

Entanglement witness for combined atom interferometer-mechanical oscillator setup

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We investigate how to entangle an atom interferometer and a macroscopic mechanical oscillator to create nonclassical states of the oscillator. This work is done in the context of an experimental-theoretical collaboration that aims to implement the proposal in the paper “Using an Atom Interferometer to Infer Gravitational Entanglement Generation,” written by Carney *et al.* [PRX Quantum 2, 030330 (2021)]. Here, we propose an entanglement witness (EW), from whose violation the generation of entanglement can be determined. We do this for three scenarios: the noiseless case, when including thermal noise, and with atomic dephasing. Thermal noise can arise from two sources: the first being when the oscillator starts in a thermal state, and the second when a continuous thermal bath is in contact with the oscillator. We find that for the appropriate oscillator quality factor Q , violation always exists for any value of coupling λ and thermal occupancy $\bar{n}$. Cooling the oscillator to its ground state provides an $O(\bar{n})$ improvement in the EW violation than when starting in a thermal state; in this case, we find that couplings of $\lambda = 10^{-4.5}$ require at least $\sim 10^{5.2}$ measurements for violation to be observed. For atomic dephasing, we find that EW violation can exist for an atom interferometer with a coherence time of 600 s, which is the parameter our experimental collaborators hope to reach. Finally, we consider how to use magnetic interactions to realize this idea within the context of the torsional pendulum that has been planned for our collaboration.

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I. INTRODUCTION

Entanglement is a core feature of quantum mechanics that precludes any hidden variable classical theory. Many groups have considered entanglement, particularly in the case of spin-spin systems [2–6], oscillator-oscillator systems [7–13], and spin-oscillator systems [14–16]. For instance, Kong *et al.* experimentally generate entanglement between the spins of atoms in a vapor [2], Ockeloen-Korppi *et al.* use a cavity to entangle two mirrors located 600 μm apart [7], and Riedinger *et al.* entangle two chips separated by 20 cm [8]. Thomas *et al.* entangle a spin ensemble with a square membrane of sides 1 mm in length [14].

Of particular interest are approaches to entangle masses and/or spins to investigate theories that incorporate quantum mechanics with gravity. For example, Bose *et al.* [3] propose using two masses, each forming a Stern-Gerlach interferometer. Each interferometer consists of one mass in a superposition of its spin states. These masses interact through a gravitational force; if the two masses become entangled,

then we can conclude that the gravitational field is capable of generating entanglement. Another paper is by Marletto and Vedral [17], where they also propose having two masses, each of them in a superposed state of its spatial locations, coupled through the gravitational field. As previously, if the two masses are entangled, then the gravitational field is capable of generating entanglement. One of the defining aspects of this proposal is that only one of the masses is measured to determine the existence of entanglement—this mass acts as the witness of entanglement. A third proposal is by Kafri and Taylor [18], who propose determining whether two torsional pendulums are entangled gravitationally by measuring the rate at which “excess” noise in the system changes. Here, “excess” noise is that which is in addition to thermal noise; the rate of its change has a bound if the interaction is classical.

A different approach by Carney *et al.* [1] suggests using an atom interferometer to create nonclassical states of a mechanical oscillator. A trapped atom interferometer employing an optical lattice interacts with a macroscopic mechanical oscillator during the time for which the atoms are trapped. The atoms, which are in spatial superposition, exert a gravitational force on the oscillator. If this interaction is entangling, the oscillator and the atoms will get entangled, which can be detected using the interferometer’s signal (for caveats and conditions, see Ref. [1]). In a similar vein to Marletto and Vedral [17], where only one of the masses is measured, this proposal requires that only the atoms are measured.

The authors of this paper work within the context of an experimental-theoretical collaboration; this collaboration

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includes all the authors of the paper by Carney *et al.* [1] and aims to implement the proposal experimentally. In the progress toward such a gravity experiment, it is more reasonable to first attempt the generation of *magnetic* entanglement between the atom interferometer and the oscillator. This has a number of uses. Long term, it was shown by Carney *et al.* [1] that creating initial nongravitational entanglement leads to a signal that is linear in the gravitational coupling, rather than quadratic as would be the case for only a gravitational interaction. Short term, it is more feasible to attempt the generation of magnetic entanglement given that a sufficiently strong magnetic coupling is easier to obtain. Furthermore, we will make some progress toward creating nonclassical states of a macroscopic object, which will be useful on its own, even if gravitational entanglement continues to be elusive.

Before proceeding with actually building the experiment, it is important to ensure that entanglement will exist, at least theoretically, and understand the physical parameters at which it will occur. In order to do this, we construct an entanglement witness (EW) for our setup. An EW is a quantity composed of observables; it has a bound satisfied by separable states, which, if violated, determines that entanglement exists between the systems. The literature on creating an entanglement witness is varied. The most well-known example of EWs might be Bell inequalities [19,20] used to distinguish between the possibility of local hidden variable theories and entanglement. Another common EW is one that is often applied to oscillator-oscillator entanglement, whose general version was proposed by Duan *et al.* [21].

To provide further concrete examples, the experiments in Refs. [2,3,7,8,14] use (or propose using) an EW; however, these EWs work for very low noise situations, where one prepares pure states of the two systems, and noise and decoherence are neglected. The exploration of an EW in our paper provides a path for violation at finite motional temperatures and with noise and dephasing. We investigate the effects of thermal noise on EW violation and establish the conditions on parameters such that violation exists; we find that for appropriate values of oscillator quality factor Q , violation always exists for any value of coupling λ and thermal occupancy $\bar{n}$. We also include atomic dephasing and find that EW violation exists for an atom interferometer with a coherence time of 600 s, which is a target parameter for our collaboration.

Section II lays out the entanglement witness that must be measured in order to determine that entanglement has indeed been generated, including in the cases of thermal noise and atomic dephasing. Section III details possible experimental implementations using magnetic interactions and the strength of the couplings.

II. ENTANGLEMENT WITNESS

A. General entanglement witness

A simplified setup that we can consider is shown in Fig. 1. The Hamiltonian of the system is (note that $\hbar = 1$ herein) [1]

$$H = \omega \hat{c}^\dagger \hat{c} + g(\hat{c} + \hat{c}^\dagger) \hat{J}_z. \quad (1)$$

$\hat{c}^\dagger$ and $\hat{c}$ are the oscillator creation and annihilation operators. $\hat{J}_z$ is a pseudospin operator describing the location

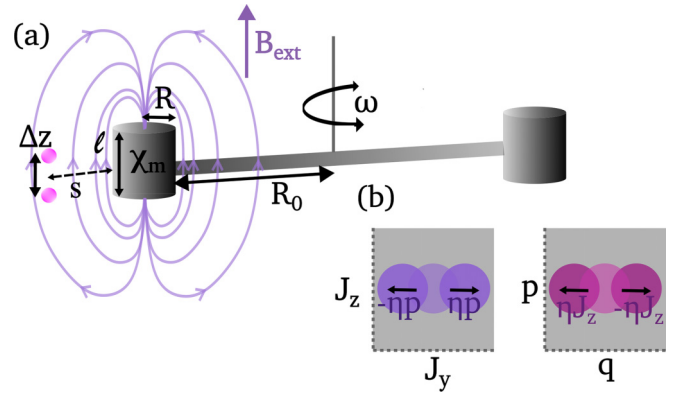


FIG. 1. (a) Simplified version of oscillator-interferometer setup. The atoms (pink) of the interferometer are in traps [22] and are located at equal distances above and below the center of the cylinder. An external magnetic field $\mathbf{B}_{\text{ext}}$ induces a magnetic field in the cylinders that have magnetic susceptibility χ_m . The oscillator rotates in the horizontal plane with frequency ω . (b) Depiction of how the oscillator and spin operators become correlated with each other, with $\eta = \sqrt{2/(m\omega)} (g/\omega)$. The explicit expressions are given in Sec. II B.

of N atoms; that is, $\hat{J}_z = \sum_{j=1}^N \hat{S}_{z,j}$, where $\hat{S}_{z,j}$ characterizes which of the two interferometer traps the j th atom is in. There are several ways to understand $\hat{S}_{z,j}$. First, we can make an analogy with a nitrogen-vacancy (NV) center; if an NV center is primarily driven between the $m = 0$ and $m = 1$ states, it can be described with the two-level $\hat{S}_z$ operator [23,24]; this will be considered a pseudospin because it describes the effective state of the NV center and is different from its general three-level version. A second method of understanding $\hat{S}_{z,j}$ is to consider a possible experimental implementation of our proposal as detailed in Sec. III C: if we were to use magnetically sensitive states in our setup, $\hat{S}_{z,j}$ would now refer to the internal states of each atom; for instance, in Sec. III C we discuss the possibility where the atoms in one trap are in the $m_F = 1$ state and the atoms in the second trap are in the $m_F = -1$ state. As an additional side note, Eq. (1) assumes that simple harmonic motion is slow compared to the Larmor precession; therefore, we exclude rotating terms.

In the upcoming sections, we tend to work with $\lambda = g/\omega$, since this is a unitless quantity. As stated earlier, the creation of entanglement between the interferometer and the oscillator can be determined by measuring an EW. We find that the following quantity is an applicable EW for our setup:

$$W = \text{Var}(\hat{J}_x) + \text{Var}(\hat{J}_y + a_y \hat{q} + b_y \hat{p}) + \text{Var}(\hat{J}_z + a_z \hat{q} + b_z \hat{p}). \quad (2)$$

Here, $\hat{J}_\mu = \sum_{j=1}^N \hat{S}_{\mu,j}$ and $\hat{q}(t) = \sqrt{1/(2m\omega)}(\hat{c} + \hat{c}^\dagger)$. The coefficients a_μ and b_μ can be chosen as appropriate. Such spin-oscillator variances allow for the characterization of spin-oscillator correlations. Equation (2) is bounded by W_b for all separable states. If this bound is violated by the experimental state of concern, we can conclude that it is an entangled state.

