_1, \bar{Z}_0)}, \quad (3)$$

*Contact author: houtaowen@tiangong.edu.cn†Contact author: yzliu@hmf.ac.cn‡Contact author: jiadong.zang@unh.edu

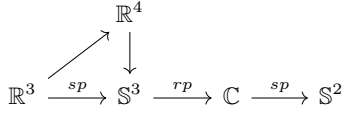


FIG. 1. Two mapping pathways from $\mathbb{R}^3$ to $\mathbb{S}^2$. The lower one is the standard construction for a single hopfion, and the upper one is the path for hopfion crystals. sp stands for stereographic projection and rp stands for rational map.

where P and Q are polynomials of $Z_1, Z_0, \bar{Z}_1$, and $\bar{Z}_0$ (the complex conjugates of Z_1 and Z_0). Depending on these two polynomials, different types of hopfions can be constructed. The final step is the stereographic projection $\mathbb{C} \cong \mathbb{S}^2$:

$$n_1 = \frac{2\text{Re}(\xi)}{1 + \xi\bar{\xi}}, \quad n_2 = \frac{2\text{Im}(\xi)}{1 + \xi\bar{\xi}}, \quad n_3 = \frac{1 - \xi\bar{\xi}}{1 + \xi\bar{\xi}}, \quad (4)$$

or equivalently

$$\xi = \frac{n_1 + in_2}{1 + n_3}. \quad (5)$$

The isomorphism $\mathbb{C} \cong \mathbb{S}^2$ can also be constructed in terms of a spinor

$$\chi = \mathcal{N} \begin{pmatrix} Q \\ P \end{pmatrix}, \quad \mathbf{n} = \langle \chi | \sigma | \chi \rangle. \quad (6)$$

Here $\mathcal{N}$ is the normalization factor. The $U(1)$ gauge degree of freedom in χ is equivalent to projective degree of freedom in Eq. (3). The entire mapping pathway for a single hopfion is shown in Fig. 1.

A simple choice for ξ is $\xi = Z_1/Z_0$, corresponding to the spinor $|\chi\rangle = (Z_1, Z_0)^T$. Equispin contour, i.e., the preimage of map $\mathbb{R}^3 \rightarrow \mathbb{S}^2$, of $\mathbf{n} = (0, 0, -1)$ locates at $x = y = 0$, where $\xi = \infty$. Meanwhile, on the preimage of $\mathbf{n} = (0, 0, 1)$, $\xi = 0$ so that $u_3 = u_4 = 0, z = \cos f = 0$. If the conventional choice $\cos f = (1 - r^2)/(1 + r^2)$ is used, which corresponds to a circle with $r = 1$ on the x - y plane.

The topological index of the hopfion is the linking number between two arbitrary preimages. On the preimage ξ is kept constant, so that its tangential direction defines an emergent magnetic field

$$\mathbf{b} = \frac{1}{2\pi i} \frac{\nabla \xi \times \nabla \bar{\xi}}{(1 + \xi\bar{\xi})^2}. \quad (7)$$

It can be easily proved that this field is equivalent to the real space Berry curvature of the spin texture

$$\mathbf{b} = \frac{1}{4\pi} \mathbf{n} \cdot (\nabla \mathbf{n} \times \nabla \mathbf{n}). \quad (8)$$

The Hopf index is thus calculated as

$$Q_H = - \int d^3r \mathbf{a} \cdot \mathbf{b}, \quad (9)$$

given $\nabla \times \mathbf{a} = \mathbf{b}$, an arbitrary vector potential. A straightforward choice is $\mathbf{a} = -\frac{i}{2\pi} \langle \chi | \nabla | \chi \rangle$. The example above has the Hopf index $Q_H = 1$.

Generalization of the aforementioned single hopfion to a hopfion crystal is inspired by the construction of the skyrmion crystal. A closely packed skyrmion crystal is a superposition

