

Frequency-dependent topology in multisublattice systems: Analytical theory of dispersive effective Hamiltonians

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(Received 8 January 2026; accepted 30 March 2026; published 20 April 2026)

Topological phases in one-dimensional lattices are usually described by fixed, frequency-independent Hamiltonians, where topology is determined solely by structural parameters. However, in many wave-based systems with internal resonances, effective couplings become intrinsically dispersive and depend on the eigenfrequency itself. Here, we develop an analytical theory of such *frequency-dependent topology* using a trimer (SSH3) mass-spring lattice as a minimal model. By exactly eliminating the internal sublattice, the three-band SSH3 dynamics are mapped onto an effective two-sublattice Hamiltonian with frequency-dependent inertial and coupling terms. Because the effective parameters depend explicitly on the eigenfrequency, the resulting eigenvalue problem is nonlinear in ω and self-consistently reproduces the full three-band spectrum despite the two-level representation. This mapping reveals that the resonant effective coupling $G_2(\omega)$ acts as a bifurcation parameter controlling band inversion, redistribution of integer winding numbers, and the emergence of edge-localized modes. We show that a sign reversal of $G_2(\omega)$ induces a frequency-selective topological transition accompanied by a winding redistribution $(\nu_1, \nu_2, \nu_3) = (-1, 2, -1)$ and analytically predictable in-gap edge states governed by closed-form localization criteria. Within this framework, the bulk-edge correspondence is not determined by a fixed Hamiltonian but by the effective Hamiltonian evaluated self-consistently along the bulk dispersion relation. The formulation is further extended to adiabatically modulated lattices, where locally nontrivial Berry curvature and alternating topological charges emerge on the (q, t) torus, while the global Chern number remains neutral. Because dispersive renormalization arising from internal degrees of freedom is ubiquitous in phononic, photonic, and other wave-based lattice systems, the present theory establishes a general analytical framework for topological phases in dispersive multisublattice media beyond conventional static tight-binding descriptions.

DOI: [10.1103/rmcp-lwsl](https://doi.org/10.1103/rmcp-lwsl)

I. INTRODUCTION

Topological phases in one-dimensional systems have traditionally been understood within tight-binding models featuring fixed, energy-independent couplings, with the Su-Schrieffer-Heeger (SSH) model providing a paradigmatic example [1–8]. Within this framework, the quantization of the Zak phase under chiral symmetry and the associated bulk-edge correspondence offer a minimal yet powerful description of boundary-localized modes in gapped bands [5,8–11].

Recent developments in multisublattice lattices, Floquet systems, and multiorbital band theory have revealed that internal degrees of freedom can enrich this picture, giving rise to higher winding numbers, symmetry-protected degeneracies, higher-order band topology, and topological charges in synthetic dimensions [12–15]. These concepts have been realized in a variety of electronic, photonic, mechanical, and topoelectrical platforms [16–21]. Furthermore,

geometric-phase and surface-impedance approaches have provided alternative bulk characterizations of one-dimensional topological systems [22]. In particular, trimer (SSH3) and related multisublattice extensions of the SSH model have been extensively studied in recent years, where multigap topology, higher winding numbers, and multiple edge states associated with three-site unit cells were demonstrated in electronic, photonic, and mechanical systems [20,23–27]. Yet, a key feature common to many wave and quantum platforms—namely, that effective interactions can become dispersive and depend on the eigenfrequency—has not been fully incorporated into standard topological band theory.

Such frequency- or energy-dependent renormalization arises naturally from local resonances, self-energy corrections, and time-periodic modulation, and thus appears widely in photonic, phononic, magnonic, and electronic systems [28–32]. When the effective couplings depend on the eigenfrequency itself, the use of a fixed Hamiltonian becomes subtle: The topology of a mode can, in principle, be selected by its own oscillation frequency. In this work, we refer to this situation as *frequency-dependent topology*, in which the topological phase of an effective Hamiltonian becomes a function of the eigenfrequency through dispersive interactions, in contrast to conventional SSH-type lattices where the topology is fixed solely by structural parameters. This provides a mechanism for frequency-selective topological transitions,

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namely, phase changes driven not by external tuning but by the frequency-dependent renormalization inherent to the system itself.

Phononic crystals and metamaterials provide a natural platform for investigating this physics. At long wavelengths, their response can be described by effective mass and stiffness parameters that are inherently frequency dependent due to local resonances [28–31], while at shorter wavelengths, where the discrete periodicity becomes essential, Bloch band theory governs the emergence of band gaps, band inversions, and symmetry-induced features [18,19,32]. Because both descriptions originate from the same underlying band structure, establishing a theoretical framework that connects continuumlike effective parameters with discrete topological classifications remains an open challenge.

A minimal multisublattice model capturing this correspondence is the trimer extension of the SSH model (SSH3), which hosts multiple band gaps and integer-valued winding numbers [20,23–25]. Although the topological properties of SSH3 lattices can be obtained directly from the full three-band Bloch Hamiltonian, existing studies have primarily considered fixed, frequency-independent parameters. In contrast, in many mechanical and wave-based systems, internal resonances and effective-medium descriptions naturally introduce dispersive effective interactions, whose impact on band topology and bulk-edge correspondence has not been systematically clarified.

In this work, we develop an analytical theory that maps a one-dimensional trimer mass-spring lattice onto an effective two-sublattice Hamiltonian formally equivalent to the SSH model, but endowed with frequency-dependent intersublattice couplings and inertial terms. Because the effective Hamiltonian obtained after eliminating the internal degree of freedom depends explicitly on the eigenfrequency, the resulting eigenvalue problem is nonlinear in ω . Consequently, the dispersion relation is determined self-consistently from a higher-order characteristic equation in ω , whose solutions reproduce the original three-band structure of the SSH3 lattice, thereby resolving the apparent discrepancy between a 2×2 reduced Hamiltonian and three dispersion branches. In this sense, the reduction to a two-sublattice description preserves the full band information while exposing the dispersive structure of the effective coupling. We show that the sign reversal of the effective coupling serves as a bifurcation point for the winding number, leading to frequency-selective topological transitions accompanied by band inversion and eigenmode-symmetry switching.

Importantly, the contribution of the present work is not to replace the full multiband description, but to provide an analytically tractable and physically transparent framework in which band inversion, winding redistribution, and edge-mode conditions can be directly expressed in terms of frequency-dependent effective parameters. This establishes an explicit analytical connection between dispersive renormalization induced by internal resonances and topological band theory, which is not evident in conventional fixed-parameter tight-binding formulations.

Although we focus on SSH3 as the *minimal* nontrivial multisublattice model, we demonstrate in Sec. S7 of the Supplemental Material [33] that the same elimination procedure

applies to generic N -sublattice chains, where SSH3 plays the role of a prototype building block. The general N -sublattice formulation shows that eliminating internal resonant sites systematically generates higher-order frequency-dependent effective couplings, indicating that the present mechanism is not restricted to a specific trimer model. Because such dispersive renormalization is ubiquitous in wave and quantum systems, our results identify a general mechanism underlying topological phases in multisublattice lattice systems.

We further extend the framework to adiabatically time-modulated systems, enabling analytical evaluation of the Berry curvature and local topological charges on the (q, t) torus spanned by wave number and time [26,27,34,35]. This extension clarifies the range of validity of the adiabatic approximation and highlights how dispersive effective parameters influence both static and dynamically driven topological phases within a unified analytical framework. Our formulation connects effective-medium mechanics with dispersive Hamiltonians within topological band theory and provides a general analytical framework for understanding topological phases in multisublattice systems with frequency-dependent interactions, applicable to both static and adiabatically driven settings. Full derivations and extended analyses are provided in the Supplemental Material [33].

