

Tunable two-dimensional energy orbital angular momentum synthetic lattices with free electrons

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Free-electron interactions with optical near fields are conventionally described as one-dimensional energy ladders, in which multicolor photon-exchange pathways can become spectrally degenerate. In this work, we show that coupling a single free electron to commensurate two-color optical vortices can unfold this degeneracy, realizing a tunable synthetic lattice spanned by the electron energy and orbital angular momentum (OAM). The resulting single-particle dynamics map the free-electron light interaction onto an effective tight-binding Hamiltonian whose primitive hopping vectors and coupling anisotropy are controlled by the optical OAM charges and near-field amplitudes. We use this construction to emulate a Lifshitz transition in the synthetic quasienergy band structure, identified by a topological reconnection of the zero-quasienergy contour and the associated Van Hove singularity in the density of states. We further show that the generated electron state exhibits tunable intraparticle energy-OAM mode nonseparability, which serves as a diagnostic of the synthetic-lattice geometry. These results establish free electrons as a tunable platform for synthetic-band engineering, combining electron-wavefunction control with ultrafast electron-microscopy and nanoscale near-field techniques.

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I. INTRODUCTION

Synthetic dimensions, in which internal or modal degrees of freedom are reinterpreted as lattice coordinates, have emerged as a powerful framework for Hamiltonian engineering and analog emulation of lattice phenomena [1–4]. By encoding higher-dimensional geometries into frequency modes, photon arrival times, or angular momenta, synthetic architectures provide access to band structures and lattice connectivities that are difficult to realize directly in real space. This concept has been extensively developed in photonics to investigate phenomena ranging from quantum walks and the quantum Hall effect to high-dimensional topological phases [5–11]. Extending synthetic-dimensional control to free electrons offers a complementary setting that combines coherent light-electron coupling, nanoscale near-field control, and ultrafast electron-microscopy readout.

Recently, quantum optics with free electrons has become a new frontier, significantly advanced by the benchmark experiment of photon-induced near-field electron microscopy (PINEM) [12–14]. In the standard framework of PINEM, a free electron traversing an optical field acquires quantized energy modulation, which creates a superposition of

equidistant energy sidebands [12,15–22]. This process naturally realizes a one-dimensional (1D) synthetic lattice in energy space, probed via electron energy-loss spectroscopy (EELS) [13,23,24]. However, the intrinsic dimensionality of free-electron quantum optics is severely limited by the “spectral collapse” problem: When driven by commensurate multicolor fields (e.g., ω and 2ω), the distinct photon-exchange pathways interfere and collapse onto a single 1D energy axis N , obscuring the underlying high-dimensional dynamics [25–28]. Unfolding these dimensions to construct controllable, higher-order synthetic lattices remains a central challenge [24,29–33].

In this work, we develop a theoretical framework for two-dimensional (2D) synthetic-lattice engineering with free electrons. By driving a single electron with commensurate two-color plasmonic vortices, we unfold the hidden photon-number space into an observable two-dimensional lattice labeled by the electron energy order N and OAM order ℓ . The optical OAM charges ℓ_1 and ℓ_2 determine the primitive hopping vectors, while the coupling strengths set the hopping anisotropy and effective walk “depth.” Within this framework, the electron-light interaction maps onto a synthetic tight-binding Hamiltonian whose connectivity, anisotropy, and quasienergy landscape are controlled by the optical near fields. We show that tuning the coupling ratio drives a Lifshitz transition of the synthetic band structure, manifested by the reconnection of the zero-quasienergy contour and a change of winding topology on the Brillouin-zone (BZ) torus. The present construction concerns single-electron Hamiltonian dynamics rather than an interacting many-body problem, and establishes a microscopy-compatible route for

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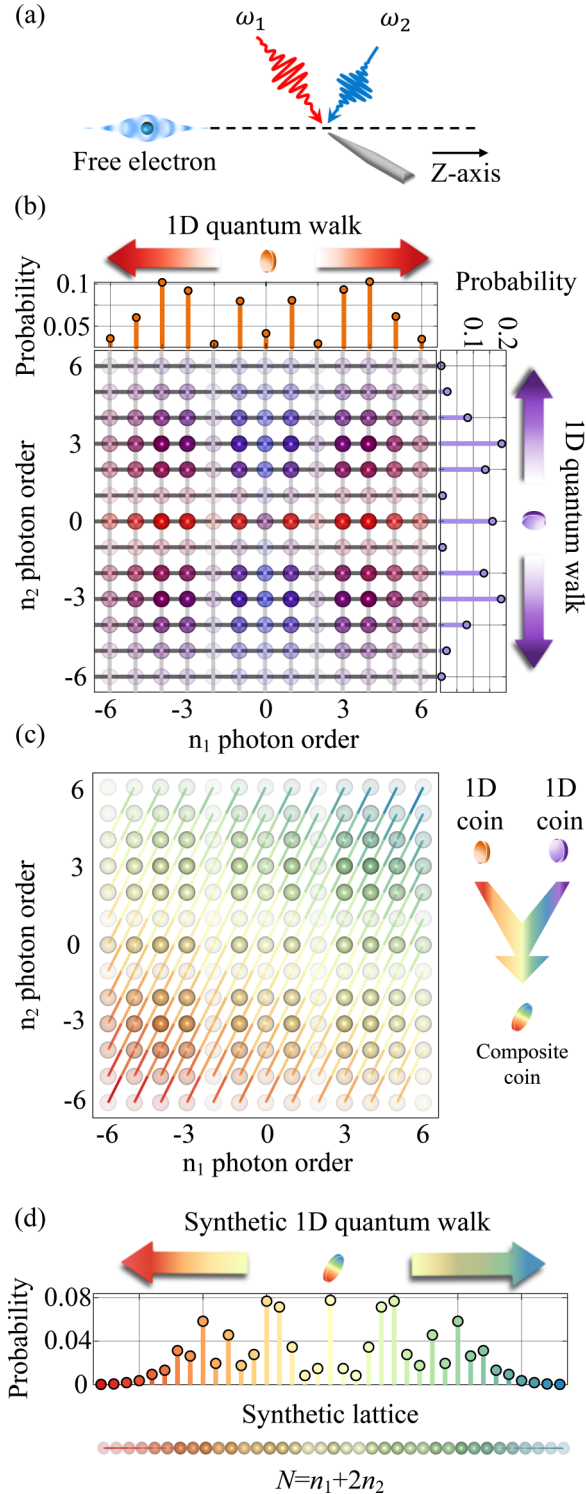


FIG. 1. Spectral collapse and synthetic unfolding in two-color free-electron quantum walks. (a) Schematic of a free electron interacting with a two-color near field (ω_1, ω_2). (b) For incommensurate frequencies, distinct photon-exchange pathways (n_1, n_2) remain spectrally distinguishable in the ideal energy spectrum. (c) For the commensurate case ($\omega_2 = 2\omega_1$), the observable energy order $N = n_1 + 2n_2$ groups different hidden photon-exchange pathways along constant- N diagonals (linked by the solid lines), producing spectral degeneracy in energy-only EELS. (d) The collapsed one-dimensional energy distribution P_N obtained after summing over the hidden constant- N pathways. Panels (b)–(d) are plotted for

engineering and probing synthetic-band phenomena in the Hilbert space of a propagating free electron.