of three spin helices: $\mathbf{n}(\mathbf{r}) = \mathbf{n}_0 + \sum_{i=1}^3 \mathbf{n}(\mathbf{k}_i) \exp(i\mathbf{k}_i \cdot \mathbf{r}) + \text{c.c.}$ [37]. Here, three wave vectors $\mathbf{k}_i$ having the same pitch length are coplanar and equal lateral. $\mathbf{n}(\mathbf{k}) = \frac{1}{2}(\hat{z} - i\hat{\mathbf{k}})$ gives a Néel skyrmion crystal. A Bloch skyrmion crystal takes another choice of $\mathbf{n}$ but it relates to the Néel skyrmion by $\pi/2$ rotation about $\hat{z}$ of all spins, so the topology is the same. $\mathbf{n}_0$ is a constant vector associated with $\mathbf{k} = 0$. Its presence is essential to avoid singularity $\mathbf{n} = 0$. This construction is able to build a crystal with nonzero topological charge $\int d^2r b_z$ because the Fourier transformation of the emergent magnetic field in Eq. (8) involves three linearly independent $\mathbf{n}(\mathbf{k})$ associated with three different $\mathbf{k}$. The topological index of hopfion in Eq. (9), however, involves six $\mathbf{n}(\mathbf{k})$. A direct search of valid $\mathbf{n}(\mathbf{k})$ likes finding a needle in a haystack.

The key idea behind the skyrmion crystal is to embed a spin vector—normally defined on $\mathbb{S}^2$ —into $\mathbb{R}^3$ instead. It can recover $\mathbb{S}^2$ space by renormalization. Therefore, an analogous way of constructing the hopfion crystal is to embed the vector $\mathbf{u}$ in Eq. (1) into $\mathbb{R}^4$ space instead. The map from $\mathbb{R}^3$ to $\mathbb{S}^2$ takes the upper path in Fig. 1. Imitating the expression for the Néel-type skyrmion crystal, we have

$$\mathbf{u}(\mathbf{r}) = \mathbf{u}_0 + \sum_{i=1}^3 \mathbf{u}(\mathbf{k}_i) \exp[i(\mathbf{k}_i \cdot \mathbf{r} + \phi_i)] + \text{c.c.}, \quad (10)$$

where ϕ_i are phase factors of each helix. Here, three independent $\mathbf{k}_i$ are taken. This is the minimal number of linearly independent vectors in three dimensions. In the skyrmion crystal in two dimensions, in principle, superposition of two wave vectors is enough to enable topological textures. However, the resulting state is a lattice of meron and antimeron with zero net topological charge [38,39]. In our current case, it will be shown shortly that three wave vectors are adequate.

We start with $\mathbf{k}_i = \hat{r}_i$ and $\mathbf{u}(\mathbf{k}_i) = \frac{1}{2}(\hat{l} - i\hat{\mathbf{k}}_i)$ in analogy to the skyrmion crystal. Here $\hat{l}$ is the unit vector in the fourth dimension. As a result, each helix takes the following forms:

$$\begin{aligned} \mathbf{u}_1 &= (\sin(\mathbf{k}_1 \cdot \mathbf{r} + \phi_1), 0, 0, \cos(\mathbf{k}_1 \cdot \mathbf{r} + \phi_1)), \\ \mathbf{u}_2 &= (0, \sin(\mathbf{k}_2 \cdot \mathbf{r} + \phi_2), 0, \cos(\mathbf{k}_2 \cdot \mathbf{r} + \phi_2)), \\ \mathbf{u}_3 &= (0, 0, \sin(\mathbf{k}_3 \cdot \mathbf{r} + \phi_3), \cos(\mathbf{k}_3 \cdot \mathbf{r} + \phi_3)), \end{aligned} \quad (11)$$

and $\mathbf{u}_0 = (0, 0, 0, c)$ is introduced to avoid singularity at which $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0}$. If a singularity were to exist, all sines in Eq. (11) are zero, and all cosines are ± 1 . Therefore, c cannot take the values of ± 1 and ± 3 . Every time these values are crossed, the Hopf index changes.

With these preparations, a hopfion crystal can be constructed using the same rational map. We again choose the simple case with Riemann coordinate $\xi = Z_1/Z_0$. The corresponding spinor state is then given by

$$|\chi\rangle = \mathcal{N} \begin{pmatrix} \sin Q_1 + i \sin Q_2 \\ \sin Q_3 + i(\cos Q_1 + \cos Q_2 + \cos Q_3 + c) \end{pmatrix}, \quad (12)$$

where $Q_i = \mathbf{k}_i \cdot \mathbf{r} + \phi_i$ and $\mathcal{N} = [(\cos Q_1 + \cos Q_2 + \cos Q_3 + c)^2 + \sin^2 Q_1 + \sin^2 Q_2 + \sin^2 Q_3]^{-\frac{1}{2}}$ is the renormalization factor. Without loss of generality, k , the amplitude of each wave vector, is chosen to be 1.