II. EXACT MAPPING TO A TWO-SUBLATTICE EFFECTIVE MODEL

We begin with the full equations of motion of the trimer chain [see Eq. (S1) and Sec. S1.1 of the Supplemental Material [33]] describing the coupled displacements $u_n^{(A,B,C)}$. Assuming Bloch-type wave solutions $u_n^{(j)} = U^{(j)} e^{i(qn - \omega t)}$ ($j = A, B, C$), we obtain algebraic relations among $U^{(A,B,C)}$, from which the intermediate site B can be eliminated analytically via Eq. (1):

$$U^{(B)} = \frac{K'U^{(A)} + K''U^{(C)}}{K' + K'' - M_B\omega^2} \quad (1)$$

[see Eq. (S3) and Sec. S1.2 of the Supplemental Material [33]]. The mapping procedure is schematically illustrated in Fig. 1, yielding the closed 2×2 eigenvalue problem [Eq. (2)] for the outer sublattices A and C .

$$\begin{bmatrix} m_1\omega^2 - G_1 - G_2 & G_1 + G_2e^{-iq} \\ G_1 + G_2e^{iq} & m_2\omega^2 - G_1 - G_2 \end{bmatrix} \begin{bmatrix} U^{(A)} \\ U^{(C)} \end{bmatrix} = \mathbf{0}, \quad (2)$$

$$G_1 = \frac{K''}{2}, \quad G_2 = \frac{K}{2} \left(1 + \frac{K''}{K'} - \frac{M_B}{K'}\omega^2 \right), \quad (3)$$

$$m_1 = \frac{M_A}{2} \left(1 + \frac{K''}{K'} - \frac{M_B}{K'}\omega^2 \right) + \frac{M_B}{2},$$

$$m_2 = \frac{M_C}{2} \left(1 + \frac{K''}{K'} - \frac{M_B}{K'}\omega^2 \right) + \frac{M_B K''}{2K'}.$$

G_1 represents a purely geometric coupling, while $G_2(\omega)$ includes a resonant correction term proportional to ω^2 that originates from the internal vibration of the B site. Equation (2) thus captures both the static and dynamic coupling between A and C in a single compact form.

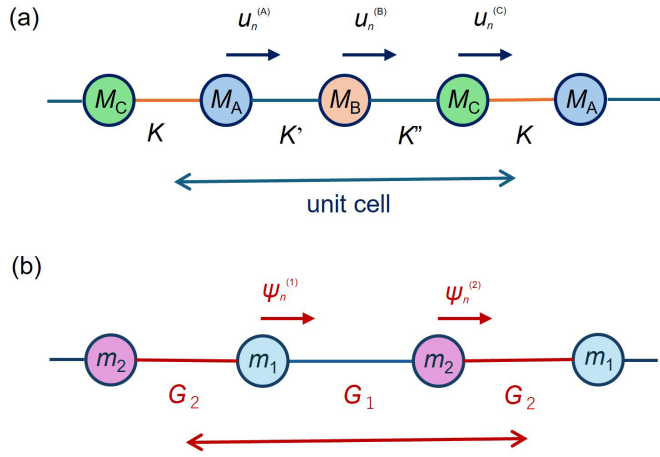


FIG. 1. Exact mapping from the trimer (SSH3) lattice to a two-sublattice effective model. (a) Unit cell consisting of sublattices A, B, and C with masses (M_A, M_B, M_C) and springs (K, K', K''). Bloch amplitudes $u_n^{(A,B,C)}$ describe the three coupled degrees of freedom. (b) Exact elimination of the internal site B yields a two-sublattice model with geometric coupling G_1 and resonant, frequency-dependent coupling $G_2(\omega)$, together with effective masses $m_{1,2}(\omega)$. The resulting Pauli-form Hamiltonian fully reproduces the bulk bands, band inversions, and edge physics of the original SSH3 lattice, and provides the analytical basis for the theory developed in this work.

To elucidate the structure, we introduce normalized variables $\psi^{(1)} = \sqrt{m_1} U^{(A)}$ and $\psi^{(2)} = \sqrt{m_2} U^{(C)}$. The eigenvalue problem can then be written as

$$H|\psi\rangle = \omega^2|\psi\rangle. \quad (4)$$

The effective Hamiltonian takes the Pauli form

$$H = \mathbf{R} \cdot \boldsymbol{\sigma} + WI, \quad (5)$$

where W denotes the mean on-site term (energy offset) and $\mathbf{R} = (X, Y, Z)$ describes the effective coupling and asymmetry between the sublattices [see Eqs. (S10)–(S12) and Sec. S2.1 of the Supplemental Material [33]]. This representation reveals that the trimer chain can be mapped exactly onto an effective two-level *Hamiltonian system*, where the “pseudospin” degrees of freedom correspond to the collective motion of the A and C sublattices. Full derivations are given in Secs. S1 and S2 of the Supplemental Material [33].

Having established the exact mapping, we next examine how the chiral symmetry and the frequency-dependent coupling $G_2(\omega)$ govern the band topology of this effective two-sublattice model.

Generality of the mapping beyond the trimer. Although the explicit derivation above was presented for the trimer SSH3 chain, the structure of Eq. (2) is a general consequence of eliminating internal resonant degrees of freedom in multisublattice periodic lattices. Whenever an N -sublattice chain contains $(N - 2)$ internal sites whose local equations couple only to their nearest neighbors, iterative elimination of those internal sites yields a reduced two-degree-of-freedom Hamiltonian for the outermost sites of the unit cell. The resulting effective couplings and inertial parameters become rational functions of ω^2 whose degree equals the number of

eliminated sites. Thus, the frequency dependence observed in $G_2(\omega)$ and $m_{1,2}(\omega)$ is not a special feature of the SSH3 chain but a structural consequence of internal-site elimination in multisublattice systems (see Sec. S7 of the Supplemental Material [33] for the general N -sublattice formulation and explicit SSH4/SSH5 examples). In this sense, the trimer SSH3 model serves as a minimal realization of a broader class of frequency-dependent effective Hamiltonians in multisublattice lattices. Higher-order systems (e.g., SSH4 and SSH5) produce second- and third-order frequency-dependent couplings, respectively, with multiple resonance zeros, and the same mechanism can govern their topological transitions.

Connection to effective-medium mechanics. The appearance of frequency-dependent effective masses and stiffnesses in Eq. (3) mirrors the classical effective-medium descriptions used for locally resonant metamaterials. The present mapping therefore provides an analytical connection between effective-medium mechanics and discrete topological band theory, clarifying that both descriptions originate from the same dynamical renormalization mechanism.

III. CHIRAL SYMMETRY, BAND EDGES, AND BAND INVERSION

A. Chiral-symmetry condition under inversion symmetry

When the inversion symmetry holds ($M_A = M_C$ and $K' = K''$), the effective masses coincide ($m_1 = m_2 \equiv m$) and the reduced Hamiltonian $\tilde{H} = H - WI = \mathbf{R} \cdot \boldsymbol{\sigma}$ satisfies the chiral-symmetry condition

$$\sigma_z \tilde{H} \sigma_z = -\tilde{H}. \quad (6)$$

This ensures that all eigenfrequencies appear in symmetric pairs $\pm(\omega^2 - W)$ around the midgap level, analogous to the electronic SSH model.

The corresponding effective parameters simplify to

$$G_1 = \frac{K'}{2}, \quad G_2 = K \left(1 - \frac{M_B}{2K'} \omega^2 \right), \quad (7)$$

$$m_1 = m_2 \equiv m = M_A \frac{G_2}{K} + \frac{M_B}{2}.$$

Substituting these into the general 2×2 form gives the compact dispersion relation

$$m\omega^2 - G_1 - G_2 = \pm |G_1 + G_2 e^{iq}|, \quad (8)$$

which self-consistently determines the three dispersion branches (acoustic and two optical) and their gap edges.

Equation (8) therefore plays the role of a closed-form dispersion relation from which all key bulk features of the SSH3 lattice—multiple band gaps, band inversions, and the conditions for edge localization—follow analytically. This expression encapsulates the central results introduced in the Introduction.