II. TWO-DIMENSIONAL ENERGY-OAM SYNTHETIC-LATTICE CONSTRUCTION

We consider a recoil-free fast electron with a velocity v propagating along the z axis, interacting with optical near fields excited by a two-color optical field at frequencies ω_1 and ω_2 , as illustrated in Fig. 1(a).

In the recoil-free PINEM regime, the interaction is described by the standard longitudinal phase-modulation operator. For two commuting optical frequency components, the postinteraction state can be expanded on the photon-exchange lattice (n_1, n_2) as [18,25]

$$|\psi\rangle = \sum_{n_1, n_2} c_{n_1, n_2} |n_1, n_2\rangle, \quad (1)$$

$$c_{n_1, n_2} = \prod_{j=1,2} i^{n_j} J_{n_j}(2|\beta_{\omega_j}|) e^{in_j \phi_{\omega_j}},$$

where J_n is the Bessel function of the first kind and $\phi_{\omega_j} = \arg \beta_{\omega_j}$. The definition of the dimensionless coupling β_{ω_j} and the derivation of Eq. (1) from the PINEM unitary are given in the Appendixes.

Each photon-exchange event corresponds to a nearest-neighbor hopping in this synthetic (n_1, n_2) space. However, the standard EELS detects only the total energy transfer $\Delta E = n_1 \hbar \omega_1 + n_2 \hbar \omega_2$. For an incommensurate frequency ratio ω_2/ω_1 , different integer pairs (n_1, n_2) correspond to distinct energy shifts within any finite sideband window, so that the hidden two-dimensional distribution is in principle spectrally resolvable [Fig. 1(b)]. In contrast, for the commensurate $\omega=2\omega$ case, we define the integer energy order

$$N \equiv \frac{\Delta E}{\hbar \omega} = n_1 + 2n_2. \quad (2)$$

All hidden sites lying on the same constant- N line become degenerate in energy-only EELS [Fig. 1(c)]. If no additional degree of freedom is resolved, the measured amplitude in the N th energy sideband is the coherent sum $A_N = \sum_{n_1+2n_2=N} c_{n_1, n_2}$, and the intrinsic two-dimensional dynamics collapses onto an effective one-dimensional energy lattice [Fig. 1(d)]. To unfold this degeneracy, we use optical OAM transfer as a second observable synthetic coordinate.

In a vortex PINEM geometry, the exchange of a photon (ω_j, ℓ_j) changes both energy and OAM. For the $\omega=2\omega$ configuration, we define the physical synthetic coordinates as the net energy order N [Eq. (2)] and the net OAM order $\ell = n_1 \ell_1 + n_2 \ell_2$. The transformation from the hidden photon-number basis (n_1, n_2) to the observable synthetic basis (N, ℓ)

$|\beta_{\omega_1}| = 2.4$, $|\beta_{\omega_2}| = 2$. In the lattice panels, the lattice and sphere colors indicate the composite coordinate label, while sphere opacity encodes the normalized amplitude magnitude; sphere size is kept fixed for visual consistency.

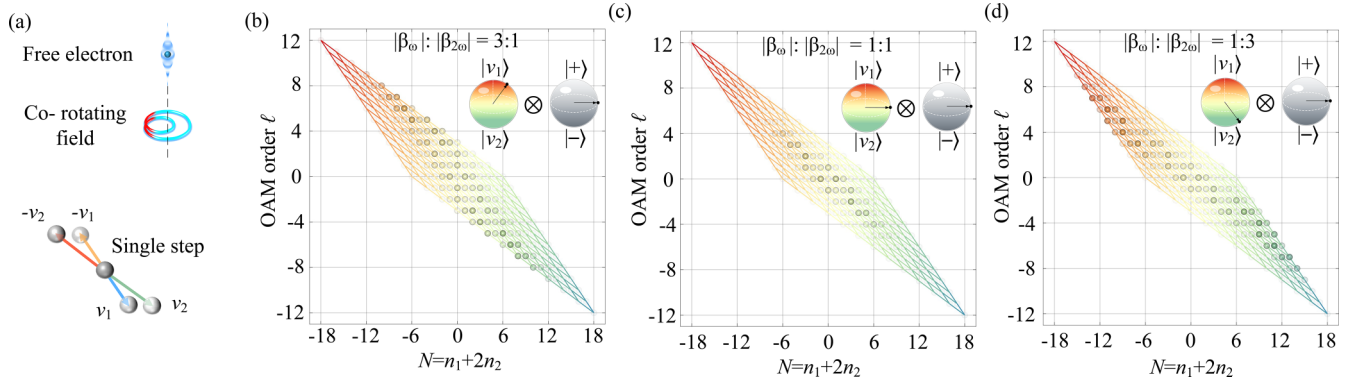


FIG. 2. Continuous-time quantum walks on an energy-OAM synthetic lattice. (a) Interaction with two-color corotating vortex fields ($\ell_1 = \ell_2 = -1$) generates a tilted rhombic Bravais lattice in the synthetic (N, ℓ) plane. The primitive vectors $\mathbf{v}_1 = (1, -1)$ and $\mathbf{v}_2 = (2, -1)$ correspond to single-photon (ω) and single-photon (2ω) exchange events, respectively. (b)–(d) Joint probability distributions $P(N, \ell)$ for coupling ratios $|\beta_{2\omega}|/|\beta_{\omega}| = 1/3, 1$, and 3 , respectively. The electron undergoes a continuous-time evolution driven by a local Hamiltonian, accumulating high-order photon exchanges (large displacement on the lattice) determined by the coupling strengths. The insets schematically indicate how the relative coupling strengths bias the walk along the two primitive hopping directions.