W_b can be derived using the theorem in the paper by Hofmann and Takeuchi [25]. They showed that for operators $\hat{A}_\mu$ of system A and $\hat{B}_\mu$ of system B , if $\sum_\mu \text{Var}(\hat{A}_\mu) \geq U_A$

and $\sum_{\mu} \text{Var}(\hat{B}_{\mu}) \geq U_B$, then $\sum_{\mu} \text{Var}(\hat{A}_{\mu} + \hat{B}_{\mu}) \geq U_A + U_B$ for any separable state. There is, however, a prerequisite that for each operator $\hat{A}_{\mu}$ there must be a $\hat{B}_{\mu}$. In the case of Eq. (2), $\hat{A}_{\mu} = \hat{J}_{\mu}$ and $\hat{B}_{\mu} = a_{\mu}\hat{q} + b_{\mu}\hat{p}$. It can be seen in Eq. (2) that there is a term with no oscillator operators and composed of only the spin operator $\hat{J}_x$; we do not measure correlations between $\hat{J}_x$ and the oscillator operators because the mean of these correlations are zero, at least for the noiseless entangled state in Eq. (16). In order to satisfy the requirement of every $\hat{A}_{\mu}$ operator having its counterpart $\hat{B}_{\mu}$, we started with the term $\text{Var}(\hat{J}_x + a_x\hat{q} + b_x\hat{p})$ and set the coefficients a_x and b_x to zero *after* the derivation of the bound. We lay out this derivation here.

Starting with $\tilde{W} = \sum_{\mu} \text{Var}(\hat{J}_{\mu} + a_{\mu}\hat{q} + b_{\mu}\hat{p})$, where the summation $\sum_{\mu}$ is over the x , y , and z components, we follow Ref. [25] to arrive at the general bound

$$\tilde{W} \geq \left[\sum_{\mu} \text{Var}(\hat{J}_{\mu}) \right]_{\min} + \left[\sum_{\mu} \text{Var}(a_{\mu}\hat{q} + b_{\mu}\hat{p}) \right]_{\min}. \quad (3)$$

To obtain the witness in Eq. (2), we set $a_x = 0$ and $b_x = 0$ on the left- and right-hand sides of this inequality.

The oscillator-operator bound can be derived as follows: We have

$$\text{Var}(A) + \text{Var}(B) = \text{Var}(A) + \frac{\text{Var}(A)\text{Var}(B)}{\text{Var}(A)}, \quad (4)$$

for which we can apply the Heisenberg uncertain principle

$$\text{Var}(A)\text{Var}(B) \geq \frac{1}{4} |\langle [A, B] \rangle|^2 \quad (5)$$

and minimize the right-hand side to give

$$\text{Var}(A) + \text{Var}(B) \geq |\langle [A, B] \rangle|. \quad (6)$$

Setting $\hat{A} = a_y\hat{q} + b_y\hat{p}$ and $\hat{B} = a_z\hat{q} + b_z\hat{p}$, we then find that the sum of the oscillator-operator variances is bounded by

$$\sum_{\mu \neq x} \text{Var}(a_{\mu}\hat{q} + b_{\mu}\hat{p}) \geq |a_y b_z - a_z b_y|. \quad (7)$$

In analogy to a derivation in Ref. [25], the general bound for the spin operators can be obtained as follows:

$$\sum_{\mu} \text{Var}(\hat{J}_{\mu}) = \sum_{\mu} (\langle \hat{J}_{\mu}^2 \rangle - \langle \hat{J}_{\mu} \rangle^2). \quad (8)$$

With $\sum_{\mu} \langle \hat{J}_{\mu}^2 \rangle = j(j+1)$ and $\sum_{\mu} \langle \hat{J}_{\mu} \rangle^2 \leq j^2$, we have

$$\sum_{\mu} \text{Var}(\hat{J}_{\mu}) \geq j. \quad (9)$$

The question now arises as to the exact value of j . For a fully symmetric state, we have

$$\sum_{\mu} \text{Var}(\hat{J}_{\mu}) \geq \frac{N}{2}. \quad (10)$$

Here, j takes the maximum possible value. Equation (10) can also be derived from the fact that $\sum_{\mu} [\text{Var}(\hat{\sigma}_{\mu})] \geq 2$ (shown in Ref. [25]),

$$\left[\sum_{\mu} \text{Var}(\hat{J}_{\mu}) \right]_{\min} = N \left[\sum_{\mu} \text{Var}(\hat{S}_{\mu}) \right]_{\min} = \frac{N}{2}. \quad (11)$$

The most general case will be

$$\sum_{\mu} \text{Var}(\hat{J}_{\mu}) \geq [j]_{\min, \exp}, \quad (12)$$

where $[j]_{\min, \exp}$ is the minimum value that j can take at the time of measurement. This gives the most general bound

$$W_b = [j]_{\min, \exp} + |a_y b_z - a_z b_y|. \quad (13)$$

For a separable state, the condition that $W \geq W_b$ will be satisfied. If we evaluate W with respect to a possibly entangled state in order to obtain the value W_{en} and find that $W_{\text{en}} < W_b$, then the bound is violated and we can conclude that the state is indeed entangled. In an experimental setting, W_b is a theoretical value that we have at hand, while W_{en} is the quantity that we estimate through measurement.

When we investigate the violation of the EW, we consider the quantity

$$W_{\text{ratio}} = [(W_b - W_{\text{en}})/W_b]_{\max}; \quad (14)$$

the EW is violated if $W_{\text{ratio}} > 0$. Since the imprecision is set by $W_b - W_{\text{en}}$, W_{ratio} informs us of the number of measurements n_{meas} required to determine the existence of entanglement, since $n_{\text{meas}} \sim W_{\text{ratio}}^{-2}$. A key step in this process is determining the coefficients a_{μ} and b_{μ} , where we attempt to minimize the variance of the EW. When $|W_b - W_{\text{en}}| \ll |W_b|$, finding a_{μ} and b_{μ} that maximizes $|W_b - W_{\text{en}}|$ suffices. Practically, however, finding $|W_b - W_{\text{en}}|_{\max}$ is complicated due to the absolute value term given by $|a_y b_z - a_z b_y|$ (for more details about this, see Appendix B). Therefore, in this paper we minimize the value W_{en} ; that is, we solve for a_{μ} and b_{μ} such that $\partial_{a_{\mu}}(W_{\text{en}}) = 0$ and $\partial_{b_{\mu}}(W_{\text{en}}) = 0$.

a_{μ} and b_{μ} contribute to obtaining $W_{\text{en}, \min}$ by reducing the value of the variance terms in Eq. (2). In the most general case, an expansion of Eq. (2) gives

$$\begin{aligned} W = & \sum_{\mu=x,y,z} \text{Var}(\hat{J}_{\mu}) + \sum_{\mu=y,z} (a_{\mu}^2 \langle \hat{q}^2 \rangle + b_{\mu} \langle \hat{p}^2 \rangle + a_{\mu} \langle \hat{J}_{\mu} \hat{q} \\ & + \hat{q} \hat{J}_{\mu} \rangle + b_{\mu} \langle \hat{J}_{\mu} \hat{p} + \hat{p} \hat{J}_{\mu} \rangle + a_{\mu} b_{\mu} \langle \hat{q}, \hat{p} \rangle \\ & - 2 \langle a_{\mu} \hat{q} + b_{\mu} \hat{p} \rangle \langle \hat{J}_{\mu} \rangle - \langle a_{\mu} \hat{q} + b_{\mu} \hat{p} \rangle^2). \end{aligned} \quad (15)$$

To obtain W_{en} , the expectation value indicated by $\langle \rangle$ is taken by evaluating the operators between the appropriate entangled state denoted by $|\psi_{\text{en}}\rangle$. Observing the terms of W , we can see that a_{μ} and b_{μ} characterize the contribution of covariances to the value of the witness; for instance, there are terms such as $a_{\mu} \langle \hat{J}_{\mu} \hat{q} + \hat{q} \hat{J}_{\mu} \rangle$ and $b_{\mu} \langle \hat{J}_{\mu} \hat{p} + \hat{p} \hat{J}_{\mu} \rangle$. Additionally, as described in the upcoming sections, the spin and oscillator operators get correlated with each other, which can be seen when investigating their evolution in the Heisenberg picture. Therefore, in optimizing for the values of a_{μ} and b_{μ} , we are primarily controlling the contribution of these covariances and correlations, such that we obtain $W_{\text{en}, \min}$.

We explore this EW in the context of three scenarios; when the systems are noiseless, when there is thermal noise, and when there is atomic dephasing.

B. Noiseless entanglement witness

First, we investigate the violation for the noiseless scenario. Here, the atomic states start in the $\hat{J}_x$ eigenstate

$|++\dots\rangle$ and the oscillator in the ground state $|0\rangle$ [1]; after evolving under Eq. (1), the final entangled state is given by [1]

$$|\psi_{\text{en}}\rangle = D^\dagger(\lambda\hat{J}_z)e^{-i\omega\hat{c}^\dagger\hat{c}}D(\lambda\hat{J}_z)|++\dots\rangle|0\rangle. \quad (16)$$

Compared with Ref. [1], we have dropped the $\exp[-i(g^2t/\omega)\hat{J}_z^2]$ squeezing term, because it is likely to be of $O(1)$ given the size of the coupling g .