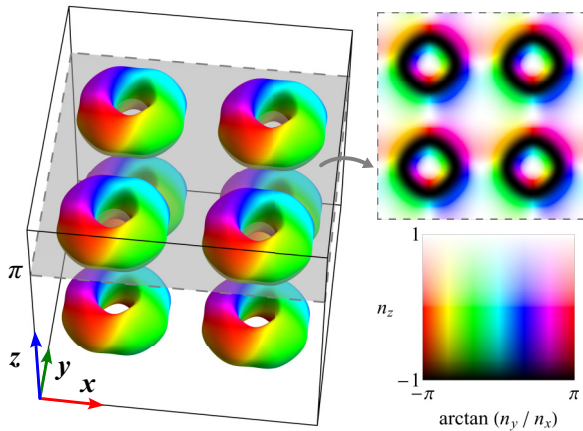


FIG. 2. $Q_H = 1$ sc hopfion crystal. The left is the sc $Q_H = 1$ hopfion crystal with $c = 2$ and x, y, z are from -2π to 2π . The right top is the spin texture of $z = \pi$ plane. The right bottom indicates the color dependence on $\arctan(n_y/n_x)$ and n_z .

The spin configuration with $c = 2$ is shown in Fig. 2, where we make $Q_1 = -x$, $Q_2 = y$, and $Q_3 = z$ (the purpose of choosing these parameters is to get a $Q_H = 1$ simple cubic hopfion crystal structure). The conventional unit cell with $-2\pi < x, y, z < 2\pi$ is chosen in the plot. Cluster of preimages with $n_z = 0$ form isolated tori in a simple cubic (sc) lattice. Two arbitrary preimages in the same torus wind around each other once after completing a full circle. The Hopf index of this structure within a unit cell is 1 (for details of the calculation of the Hopf index, the choice of c value and ϕ_i , see Secs. I, III, and IV of the Supplemental Material [40]).

One can try to write the resulting spin configuration into a multi- Q state as the skyrmion crystal $\mathbf{n}(\mathbf{r}) = \mathbf{p} \sin(\mathbf{k} \cdot \mathbf{r}) + \mathbf{q} \cos(\mathbf{k} \cdot \mathbf{r})$. As shown in Sec. II of the Supplemental Material [40], 12 wave vectors are needed to rebuild the simple cubic hopfion crystal. Among them, six wave vectors are associated with Bloch helix with both polarization vectors $\mathbf{p}$ and $\mathbf{q}$ perpendicular to the wave vector $\mathbf{k}$, so that spins rotate in a plane that is perpendicular to the direction of propagation. Four of them are associated with Néel helix with $\mathbf{k}$ parallel with one of the polarization vectors. In this case, spin rotates in a plane containing the propagation direction. Other than these, there is one component $-2c\hat{z} \sin z$, whose polarization is collinear with the wave vector. This resembles a spin density wave state. This component is responsible for the equivalency between two hopfions in the simple cubic lattice mentioned above. Due to the complexity of this multi- Q decomposition, realization of a hopfion crystal in real materials poses a challenge. Nevertheless, the analysis of each component above sheds light on the materials' design and provides a hint for the experimental detection of hopfion crystals with methods such as neutron and x-ray scattering [37,41]. Moreover, if any of the two Q_i are linearly dependent, no hopfion is formed. The configurations with different choice of ϕ_i can be connected by the spatial translation. Some configurations with different combinations of ϕ_i ($i = 1, 2, 3$) are shown in Sec. III of the Supplemental Material [40].

Choosing other sets of $\mathbf{k}_i$ vectors, hopfion crystals with other geometries can also be constructed. One may

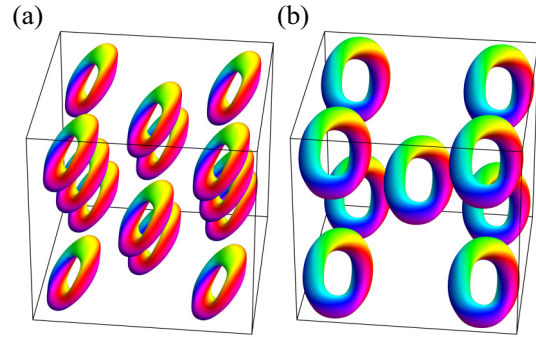


FIG. 3. $Q_H = 1$ fcc [panel (a)] and bcc [panel (b)] crystals with $c = 2$.