is governed by the linear map

$$\begin{pmatrix} N \\ \ell \end{pmatrix} = \mathbf{A} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ \ell_1 & \ell_2 \end{pmatrix}.$$

Provided that $\det \mathbf{A} = \ell_2 - 2\ell_1 \neq 0$, this mapping is invertible, establishing (N, ℓ) as a pair of independent synthetic coordinates. For configurations where $|\det \mathbf{A}| > 1$, the transformation enforces a parity selection rule, confining the walk to a specific sublattice within the joint (N, ℓ) space. The two-color unitary U , defined explicitly in the Appendixes, is translationally invariant in the (n_1, n_2) lattice and can be expanded in terms of displacement operators:

$$U|n_1, n_2\rangle = \sum_{\Delta n_1, \Delta n_2} c_{\Delta n_1, \Delta n_2} |n_1 + \Delta n_1, n_2 + \Delta n_2\rangle, \quad (3)$$

where $c_{\Delta n_1, \Delta n_2}$ are the Bessel-weighted population amplitudes defined in Eq. (1). By applying the inverse basis transformation $\mathbf{A}^{-1}$, we derive the hopping rules on the observable (N, ℓ) lattice. A displacement $(\Delta N, \Delta \ell)$ is allowed if and only if it corresponds to an integer photon exchange $(\Delta n_1, \Delta n_2)^T = \mathbf{A}^{-1}(\Delta N, \Delta \ell)^T$. The effective hopping amplitude in the synthetic space is given by

$$t_{\Delta N, \Delta \ell} \equiv \langle N + \Delta N, \ell + \Delta \ell | U | N, \ell \rangle = c_{\Delta n_1, \Delta n_2}. \quad (4)$$

This allows us to program the connectivity and topology of the 2D synthetic lattice through the choice of optical OAM charges ℓ_j and coupling parameters β_{ω_j} .

We explicitly construct this synthetic topology by considering a corotating ω – 2ω configuration with OAM charges $\ell_1 = \ell_2 = -1$ [Fig. 2(a)]. Applying the basis transformation $\mathbf{A}$, the elementary photon-exchange events $(\Delta n_1, \Delta n_2) = (1, 0)$ and $(0, 1)$ map onto the primitive lattice vectors $\mathbf{v}_1 = (1, -1)$ and $\mathbf{v}_2 = (2, -1)$ in the synthetic (N, ℓ) plane. While the final electron state involves multiple photon exchanges (spanning many lattice sites), the underlying interaction is controlled by a local generator. One can write the PINEM unitary operator (see Appendixes) exactly as the time-evolution operator of an effective tight-binding Hamiltonian, $U = \exp(-i\hat{H}_{\text{eff}})$, where the effective tight-binding Hamiltonian

can be written as

$$\hat{H}_{\text{eff}} = \sum_{\mathbf{R}} (\beta_{\omega} |\mathbf{R} + \mathbf{v}_1\rangle \langle \mathbf{R}| + \beta_{2\omega} |\mathbf{R} + \mathbf{v}_2\rangle \langle \mathbf{R}| + \text{H.c.}). \quad (5)$$

Here, $|\mathbf{R}\rangle = |N, \ell\rangle$ denotes the lattice sites. Equation (5) identifies the free-electron interaction with a continuous-time quantum walk generated by a nearest-neighbor Hamiltonian on the synthetic Bravais lattice. The primitive hops are fixed by the optical selection rules, $\mathbf{v}_1 = (1, \ell_1)$ and $\mathbf{v}_2 = (2, \ell_2)$, while the complex coupling amplitudes β_{ω} and $\beta_{2\omega}$ set the corresponding hopping strengths. Their ratio $\eta \equiv |\beta_{2\omega}|/|\beta_{\omega}|$ determines the anisotropy of the walk, steering the electron flux preferentially along $\mathbf{v}_1$ or $\mathbf{v}_2$, as shown in Figs. 2(b)–2(d). This directional control can be visualized as a programmable Hamiltonian vector on the Bloch sphere (insets in Fig. 2). Their absolute values determine the “depth” of the walk. In the $|\beta| \gg 1$ regime, the system evolves far beyond the perturbative limit. The observed high-order photon sidebands then arise from the finite-time propagator $U = \exp(-i\hat{H}_{\text{eff}})$: Large displacements correspond to paths composed of successive primitive hops. Thus, the coupling magnitude controls the effective walk depth, whereas the lattice connectivity is determined by the two elementary optical transitions.

We now analyze the spectral properties of this Hamiltonian in the synthetic reciprocal space. Since $\hat{H}_{\text{eff}}$ [Eq. (5)] commutes with the discrete translation operators $T_{\mathbf{v}_1}$ and $T_{\mathbf{v}_2}$, the eigenstates are Bloch waves $|\mathbf{K}\rangle = \sum_{\mathbf{R}} e^{i\mathbf{K} \cdot \mathbf{R}} |\mathbf{R}\rangle$, parametrized by the synthetic crystal momentum $\mathbf{K} = (K_N, K_{\ell})$ within the BZ $[-\pi, \pi]^2$. Diagonalizing $\hat{H}_{\text{eff}}$ in this basis yields the Floquet quasienergy band dispersion $\varepsilon(\mathbf{K})$:

$$\varepsilon(\mathbf{K}) = 2|\beta_{\omega}| \cos(\mathbf{K} \cdot \mathbf{v}_1 + \phi_{\omega}) + 2|\beta_{2\omega}| \cos(\mathbf{K} \cdot \mathbf{v}_2 + \phi_{2\omega}). \quad (6)$$

Here, $\phi_{\omega} = \arg \beta_{\omega}$ and $\phi_{2\omega} = \arg \beta_{2\omega}$ are the hopping phases inherited from the optical near fields. For the unfolded two-dimensional lattice considered here, where $\det \mathbf{A} = \ell_2 - 2\ell_1 \neq 0$, these phases are gauge degrees of freedom (the explicit gauge transformation is given in the Appendixes). The topology of the electron dynamics is thus encoded in