We can understand the evolution of the system described by Eq. (1) within the Heisenberg picture. The Heisenberg operator $\hat{c}(t)$ is given by

$$\hat{c}(t) = e^{-i\omega t}\hat{c}(0) - i\int_0^t dt' g\hat{J}_ze^{-i\omega(t-t')}. \quad (17)$$

The spin operators $\hat{J}_x$ and $\hat{J}_y$ rotate about the z axis by angle $\epsilon(t) = g\int_0^t dt' [\hat{c}(t') + \hat{c}(t')^\dagger]$, due to the term $g\hat{J}_z(\hat{c} + \hat{c}^\dagger)$ in the Hamiltonian; this is visually shown in Fig. 1. We find the operator evolution of $\hat{J}_x$, $\hat{J}_y$, $\hat{q}$, and $\hat{p}$ to be

$$\hat{J}_x(t) \approx \hat{J}_x \cos(\theta_{\hat{p}} + \theta_{\hat{q}}) - \hat{J}_y \sin(\theta_{\hat{p}} + \theta_{\hat{q}}), \quad (18)$$

$$\hat{J}_y(t) \approx \hat{J}_y \cos(\theta_{\hat{p}} + \theta_{\hat{q}}) + \hat{J}_x \sin(\theta_{\hat{p}} + \theta_{\hat{q}}), \quad (19)$$

$$\hat{q}(t) = \hat{q} \cos(\omega t) + \frac{\hat{p}}{m\omega} \sin(\omega t) - \sqrt{\frac{2}{m\omega}} \lambda \hat{J}_z [1 - \cos(\omega t)], \quad (20)$$

$$\hat{p}(t) = \hat{p} \cos(\omega t) - m\omega \hat{q} \sin(\omega t) - \sqrt{2m\omega} \lambda \hat{J}_z \sin(\omega t), \quad (21)$$

where

$$\theta_{\hat{p}} = \sqrt{\frac{2}{m\omega}} \lambda \hat{p} [1 - \cos(\omega t)], \quad (22)$$

$$\theta_{\hat{q}} = \sqrt{2m\omega} \lambda \hat{q} \sin(\omega t). \quad (23)$$

Note that we have dropped the squeezing term, which goes as $\lambda^2 \hat{J}_z^2$ inside the cosine and sine functions of Eqs. (18) and (19). As observed in these equations, the operators become correlated with each other [for instance, $\hat{q}(t)$ and $\hat{J}_z$]. Optimizing the values of a_μ and b_μ translates to controlling the contributions of these correlations such that the variances of Eq. (2) are minimized for the entangled state. For instance, if we consider the operator combination $\hat{J}_y(t) + a_y \hat{q}(t) + b_y \hat{p}(t)$, to first order in λ (where $\lambda = g/\omega$) for simplicity, we have

$$\begin{aligned} & \hat{J}_y(t) + a_y \hat{q}(t) + b_y \hat{p}(t) \\ & \approx \hat{J}_y + \lambda \hat{J}_x \left[\sqrt{\frac{2}{m\omega}} [\hat{p} - \hat{p} \cos(\omega t)] + \sqrt{2m\omega} \hat{q} \sin(\omega t) \right] \\ & + a_y \left[\hat{q} \cos(\omega t) + \frac{\hat{p} \sin(\omega t)}{m\omega} \right] + b_y [\hat{p} \cos(\omega t) \\ & - m\omega \hat{q} \sin(\omega t)] + O(\lambda \hat{J}_z). \end{aligned} \quad (24)$$

The first line of Eq. (24) is the Heisenberg-evolved operator $\hat{J}_y(t)$ expanded to $O(\lambda)$ (with the squeezing term dropped). The term $a_y \hat{q}(t) + b_y \hat{p}(t)$ added to $\hat{J}_y(t)$ contributes the coefficients a_y and b_y ; given the explicit form of $a_y \hat{q}(t) + b_y \hat{p}(t)$ as shown in the second line of Eq. (24), it seems plausible that values for a_y and b_y can be chosen such that the contribution to $\hat{J}_y(t)$ from $\hat{J}_x$ can be canceled. Note that, since we are starting

in the state $|++\dots\rangle$, the $\hat{J}_x$ operators here evaluate to $N/2$ and the $O(\lambda \hat{J}_z)$ terms evaluate to 0.

If we proceed with solving for a_μ and b_μ , we see that this does indeed occur. We first expand various products of the operators $[\hat{J}_x(t)\hat{q}(t)$, etc.] to $O(\lambda^2)$; $O(\lambda^2)$ should be appropriate, since the only other parameter that can greatly “cancel out” the effects of λ is the number of atoms N and the highest order of N is N^2 (which arises from the squaring of the spin operators). Once we have these Heisenberg-evolved operators and their products, we can evaluate their expectation values with respect to $|++\dots\rangle|0\rangle$ (which is the remainder of $|\psi_{\text{en}}\rangle$) to obtain W_{en} .

Now we can minimize W_{en} with respect to a_μ and b_μ . We find that

$$a_{y,0,\lambda} = -\lambda N \sqrt{\frac{m\omega}{2}} \sin(\omega t), \quad (25)$$

$$b_{y,0,\lambda} = \lambda N \sqrt{\frac{1}{2m\omega}} [1 - \cos(\omega t)], \quad (26)$$

$$a_{z,0,\lambda} = \lambda N \sqrt{\frac{m\omega}{2}} [1 - \cos(\omega t)], \quad (27)$$

$$b_{z,0,\lambda} = \lambda N \sqrt{\frac{1}{2m\omega}} \sin(\omega t). \quad (28)$$

As can be seen, these coefficients and, thereby, the EW of Eq. (2), depend on the following experimental parameters: the coupling λ , the number of atoms N , the mass of the oscillator m , oscillator frequency ω , and the interaction time t . Furthermore, these coefficients satisfy $\mathbf{a} \cdot \mathbf{b} = 0$.

We can substitute these coefficients into Eq. (24) to see how they can be used to cancel out correlations between operators. If we were to set $\langle ++\dots | \hat{J}_x | ++\dots \rangle = N/2$, we find that $\langle \psi_{\text{en}} | \hat{J}_y + a_y \hat{q} + b_y \hat{p} | \psi_{\text{en}} \rangle = \langle 0 | \langle ++\dots | \hat{J}_y | ++\dots \rangle | 0 \rangle$.

We can now finally obtain the bound

$$W_{b,0} = \frac{N}{2} + \lambda^2 N^2 [1 - \cos(\omega t)] \quad (29)$$

and the entangled-state value

$$W_{\text{en},0} = \frac{N}{2} - \frac{\lambda^2 N^2}{2} [1 - \cos(\omega t)]. \quad (30)$$

We see that $|\psi\rangle_{\text{en}}$ violates the EW bound by a value of $(3/2)\lambda^2 N^2 [1 - \cos(\omega t)]$. It must be emphasized again that these calculations hold for $\lambda \ll 1$. The exact limit on λ is such that Eq. (30) is always positive; that is, the result is valid for $\lambda < 1/\sqrt{2N}$, which for $N = 10^6$ would be $\lambda < 7 \times 10^{-4}$. Figure 2 shows a plot of the violation (black line); it is highest at $t = \pi/\omega$, which is consistent with the results in Ref. [1], where the oscillator is completely entangled with the atoms at $t = \pi/\omega$.

Note that $W_{\text{en},0}$ and $W_{b,0}$ are obtained by minimizing the value of the EW for state $|\psi\rangle_{\text{en}}$. While this means that Eq. (30) is the lowest value for the EW that can be obtained with $|\psi\rangle_{\text{en}}$, it may not necessarily provide us with the greatest difference between the separable-state bound and the entangled-state value, which would be obtained by maximizing the quantity $W_b - W_{\text{en}}$ with respect to a_μ and b_μ . The reason that we have

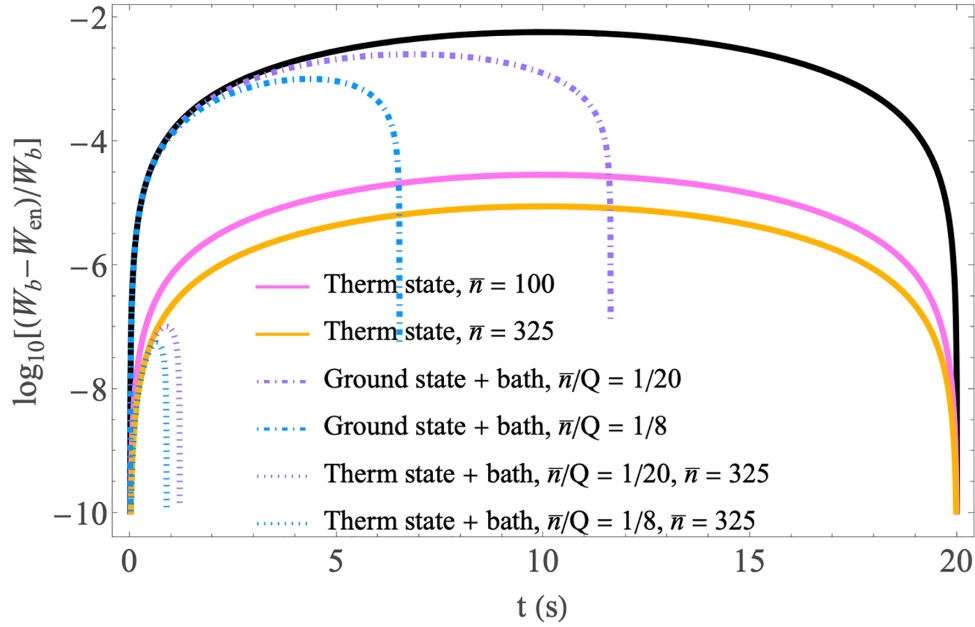


FIG. 2. The black line depicts the noiseless case. We have set $\lambda = 10^{-4.5}$, $N = 10^6$, and $\omega = 2\pi/20$ rad/s. White noise depends only on $\bar{n}/Q$. Thus, the scenario where the oscillator starts in a ground state and then remains in contact with a bath can be characterized solely using $\bar{n}/Q$. However, a value for $\bar{n}$ must be set when starting in a thermal state. Since the condition $\bar{n}\lambda^2 \ll 1$ must be satisfied, we consider $\bar{n} = 325$ to be the highest appropriate value. Purple lines correspond to $\bar{n}/Q = 1/20$ and blue lines correspond to $\bar{n}/Q = 1/8$. As can be seen in the plots, the lower the value of $\bar{n}/Q$, the better the violation. Note that introducing a bath results in the violation *not* being seen for the entire oscillator period; where the plots get truncated is when $W_b - W_{\text{en}} < 0$. In Fig. 8, we present the same graph split into two panels in case the reader may prefer fewer plots in a single figure.

not presented results from the optimization of $W_b - W_{\text{en}}$ is due to certain mathematical complexities.

Also note that, though we present the coefficients to $O(\lambda)$ in Eqs. (25)–(28), when calculating $W_{b,0}$ and $W_{\text{en},0}$, we used the nonsimplified expressions given in Appendix B 1.

If experimentally measuring the entanglement witness, measurements of *all* the operators of a single variance term must be made in a single experimental run; that is, in a single run we must measure all the operators in the set $\{\hat{J}_y, \hat{J}_y^2, \hat{q}^2, \hat{q}\hat{J}_y + \hat{J}_y\hat{q}, \hat{p}\hat{J}_y + \hat{J}_y\hat{p}\}$, with analogous sets for the $\mu = x$ and $\mu = z$ variance terms. After the completion of all runs, appropriate averaging over these runs is conducted to obtain $\langle \hat{J}_x \rangle$, $\langle \hat{J}_x^2 \rangle$, etc. Due to the existence of noise, these averages are not purely the expectation values of operators taken over a quantum state. We will also be averaging over noise, so it is in fact more accurate to state that the final experimental values we obtain are those for $\langle \langle \hat{J}_x \rangle \rangle$, $\langle \langle \hat{J}_x^2 \rangle \rangle$, etc., where double-angle brackets denote the inclusion of classical noise. We look at the cases of thermal noise and atomic dephasing in the upcoming sections.