use $\mathbf{u}(\mathbf{k}_i) = \frac{1}{2}(\hat{\mathbf{i}} - i\hat{\mathbf{k}}_i)$. Alternatively, one can continue to use Eq. (1). Two approaches just differ by a spin rotation from each other, but the structure of hopfions and the value of the Hopf index are unchanged. We will use the latter approach in what follows. Selecting $\mathbf{k}_1 = \frac{1}{\sqrt{3}}(1, 1, 1)$, $\mathbf{k}_2 = \frac{1}{\sqrt{3}}(1, 1, -1)$, and $\mathbf{k}_3 = \frac{1}{\sqrt{3}}(1, -1, -1)$, the unit vectors of a bcc cubic lattice, one can yield a face-centered cubic (fcc) hopfion crystal. Similarly, choosing $\mathbf{k}_1 = \frac{1}{\sqrt{2}}(-1, -1, 0)$, $\mathbf{k}_2 = \frac{1}{\sqrt{2}}(1, 0, 1)$, and $\mathbf{k}_3 = \frac{1}{\sqrt{2}}(0, 1, 1)$ yields a bcc hopfion crystal. These structures are shown in Fig. 3.

A central question is then how to stabilize these hopfion crystals in a realistic model Hamiltonian. This question is translated to the possibility of minimizing the Hamiltonian with finite k in Eq. (11). When k approaches zero, the hopfion crystal is melted to become a conventional ferromagnetic state. On the other hand, if k approaches infinity with an atomic cutoff of $1/a$, the continuum limit presumed in Eq. (11) is no longer respected. The state then becomes antiferromagnetic instead. We propose the following model as a candidate:

$$H = \int d^3r J[(\nabla \mathbf{n})^2 - \kappa(\Delta \mathbf{n})^2], \quad (13)$$

where Δ is the Laplacian operator. This model is the continuum limit of spin models with competing higher-order exchange interactions [16,25]. It is important to fine tune the amplitudes of these competing interactions to ensure that J and κ are both positive parameters. The reason for this is the following. Given a finite $k = 2\pi/\lambda$, one can perform a scaling transformation $\mathbf{r} = \lambda \tilde{\mathbf{r}}$. Under this transformation, the first term in Eq. (13) becomes $E_1 = \lambda \int d^3\tilde{r} J(\tilde{\nabla} \mathbf{n}(\tilde{\mathbf{r}}))^2$ and the second term becomes $E_2 = -\kappa(1/\lambda) \int d^3\tilde{r} J(\tilde{\Delta} \mathbf{n}(\tilde{\mathbf{r}}))^2$. k or equivalently λ is optimized to minimize the energy density $\frac{1}{\lambda^3} \int_{\text{cube}} d^3\tilde{r}$, where integral is carried out in a $1 \times 1 \times 1$ cube. Given both integrals involved are positive definite, the first term E_1 is proportional to $1/\lambda^2$ and the second term E_2 is proportional to $-\kappa/\lambda^4$. The energy density, which can be expressed as $E = E_1 + E_2$, is minimized with a finite value of λ only when κ is positive.

The energy density as a function of periodicity λ for sc, fcc, and bcc hopfion crystals is shown in Fig. 4(a), where the variables of vertical and horizontal axes have been transformed into dimensionless ones. Without loss of generality,

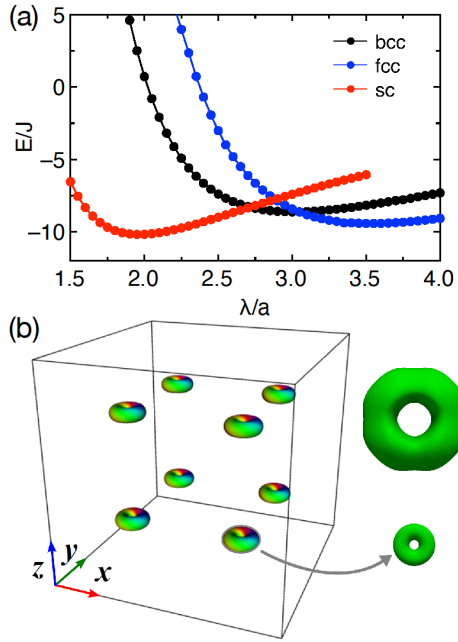


FIG. 4. (a) E/J vs λ/a of sc, fcc, and bcc hopfion crystals, where a is the lattice constant. (b) $|\mathbf{k}| = 1$ and $Q_H = 1$ sc hopfion crystal after relaxation. Left is the sc crystal structure. Right top (bottom) is the shape of one hopfion before (after) relaxation.