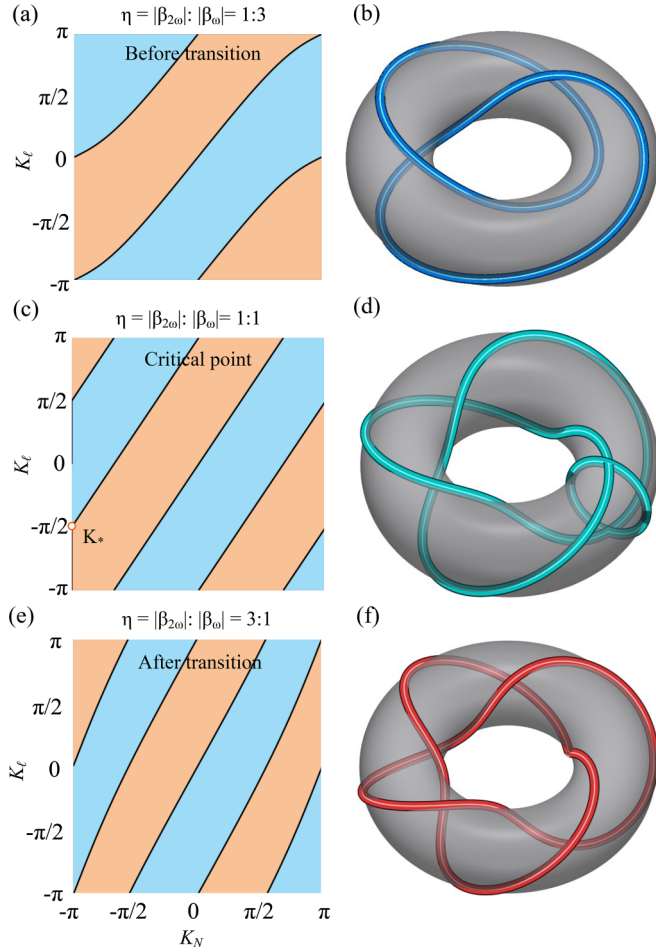


FIG. 3. Lifshitz transition on the BZ torus. Left column: Planar BZ $(K_N, K_\ell) \in [-\pi, \pi]^2$, where the color indicates the sign of the quasienergy $\varepsilon(K_N, K_\ell)$ and the black curve is the zero-quasienergy contour $\varepsilon = 0$. The critical panel marks the saddle point K_* by a circle, where $\varepsilon(K_*) = 0$ and $\nabla_{\mathbf{K}}\varepsilon(K_*) = 0$. Right column: The same $\varepsilon = 0$ contour drawn on the glued BZ (a torus), revealing its global connectivity under periodic boundary identification. (a), (b) Before the transition, $|\beta_{2\omega}| : |\beta_\omega| = 1:3$, the dominant contour forms a noncontractible loop with winding class $(w_N, w_\ell) = \pm(1, 1)$ (blue). (c), (d) At the critical point, $|\beta_{2\omega}| : |\beta_\omega| = 1:1$, the $\varepsilon = 0$ set passes through a saddle point and undergoes a topological reconnection (cyan). (e), (f) After the transition, $|\beta_{2\omega}| : |\beta_\omega| = 3:1$, the dominant noncontractible loop switches to a higher-order winding class $(w_N, w_\ell) = \pm(1, 2)$ (red), providing a direct topological signature of the Lifshitz transition.

the geometry of the “Floquet Fermi surface,” defined by the zero-quasienergy contour $\varepsilon(\mathbf{K}) = 0$.

III. FLOQUET-BAND TOPOLOGY AND LIFSHITZ TRANSITION

Figure 3 summarizes the quasienergy dispersion $\varepsilon(\mathbf{K})$ and its associated Lifshitz transition under tuning of the optical coupling ratio $\eta \equiv |\beta_{2\omega}|/|\beta_\omega|$. In the left column, we show the planar BZ $(K_N, K_\ell) \in [-\pi, \pi]^2$, where the sign of $\varepsilon(K_N, K_\ell)$ is color coded, and the black curve denotes the zero-

quasienergy contour $\varepsilon = 0$. The right column plots the same contour on the glued BZ, i.e., a torus obtained by the periodic identification of opposite BZ edges. Numerically, we extract the $\varepsilon = 0$ contours from a dense (K_N, K_ℓ) grid and “stitch” segments across the BZ boundaries by noting that successive points are allowed to shift by integer multiples of $(2\pi, 0)$ and $(0, 2\pi)$. Each resulting loop is assigned a topological label by its winding numbers $(w_N, w_\ell) \in \mathbb{Z}^2$, calculated from the net 2π advance of the stitched coordinates along the loop. Noncontractible loops correspond to open orbits in the torus representation, with $(w_N, w_\ell) \neq (0, 0)$.

For $\eta < \eta_c \equiv 1$, the $\varepsilon = 0$ loop is noncontractible and exhibits a characteristic winding class $(w_N, w_\ell) = \pm(1, 1)$, which indicates a quasi-1D drift direction dictated by the group velocity $\mathbf{v}_g = \nabla_{\mathbf{K}}\varepsilon$ [Figs. 3(a) and 3(b)]. At the critical point $\eta = \eta_c$, the zero-quasienergy set passes through a saddle point of the band structure, satisfying $\varepsilon(\mathbf{K}_*) = 0$ and $\nabla_{\mathbf{K}}\varepsilon(\mathbf{K}_*) = 0$. We locate $\mathbf{K}_*$ and η_c using fast Fourier transform-based spectral gradients on a periodic grid. This saddle-point crossing enables a topological reconnection of the $\varepsilon = 0$ contours on the torus, reorganizing the set of noncontractible winding classes [Figs. 3(c) and 3(d)]. For $\eta > \eta_c$, the dominant noncontractible loop abruptly switches to a higher-order winding class $\pm(1, 2)$ [Figs. 3(e) and 3(f)]. Physically, this corresponds to a global reorientation of the open-orbit topology: The ballistic transport channel is redirected such that the electron accumulates twice the OAM per synthetic energy site traversed. This winding-class reorganization on the torus constitutes a direct, high-dimensional topological signature of the Lifshitz transition in our Floquet synthetic band structure.