C. Entanglement witness with a thermal state and white noise bath

We can now consider the case of thermal noise. There are three scenarios that can be considered; the first when the oscillator starts in a thermal state, the second when the oscillator starts in a thermal state and remains in contact with a white noise bath, and the third when the oscillator starts in the *ground* state and remains in contact with a bath.

For a thermal state $\hat{\rho} = (1 - e^{-\beta\omega}) \sum_{n=0}^{\infty} e^{-n\beta\omega} |n\rangle \langle n|$ [26], introducing the initial-thermal-state-only effect into the previous noiseless calculation is rather straightforward. For average thermal occupancy $\bar{n}$, we need only replace the expectation values of any $\hat{q}^2$ and $\hat{p}^2$ terms that appear in the evaluation of the Heisenberg picture operators as follows:

$$\langle q^2 \rangle : \frac{1}{2m\omega} \rightarrow \frac{1}{2m\omega} (2\bar{n} + 1), \quad (31)$$

$$\langle p^2 \rangle : \frac{m\omega}{2} \rightarrow \frac{m\omega}{2} (2\bar{n} + 1). \quad (32)$$

With this, the updated coefficients to $O(\lambda)$ are

$$a_{y,\bar{n}} = -\lambda N \sqrt{\frac{m\omega}{2}} \sin(\omega t), \quad (33)$$

$$b_{y,\bar{n}} = \lambda N \sqrt{\frac{1}{2m\omega}} [1 - \cos(\omega t)], \quad (34)$$

$$a_{z,\bar{n}} = \frac{\lambda N}{2\bar{n} + 1} \sqrt{\frac{m\omega}{2}} [1 - \cos(\omega t)], \quad (35)$$

$$b_{z,\bar{n}} = \frac{\lambda N}{2\bar{n} + 1} \sqrt{\frac{1}{2m\omega}} \sin(\omega t). \quad (36)$$

The next order of these coefficients is $O(\lambda^3)$. Comparing with Eqs. (25)–(28), it can be seen that $a_{z,\bar{n}}$ and $b_{z,\bar{n}}$ have been modified by a factor of $1/(2\bar{n} + 1)$. However, they no longer satisfy $\mathbf{a} \cdot \mathbf{b} = 0$. As explained previously, and as can be seen by their similar forms, these coefficients work together with the covariances and correlations to minimize the value of W_{en} when starting in a thermal state.

The bound to $O(\lambda^2)$ is

$$W_{b,\bar{n}} = \frac{N}{2} + \frac{1}{2\bar{n}+1} \lambda^2 N^2 [1 - \cos(\omega t)] \quad (37)$$

and the entangled-state value of the EW to $O(\lambda^2)$ is

$$W_{\text{en},\bar{n}} = \frac{N}{2} - \frac{1}{2\bar{n}+1} \frac{\lambda^2 N^2}{2} [1 - \cos(\omega t)]. \quad (38)$$

The expansion to $O(\lambda^2)$ is only valid if $\bar{n}\lambda^2 \ll 1$; larger $\bar{n}\lambda^2$ leads to key corrections associated with atomic phase evolution of order unity. Plots for different values of $\bar{n}$ are shown in Fig. 2; the higher the value of $\bar{n}$, the weaker the violation.

Next, we consider a scenario where the oscillator is in continuous contact with white noise, which approximates a thermal bath at high temperature. The Hamiltonian is now given by

$$H = \omega \hat{c}^\dagger \hat{c} + [gJ_z + F_{\text{in}}(t)](\hat{c} + \hat{c}^\dagger), \quad (39)$$

where $F_{\text{in}}(t)$ is a classical white noise force; this Hamiltonian is simply that of Eq. (1) to which the effect from $F_{\text{in}}(t)$ has been added. The autocorrelation function is a delta function with an amplitude set by the fluctuation-dissipation theorem

$$\langle F_{\text{in}}(t) F_{\text{in}}(t') \rangle = 2\bar{n}\gamma \delta(t - t'). \quad (40)$$

Proceeding to understand within the Heisenberg picture as previously, we can solve for $\hat{c}(t)$ to obtain

$$\hat{c}(t) = e^{-i\omega t} \hat{c}(0) - i \int_0^t dt' (g\hat{J}_z + F_{\text{in}}(t')) e^{-i\omega(t-t')}. \quad (41)$$

With $\hat{q}(t) = q_{\text{zpf}}[\hat{c}(t) + \hat{c}^\dagger(t)]$ and $\hat{p}(t) = ip_{\text{zpf}}[-\hat{c}(t) + \hat{c}^\dagger(t)]$ (where $q_{\text{zpf}} = \sqrt{1/(2m\omega)}$ and $p_{\text{zpf}} = \sqrt{m\omega/2}$), the noise contributions to $\hat{q}(t)$ and $\hat{p}(t)$ can be expressed as

$$Q(t) = -2q_{\text{zpf}} \int_0^t dt' F_{\text{in}}(t') \sin[\omega(t - t')] \quad (42)$$

and

$$P(t) = -2p_{\text{zpf}} \int_0^t dt' F_{\text{in}}(t') \cos[\omega(t - t')]. \quad (43)$$

Further, $\hat{J}_x$ and $\hat{J}_y$ will rotate about the z axis by an angle $\xi(t) = g \int_0^t dt' [\hat{c}(t') + \hat{c}^\dagger(t')]^\dagger$ as follows:

$$\hat{J}_x(t) = \hat{J}_x \cos[\xi(t)] - \hat{J}_y \sin[\xi(t)], \quad (44)$$

$$\hat{J}_y(t) = \hat{J}_y \cos[\xi(t)] + \hat{J}_x \sin[\xi(t)]. \quad (45)$$

The white noise contribution to this angle is given by

$$\begin{aligned} \Xi(t) &= \sqrt{2m\omega g} \int_0^t dt' Q(t') \\ &= -2g \int_0^t \int_0^{t'} dt'' dt' F_{\text{in}}(t'') \sin[\omega(t' - t'')]. \end{aligned} \quad (46)$$

Note that $Q(t)$, $P(t)$, and $\Xi(t)$ are not operators since they arise from the classical noise $F_{\text{in}}(t)$; therefore, their covariances consist of averages over classical noise and are hence represented with the notation $\langle \langle \dots \rangle \rangle$. For instance, $\langle \langle \Xi(t) \Xi(t) \rangle \rangle$ contributes to the noise in the EW term $\Delta \langle \langle \hat{J}_x \rangle \rangle^2$ and $\langle \langle Q(t) Q(t) \rangle \rangle$ contributes to the noise in $\langle \langle \hat{q}^2 \rangle \rangle$ (see Appendix B 2). Therefore, the expectation values that compose

the EW must be modified by adding the noise contributions $\Delta \langle \langle \hat{J}_x \rangle \rangle^2$, $\Delta \langle \langle \hat{q}^2 \rangle \rangle$, $\Delta \langle \langle \hat{p}^2 \rangle \rangle$, $\Delta \langle \langle \hat{p} \hat{J}_y + \hat{J}_y \hat{p} \rangle \rangle$, $\Delta \langle \langle \hat{q} \hat{J}_y + \hat{J}_y \hat{q} \rangle \rangle$, and $\Delta \langle \langle \hat{q} \hat{p} + \hat{q} \hat{p} \rangle \rangle$; an appropriate linear combination of these gives $\Delta W_{\bar{n},\text{noise}}$. Thus, we now have, $W_{\text{en},\bar{n},\text{noise}} = W_{\text{en},\bar{n}} + \Delta W_{\bar{n},\text{noise}}$, where

$$\begin{aligned} \Delta W_{\bar{n},\text{noise}} &= \frac{N^2}{4} \frac{2\lambda^2 \bar{n}}{Q} [6\omega t - 8 \sin(\omega t) + \sin(2\omega t)] \\ &\quad - a_y \sqrt{\frac{1}{2m\omega}} \frac{N}{2} \frac{32\lambda \bar{n}}{Q} \sin^4\left(\frac{\omega t}{2}\right) \\ &\quad + b_y \sqrt{\frac{m\omega}{2}} \frac{N}{2} \frac{16\lambda \bar{n}}{Q} \left[\frac{\omega t}{2} - \sin(\omega t) + \frac{\sin(2\omega t)}{4} \right] \\ &\quad + (a_y^2 + a_z^2) \frac{1}{2m\omega} \frac{2\bar{n}}{Q} [2\omega t - \sin(2\omega t)] \\ &\quad + (b_y^2 + b_z^2) \frac{m\omega}{2} \frac{2\bar{n}}{Q} [2\omega t + \sin(2\omega t)] \\ &\quad + (a_y b_y + a_z b_z) \frac{4\bar{n}}{Q} \sin^2(\omega t), \end{aligned} \quad (47)$$

with the coefficients being those in Eqs. (33)–(36). Note that the terms where λ appears have been truncated to $O(\lambda^2)$. The term $\langle \langle \hat{J}_x^2 + \hat{J}_y^2 \rangle \rangle$ is unaffected by noise as it commutes through the Hamiltonian. Therefore, for the spin variances, only the term $\Delta \langle \langle \hat{J}_x \rangle \rangle^2$ comes into play; starting in $\hat{J}_x$ eigenstate means that, to $O(\lambda^2)$, there is no noise contribution from $\langle \langle \hat{J}_y \rangle \rangle$ either.

Using the previous bound in Eq. (37), we plot the EW violation in Fig. 2 (dotted lines). For violation to take place, the condition that

$$\frac{Q}{\bar{n}} > \frac{\pi [30 + 88\bar{n}(1 + \bar{n})]}{6(1 + 2\bar{n})} \sim 7\pi \bar{n} \quad (48)$$

is required when $t = \pi/\omega$. The bound is approximately $7\pi \bar{n}$ for $\bar{n} \gg 1$.