B. Band-edge eigenfrequencies and symmetry of eigenmodes

For equal masses $M_A = M_B = M_C = M$, analytical band-edge frequencies at $q = 0$ and π are obtained from Eq. (8) as

$$q = 0: \quad \omega_1^2 = 0, \quad \omega_2^2 = \frac{K' + 2K}{M}, \quad \omega_3^2 = \frac{3K'}{M},$$

$$q = \pi : \quad \omega_{4,6}^2 = \frac{3K' + 2K \mp \sqrt{9K'^2 - 4K'K + 4K^2}}{2M},$$

$$\omega_5^2 = \frac{K'}{M}. \quad (9)$$

The corresponding eigenvectors take simple symmetric or antisymmetric forms:

$$|\psi_{1,3,4,6}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |\psi_{2,5}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

[Eqs. (S24) and (S25), and Sec. S3.1 in the Supplemental Material], describing in-phase (*S*) or out-of-phase (*A*) motion between the *A* and *C* sites of each trimer. The ordering of these eigenmodes changes as K'/K crosses unity, signaling a band inversion.

Indeed, Eq. (9) gives the two characteristic sequences:

$$\begin{cases} \omega_1 < \omega_2 < \omega_3, & \omega_4 < \omega_5 < \omega_6 & (K' > K), \\ \omega_1 < \omega_3 < \omega_2, & \omega_5 < \omega_4 < \omega_6 & (K' < K). \end{cases} \quad (10)$$

Thus, the symmetry of the second band—*A*-type for $K' > K$ and *S*-type for $K' < K$ —is reversed across $K' = K$. This interchange anticipates a qualitative change in the band topology, whose formal characterization requires examining the phase evolution across the entire band, as done in Sec. IV.

C. Resonance frequency ω_S and internal phase evolution

A distinctive feature of the SSH3 lattice is the frequency dependence of the effective coupling $G_2(\omega)$. It vanishes at the local-resonance frequency of the intermediate mass M_B ,

$$\omega_S^2 = \frac{2K'}{M_B}, \quad (11)$$

as derived in Sec. S3.2 of the Supplemental Material [33] (“Effective coupling zero and local resonance”). At $\omega = \omega_S$, the coupling between the outer sites *A* and *C* is momentarily switched off, and the two-level eigenvector reduces to the real antisymmetric form

$$|\Psi_n(\omega_S)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

while the lifted trimer displacement exhibits a $\pi/2$ phase shift on the *B* site. Thus, whenever ω_S lies inside the second band ($K' < 2K$), the vibration pattern must pass through an antisymmetric configuration in its interior.

The qualitative phase evolution across the second band is then fixed by the symmetry of its two edges:

(1) *Normal regime* ($K' > K$). Both edges are antisymmetric (*A*-type). If ω_S lies inside the band, the eigenmode at ω_S is also antisymmetric, and the phase evolution follows an *A*–*A* sequence.

(2) *Inverted regime* ($K' < K$). Both edges are symmetric (*S*-type). In this regime, the resonance frequency ω_S always lies inside the second band, forcing the eigenmode to pass through an antisymmetric configuration at $\omega = \omega_S$, and producing the internal *S*–*A*–*S* sequence.

In both cases, the band-edge symmetries remain fixed, while the internal configuration changes only when ω_S is located within the band. This leads to three distinct regimes (Figs. 2–4):

(1) *Regime I* ($K' > 2K$). The resonance frequency ω_S lies above the second band, so $G_2(\omega)$ never vanishes. Its sign remains constant across the entire band, both band edges are antisymmetric, and no internal symmetry switching occurs.

(2) *Regime II* ($K < K' < 2K$). Here, ω_S lies inside the second band and $G_2(\omega)$ changes sign once as ω passes through ω_S . Nevertheless, both band edges remain antisymmetric, so no net symmetry change occurs across the band.

(3) *Regime III* ($K' < K$). Again ω_S lies inside the second band, but now both band edges are symmetric. The mid-band state at $\omega = \omega_S$ is forced to be antisymmetric, producing the characteristic *S*–*A*–*S* sequence and yielding the nontrivial winding number $\nu_2 = 2$.

These symmetry sequences compactly encode the internal structure of the second band and provide the structural input for the topological analysis in Sec. IV.

Self-induced topological transition. Because the effective coupling $G_2(\omega)$ depends explicitly on the eigenfrequency through the internal resonance of the *B* site, the winding number of the reduced two-sublattice Hamiltonian can change without any modification of structural parameters. This constitutes a genuine self-induced topological transition: The topology is determined by the frequency of the mode itself via the dispersive renormalization of $G_2(\omega)$. In other words, the SSH3 lattice provides a concrete realization of the *frequency-dependent topology* introduced in the Introduction, where the topological phase becomes a function of the eigenfrequency.

IV. TOPOLOGICAL INVARIANTS FROM THE TWO-LEVEL PHASE

We now formulate the topological characterization of the SSH3 lattice using the two-level representation developed in Sec. II. In particular, we show how the frequency dependence of the effective coupling $G_2(\omega)$ leads to a genuine form of *frequency-dependent topology*, in which the winding number becomes a function of the eigenfrequency rather than being fixed by static structural parameters.

Under chiral symmetry, the reduced Hamiltonian takes the planar Pauli form

$$\tilde{H} = \mathbf{R}(q) \cdot \boldsymbol{\sigma},$$

where $\mathbf{R}(q) = (X, Y, Z)$ and the chiral condition enforces $Z = 0$. The in-plane components are

$$x(q) = G_1 + G_2 \cos q, \quad y(q) = G_2 \sin q, \quad \tan \phi(q) = \frac{y}{x}. \quad (12)$$

Here, the dispersive coupling is evaluated self-consistently on the n th bulk dispersion relation, i.e., $G_2 \equiv G_2(\omega_n(q))$, when discussing the winding of band n . Accordingly, $\phi(q)$ denotes the argument of the in-plane vector $(x(q), y(q))$ traced as q varies over the Brillouin zone.

As the Bloch wave number q varies continuously from $-\pi$ to π , the vector $\mathbf{R}(q)$ traces a closed curve C_n in the (x, y) plane, and the geometric phase $\phi(q)$ represents its accumulated rotation angle.