IV. SYNTHETIC-LATTICE GEOMETRIES AND ENERGY-OAM MODE NONSEPARABILITY

Beyond the specific case considered here, the connectivity of the synthetic tight-binding graph is set by the primitive hopping vectors determined by the chosen integer charges ℓ_j and the associated frequency harmonics, which reshapes the effective Bravais lattice in synthetic space and thereby the resulting band topology encoded in $\varepsilon(K_N, K_\ell)$. Figure 4 illustrates three distinct lattice classes accessible within this framework. When one of the driving fields carries zero net OAM (linearly polarized near field at 45° azimuth in our calculations), the transport along that frequency channel decouples from the OAM dimension. For instance, the configuration $(\ell_1, \ell_2) = (0, 1)$ [Fig. 4(a)] yields $\mathbf{v}_1 = (1, 0)$, creating horizontal hopping channels where energy changes but OAM is conserved. The $(\ell_1, \ell_2) = (1, 0)$ case [Fig. 4(c)] produces a sheared lattice geometry. When both fields carry nonzero and counterrotating OAM, such as $(\ell_1, \ell_2) = (-1, 1)$ [Fig. 4(e)], the primitive vectors span a widely open rhombic cell, maximizing the mixing between the two synthetic coordinates (n_1, n_2) .

These geometric variations imprint onto the synthetic quasienergy landscape. As shown in Figs. 4(b), 4(d), and 4(f) (calculated at $|\beta_\omega| = |\beta_{2\omega}| = 2$), changing the OAM charges modifies how the primitive photon-exchange directions are embedded in the observable (N, ℓ) coordinates, producing

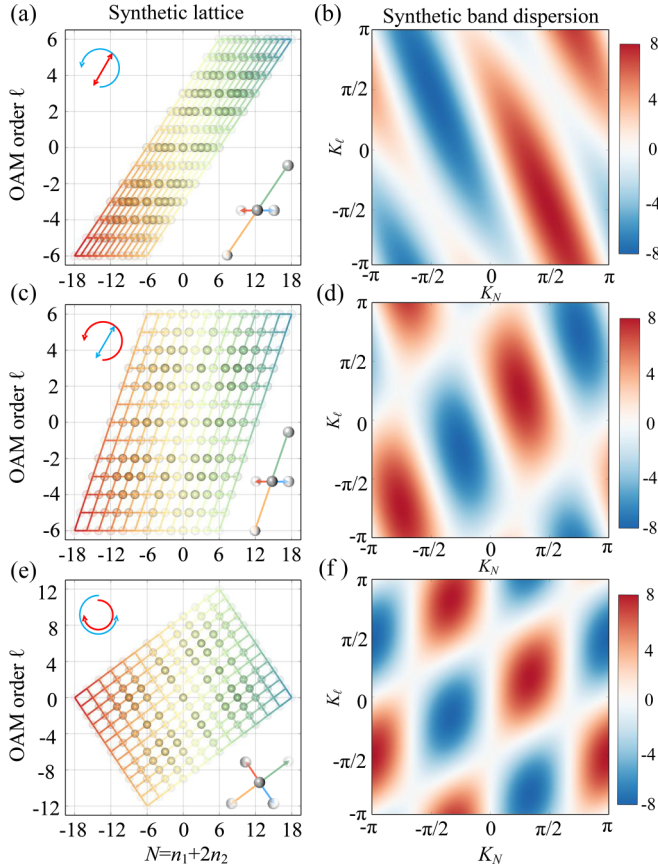


FIG. 4. Engineering synthetic-lattice geometries and Floquet band textures. (a), (c), (e) Real-space structure of synthetic (N, ℓ) lattices generated by two-color vortex fields with OAM charges $(\ell_1, \ell_2) = (0, 1)$, $(1, 0)$, and $(-1, 1)$, respectively. The primitive vectors $\mathbf{v}_1 = (1, \ell_1)$ and $\mathbf{v}_2 = (2, \ell_2)$ dictate the connectivity graph. (b), (d), (f) Corresponding synthetic-band dispersion $\varepsilon(K_N, K_\ell)$ calculated over the first BZ, and the color bars specify the quasienergy scale used in the calculations. We calculate the band textures with coupling strengths $|\beta_\omega| = |\beta_{2\omega}| = 2$ to visualize the full dispersion depth. The interplay between the OAM-dependent lattice vectors and the strong optical driving creates a diverse family of tunable band structures, ranging from quasi-1D stripes [panel (b)] to complex 2D saddle-point landscapes [panel (f)]. In the lattice panels, the background hue guides the synthetic coordinate, while the overlaid sphere opacity encodes the normalized amplitude magnitude; sphere size is kept fixed across panels.

band representations ranging from quasi-1D stripelike patterns to fully two-dimensional saddle-point landscapes. The OAM charges, therefore, provide geometric control over the observable synthetic lattice, while the optical coupling amplitudes independently tune the hopping anisotropy.

A further consequence of the lattice shear is the emergence of high-dimensional energy-OAM mode nonseparability in the outgoing single-electron state. Because the two frequency components commute, the postinteraction state is separable in the hidden photon-number partition, $|\psi\rangle = \sum_{n_1, n_2} c_{n_1, n_2} |n_1\rangle \otimes |n_2\rangle$. The state is therefore separable with respect to the hidden $n_1 \otimes n_2$ partition. However, the experimentally resolved synthetic coordinates (N, ℓ) correspond

to a different tensor-product factorization $\mathcal{H} \simeq \mathcal{H}_N \otimes \mathcal{H}_\ell$ whenever $\det \mathbf{A} \neq 0$. Expressed in the physical basis,

$$|\psi\rangle = \sum_{N, \ell} C_{N, \ell} |N\rangle \otimes |\ell\rangle, \quad C_{N, \ell} \equiv c_{n_1(N, \ell), n_2(N, \ell)}. \quad (7)$$

Because the linear transformation from (n_1, n_2) to (N, ℓ) shears the two axes, the coefficient matrix $C_{N, \ell}$ generally cannot be decomposed as $f(N) \times h(\ell)$. This nonseparability is therefore controlled by the synthetic-lattice geometry itself: The optical OAM charges determine how the photon-exchange axes are projected onto the physical energy-OAM basis.

To quantify this structure, we compute the von Neumann mode entropy $S_{N|\ell}$ associated with the energy-OAM factorization of the single-electron state, using the reduced density matrix $\rho_N = \text{Tr}_\ell |\psi\rangle\langle\psi|$,

$$S_{N|\ell} = -\text{Tr}(\rho_N \log_2 \rho_N) = -\sum_k \lambda_k \log_2 \lambda_k, \quad (8)$$

where λ_k are the Schmidt eigenvalues of ρ_N . For the representative choice $|\beta_\omega| = |\beta_{2\omega}| = 2$, geometries in which one frequency channel carries no OAM, such as $(\ell_1, \ell_2) = (0, 1)$ or $(1, 0)$, yield $S_{N|\ell} \simeq 2.99$ bits. When both colors carry OAM, the lattice shear enhances the energy-OAM mixing; for the corotating $(-1, -1)$ and counterrotating $(-1, 1)$ configurations, we obtain $S_{N|\ell} \simeq 3.64$ and 3.62 bits, respectively. The corresponding effective Schmidt rank $K_{\text{eff}} = 2^{S_{N|\ell}} \approx 12$ illustrates that the synthetic lattice supports a large single-electron energy-OAM mode space whose structure is optically programmable.