We can further consider the scenario where the oscillator is cooled to its ground state at the start, but remains in contact with a bath for the rest of time; this is given by $W_{\text{en},0,\text{noise}} = W_{\text{en},0} + \Delta W_{\bar{n},\text{noise}}$, with the coefficients of Eqs. (25)–(28) being used in Eq. (47) (see Appendix B 2 for “reoptimized” coefficients). The dot-dashed lines in Fig. 2 show the violation for different values of $\bar{n}/Q$. The condition for violation to take place is now

$$\frac{Q}{\bar{n}} > \frac{\csc\left(\frac{\omega t}{2}\right)^2}{2} [6\omega t - 4\omega t \cos(\omega t) - 4 \sin(\omega t) + \sin(2\omega t)], \quad (49)$$

which for $t = \pi/\omega$ is $\frac{Q}{\bar{n}} > 5\pi$. This is $O(\bar{n})$ better than for an initial thermal state. Nevertheless, by the fact that $n_{\text{meas}} \sim [(W_b - W_{\text{en}})/W_b]^{-2}$, as discussed in Sec. II A, the number of measurements required to observe violation at a coupling of $10^{-4.5}$ is $\sim 10^{5.2}$.

We add some commentary on the shape of the plots in Fig. 2. If we plot the violation directly (i.e., without taking the logarithm of the value), we will obtain plots with shapes as shown in Fig. 7 for some specific scenarios. In particular, as discussed previously, we find that the ratio of $(W_b - W_{\text{en}})/W_b$ for the noiseless scenario is highest at half a period; as explained in Ref. [1], this is when we expect

the system to have the greatest entanglement. Essentially, the “cosinusoidal” shape describes the system going from being unentangled to entangled to unentangled as the oscillator completes a full period. When we have thermal noise, we note that the logarithmic plots get cutoff—this is because the violation becomes negative once noise takes over.

We now investigate the value of the magnetic coupling in Fig. 1. A large diamagnetic coupling can be attained by using superconducting masses and placing the atoms in different magnetically sensitive states. For instance, we can place one atom in the $m_F = 1$ state and the other in $m_F = -1$. In such a scenario, the coupling would primarily be due to the first-order Zeeman shift, which is linear in B [27]. For a cylinder with $2R = l = 5$ mm, $\chi = -1$, density = 11.34 g/cm³ [28], $\Delta z = 5$ μ m, $s = 8$ mm, $R_0 = 4.5$ cm, $B_{\text{ext}} = 50$ mT, $\omega = 2\pi/20$, and $N = 10^6$, we get $\lambda = 6.3 \times 10^{-4}$. A coupling approximately $\times 10^2$ weaker is available for atomic clock ($m_F = 0$) states. For details on this calculation, see Sec. III.

D. Entanglement witness with atomic dephasing

Atomic dephasing can be introduced to the entangled state by introducing a spin rotation a_j about the z axis for each atom j :

$$|\psi\rangle_{\text{en},\alpha_j} = D^\dagger(\lambda J_z) e^{-i\omega c t} D(\lambda J_z) \times \exp \left[i \sum_j \hat{S}_{z,j} \alpha_j \right] |++\cdots\rangle |0\rangle. \quad (50)$$

We evaluate each of the expectation value terms in Eq. (15) for $|\psi\rangle_{\text{en},\alpha_j}$; as previously, we evolve the operators in the Heisenberg picture first, with appropriate modifications made to Eqs. (18)–(21) as dictated by the spin rotation α_j , after which we evaluate with respect to $|++\cdots\rangle |0\rangle$. Taking each α_j to be independent of each other and having a Gaussian distribution of

$$p(\alpha_j) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp \left[-\frac{\alpha_j^2}{2\sigma^2} \right], \quad (51)$$

we obtain the classical mean of each of these terms (that is, expressions for $\langle\langle \hat{J}_x \rangle\rangle$, $\langle\langle \hat{J}_x^2 \rangle\rangle$, etc.) with respect to the angles α_j . Here, the notation $\langle\langle \cdots \rangle\rangle$ refers to the process of first evaluating the expectation value of the operators, and afterward the classical mean.

Compared to the noiseless case, we have the following change Δ to the variances of the spin operators for $|\psi_{\text{en}}\rangle$, to $O(\sigma^2)$:

$$\Delta \text{Var}(\hat{J}_x) = \left\{ \frac{N}{4} - \frac{\lambda^2 N}{2} [1 - \cos(\omega t)] \right\} \sigma^2, \quad (52)$$

$$\Delta \text{Var}(\hat{J}_y) = \left\{ \frac{\lambda^2 N}{2} [1 - \cos(\omega t)] - \frac{\lambda^2 N^2}{2} [1 - \cos(\omega t)] \right\} \sigma^2. \quad (53)$$

Since $\hat{J}_z$ commutes with the dephasing operators $\hat{S}_{z,j}$, its variance remains intact.

With these, the value of the EW with the inclusion of atomic dephasing is

$$\langle\langle W_{\text{en},\sigma,\text{non-opt}} \rangle\rangle = \frac{N}{2} \left(1 + \frac{\sigma^2}{2} \right) - \frac{\lambda^2 N^2}{2} [1 - \cos(\omega t)]. \quad (54)$$

Note that the expressions are to $O(\sigma^2)$; $\sigma = \sqrt{t/\tau_c}$, where τ_c is the coherence time of interferometer that we require to be large for interferometry; therefore, we expect $\sigma \ll 1$. Here, we have used the same values of a_μ and b_μ as for the noiseless case. The $(N/4)\sigma^2$ difference between Eqs. (54) and (30) arises from dephasing-induced changes to the sum of the spin variances $\sum_\mu \text{Var}(\hat{J}_\mu)$. If we break down the contributions from the different operators, we have

$$\sum_\mu \text{Var}(\hat{J}_\mu) = \frac{N}{2} \left(1 + \frac{\sigma^2}{2} \right) + \frac{\lambda^2 N^2}{2} [1 - \cos(\omega t)] (1 - \sigma^2), \quad (55)$$

while the rest of the operators give

$$\begin{aligned} \langle\langle W_{\text{en},\sigma,\text{non-opt}} \rangle\rangle - \sum_\mu \text{Var}(\hat{J}_\mu) \\ = -\lambda^2 N^2 [1 - \cos(\omega t)] \left(1 - \frac{\sigma^2}{2} \right). \end{aligned} \quad (56)$$

It can be seen that $\langle\langle W_{\text{en},\sigma,\text{non-opt}} \rangle\rangle > W_{\text{en},0}$ [given in Eq. (30)], showing that atomic dephasing can make it harder to violate the separable-state bound. Since we used the same coefficients as for the noiseless case, the EW bound is partially given by $W_{b,0}$ of the noiseless case,

$$\begin{aligned} \langle\langle W_{b,\sigma,\text{non-opt}} \rangle\rangle = [j]_{\text{min,exp,start}} \left(1 - \frac{\sigma^2}{2} \right) \\ + \lambda^2 N^2 [1 - \cos(\omega t)]. \end{aligned} \quad (57)$$

$[j]_{\text{min,exp,start}}$ is the minimum value of the total spin variances for a separable state as can be experimentally achieved at the *start* of the experiment; its occurrence in this equation, which reduces the violation by $\sigma^2/2$, indicates the value we expect it to reduce to just before the measurement as atomic dephasing reduces the total j quantum number.

We can further reoptimize the coefficients a_μ and b_μ for the case of dephasing by applying the constrain that $a_{y,\sigma} = -(a_{z,\sigma} b_{z,\sigma})/b_{y,\sigma}$. To $O(\sigma^2)$ and $O(\lambda)$, we have

$$a_{y,\sigma} = -\lambda N \sqrt{\frac{m\omega}{2}} \sin(\omega t) \frac{\{4 - \sigma^2 [1 + \cos(\omega t)]\}}{4}, \quad (58)$$

$$a_{z,\sigma} = \lambda N \sqrt{\frac{m\omega}{2}} [1 - \cos(\omega t)] \frac{\{4 - \sigma^2 [1 + \cos(\omega t)]\}}{4}, \quad (59)$$

$$b_{y,\sigma} = \lambda N \sqrt{\frac{1}{2m\omega}} [1 - \cos(\omega t)] \frac{\{4 - \sigma^2 [1 - \cos(\omega t)]\}}{4}, \quad (60)$$

$$b_{z,\sigma} = \lambda N \sqrt{\frac{1}{2m\omega}} \sin(\omega t) \frac{\{4 - \sigma^2 [1 - \cos(\omega t)]\}}{4}. \quad (61)$$

Note that in the succeeding calculations, we used less simplified expressions; we do not present them here due to their

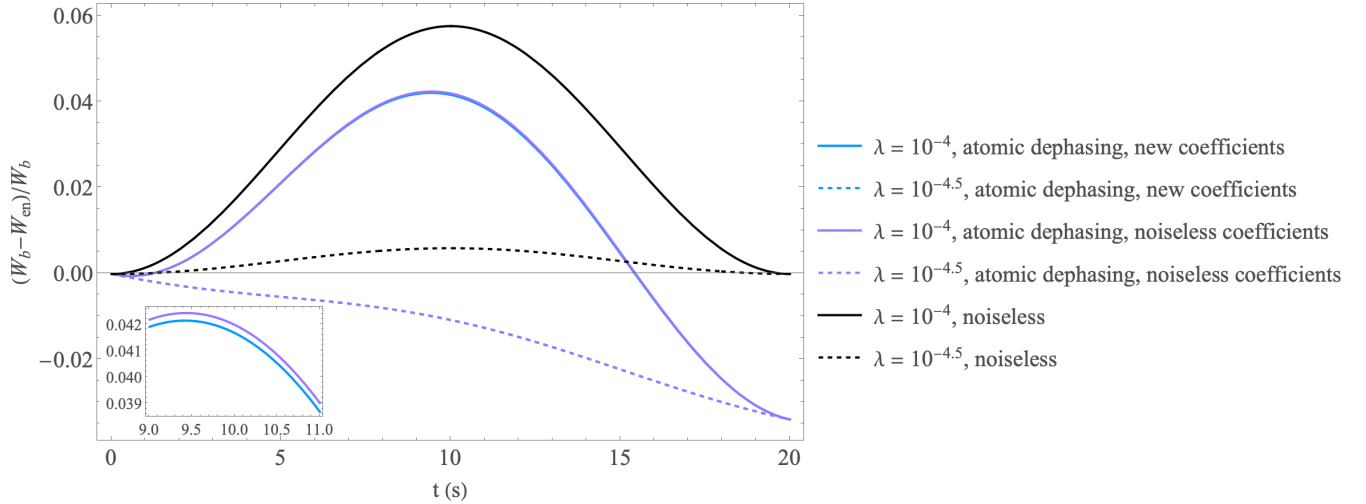


FIG. 3. Here, $\omega = 2\pi/20$ rad/s, $\sigma^2 = t/600$, $N = 10^6$, and we have set $[j]_{\min, \exp, \text{start}} = N/2$. Note that the coupling satisfies $\lambda < 7 \times 10^{-4}$. As expected, dephasing reduces the violation.

complexity. As can be verified, these coefficients satisfy $\mathbf{a} \cdot \mathbf{b} = 0$. The bound is now

$$\langle\langle W_{b,\sigma} \rangle\rangle = [j]_{\min, \exp, \text{start}} \left(1 - \frac{\sigma^2}{2}\right) + \lambda^2 N^2 [1 - \cos(\omega t)] \left(1 - \frac{\sigma^2}{2}\right). \quad (62)$$

The entangled-state value for the optimized coefficients remains the same as that of Eq. (54) to $O(\lambda^2)$ and $O(\sigma^2)$; that is, the “optimized” coefficients give the same entangled-state value as when using noiseless coefficients [given in Eq. (54)] to $O(\sigma^2)$. Thus, we may as well use the noiseless coefficients for the case of atomic dephasing, because the “optimized” coefficients only serve to decrease $\langle\langle W_{b,\sigma} \rangle\rangle$ while keeping $\langle\langle W_{\text{en}, \sigma, \text{non-opt}} \rangle\rangle$ unchanged.