κ is chosen to be 0.01. The energy is minimized at finite periodicity. The minimal energy is negative for all crystals, indicating the preference over ferromagnetic state whose energy is zero. Interestingly, the sc lattice has the lowest energy density, so the hopfion crystal does not prefer the most closed packed state. This is related to the geometric anisotropy of the hopfions.

To further validate the numerical calculations, we have performed atomic spin simulations based on a frustrated magnet model using MicroMagnetic.jl, an open access multiscale magnetic simulation package with graphics processing unit acceleration [42]. We discretize the spin texture shown in Fig. 2 into a cubic lattice with size $129 \times 129 \times 129$. The lattice constant is set to 1. The Hamiltonian of the model is

$$H = - \sum_{i < j} J_{ij} \mathbf{n}_i \cdot \mathbf{n}_j, \quad (14)$$

in which $\mathbf{n}_i$ is the spin vector on site i . The exchange interaction between site i and j is J_{ij} . The interactions include the first four nearest-neighbor shells, with dimensionless couplings $J_1 = 61$, $J_2 = -10$, $J_3 = 0$, and $J_4 = -5$ [16,43]. This set of parameters approximates a discretized model of Eq. (13) with $J = 1$ and $\kappa = 0.01$. Using the steepest descent method under periodic boundary conditions, we relax the initial spin texture to find stable hopfion crystals. The initial $Q_H = 1$ hopfion state, defined by wave vectors $\mathbf{k}_1 = (-1, 0, 0)$, $\mathbf{k}_2 = (0, 1, 0)$, and $\mathbf{k}_3 = (0, 0, 1)$, exhibits a simple cubic lattice of C_4 symmetric hopfions. After relaxation, the lattice structure persists, but each hopfion evolves into a smaller and more round form. The result is shown in Fig. 4(b). As the scaling analysis, the final state should have a fixed periodicity. Changing the magnitudes of wave vectors k_i of the initial structures may lead the hopfions to merge with each other during the

relaxation process. The final states are the hopfion crystals with higher Hopf charge and the periodicities are different from the initial states. For example, for a hopfion crystals with $k_1 = (-2, 0, 0)$, $k_2 = (0, 1, 0)$, and $k_3 = (0, 0, 1)$ as the initial state, two $Q_H = 1$ hopfions merge into one $Q_H = 2$ hopfion after relaxation. Increasing the magnitude of wave vector $k = |\mathbf{k}|$ beyond a threshold k_0 (about 2.67 in our model) causes the hopfions in the crystal structure to melt (see Sec. V of the Supplemental Material [40]). Such a model Hamiltonian may be realized in itinerant ferromagnets where the magnetic interactions are primarily described by the Ruderman-Kittel-Kasuya-Yosida mechanism. So far only crystal of hopfions with $Q_H = 1$ is concerned. Using complex rational map, our method can be used to generate crystals of other hopfion textures with higher Hopf index, such as axially symmetric tori, torus links, and torus knots.

The axially symmetric torus is named as (m, n) -torus or $\mathcal{A}_{m,n}$, where m and n are the winding numbers of the spin fields in two orthogonal directions. To construct an $\mathcal{A}_{m,n}$ spin texture, one can choose the Riemann coordinate

$$\xi = \frac{Z_1^m}{Z_0^n}. \quad (15)$$

The resulting structure is a crystal of $Q_H = mn$ hopfions. The aforementioned $Q_H = 1$ hopfions correspond to $m = n = 1$. The hopfion crystal with combinations of $(m, n) = (3, 2)$ is shown in Fig. 5(a). For simplicity, only the simple cubic form is demonstrated; other forms of the crystal can be constructed in a similar way as discussed above.

For torus links $\mathcal{L}_{n,n}^{1,1}$ (the number of subscripts denotes the number of linking components, each subscript describes the charge of the linking component, and the superscript denotes the times of linking between different components), the Riemann coordinate is given by

$$\xi = \frac{Z_1^{n+1}}{Z_1^2 - Z_0^2}. \quad (16)$$

When $n = 2$, this results in a torus link with $Q_H = 2n + 2 = 6$, which has two toruslike hopfions, whose Hopf indices are both 2, linked with each. The $Q_H = 6$ simple cubic crystal is shown in Fig. 5(b). For even more complex link topologies with additional linking components (e.g., $\mathcal{L}_{n,n,n}^{2,2,2}$), similar rational map formulations can be developed by extending the polynomial expressions in the numerator and denominator.