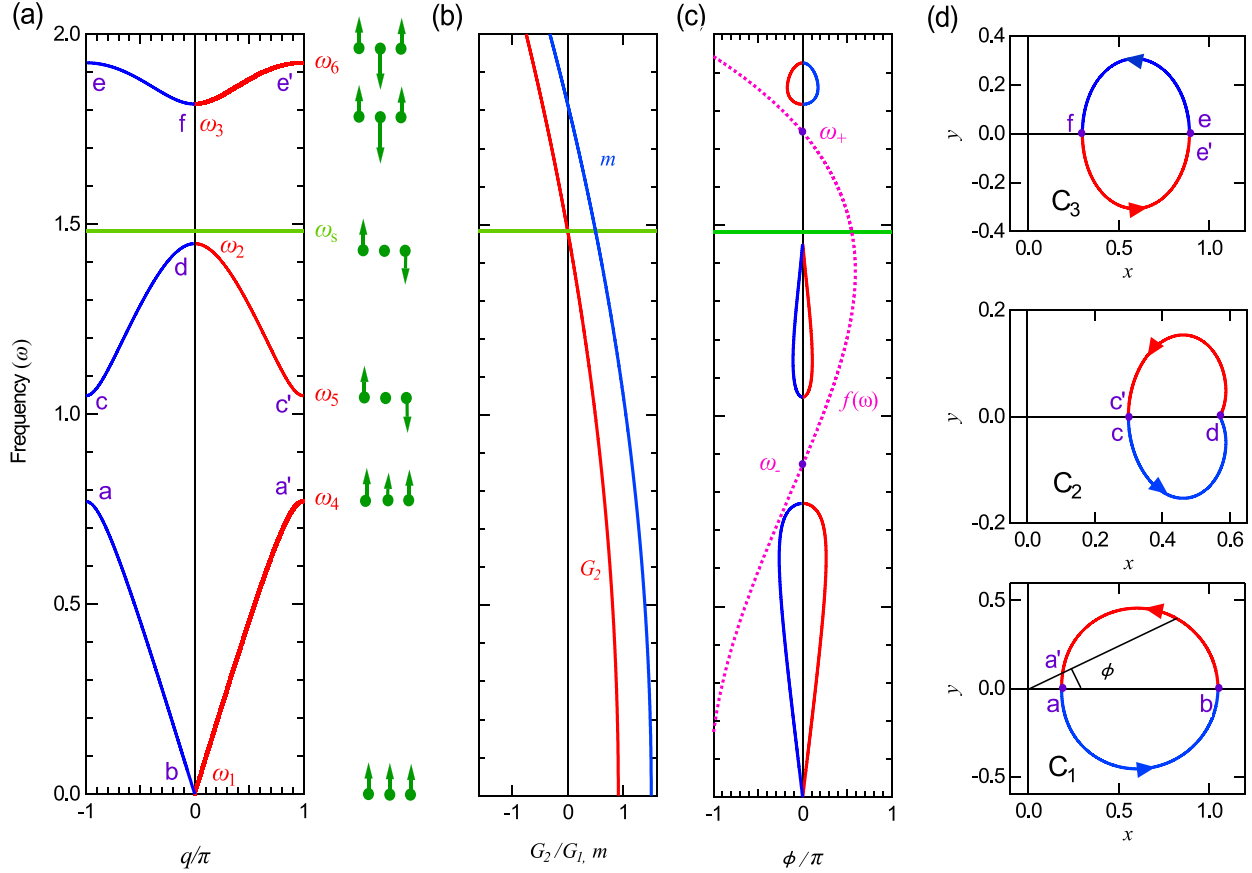


FIG. 2. Regime I ($K' > 2K$): Normal (topologically trivial) phase. Case $K = 0.5$, $K' = 1.1$, $M_A = M_B = M_C = 1$. (a) Dispersion curves $\omega_n(q)$ of the SSH3 lattice. Band-edge markers $a-f$ and $a'-f'$ link the dispersion to the phase trajectories in panel (d); arrows on the right illustrate the in-phase (S) or out-of-phase (A) displacement patterns on the (A, B, C) sublattices. Because $K' > 2K$, the resonance frequency ω_S (green line) lies above the second band, and the band-edge symmetries remain uniformly antisymmetric. (b) Frequency dependence of $G_2(\omega)/G_1$ (red) and effective mass $m(\omega)$ (blue). $G_2(\omega)$ keeps a fixed sign throughout the second band. (c) Profiles of $f(\omega) = m\omega^2 - G_1 - G_2$ and the geometric phase $\phi(q)$ evaluated along the dispersion $\omega = \omega_n(q)$. Small excursions of $\phi(q)$ do not accumulate, giving zero winding. (d) Phase-vector trajectories C_1 , C_2 , and C_3 in the (x, y) plane, where $(x, y) = (G_1 + G_2 \cos q, G_2 \sin q)$. No trajectory encircles the origin; all bands have $(\nu_1, \nu_2, \nu_3) = (0, 0, 0)$. Panels (a)–(d) consistently show that Regime I realizes a trivial phase with no band inversion and no edge states.

The band-resolved winding number is then defined as

$$\nu_n = \frac{(-1)^{n-1}}{2\pi} \oint_{C_n} d\phi, \quad (13)$$

where the prefactor $(-1)^{n-1}$ alternates the sign of the contribution for successive bands, ensuring that the total winding over all bands is conserved (see Sec. S4 of the Supplemental Material [33] for the derivation from the Berry connection and Zak phase).

A. Geometric role of $G_2(\omega)$ and phase evolution

The trajectory $(x(q), y(q)) = (G_1 + G_2 \cos q, G_2 \sin q)$ represents the projection of $\mathbf{R}(q)$ in the (x, y) plane. The shape and position of this curve are determined by both G_1 and $G_2(\omega)$. When $|G_2| < |G_1|$, the curve remains entirely on one side of the origin, whereas for $|G_2| > |G_1|$, it can wrap around the origin, producing a finite winding.

If $G_2(\omega)$ vanishes at a certain frequency $\omega = \omega_S$, the even symmetry $\omega(q) = \omega(-q)$ gives two equivalent wave

numbers $\pm q_S$ corresponding to the same frequency. During one Brillouin-zone cycle ($q = -\pi \rightarrow \pi$), the trajectory $(x(q), y(q))$ crosses the x axis at $q = 0, \pi$, and $\pm q_S$. Since the crossings at $\pm q_S$ coincide, the curve intersects the x axis at three distinct positions: $G_1 - G_2$, G_1 , and $G_1 + G_2$. As q evolves from $-\pi$ to π , the trajectory starts at $(G_1 - G_2, 0)$, passes $(G_1, 0)$ twice, and reaches $(G_1 + G_2, 0)$ before returning to its origin. This geometric motion determines the total phase evolution $\phi(q)$ and hence the winding number ν_n .

The qualitative behavior of $\phi(q)$ differs across the three regimes, as illustrated in Figs. 2(c)–4(c):

(1) *Regime I* ($K' > 2K$). $G_2(\omega)$ keeps a constant sign across the entire second band. As q varies from $-\pi$ to π , the phase ϕ makes only small, symmetric excursions about zero: It departs slightly into the negative region, returns to $\phi = 0$, and then slightly into the positive region before closing again at $\phi = 0$. The net (principal-value) change is $\Delta\phi = 0$; hence, $\nu_2 = 0$. Both edges of the second band are A -symmetric, consistent with the trivial configuration in Figs. 2(c) and 2(d).