Due to our initial expansions to $O(\lambda^2)$, there is a maximum value that λ can take such that Eq. (54) is always positive; this condition is given by

$$\lambda < \sqrt{\frac{(1 + \frac{\sigma^2}{2})}{2N}}. \quad (63)$$

For $\sigma = \sqrt{t/600}$, $t = 20$ s, and $N = 10^6$, it is required that $\lambda < 7 \times 10^{-4}$. We have plotted the violation in Fig. 3. As expected, atomic dephasing reduces the violation.

The heating and atomic dephasing models we considered allow many practical sources of noise to be mapped. For example, seismic noise can be mapped to our heating model. The interferometer’s coherence time will be affected by noise in the magnetic field, nonuniform density/impurities in the material (which would change the generated magnetic field), and atom trap noise, and can thus be mapped to our dephasing model.

Other sources of noise, such as parametric noise, $1/f$ noise, thermoelastic noise, and suspension noise are not captured by our models. Since this paper is in the context of an experimental-theoretical collaboration, we expect to address explicit physical sources of noise in future work, once prototypes of our apparatus have been built.

III. MAGNETIC INTERACTIONS

We now explain in detail the interaction between an atom interferometer and a mechanical oscillator, and how to obtain the magnetic coupling between them.

Atom interferometers measure the phase difference accumulated by the components of an atom’s wave function that travel along different trajectories between two points in spacetime. In general, the phase accumulated by each component is the integral of the Lagrangian along the classical trajectory it traverses. In an optical lattice atom interferometer, the trajectories include a hold period where the atom is in a superposition of being trapped at different sites in an optical lattice. When this hold period is much longer than the time for the trajectories to separate and recombine, the dominant contribution to the relative phase is the integral of the force, $F_a(t)$, across the atoms at r_a through the hold time, T , multiplied by the atom separation Δr_a . In units with $\hbar = 1$, the interferometer phase has the following form [22]:

$$\Delta\phi = \Delta r_a \int_0^T dt F_a(t). \quad (64)$$

A. Interactions through the quadratic Zeeman effect

If the atoms are exposed to a spatially varying magnetic field, then a force across the atoms is generated due to the gradient in the Zeeman splitting of the atom’s energy levels. For Cesium-133 atoms in the $m_F = 0$ ground state, the leading order Zeeman effect is quadratic in the magnetic field [27], yielding a force

$$F_a \propto \frac{\partial B^2(r_a)}{\partial r_a}. \quad (65)$$

Thus, atom interferometers can act as magnetometers, measuring the gradient in the magnitude of the magnetic field. If the source of the magnetic field is an oscillator composed of magnetic material, then the force across the atoms will encode the displacement of the oscillator, r_o .

$$F_a(r_o) \propto \frac{\partial B^2(r_o, r_a)}{\partial r_a}. \quad (66)$$

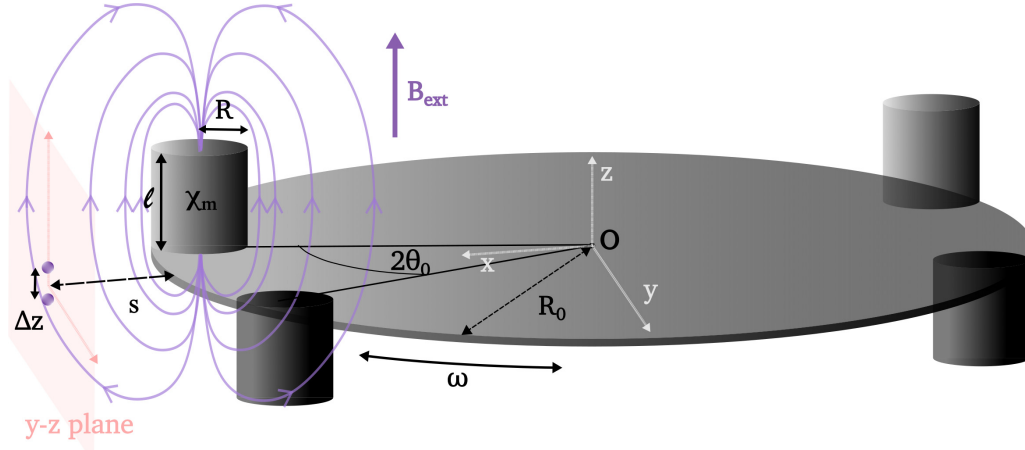


FIG. 4. More advanced version of oscillator-atom setup. The parameter definitions are given in Table I. An external magnetic field $\mathbf{B}_{\text{ext}}$ induces a magnetic field in the cylinders that have magnetic susceptibility χ_m . Note that, while only the field lines for one of the cylinders are shown for presentation purposes, both cylinders next to the atoms will be magnetized in reality. The direction of the induced magnetic field depends on the sign of χ_m . The oscillator rotates in the x - y plane with frequency ω . The cylinders are separated by angle $2\theta_0$. The reason for this configuration is to translate a force along the z direction (as applied to the atoms) to a torque in the x - y plane. The two cylinders on the right are to balance the oscillator.

The atoms will feel a force that is modulated by the oscillator's motion. This results in an interferometer phase that is sensitive to the oscillators position. There will also be a back action force on the oscillator, F_o , from the dependence of the magnetic field on the oscillator's position. This force will be sensitive to the atom's position.

$$F_o(r_a) \propto \frac{\partial B^2(r_o, r_a)}{\partial r_o}. \quad (67)$$

This physics is captured by the interaction potential for small displacements in the atom and oscillator coordinates.

$$U(r_o, r_a) = \phi_0 \frac{\partial^2 B^2(0, 0)}{\partial r_o \partial r_a} r_a r_o. \quad (68)$$

The oscillator motion is now subject to a linear potential that modifies its motion and the potential interacts with the atoms, spatially separated by Δr_a , to accumulate interferometer phase at a rate $\dot{\phi}$,

$$\dot{\phi} = \phi_0 \frac{\partial^2 B^2(0, 0)}{\partial r_o \partial r_a} r_o \Delta r_a. \quad (69)$$

Here, ϕ_0 is an atom's energy difference between the $m_F = 0$ states of the ground state. To understand the magnitude of this interaction achievable with current technologies and how it can be increased, we model a particular experimental setup in the next section.

B. Proposed setup

Section II introduced a simplified version of the torsional pendulum. Since we plan to gradually proceed to a gravity experiment, we investigated a more advanced version of a pendulum as depicted in Fig. 4. The pendulum that will oscillate at frequency ω about the vertical axis with two diamagnetic cylinders with magnetic susceptibility χ_m on its base. The atom interferometer sequence will trap atoms in a superposition of two lattice sites that are horizontally

separated from the pendulum base by a distance, s , and vertically separated above and below the base by $\Delta z/2$; that is, $\Delta r_a = \Delta z$, since our atoms only have a vertical separation. In a uniform external magnetic field, $\mathbf{B}_{\text{ext}}$ (which can, for example, be created by a solenoid), these cylinders will produce an induced magnetic field. The gradients in the induced field will couple to the atom interferometer through the quadratic Zeeman effect to produce an interaction potential that is sensitive to the pendulum's rotation angle and the atom's location.

The magnetic field from a finite length cylinder with uniform magnetization has been calculated in Ref. [31]. The total magnetic field is a superposition of the induced field from each cylinder and the external magnetic field. The resulting phase accumulation rate for a range of atom locations and oscillator rotation angles is displayed in Figs. 5(a) and 5(b). With the atoms centered at $y = z = 0$, the interferometer phase is calculated for the oscillator in classical motion: $\theta(t) = \theta_{\text{max}} \sin(\omega t)$. The result is plotted in Fig. 6 along with the rotation angle of the oscillator. The interferometer phase, and therefore the probability to find the atom in each lattice site, is clearly correlated with the oscillators position. If the oscillator is quantum mechanical and its position and momentum become sufficiently correlated with the atoms, then there is the possibility of observing the system in a nonclassical state.