V. CONCLUSIONS

In conclusion, we have established a theoretical framework for two-dimensional synthetic-lattice engineering in free-electron quantum optics that unfolds the spectral degeneracy of commensurate multicolor interactions. By promoting OAM from a passive readout variable to an active synthetic coordinate, we construct tunable energy-OAM lattices whose primitive hopping directions are determined by the optical selection rules and whose anisotropy is controlled by the near-field coupling strengths. Within this single-electron setting, we emulate a Lifshitz transition and its associated Van Hove singularity in the synthetic quasienergy band structure, demonstrating access to band-structure phenomena beyond a one-dimensional energy-sideband quantum walk. We further identify energy-OAM mode nonseparability as a useful diagnostic of the engineered single-electron state.

As an experimental outlook, recent OAM-EELS measurements using a postspecimen OAM sorter followed by an energy spectrometer have demonstrated simultaneous recording of electron energy loss and OAM, providing a possible readout architecture for the joint $P(N, \ell)$ distribution considered here [34]. Together with free-electron vortex generation by chiral plasmonic near fields, tunable surface plasmon polariton-mediated electron shaping, and phase-controlled free-electron interferometry [35–37], these developments provide ingredients for a proof-of-principle test of the low-order synthetic-lattice selection rules. The present work is

restricted to single-electron dynamics; extending synthetic-dimensional control to sequential interactions, propagation between interaction regions, and interacting electron ensembles provides a natural route toward genuinely interacting synthetic-lattice dynamics.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: PINEM PHASE OPERATOR AND PHOTON-EXCHANGE COEFFICIENTS

Here, we summarize the convention used for the recoil-free PINEM unitary and derive the photon-exchange coefficients used in the main text. A fast electron moving along z with velocity v samples the longitudinal optical near field along its classical trajectory. For a monochromatic near-field component written as

$$E_z(\mathbf{R}_e, z, t) = E_{z,j}(\mathbf{R}_e, z)e^{-i\omega_j t} + E_{z,j}^*(\mathbf{R}_e, z)e^{+i\omega_j t}, \quad (\text{A1})$$

the recoil-free interaction gives the longitudinal phase modulation

$$U_j = \exp[i(\beta_{\omega_j} e^{-i\omega_j z/v} + \beta_{\omega_j}^* e^{+i\omega_j z/v})], \quad (\text{A2})$$

where the dimensionless PINEM coupling is

$$\beta_{\omega_j} = \frac{e}{\hbar\omega_j} \int_{-\infty}^{+\infty} dz E_{z,j}(\mathbf{R}_e, z) e^{-i\omega_j z/v}. \quad (\text{A3})$$

The overall sign of β_{ω_j} depends on the optical phasor convention and can be absorbed into the definition of $\phi_{\omega_j} = \arg \beta_{\omega_j}$. The measurable sideband probabilities are unaffected by this convention.

For two optical components at ω_1 and ω_2 , the longitudinal phase factors commute. The total unitary is therefore

$$U = U_1 U_2 = \exp \left[i \sum_{j=1,2} (\beta_{\omega_j} e^{-i\omega_j z/v} + \beta_{\omega_j}^* e^{+i\omega_j z/v}) \right]. \quad (\text{A4})$$

Using $\beta_{\omega_j} = |\beta_{\omega_j}| e^{i\phi_{\omega_j}}$, Eq. (A2) can be written as

$$U_j = \exp[i2|\beta_{\omega_j}| \cos(\phi_{\omega_j} - \omega_j z/v)]. \quad (\text{A5})$$

The Jacobi-Anger identity,

$$e^{ix \cos \theta} = \sum_{n=-\infty}^{+\infty} i^n J_n(x) e^{in\theta}, \quad (\text{A6})$$

then gives

$$U_j = \sum_{n_j=-\infty}^{+\infty} i^{n_j} J_{n_j}(2|\beta_{\omega_j}|) e^{in_j \phi_{\omega_j}} e^{-in_j \omega_j z/v}. \quad (\text{A7})$$

The factor $e^{-in_j \omega_j z/v}$ represents an electron energy change of $n_j \hbar \omega_j$. Hence, for two colors, the outgoing state can be expanded as

$$|\psi\rangle = \sum_{n_1, n_2} c_{n_1, n_2} |n_1, n_2\rangle, \quad (\text{A8})$$

with

$$c_{n_1, n_2} = \prod_{j=1,2} i^{n_j} J_{n_j}(2|\beta_{\omega_j}|) e^{in_j \phi_{\omega_j}}. \quad (\text{A9})$$

This is the coefficient used in Eq. (1). The probability in the hidden photon-exchange basis is therefore

$$P(n_1, n_2) = |c_{n_1, n_2}|^2 = \prod_{j=1,2} J_{n_j}^2(2|\beta_{\omega_j}|), \quad (\text{A10})$$

which depends only on the coupling magnitudes in the fully resolved (n_1, n_2) basis.

APPENDIX B: OPTICAL PHASES AS GAUGE DEGREES OF FREEDOM

We show here why the optical phases ϕ_{ω} and $\phi_{2\omega}$ do not change the unfolded synthetic-band topology when the map from photon-exchange indices to observable coordinates is invertible. The effective Hamiltonian in the observable synthetic lattice can be written as

$$H_{\text{eff}} = \sum_{\mathbf{R}} (\beta_{\omega} |\mathbf{R} + \mathbf{v}_1\rangle \langle \mathbf{R}| + \beta_{2\omega} |\mathbf{R} + \mathbf{v}_2\rangle \langle \mathbf{R}| + \text{H.c.}), \quad (\text{B1})$$

where $\mathbf{R} = (N, \ell)$, $\mathbf{v}_1 = (1, \ell_1)$, and $\mathbf{v}_2 = (2, \ell_2)$. Let $\beta_{\omega_j} = |\beta_{\omega_j}| e^{i\phi_{\omega_j}}$. Under the site-dependent basis transformation

$$|\mathbf{R}\rangle \rightarrow e^{i\mathbf{q} \cdot \mathbf{R}} |\mathbf{R}\rangle, \quad \mathbf{q} = (q_N, q_\ell), \quad (\text{B2})$$

the hopping coefficient along $\mathbf{v}_j$ transforms as

$$\beta_{\omega_j} \rightarrow \beta_{\omega_j} e^{-i\mathbf{q} \cdot \mathbf{v}_j}. \quad (\text{B3})$$