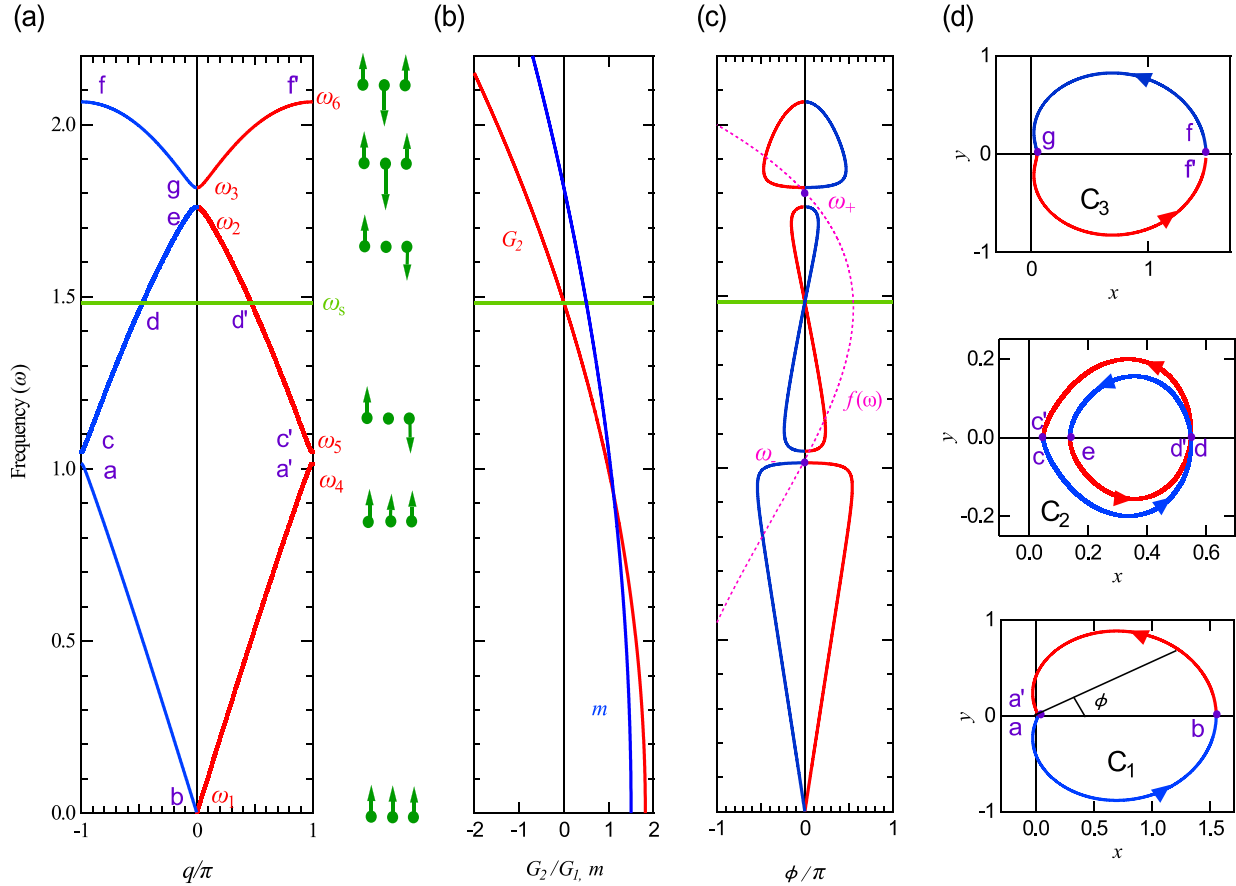


FIG. 3. Regime II ($K < K' < 2K$): Trivial phase with internal sign reversal. Case $K = 1.0, K' = 1.1, M_A = M_B = M_C = 1$. (a) Dispersion relation $\omega_n(q)$. Band-edge markers again correspond to panel (d), while the arrows to the right indicate the symmetry patterns on (A, B, C). Here, the resonance frequency ω_s lies *inside* the second band, enforcing an internal antisymmetric configuration. (b) Frequency dependence of $G_2(\omega)$ and $m(\omega)$. $G_2(\omega)$ changes sign once within the second band. (c) Profiles of $f(\omega)$ and the geometric phase $\phi(q)$. Although $\phi(q)$ reverses sign at $\omega = \omega_s$, the net evolution is zero. (d) Phase trajectories C_1 – C_3 in the (x, y) plane. None encloses the origin, giving $(\nu_1, \nu_2, \nu_3) = (0, 0, 0)$. These panels establish that Regime II remains topologically trivial, despite the internal sign reversal of the effective coupling.

(2) *Regime II* ($K < K' < 2K$). Here, $G_2(\omega)$ changes sign once within the second band at the resonance point $\omega = \omega_s$ with $G_2(\omega_s) = 0$. As q runs from $-\pi$ to π , the *sign* of the principal-value phase ϕ alternates *exactly twice* (“ $- \rightarrow + \rightarrow - \rightarrow +$ ”), corresponding to two internal zero crossings at $q = \pm q_s$ in addition to the band-edge zeros. Nevertheless, the total principal-value accumulation still cancels, $\Delta\phi = 0$, yielding $\nu_2 = 0$. This is the small, near-zero excursion pattern (A–A–A) shown in Figs. 3(c) and 3(d).

(3) *Regime III* ($K' < K$). In the inverted regime, $G_2(\omega)$ also changes sign once within the second band. Let $\phi \in (-\pi, \pi]$ denote the principal-value phase and let $\tilde{\phi}$ be its continuously unwrapped branch. Because $G_2(\omega_n(q)) = 0$ at $\omega = \omega_s$, the phase satisfies $\phi(q_s) = 0$ at the corresponding wave numbers $q = \pm q_s$ on the same band. Then, as q increases from $-\pi$ to 0, $\phi(q)$ runs from $-\pi$ to $+\pi$ (passing through $\phi(q_s) = 0$), and as q increases further from 0 to π , $\phi(q)$ again runs from $-\pi$ to $+\pi$. Choosing the continuous branch $\tilde{\phi}(q)$ removes the 2π ambiguity, so that $\tilde{\phi}(q)$ advances monotonically by a total of 4π over $q \in [-\pi, \pi]$, which gives $\nu_2 = 2$. In this case, the band-bottom and band-top configurations are both in-phase (S-type), with the intermediate state at

$\omega = \omega_s$ being antisymmetric (A-type)—the S–A–S sequence seen in Figs. 4(c) and 4(d).

These distinct phase evolutions explain how the same apparent band-edge symmetry can coexist with different internal phase windings across the three regimes. In particular, only in the inverted regime (III) does ϕ advance by a full 4π , giving $\nu_2 = 2$, whereas in Regimes I and II the total phase change remains zero.

B. Physical meaning of the winding redistribution

The winding number ν_n quantifies the total phase accumulation of the effective two-level system over one Brillouin period. The emergence of $\nu_2 = 2$ indicates that the phase vector $\mathbf{R}(q)$ rotates twice around the origin, corresponding to a complete inversion of the bonding and antibonding character between the A and C sublattices as the frequency passes through ω_s . The accompanying alternation of ν_1 and ν_3 ensures that the overall chirality of the system remains conserved. Thus, the resonance of the intermediate B site induces a redistribution of winding numbers among the three bands, producing the characteristic pattern

$$(\nu_1, \nu_2, \nu_3) = (-1, 2, -1).$$

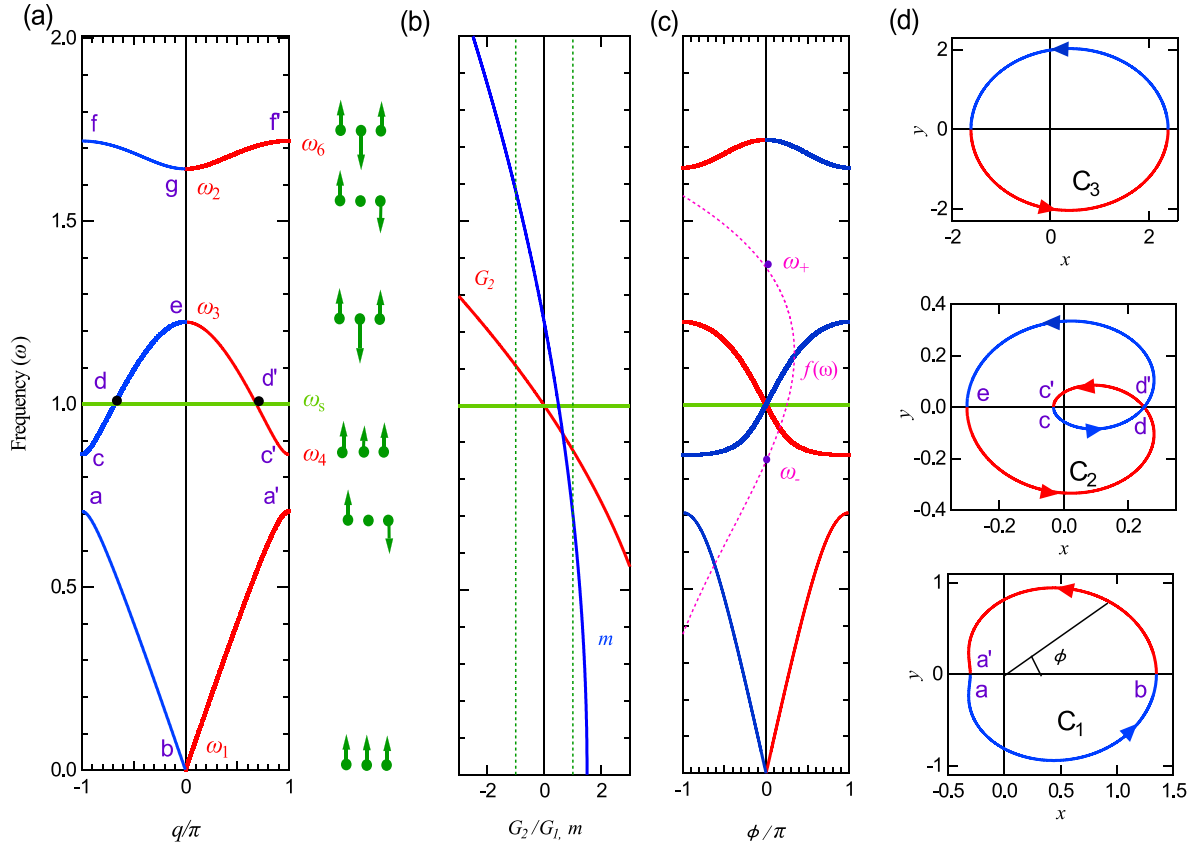


FIG. 4. Regime III ($K' < K$): Inverted and topologically nontrivial phase. Case $K = 1.1$, $K' = 0.5$, $M_A = M_B = M_C = 1$. (a) Dispersion $\omega_n(q)$, with symmetry patterns shown on the right. Because ω_5 lies inside the second band and the band edges are S -type, the sequence becomes S – A – S , indicating an inverted configuration. (b) Frequency dependence of $G_2(\omega)/G_1$ and $m(\omega)$. $G_2(\omega)$ changes sign once within the second band. (c) Profiles of $f(\omega)$ and the geometric phase $\phi(q)$. The unwrapped phase advances by 4π across the second band, yielding $\nu_2 = 2$. (d) Phase-vector trajectories C_1 , C_2 , and C_3 . Each trajectory encircles the origin, with C_2 completing two counterclockwise rotations, giving $(\nu_1, \nu_2, \nu_3) = (-1, 2, -1)$. Together these panels identify the inverted, topologically nontrivial phase arising from the frequency-dependent sign reversal of the effective coupling.