C. Interaction with a quantum harmonic oscillator

For readability, we reiterate and expand on the physical reasoning behind the combined oscillator-atom Hamiltonian here [1]:

$$H = \omega \hat{c}^\dagger \hat{c} + g(\hat{c} + \hat{c}^\dagger) \hat{S}_z. \quad (70)$$

As mentioned before, $\hat{c}^\dagger$ and $\hat{c}$ are the oscillator creation and annihilation operators, respectively, while $\hat{S}_z$ is a pseudospin operator describing the atom's location. The second term is the interaction potential in Eq. (68) with quantized oscillator

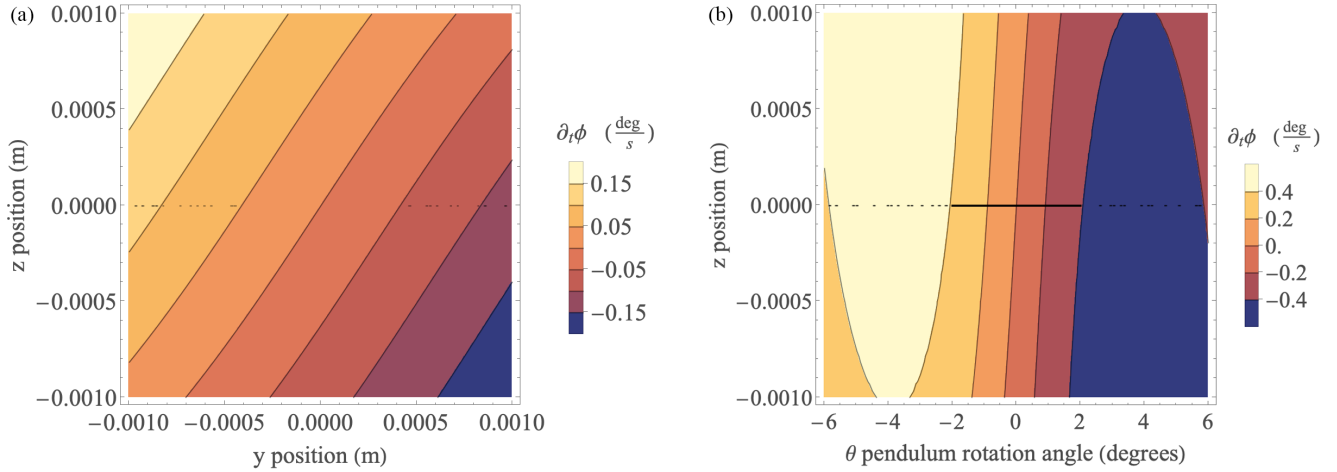


FIG. 5. The physical parameters used are given in Table I. In panel (a), the interferometer phase accumulation rate is plotted for different locations of the atoms in the y - z plane shown in Fig. 4. In panel (b), the phase accumulation rate is plotted for different z locations of the atoms and rotation angles of the oscillator. The black line is the path the atom oscillator system follows to produce the interferometer phase plotted in Fig. 6.

and atom coordinates. g is defined as follows:

$$g = \sqrt{\frac{1}{2 n_{\text{cyl}} \pi R^2 l \rho_{\text{cyl}} \omega}} \Delta z \phi_0 \frac{1}{R_0 - R} \frac{\partial^2 B^2(0, 0)}{\partial z \partial \theta}. \quad (71)$$

Here, n_{cyl} is the number of cylinders. The factor of $1/(R_0 - R)$ simply arises from taking the derivative in the direction of θ . For oscillator mass and frequency m and ω , the harmonic oscillator position operator is $\hat{r}_o = (R_0 - R)\hat{\theta} = \sqrt{\frac{1}{2m\omega}}(\hat{c} + \hat{c}^\dagger)$. Since the atom is confined to two sites in an optical lattice separated by Δz , the atom's position operator can be expressed as a spin operator in a pseudospin-1/2 Hilbert space.

$$\hat{r}_a = \Delta z/2 |0\rangle \langle 0| - \Delta z/2 |1\rangle \langle 1| = \Delta z \hat{S}_z. \quad (72)$$

For the more realistic case of an atom interferometer with N atoms, the interaction potential can be linearized in each of their coordinates to produce an interaction linear in the

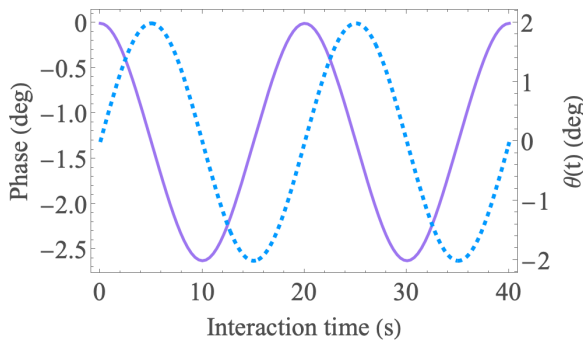


FIG. 6. The physical parameters used are given in Table I. The interferometer phase from magnetic interaction with the oscillator in classical motion is plotted with a solid line. The rotation angle of the oscillator is plotted in a dashed line. The oscillator motion is clearly encoded in the interferometer phase.

collective $N/2$ pseudospin operator $\hat{J}_z$ [1].

$$U_{\text{multi-atom}} = g(\hat{c} + \hat{c}^\dagger)\hat{J}_z. \quad (73)$$

The system Hamiltonian, Eq. (70), is capable of generating entanglement between the atoms and oscillator. For the oscillator initialized in its ground state and the atom in a symmetric spatial superposition, the system evolves to an entangled state where the displacement of the oscillator is correlated with the atom's location [1],

$$\frac{|0\rangle_a |\delta\rangle_o + |1\rangle_a |-\delta\rangle_o}{\sqrt{2}}. \quad (74)$$

$|\delta|$ is the coherent state amplitude proportional to $\lambda = g/\omega$. The mediators of this entanglement are photons; specifically, it is caused by the interaction between the magnetic dipole of the atom and the magnetic dipole of the diamagnetic oscillator and is therefore a two-photon interaction. For an experiment to resolve the correlations between the oscillator and the atom, λ must be sufficiently large.

We can readily calculate λ with Eq. (71) for our proposed setup. With the parameter values in Table I, we find the fol-

TABLE I. The model of the atom-oscillator system has the following parameters, unless otherwise stated.

Parameter	Symbol	Value
$m_F = 0$ Cs ground-state shift [27]	ϕ_0	268.575 rad/(ns T ²)
External magnetic field strength	B_{ext}	5 mT
Magnetic susceptibility [29]	χ_m	6.8×10^{-5}
Pendulum radius	R_0	10 cm
Pendulum frequency	ω	$2\pi/20$ rad/s
Pendulum density [30]	ρ_{cyl}	19.3 g/cm ³
Cylinder height	l	10 mm
Cylinder radius	R	5 mm
Half of cylinder separation angle	θ_0	2°
Atom to pendulum distance	s	8 mm
Atom wave packet separation	Δz	5 μ m

lowing value for λ (for $n_{\text{cyl}} = 4$) to first order in χ_m :

$$\lambda = -4.0 \times 10^{-16} \times \left(\frac{B_{\text{ext}}}{5 \text{ mT}} \right)^2 \left(\frac{\chi_m}{6.8 \times 10^{-5}} \right) \times \left(\frac{19.3 \text{ g/cm}^3}{\rho_{\text{cyl}}} \right)^{\frac{1}{2}} \left(\frac{1}{\alpha} \right)^{\frac{7}{2}}. \quad (75)$$

The susceptibility of 6.8×10^{-5} and density of 19.3 g/cm^3 correspond to those of tungsten [29,30]. α parametrizes the length scale of the system as $L \rightarrow \alpha L$; that is, α is the scaling factor for all the *length* parameters in Table I, with the exception of Δz (i.e., the parameters R_0 , l , R , and s). This scaling implies that g is increased by decreasing the system size. In the parameter space we are at, recalling that $\mathbf{B} = \mathbf{B}_{\text{ext}} + \mathbf{B}_{\text{cyl}}$ (with $\mathbf{B}_{\text{cyl}}$ being the induced magnetic field of the cylinders), the term dB^2/dz of Eq. (71) can be approximated to be $2 \mathbf{B}_{\text{ext}} \cdot (\partial \mathbf{B}_{\text{cyl}}/\partial z)$. This is because $|\mathbf{B}_{\text{ext}}|^2$ has no gradient and $\partial |\mathbf{B}_{\text{cyl}}|^2/\partial z$ is negligible compared to the cross term. This allows the scalings of χ_m and α to be expressed as “single” factors. As a side note, the value of -4.0×10^{-16} was obtained using the expressions in Ref. [31].

The value of the coupling in Eq. (75) is extremely small. There are several ways through which we can increase λ . One of the primary changes is to switch to a superconducting mass. Though one might think that this will reintroduce the contribution from $\partial |\mathbf{B}_{\text{cyl}}|^2/\partial z$, it is still $\times 10^2$ smaller than the cross term (at least for the physical parameters we use). Therefore, we can still use Eq. (75) to inform us of how the values change with rescaling the parameters; having said that, the upcoming calculation was done exactly using the expressions in Ref. [31]. We now have $\chi = -1$, which will increase the coupling by a factor of 10^5 ; increasing the field to 50 mT will contribute a factor of 10^2 (note that at this field, we are still well before the Paschen-Back regime for cesium, as seen in Ref. [27]); the system size can be reduced by a factor of 2 (taking $\alpha = 0.5$); a cloud of N atoms leads to a $\sqrt{N}$ increase—we set $N = 10^6$. If we take the material to be lead, then the density will be $\rho_{\text{cyl}} = 11.34 \text{ g/cm}^3$ [28]. Together, these factors increase the coupling to $\lambda_N = 8.8 \times 10^{-6}$. The main concern would be the experimental challenges for some of these changes; in particular, scaling the atom to pendulum distance from 8 to 4 mm would be pushing the limits of existing systems, since there is a requirement that the atom trap laser not interact with the pendulum.

We can also gain improvements in coupling by having the cesium atoms be magnetically sensitive states; for instance, we can place one atom in the $m_F = 1$ state and the other in the $m_F = -1$. In such a scenario, the coupling would primarily be due to the first-order Zeeman shift, which is linear in B [27]. The energy difference in the atoms now arises due to the difference in their internal states, in addition to the magnetic field differences between their two locations (that is, a coupling will exist even if $\Delta z = 0$). In particular, the term of the Zeeman energy shift that contributes to the coupling is [27]

$$\omega_{\text{1st, Zeeman}} = \frac{(g_J - g_I) \mu_B}{(2I + 1)\hbar} m_F B \quad (76)$$

where I is the nuclear spin, which is $7/2$ for Cs; we take the difference of this quantity between the two atoms, and then the gradient in the direction of oscillator motion to obtain the coupling. However, having atoms be located symmetrically in the top and bottom halves of the cylinder is detrimental to the coupling, because now it arises solely from $dB/d\theta$. Thus, we shift the positions of both atoms such that they are both in $z > 0$ plane. If we place the top atom to be at *around* $l/2$, we obtain the coupling to be

$$\lambda = -4.4 \times 10^{-12} \times \left(\frac{B_{\text{ext}}}{5 \text{ mT}} \right) \left(\frac{\chi_m}{6.8 \times 10^{-5}} \right) \times \left(\frac{19.3 \text{ g/cm}^3}{\rho_{\text{cyl}}} \right)^{\frac{1}{2}} \left(\frac{1}{\alpha} \right)^{\frac{5}{2}}; \quad (77)$$

we can see that it is now 4 orders of magnitude higher. Note that the α scaling here also incorporates changing the position of the top atom location by a factor of α ; the separation between the two atom locations is still kept constant, so the position of the second location changes too. For 50 mT, $\chi = -1$, $\rho_{\text{cyl}} = 11.34 \text{ g/cm}^3$ [28], $N = 10^6$, and $\alpha = 0.5$, we will get $\lambda_N = 4.8 \times 10^{-3}$, which is a significant improvement from having purely $m_F = 0$ states. Note that scaling α also decreases the atom-oscillator separation from 8 to 4 mm.