Finally, the Riemann coordinate of torus knot $\mathcal{K}_{a,b}$ is expressed as

$$\xi = \frac{Z_1^\alpha Z_0^\beta}{Z_1^\alpha + Z_0^\beta}. \quad (17)$$

The trefoil knot is characterized by the parameters as $a = 3$, $b = 2$, $\alpha = 2$, and $\beta = 1$, yielding a Hopf index $Q_H = a\beta + \alpha b = 7$. The simple cubic trefoil crystal is illustrated in Fig. 5(c). Similarly, knot structures with higher Hopf indices can be constructed by modifying the values of a , b , α , and β .

In summary, we have demonstrated a systematic approach for constructing hopfion crystals through the superposition of $\mathbb{R}^4$ helical waves and rational maps. Our framework enables the creation of various cubic lattices (sc, fcc, and bcc) with tunable topology. By controlling winding numbers and geometric parameters, we can systematically generate

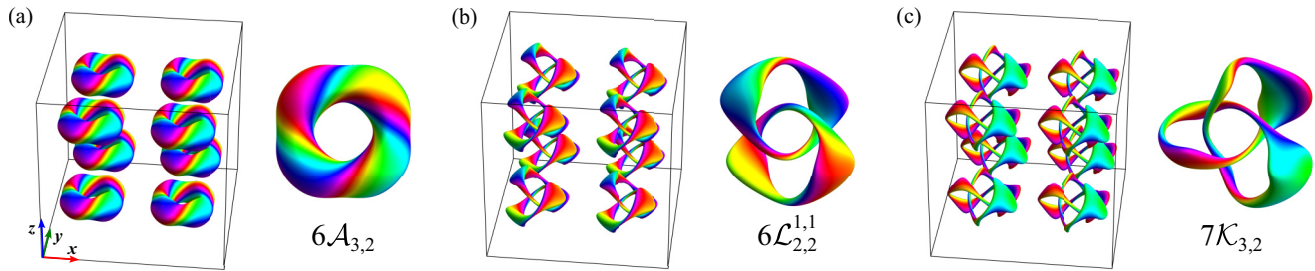


FIG. 5. Higher-charge hopfion crystals. The left is the sc structure. The right is one hopfion in the crystal structure. (a) $(m, n) = (3, 2)$ $Q_H = 6$ torus. (b) $Q_H = 6$ torus link. (c) $Q_H = 7$ trefoil.

hopfion crystals with different Hopf indices and complex topological structures, including axially symmetric tori, torus links, and torus knots. Our theoretical framework provides a foundation for studying hopfion crystals using variational methods and numerical simulations, which could lead to the identification of real materials that can host hopfion crystals as ground states. Our method can also be applied to construct heliknoton crystals by adding a spatial-dependent helix rotation to all spins [44]. Furthermore, our approach opens avenues for investigating collective behaviors of hopfion ensembles, such as phase transitions, internal modes, magnon bands, magnonic crystals, and universality classes,

which would deepen the understanding of 3D topological solitons and advance the development of hopfion-based devices [30,45].

W.-T.H. and Z.X. thank Prof. Yijie Shen for the delighted discussion on structures of hopfions. We are grateful for fruitful discussions with Prof. Naoto Nagaosa. This work was supported by the Office of Basic Energy Sciences, Division of Materials Sciences and Engineering, U.S. Department of Energy, under Award No. DE-SC0020221.

Data availability. The data supporting this study's findings are available within the article.