These topological indices serve as bulk invariants that govern the emergence of edge states in finite SSH3 chains, as demonstrated in the following section.

V. EDGE MODES AND BULK-EDGE CORRESPONDENCE

Since the winding numbers (ν_1, ν_2, ν_3) arise directly from the internal phase evolution of the bulk bands described in Sec. IV, the emergence or absence of edge-localized states in finite SSH3 chains follows immediately from this bulk structure. In particular, the nontrivial S – A – S evolution of the second band for $K' < K$ enforces $\nu_2 = 2$, which guarantees the existence of in-gap edge modes analyzed below.

Having established the bulk topological invariants and the winding redistribution induced by the sign change of $G_2(\omega)$ (Sec. IV), we now examine how they manifest as localized edge states in finite SSH3 chains over the full range of stiffness ratio K/K' .

Figure 5 presents both the trivial ($K/K' < 1$) and nontrivial ($K/K' > 1$) regimes for equal masses $M_A = M_B = M_C = 1$. Panels (a) and (b) correspond to a symmetric 120-site chain ($N = 40$ unit cells), while panels (c) and (d) correspond to an asymmetric 119-site chain with the rightmost site removed.

In the symmetric case, two degenerate edge branches appear in the first and second gaps, producing left-right pairs (A–D). In the asymmetric case, only one localized mode remains per gap, as labeled E–H, confirming that the edge localization depends on the termination and that the emergence of these states follows the condition $|G_2(\omega)/G_1| > 1$ [Eq. (15)]. These results quantitatively verify the bulk-edge correspondence of the SSH3 lattice derived from the topological winding $(\nu_1, \nu_2, \nu_3) = (-1, 2, -1)$.

The analytical origin of these numerically observed in-gap branches can be clarified by returning to the effective two-sublattice model. In analogy with zero-energy edge states in chiral lattices, a midgap edge state at a chiral-symmetric boundary satisfies the condition

$$f(\omega) \equiv m\omega^2 - G_1 - G_2 = 0, \quad (14)$$

as derived in Sec. S5.1 of the Supplemental Material [33] (“Boundary condition and recurrence relations”). In this case, the bulk Bloch factor is eliminated and the lattice recurrences reduce to first order, decoupling the A and C channels. Normalizability requires the geometric damping

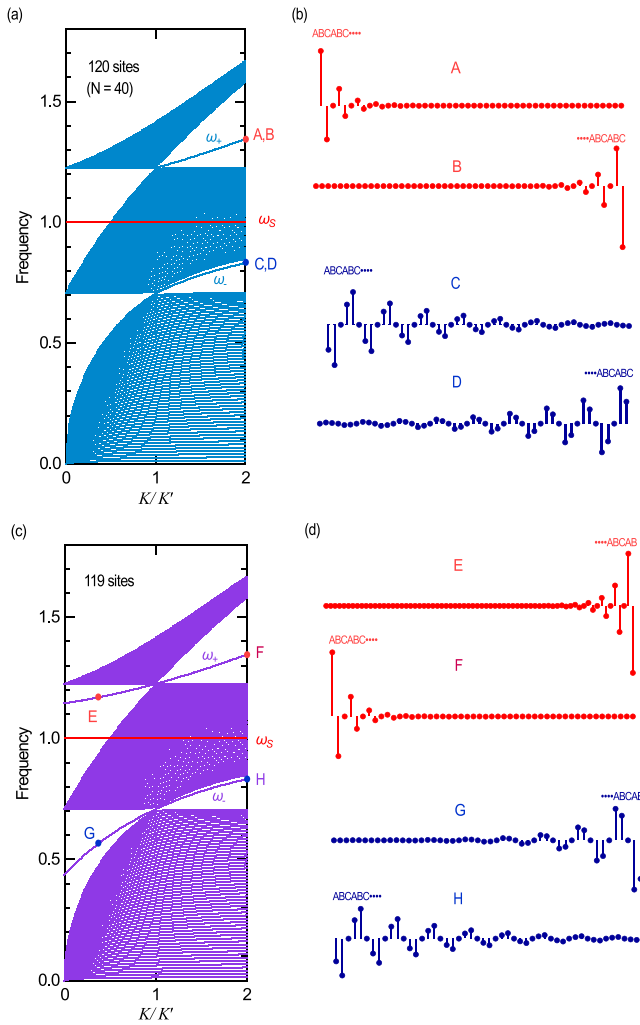


FIG. 5. Edge states and localization in finite SSH3 chains. (a) Eigenfrequency spectra of a symmetric 120-site chain ($N = 40$ unit cells) as a function of K/K' . Two in-gap branches appear in each band gap, forming left-right symmetric edge-mode pairs. (b) Displacement profiles of representative edge states (A–D), confirming exponential localization at opposite ends. (c) Eigenfrequencies of an asymmetric 119-site chain obtained by removing the terminal C site. Only one in-gap branch remains in each gap, reflecting the broken left-right symmetry. (d) Localization profiles of the surviving modes (E–H). All edge states obey the midgap condition $f(\omega) = 0$ and the localization criterion $|G_2(\omega)/G_1| > 1$, in agreement with the analytic expressions for $\omega_{\pm}$ and the complex Bloch number in Eqs. (14)–(17).

condition

$$\left| \frac{G_2(\omega)}{G_1} \right| > 1, \quad (15)$$

which is obtained by solving the first-order recurrence relations in Sec. S5.1 of the Supplemental Material [33], and holds throughout the gaps for $K > K'$.

Closed-form edge frequencies follow for equal masses ($M_A = M_B = M_C = M$) in the chiral case:

$$M\omega_{\pm}^2 = \frac{K + 3K'}{2} \pm \frac{1}{2}\sqrt{(K - K')^2 + 4K'^2}, \quad (16)$$

one in each of the first and second gaps (ω_- : first gap, ω_+ : second gap; see Sec. S5.2 of the Supplemental Material [33] for the derivation). The complex Bloch number of the edge solution is

$$q = n\pi + i \log \left| \frac{G_2(\omega_{\pm})}{G_1} \right|, \quad n = 0 (\omega_+), \quad n = 1 (\omega_-), \quad (17)$$

whose imaginary part determines the exponential decay rate of the edge amplitude in units of lattice cells, as shown explicitly in Sec. S5.2 of the Supplemental Material [33].

Boundary dependence. For a finite SSH3 chain with $3N$ sites, imposing the boundary conditions $u_0^{(C)} = 0$ and $u_{N+1}^{(A)} = 0$ yields two degenerate edge solutions: a left-edge mode with $u_n^{(C)} = 0$ [Eq. (S47) of the Supplemental Material] and a right-edge mode with $u_n^{(A)} = 0$ [Eq. (S48) of the Supplemental Material] [33]. Both modes satisfy the localization condition $|G_2/G_1| > 1$ and decay exponentially into the bulk.