Both phases are removed if

$$\mathbf{q} \cdot \mathbf{v}_1 = \phi_1, \quad \mathbf{q} \cdot \mathbf{v}_2 = \phi_2, \quad (\text{B4})$$

or explicitly

$$\begin{pmatrix} 1 & \ell_1 \\ 2 & \ell_2 \end{pmatrix} \begin{pmatrix} q_N \\ q_\ell \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \quad (\text{B5})$$

This system has a solution whenever

$$\det \mathbf{A} = \ell_2 - 2\ell_1 \neq 0. \quad (\text{B6})$$

Therefore, in the genuinely two-dimensional unfolded case, the Hamiltonian with complex hoppings is unitarily equivalent to the Hamiltonian with real hoppings $|\beta_1|$ and $|\beta_2|$.

The same result is visible in reciprocal space:

$$\varepsilon_\phi(\mathbf{K}) = \varepsilon_0(\mathbf{K} + \mathbf{q}). \quad (\text{B7})$$

The phases shift the origin of $\mathbf{K}$ but do not change the topology of the zero-quasienergy contour. Consequently, the unfolded probability distribution $P(N, \ell)$, the density of states, the Lifshitz transition, and the winding class of the zero-quasienergy contour are phase invariant. Since the gauge transformation in Eq. (B2) factorizes as $e^{iq_N N} e^{iq_\ell \ell}$, the Schmidt

spectrum in the $N \otimes \ell$ partition is also invariant under the hopping phases.

APPENDIX C: LIFSHITZ TRANSITION AND VAN HOVE SINGULARITY

For the corotating case $(\ell_1, \ell_2) = (-1, -1)$, the primitive vectors are

$$\mathbf{v}_1 = (1, -1), \quad \mathbf{v}_2 = (2, -1). \quad (\text{C1})$$

Introducing

$$q_1 = \mathbf{K} \cdot \mathbf{v}_1 + \phi_1, \quad q_2 = \mathbf{K} \cdot \mathbf{v}_2 + \phi_2, \quad (\text{C2})$$

the dispersion is equivalent to

$$\varepsilon(q_1, q_2) = 2|\beta_1| \cos q_1 + 2|\beta_2| \cos q_2. \quad (\text{C3})$$

The saddle points occur at $q_1, q_2 = 0, \pi$. At zero quasienergy, a saddle-point crossing requires

$$2|\beta_1| \cos q_1 + 2|\beta_2| \cos q_2 = 0, \quad (\text{C4})$$

which is satisfied at $|\beta_1| = |\beta_2|$ for opposite signs of $\cos q_1$ and $\cos q_2$. Thus, the critical coupling ratio is

$$\eta_c = \frac{|\beta_2|}{|\beta_1|} = 1. \quad (\text{C5})$$

At this point, the zero-quasienergy contour reconnects at the saddle point, producing the Lifshitz transition shown in Fig. 3.

The corresponding density of states is

$$\rho(\varepsilon) = \frac{1}{(2\pi)^2} \int_{\text{BZ}} d^2K \delta[\varepsilon - \varepsilon(\mathbf{K})]. \quad (\text{C6})$$

A saddle point of $\varepsilon(\mathbf{K})$ produces a Van Hove singularity in $\rho(\varepsilon)$. For the transition studied in the main text, this singularity crosses $\varepsilon = 0$ at $\eta = \eta_c$, coinciding with the topological reconnection of the zero-quasienergy contour.