- [1] H. Hopf, Über die abbildungen der dreidimensionalen Sphäre auf die Kugelfläche, *Math. Ann.* **104**, 637 (1931).
- [2] V. Korepin and L. Faddeev, Quantization of solitons, *Theor. Math. Phys.* **25**, 1039 (1975).
- [3] L. Faddeev, Some comments on the many-dimensional solitons, *Lett. Math. Phys.* **1**, 289 (1976).
- [4] C. Wan, Y. Shen, A. Chong, and Q. Zhan, Scalar optical hopfions, *eLight* **2**, 22 (2022).
- [5] D. Ehrmanntraut, R. Droop, D. Sugic, E. Otte, M. Dennis, and C. Denz, Optical second-order skyrmionic hopfion, *Optica* **10**, 725 (2023).
- [6] Y. Shen, B. Yu, H. Wu, C. Li, Z. Zhu, and A. Zayats, Topological transformation and free-space transport of photonic hopfions, *Adv. Photon.* **5**, 015001 (2023).
- [7] R. Tamura, S. Allam, N. Litchinitser, and T. Omatsu, Three-dimensional projection of optical hopfion textures in a material, *ACS Photonics* **11**, 4958 (2024).
- [8] W. Lin, N. Mata-Cervera, Y. Ota, Y. Shen, and S. Iwamoto, Space-time optical hopfion crystals, *Phys. Rev. Lett.* **135**, 083801 (2025).
- [9] E. Babaev, L. D. Faddeev, and A. J. Niemi, Hidden symmetry and knot solitons in a charged two-condensate Bose system, *Phys. Rev. B* **65**, 100512 (2002).
- [10] R. Bisset, W. Wang, C. Ticknor, R. Carretero-González, D. Frantzeskakis, L. Collins, and P. Kevrekidis, Robust vortex lines, vortex rings, and hopfions in three-dimensional Bose-Einstein condensates, *Phys. Rev. A* **92**, 063611 (2015).
- [11] Y. Bidasyuk, A. Chumachenko, O. Prikhodko, S. Vilchinskii, M. Weyrauch, and A. Yakimenko, Stable Hopf solitons in rotating Bose-Einstein condensates, *Phys. Rev. A* **92**, 053603 (2015).
- [12] Y. Li, X. Zhou, and C. Wu, Three-dimensional quaternionic condensations, Hopf invariants, and skyrmion lattices with synthetic spin-orbit coupling, *Phys. Rev. A* **93**, 033628 (2016).
- [13] S. Zou, W.-K. Bai, T. Yang, and W.-M. Liu, Formation of vortex rings and hopfions in trapped Bose-Einstein condensates, *Phys. Fluids* **33**, 027105 (2021).
- [14] P. Ackerman, J. van de Lagemaat, and I. Smalyukh, Self-assembly and electrostriction of arrays and chains of hopfion particles in chiral liquid crystals, *Nat. Commun.* **6**, 6012 (2015).
- [15] J.-S. Wu and I. Smalyukh, Hopfions, heliknotons, skyrmions, torons and both abelian and nonabelian vortices in chiral liquid crystals, *Liq. Cryst. Rev.* **10**, 34 (2022).
- [16] P. Sutcliffe, Skyrmion knots in frustrated magnets, *Phys. Rev. Lett.* **118**, 247203 (2017).
- [17] Y. Liu, R. Lake, and J. Zang, Binding a hopfion in a chiral magnet nanodisk, *Phys. Rev. B* **98**, 174437 (2018).
- [18] X. Yu, Y. Liu, K. Iakoubovskii, K. Nakajima, N. Kanazawa, N. Nagaosa, and Y. Tokura, Realization and current-driven dynamics of fractional hopfions and their ensembles in a helimagnet FeGe, *Adv. Mater.* **35**, 2210646 (2023).
- [19] N. Kent, N. Reynolds, D. Raftrey, I. Campbell, S. Virasawmy, S. Dhuey, R. Chopdekar, A. Hierro-Rodriguez, A. Sorrentino, E. Pereiro, S. Ferrer, F. Hellman, P. Sutcliffe, and P. Fischer, Creation and observation of hopfions in magnetic multilayer systems, *Nat. Commun.* **12**, 1562 (2021).
- [20] F. Zheng, N. Kiselev, F. Rybakov, L. Yang, W. Shi, S. Blgel, and R. Dunin-Borkowski, Hopfion rings in a cubic chiral magnet, *Nature (London)* **623**, 718 (2023).
- [21] I. Luk'yanchuk, Y. Tikhonov, A. Razumnaya, and V. Vinokur, Hopfions emerge in ferroelectrics, *Nat. Commun.* **11**, 2433 (2020).