When the terminal C site is removed, giving a chain of $3N - 1$ sites, the boundary conditions change to $u_0^{(C)} = 0$ and $u_N^{(C)} = 0$. In this case, Eq. (S48) of the Supplemental Material [33] admits only the trivial solution $u_n^{(C)} = 0$, so the right-edge state cannot form, whereas the nontrivial solution given by Eq. (S47) of the Supplemental Material [33] remains and yields a left-edge mode localized on the A – B pair. Thus, removing the terminal C breaks the left-right symmetry and leaves only a single edge state on the left boundary.

The explicit recurrence relations and boundary conditions underlying these edge states are derived in Sec. S5 of the Supplemental Material [33], where the same localization criterion and closed-form eigenfrequencies are obtained analytically. This confirms that the physical emergence of edge-localized modes in finite chains directly follows from the mathematical recurrence framework.

From the effective-Hamiltonian viewpoint, the existence of edge modes can be interpreted geometrically through the ratio $|G_2(\omega)/G_1|$, which determines whether the phase trajectory of the effective Hamiltonian encircles the origin. This provides a direct connection between the winding-number redistribution described in Sec. IV and the localization condition for finite chains.

The edge states obtained here therefore admit a simple interpretation in terms of the dispersive effective Hamiltonian. Because the reduced two-sublattice description retains the chiral structure of the SSH model, the existence of edge-localized modes is controlled by the sign of the effective intersublattice coupling $G_2(\omega)$ evaluated along the bulk dispersion relation. When $G_2(\omega)$ changes sign across a band gap, the phase trajectory can reconfigure so as to change the winding number, and, provided the localization criterion $|G_2(\omega)/G_1| > 1$ is satisfied, an edge-localized mode emerges inside the gap.

In this sense, the edge modes in the SSH3 lattice are not determined by a fixed parameter set, but by the frequency-dependent effective coupling selected self-consistently by the eigenfrequency. This establishes a direct connection between dispersive renormalization and bulk-edge correspondence in multisublattice lattices.

VI. ADIABATIC MODULATION AND LOCAL TOPOLOGICAL CHARGE ON (q, t)

Having established the static topological phases and bulk-edge correspondence, we next explore how these features evolve under a slow temporal modulation. Here, we extend the SSH3 Hamiltonian into the time domain and show that adiabatic stiffness modulation generates a pattern of locally nontrivial Berry curvature, while the global Chern number remains neutral.

A. Time-periodic Hamiltonian and adiabatic parameter space

We introduce a slow stiffness modulation

$$K(t) = K' [1 + \delta \cos(2\pi t/T)],$$

which yields the instantaneous eigenproblem

$$H(q, t)|\Psi_n(q, t)\rangle = \omega_n^2(q, t)|\Psi_n(q, t)\rangle, \\ (q, t) \in \text{BZ} \times [0, T].$$

The modulation amplitude δ is assumed to be sufficiently small to satisfy the adiabatic condition, so that interband transitions remain negligible over one modulation cycle.

This defines a parameter-space torus, where the Berry connection and Berry curvature take their standard adiabatic forms. The derivation and symmetry properties ensuring global neutrality are given in Sec. S6 of the Supplemental Material [33].

B. Berry curvature and global neutrality

The Berry connections

$$A_{q,n} = i\langle\Psi_n|\partial_q|\Psi_n\rangle, \quad A_{t,n} = i\langle\Psi_n|\partial_t|\Psi_n\rangle$$

yield the Berry curvature

$$\Omega_n(q, t) = \partial_q A_{t,n} - \partial_t A_{q,n}.$$

Although $\Omega_n(q, t)$ exhibits sharp peaks in the (q, t) plane, the integrated Chern number

$$Q_n = \frac{1}{2\pi} \int_0^T dt \oint_{\text{BZ}} \Omega_n(q, t) dq$$

vanishes for inversion-symmetric SSH3 chains. Thus, the adiabatic cycle is globally topologically neutral ($Q_n = 0$), in agreement with the derivation in the Supplemental Material [3,33].

C. Local degeneracies and alternating monopole charges

Even though the global Chern number satisfies $Q_n = 0$, the instantaneous bands of the adiabatically modulated SSH3 chain exhibit four isolated degeneracies on the parameter-space torus $(q, t) \in \text{BZ} \times [0, T]$:

$$(q, t) = (0, T/4), (0, 3T/4), (\pi, T/4), (\pi, 3T/4).$$

At each of these points, two adjacent bands touch and exchange their character, producing a localized source or sink of Berry curvature.

For a closed loop D_j on the (q, t) torus encircling a single degeneracy [Fig. 6(b)], the Berry phase of band n is

$$\gamma_n(D_j) = \oint_{D_j} (A_{q,n} dq + A_{t,n} dt),$$

which measures the circulation of the geometric angle $\phi = \arctan(y/x)$ of the phase vector $(x(q, t), y(q, t))$ defined by Eq. (12).

Using the analytical expressions above and the formulation in Sec. S6 of the Supplemental Material [33], one finds that each degeneracy carries a quantized topological charge of magnitude π , with alternating signs enforced by inversion symmetry. For example, for the second band,

$$\gamma_2(D_1) = -\pi, \quad \gamma_2(D_2) = +\pi, \\ \gamma_2(D_3) = -\pi, \quad \gamma_2(D_4) = +\pi,$$

while the first and third bands distribute their Berry phases such that the sum of monopole charges cancels at each degeneracy point. These explicit values confirm the alternating (monopole-antimonopole) structure indicated by the opposite rotation senses of the trajectories C_{1-4} in Fig. 6(c).

Thus, the Berry curvature of the adiabatically modulated SSH3 lattice is *locally* nontrivial—each degeneracy acts as a topological charge $\pm\pi$ —yet *globally* neutral, in agreement with $Q_n = 0$.

VII. DISCUSSION

The analytical approach developed in this work provides an exactly tractable framework that connects effective-medium mechanics with topological band theory in multisublattice lattice systems. By eliminating the internal degree of freedom of the trimer lattice, the full three-sublattice dynamics reduce to a two-level (Pauli-form) Hamiltonian with frequency-dependent inertial and coupling terms. As illustrated in Fig. 1, the resonant intersublattice coupling $G_2(\omega)$ acts as a physically transparent parameter that governs band inversion, redistribution of integer winding numbers, and the emergence of edge-localized modes. These features follow directly from the closed-form criteria [Eqs. (14)–(17)], which reveal their geometric origin in the complex plane.

Unlike conventional SSH-type tight-binding models with fixed, frequency-independent couplings, the present formulation leads to a nonlinear eigenvalue problem in ω , where the effective Hamiltonian itself depends self-consistently on the eigenfrequency through dispersive renormalization. This structure allows the reduced two-level description to retain the full three-band information of the SSH3 lattice while providing a transparent analytical criterion for band inversion and winding redistribution in terms of the sign change of $G_2(\omega)$.