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- [1] I. M. Georgescu, S. Ashhab, and F. Nori, Quantum simulation, *Rev. Mod. Phys.* **86**, 153 (2014).
 - [2] L. Yuan, Q. Lin, M. Xiao, and S. Fan, Synthetic dimension in photonics, *Optica* **5**, 1396 (2018).
 - [3] M. Yang, J.-S. Xu, C.-F. Li, and G.-C. Guo, Simulating topological materials with photonic synthetic dimensions in cavities, *Quantum Front.* **1**, 10 (2022).
 - [4] D. Yu, W. Song, L. Wang, R. Srikanth, S. Kaushik Sridhar, T. Chen, C. Huang, G. Li, X. Qiao, X. Wu, Z. Dong, Y. He, M. Xiao, X. Chen, A. Dutt, B. Gadway, and L. Yuan, Comprehensive review on developments of synthetic dimensions, *Photonics Insights* **4**, R06 (2025).
 - [5] A. Schreiber, A. Gábris, P. P. Rohde, K. Laiho, M. Štefaňák, V. Potoček, C. Hamilton, I. Jex, and C. Silberhorn, A 2D quantum walk simulation of two-particle dynamics, *Science* **336**, 55 (2012).
 - [6] Q. Lin, M. Xiao, L. Yuan, and S. Fan, Photonic Weyl point in a two-dimensional resonator lattice with a synthetic frequency dimension, *Nat. Commun.* **7**, 13731 (2016).
 - [7] E. Lustig, S. Weimann, Y. Plotnik, Y. Lumer, M. A. Bandres, A. Szameit, and M. Segev, Photonic topological insulator in synthetic dimensions, *Nature (London)* **567**, 356 (2019).
 - [8] A. Dutt, Q. Lin, L. Yuan, M. Minkov, M. Xiao, and S. Fan, A single photonic cavity with two independent physical synthetic dimensions, *Science* **367**, 59 (2020).
 - [9] C. Esposito, M. R. Barros, A. D. Hernández, G. Carvacho, F. Di Colandrea, R. Barboza, F. Cardano, N. Spagnolo, L. Marrucci, and F. Sciarrino, Quantum walks of two correlated photons in a 2D synthetic lattice, *npj Quantum Inf.* **8**, 34 (2022).
 - [10] Q. Lin, W. Yi, and P. Xue, Manipulating directional flow in a two-dimensional photonic quantum walk under a synthetic magnetic field, *Nat. Commun.* **14**, 6283 (2023).
 - [11] G. Li, L. Wang, R. Ye, Y. Zheng, D.-W. Wang, X.-J. Liu, A. Dutt, L. Yuan, and X. Chen, Direct extraction of topological Zak phase with the synthetic dimension, *Light Sci. Appl.* **12**, 81 (2023).
 - [12] B. Barwick, D. J. Flannigan, and A. H. Zewail, Photon-induced near-field electron microscopy, *Nature (London)* **462**, 902 (2009).
 - [13] A. Feist, K. E. Echternkamp, J. Schauss, S. V. Yalunin, S. Schäfer, and C. Ropers, Quantum coherent optical phase modulation in an ultrafast transmission electron microscope, *Nature (London)* **521**, 200 (2015).
 - [14] R. Dahan, A. Gorlach, U. Haeusler, A. Karnieli, O. Eyal, P. Yousefi, M. Segev, A. Arie, G. Eisenstein, P. Hommelhoff, and I. Kaminer, Imprinting the quantum statistics of photons on free electrons, *Science* **373**, eabj7128 (2021).
 - [15] F. J. G. De Abajo, A. Asenjo-García, and M. Kociak, Multiphoton absorption and emission by interaction of swift electrons with evanescent light fields, *Nano Lett.* **10**, 1859 (2010).
 - [16] F. J. G. De Abajo, Optical excitations in electron microscopy, *Rev. Mod. Phys.* **82**, 209 (2010).
 - [17] L. Piazza, T. Lummen, E. Quiñonez, Y. Murooka, B. Reed, B. Barwick, and F. Carbone, Simultaneous observation of the quantization and the interference pattern of a plasmonic near-field, *Nat. Commun.* **6**, 6407 (2015).
 - [18] K. E. Echternkamp, A. Feist, S. Schäfer, and C. Ropers, Ramsey-type phase control of free-electron beams, *Nat. Phys.* **12**, 1000 (2016).
 - [19] O. Kfir, H. Lourenço-Martins, G. Storeck, M. Sivilis, T. R. Harvey, T. J. Kippenberg, A. Feist, and C. Ropers, Controlling free electrons with optical whispering-gallery modes, *Nature (London)* **582**, 46 (2020).
 - [20] J.-W. Henke, A. S. Raja, A. Feist, G. Huang, G. Arend, Y. Yang, F. J. Kappert, R. N. Wang, M. Möller, J. Pan, J. Liu, O. Kfir, C. Ropers, and T. J. Kippenberg, Integrated photonics enables continuous-beam electron phase modulation, *Nature (London)* **600**, 653 (2021).
 - [21] W. Wen, J. Li, and Y. Liu, Nanoscale femtosecond coherent radiation and spatiotemporally shaped free electron wave function, *Phys. Rev. Lett.* **136**, 223801 (2026).

- [22] D. Wu, J. Li, Y. Jiang, and Y. Liu, Coherent control of three-level system using shaped free electrons, *Phys. Rev. Lett.* **137**, 063601 (2026).
- [23] F. Carbone, An electron walks into a quantum bar..., *Science* **373**, 1309 (2021).
- [24] Y. Pan, B. Zhang, and D. Podolsky, Synthetic dimensions using ultrafast free electrons, *ACS Photonics* **11**, 3563 (2024).
- [25] K. E. Priebe, C. Rathje, S. V. Yalunin, T. Hohage, A. Feist, S. Schäfer, and C. Ropers, Attosecond electron pulse trains and quantum state reconstruction in ultrafast transmission electron microscopy, *Nat. Photonics* **11**, 793 (2017).
- [26] J. Li, Y. Fang, and Y. Liu, Topologically protected strong-interaction of photonics with free electrons, *Phys. Rev. Lett.* **132**, 073801 (2024).
- [27] Y. Fang, Z. Lyu, and Y. Liu, Ultrafast physics with structured light, *Nat. Rev. Phys.* **7**, 713 (2025).
- [28] C. I. Velasco and F. J. G. De Abajo, Free-space optical modulation of free electrons in the continuous-wave regime, *Phys. Rev. Lett.* **134**, 123804 (2025).
- [29] G. Braiman, O. Reinhardt, C. Mechel, O. Levi, and I. Kaminer, The synthetic Hilbert space of laser-driven free-electrons, *Quantum* **7**, 888 (2023).
- [30] C. W. Johnson, A. E. Turner, F. J. García De Abajo, and B. J. McMorran, Inelastic Mach-Zehnder interferometry with free electrons, *Phys. Rev. Lett.* **128**, 147401 (2022).
- [31] Y. Pan, E. Cohen, E. Karimi, A. Gover, N. Schönenberger, T. Chlouba, K. Wang, S. Nehemia, P. Hommelhoff, I. Kaminer, and Y. Aharonov, Weak measurements and quantum-to-classical transitions in free electron–photon interactions, *Light Sci. Appl.* **12**, 267 (2023).
- [32] A. Karnieli, S. Tsesses, R. Yu, N. Rivera, A. Arie, I. Kaminer, and S. Fan, Universal and ultrafast quantum computation based on free-electron-polariton blockade, *PRX Quantum* **5**, 010339 (2024).
- [33] R. Ruimy, A. Karnieli, and I. Kaminer, Free-electron quantum optics, *Nat. Phys.* **21**, 193 (2025).
- [34] A. H. Tavabi, P. Rosi, G. Bertoni, E. Rotunno, L. Belsito, A. Roncaglia, S. Frabboni, G. C. Gazzadi, E. Karimi, P. Tiemeijer, R. E. Dunin-Borkowski, and V. Grillo, Demonstration of angular-momentum-resolved electron energy-loss spectroscopy, *Nat. Commun.* **16**, 6601 (2025).
- [35] G. M. Vanacore, G. Berruto, I. Madan, E. Pomarico, P. Biagioni, R. J. Lamb, D. McGrouther, O. Reinhardt, I. Kaminer, B. Barwick, H. Larocque, V. Grillo, E. Karimi, F. J. G. De Abajo, and F. Carbone, Ultrafast generation and control of an electron vortex beam via chiral plasmonic near fields, *Nat. Mater.* **18**, 573 (2019).
- [36] S. Tsesses, R. Dahan, K. Wang, T. Bucher, K. Cohen, O. Reinhardt, G. Bartal, and I. Kaminer, Tunable photon-induced spatial modulation of free electrons, *Nat. Mater.* **22**, 345 (2023).
- [37] T. Bucher *et al.*, Coherently amplified ultrafast imaging using a free-electron interferometer, *Nat. Photonics* **18**, 809 (2024).