- [22] F. N. Rybakov, J. Garaud, and E. Babaev, Stable Hopf-Skyrme topological excitations in the superconducting state, *Phys. Rev. B* **100**, 094515 (2019).
- [23] G. Volovik and V. Mineev, Particle-like solitons in superfluid ^3He phases, *Zh. Eksp. Teor. Fiz.* **73**, 767 (1977).
- [24] X. Wang, A. Qaiumzadeh, and A. Brataas, Current-driven dynamics of magnetic hopfions, *Phys. Rev. Lett.* **123**, 147203 (2019).
- [25] Y. Liu, W. Hou, X. Han, and J. Zang, Three-dimensional dynamics of a magnetic hopfion driven by spin transfer torque, *Phys. Rev. Lett.* **124**, 127204 (2020).
- [26] D. Raftrey and P. Fischer, Field-driven dynamics of magnetic hopfions, *Phys. Rev. Lett.* **127**, 257201 (2021).
- [27] S. S. Pershoguba, D. Andreoli, and J. Zang, Electronic scattering off a magnetic hopfion, *Phys. Rev. B* **104**, 075102 (2021).
- [28] Y. Liu, H. Watanabe, and N. Nagaosa, Emergent magnetomultipoles and nonlinear responses of a magnetic hopfion, *Phys. Rev. Lett.* **129**, 267201 (2022).
- [29] W.-T. Hou and J. Zang, Microscopic theory of nonlinear Hall effect in three-dimensional magnetic systems, *Chin. Phys. Lett.* **41**, 117502 (2024).
- [30] G. Gubbiotti, A. Barman, S. Ladak, C. Bran, D. Grundler, M. Huth, H. Plank, G. Schmidt, S. van Dijken, R. Streubel *et al.*, 2025 roadmap on 3D nanomagnetism, *J. Phys.: Condens. Matter* **37**, 143502 (2025).
- [31] A. Fernández-Pacheco, R. Streubel, O. Fruchart, R. Hertel, P. Fischer, and R. P. Cowburn, Three-dimensional nanomagnetism, *Nat. Commun.* **8**, 15756 (2017).
- [32] C. Donnelly, P. Fischer, F. Kronast, A. Lubk, D. Wolf, V. Scagnoli, R. Schäfer, and I. Soldatov, Imaging of curved magnetic architectures, in *Curvilinear Micromagnetism: From Fundamentals to Applications*, edited by D. Makarov and D. D. Sheka (Springer International Publishing, Cham, 2022), pp. 269–304.
- [33] L. Faddeev and A. J. Niemi, Stable knot-like structures in classical field theory, *Nature (London)* **387**, 58 (1997).
- [34] C. Houghton, N. Manton, and P. Sutcliffe, Rational maps, monopoles and skyrmions, *Nucl. Phys. B* **510**, 507 (1998).
- [35] R. Balakrishnan, R. Dandoloﬀ, and A. Saxena, Exact hopfion vortices in a 3D Heisenberg ferromagnet, *Phys. Lett. A* **480**, 128975 (2023).
- [36] H. Kedia, Weaving knotted vector fields with tunable helicity, *Phys. Rev. Lett.* **117**, 274501 (2016).
- [37] S. Mülbauer, B. Binz, F. Jonietz, C. Pfleiderer, A. Rosch, A. Neubauer, R. Georgii, and P. Böni, Skyrmion lattice in a chiral magnet, *Science* **323**, 915 (2009).
- [38] X. Z. Yu, W. Koshibae, Y. Tokunaga, K. Shibata, Y. Taguchi, N. Nagaosa, and Y. Tokura, Transformation between meron and skyrmion topological spin textures in a chiral magnet, *Nature (London)* **564**, 95 (2018).
- [39] S. Hayami, T. Okubo, and Y. Motome, Phase shift in skyrmion crystals, *Nat. Commun.* **12**, 6927 (2021).
- [40] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/rr4k-x9m5> for details of the Hopf index calculation and decomposition of the hopfion crystals.
- [41] S. Zhang, G. van der Laan, W. Wang, A. Haghighirad, and T. Hesjedal, Direct observation of twisted surface skyrmions in bulk crystals, *Phys. Rev. Lett.* **120**, 227202 (2018).
- [42] W. Wang, B. Lyu, L. Kong, H. Fangohr, and H. Du, Micro-Magnetic.jl: A Julia package for micromagnetic and atomistic simulations with GPU support, *Chin. Phys. B* **33**, 107508 (2024).
- [43] S. Kasai, K. Shimizu, S. Okumura, Y. Kato, and Y. Motome, Three-dimensional topological superstructure of magnetic hopfions threaded by meron strings in easy-plane magnets, *Phys. Rev. B* (2025).
- [44] R. Voinescu, J.-S. B. Tai, and I. I. Smalyukh, Hopf solitons in helical and conical backgrounds of chiral magnetic solids, *Phys. Rev. Lett.* **125**, 057201 (2020).
- [45] I. Medlej, J. Wang, C. Hu, and K. Yu, Hopfion based magnonic crystal, *J. Magn. Magn. Mater.* **591**, 171726 (2024).