Because the same mapping remains valid in the presence of complex-valued (stiffness or damping) parameters and can be extended to adiabatically time-modulated lattices, the formulation applies to both non-Hermitian and Floquet mechanical systems. Within this viewpoint, dispersive renormalization acts as an intrinsic selector of topology: The eigenfrequency determines the effective Hamiltonian, which in turn determines the topological phase. Accordingly, frequency-dependent interactions provide a natural

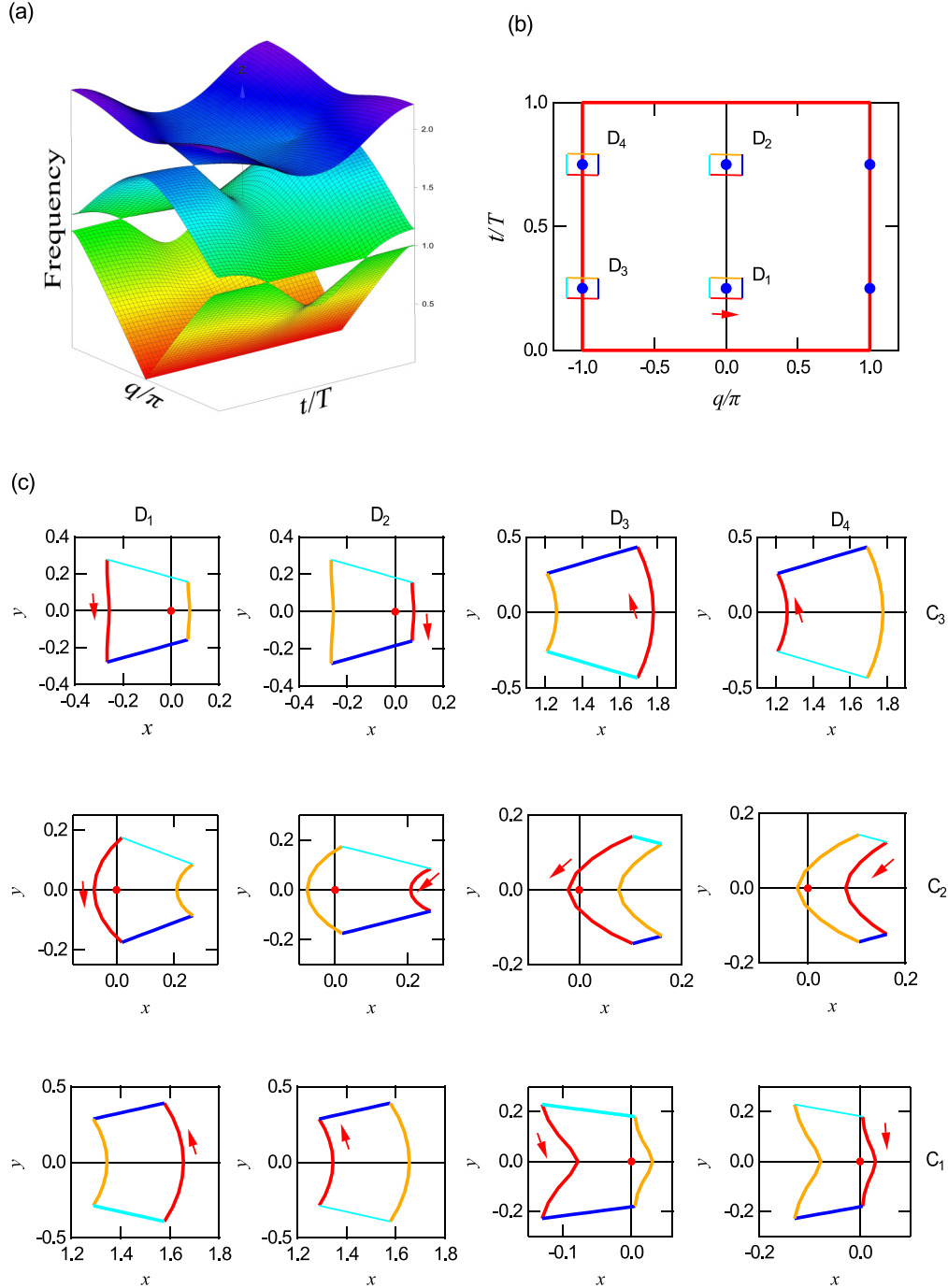


FIG. 6. Adiabatic modulation and local Berry curvature on the (q, t) torus. (a) Instantaneous dispersion $\omega_n(q, t)$ of the SSH3 lattice under slow stiffness modulation $K(t) = K'[1 + \delta \cos(2\pi t/T)]$, plotted over the parameter-space torus $(q, t) \in [-\pi, \pi] \times [0, T]$. (b) Positions of the four symmetry-related degeneracies at $(q, t) = (0, T/4), (\pi, T/4), (0, 3T/4),$ and $(\pi, 3T/4)$, labeled D_1 – D_4 . The closed paths D_1 – D_4 illustrate loops in the (q, t) plane that encircle individual band touchings. (c) Corresponding trajectories C_1 – C_4 of the phase vector $(x, y) = (G_1 + G_2 \cos q, G_2 \sin q)$ defined in Eq. (12), evaluated along the loops D_1 – D_4 . Around each degeneracy, the trajectories wind once about the origin with alternating rotation sense, yielding quantized Berry phases of magnitude π for the encircled band. The four degeneracies therefore carry alternating monopole-antimonopole charges and generate a locally nontrivial yet globally neutral Berry-curvature texture with vanishing Chern number $Q_n = 0$ for all bands.

route toward realizing frequency-selective topological transitions in multisublattice lattices.

It is important to emphasize that the topology of the SSH3 lattice can, in principle, be obtained directly from the full three-band Bloch Hamiltonian. The contribution of

the present work is not to replace this full description, but to provide an analytical framework in which the dispersive renormalization induced by internal resonances is made explicit at the level of an effective Hamiltonian. This clarifies the physical mechanism by which the sign reversal of the effective

coupling governs band inversion, winding redistribution, and the emergence of edge modes, which is not transparent in fixed-parameter formulations.

From a broader perspective, this mechanism is not restricted to mechanical lattices. Frequency- or energy-dependent effective parameters naturally arise in phononic metamaterials, photonic crystals, magnonic systems, and Floquet-engineered platforms, where local resonances, self-energy corrections, or temporal modulation introduce dispersive renormalization of the effective Hamiltonian. Therefore, the frequency-dependent topology identified here should be regarded as a general feature of dispersive multisublattice wave systems rather than a model-specific property of the SSH3 chain.

Overall, the theory provides a compact analytical connection between effective-medium descriptions and discrete topological classifications, clarifying how dispersive internal degrees of freedom influence band topology in phononic metamaterials and related mechanical systems. The SSH3 lattice serves as a minimal nontrivial realization of frequency-dependent topology, already capturing the essential role of internal resonances, dispersive couplings, and integer winding-number redistribution. The general N -sublattice mapping outlined in Sec. S7 of the Supplemental Material [33] further indicates that higher-sublattice models inherit the same algebraic structure with higher-order frequency dependence and multiple resonance zeros, establishing a unified analytical framework for dispersive topological phases in multisublattice systems.

VIII. CONCLUSION

This work develops an analytical framework for understanding how frequency-dependent effective parameters dictate the topological properties of one-dimensional multisublattice lattices. By mapping the SSH3 trimer lattice onto an effective two-sublattice Hamiltonian with dispersive couplings and inertial terms, we showed that the sign reversal of the effective intersublattice coupling originating from internal resonances acts as a bifurcation point of the winding number, leading to distinct integer topological phases.

A key conceptual outcome is that the reduced Hamiltonian depends explicitly on the eigenfrequency, resulting in a nonlinear eigenvalue problem that self-consistently reproduces the full three-band spectrum. This clarifies how multiple band gaps, band inversions, and edge-localized modes emerge from a single closed-form dispersion relation and reveals their geometric origin in the complex plane.

The frequency dependence of the effective stiffness provides a direct and experimentally accessible mechanism for inducing transitions between trivial and nontrivial phases, constituting a mechanical realization of frequency-dependent topology. When extended to adiabatically time-modulated lattices, the same formalism enables analytical evaluation of the Berry curvature and local topological charges on the (q, t) torus, demonstrating how dynamically driven systems can exhibit locally nontrivial yet globally neutral topological structures.

More generally, because dispersive renormalization arising from internal degrees of freedom is ubiquitous in phononic, photonic, and other wave-based lattice systems, the concept of frequency-dependent topology established here provides a unified analytical paradigm for dispersive multisublattice media beyond the specific SSH3 model.

ACKNOWLEDGMENTS

This work was supported by the Japan Society for the Promotion of Science (JSPS) through Grants-in-Aid for Scientific Research (KAKENHI) under Grant No. 24K08261. The author gratefully acknowledges Prof. Katsunori Wakabayashi for valuable discussions on the theoretical aspects of topological materials. The author also wishes to thank Prof. Motonobu Tomoda for insightful discussions on the experimental aspects of phononic lattices. Fruitful discussions with students, including Mr. Takuya Akisawa, have also been of great help in the development of this work.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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