If we recalculate the superconducting case for the simplified setup in Sec. II, with changes such that $n_{\text{cyl}} = 2$, the atom-oscillator separation is 8 mm (which is the more feasible situation), and the atoms are symmetrically located at $\pm \Delta z/2$, we get $\lambda_N = 6.3 \times 10^{-4}$. Potential issues of having the atoms be in magnetically sensitive states is that it can introduce sensitivities to stray magnetic fields.

Note that in all our calculations of couplings, we only considered the magnetization of the two cylinders (or single cylinder for the $n_{\text{cyl}} = 2$ case) closest to the atoms (i.e., the cylinders on the side of the atoms).

Our entanglement witness is also applicable to other setups, such as those where an NV center(s) is coupled to a mechanical oscillator. Such a system can be described by a Hamiltonian that is of similar form to Eq. (1), and thus, our EW in Eq. (2) is applicable. To highlight such expanded applications of our work, we present in Table II the couplings and parameters of some nanodiamond setups found in existing literature. One can see that these couplings tend to be much higher than the values for the interferometer-pendulum setup we have presented in this section; this is primarily due to the fact that the magnetic dipole-dipole interaction we use in our setup tends to be weaker. It must also be pointed out that for the nanodiamond setups, the mass of the oscillators tends to be orders of magnitude smaller and the frequencies orders of magnitude higher than the parameters of our apparatus.

IV. CONCLUSION

Within the context of a combined atom interferometer and oscillator system, we can use an EW to show that entanglement exists even in the case where the oscillator starts in a thermal state and remains in contact with a white noise bath. In fact, we find that for the appropriate oscillator quality factor Q and thermal occupancy $\bar{n}$, violation always exists for any value of coupling λ . Additionally, there is an $O(\bar{n})$

improvement in the EW violation if the oscillator is cooled to its ground state at the start of the experiment. With a coupling of $\lambda = 10^{-4.5}$, we need at least $\sim 10^{5.2}$ measurements to observe this violation. We find that a magnetic interaction between a superconducting mass and atoms in different magnetically sensitive states can result in couplings of $\lambda \sim 6 \times 10^{-4}$; however, implementing such a setup will be experimentally challenging. We also model atomic dephasing and find that EW violation can exist for target interferometer parameters.

Looking ahead, while it is true that a gravitational coupling will also satisfy the EW (we reiterate our conclusion that says that, under thermal noise, for appropriate values of Q and $\bar{n}$, violation always exists for any coupling λ), it will result in couplings orders of magnitude smaller, thus increasing the number of required measurements to unachievable numbers; as such, we do not expect to use this EW in a gravitational coupling experiment. However, in the context of our collaboration, where we intend to incorporate magnetic entanglement to make the detection of potential gravitational entanglement more accessible [1], our results have enlightened us of suitable parameter regimes. Finally, our EW has broader applications in nanodiamond setups, in particular where NV centers are coupled to oscillators.

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DATA AVAILABILITY

The data supporting this study’s findings are available within the article.

APPENDIX A: ADDITIONAL REMARKS ON THE GENERAL HAMILTONIAN

In explaining the form of the general Hamiltonian given in Eq. (1), we made an analogy with the Hamiltonian of an NV center. To provide an explicit example, we shall consider the case where a single NV center is coupled to the motion of its host diamond, which may be levitated in a magnetic field $\vec{B}$, for which the Hamiltonian can be given by

$$H = \omega \hat{c}^\dagger \hat{c} + D \hat{S}_z^2 + \gamma_e B_z(x) \hat{S}_z. \quad (\text{A1})$$

Here, ω is the harmonic oscillator frequency for a single mode of the levitated diamond’s motion (we neglect other modes, focusing only on the lowest frequency mode), c is the annihilation operator for the harmonic oscillator, $\hat{S}_z$ is the spin-1 operator for the three ground-state spin levels of the NV center, D is the ground-state splitting of $2\pi \times 2.87$ GHz, γ_e is the gyromagnetic ratio, and the magnetic field $\vec{B}(x)$ depends upon the position of the NV center, as occurs in levitated diamond systems. We can write this position as $x_0 + q_{\text{zpf}}(c + c^\dagger)$, where x_0 is the classical equilibrium position. As we expect small oscillations, we expand $\vec{B}(x)$ around this position and get $\vec{B}(x) \approx \vec{B}(x_0) + \partial_x \vec{B} q_{\text{zpf}}(c + c^\dagger)$.

When we use this small position expansion in the Hamiltonian, we get

$$\gamma_e B_z(x) \hat{S}_z \rightarrow \gamma_e B_z(x_0) \hat{S}_z + 2g(\hat{c} + \hat{c}^\dagger) \hat{S}_z, \quad (\text{A2})$$

where $g = \gamma_e \partial_x B_z q_{\text{zpf}}/2$. Finally, we consider experiments that only address two of the three spin levels, typically $m_s = +1$ and $m_s = 0$, which gives

$$\tilde{H} = \omega \hat{c}^\dagger \hat{c} + g \sigma_z(\hat{c} + \hat{c}^\dagger) + (\gamma_e B_z(x_0) + D) \hat{\sigma}_z/2, \quad (\text{A3})$$

where the $D \hat{S}_z^2$ term is changed to $D \hat{\sigma}_z/2$ in the two-level approximation.

Similar forms for a spin-oscillator coupled Hamiltonian in the case of NV centers can be found in Refs. [33–35], for which we have provided a summary in Table II.

TABLE II. Couplings of existing nanodiamond setups compared to the interferometer-pendulum couplings in this paper. The mass and frequency of the oscillators are also given. Some of the values are explicitly stated in the cited papers, while we have approximated others. Where necessary, we have taken the density of diamond to be 3.5 g/cm^3 [32]. The last two rows are simply the couplings and parameters for the superconducting mass scenarios mentioned after Eq. (77). For nuanced differences between the setups of the nanodiamond systems and our interferometer-pendulum system, please see the cited references.

Coupling between...	Interaction	Coupling strength	Mass	Frequency
NV center, external resonator [33]	Magnetic	$2\pi \times 100 \text{ kHz}$	$6.7 \times 10^{-18} \text{ kg}$	$2\pi \times 5 \text{ MHz}$
NV center, host diamond [34]	Strain	$2\pi \times 0.25 \text{ Hz}$	$\sim 10^{-13} \text{ kg}$	$2\pi \times 6.7 \text{ MHz}$
NV center, host diamond [35]	Magnetic	$\sim 2\pi \times 1.8 \text{ Hz}$	$\sim 6 \times 10^{-12} \text{ kg}$	$2\pi \times 0.5 \text{ kHz}$
Two-cylinder pendulum and atoms	Magnetic	$2\pi \times 0.032 \text{ mHz}$	$2.2 \times 10^{-3} \text{ kg}$	$2\pi \times 0.05 \text{ Hz}$
Four-cylinder pendulum and atoms	Magnetic	$2\pi \times 0.24 \text{ mHz}$	$4.4 \times 10^{-3} \text{ kg}$	$2\pi \times 0.05 \text{ Hz}$

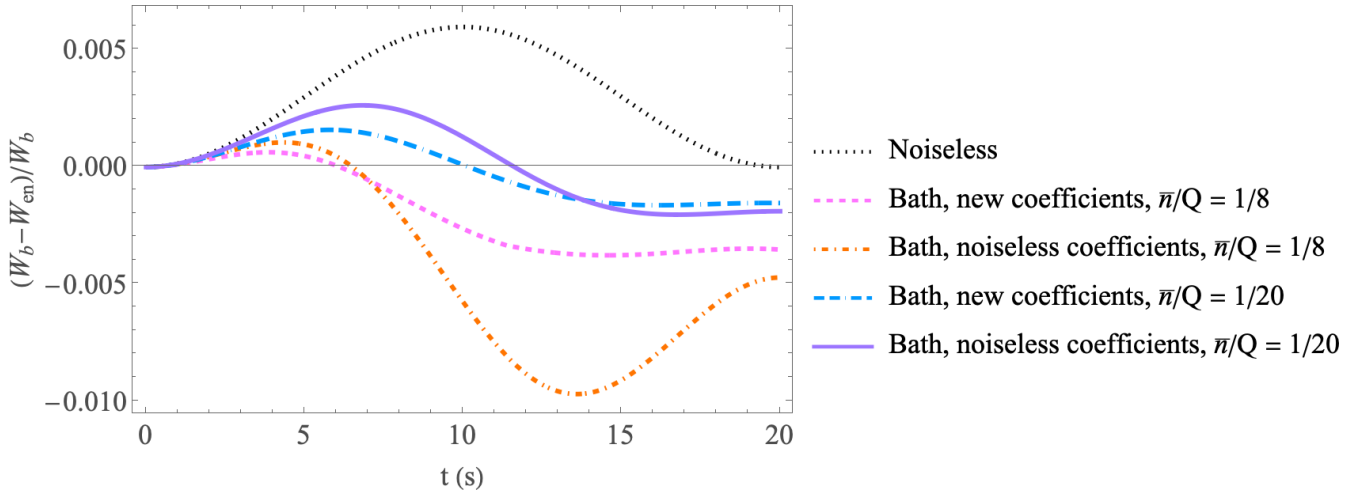


FIG. 7. EW violation when the oscillator starts in a ground state and remains in contact with a thermal bath. $\lambda = 10^{-4.5}$, $N = 10^6$, and $\omega = 2\pi/20$ rad/s. We can see that for the region where the violation is positive, the noiseless coefficients are still better than the reoptimized ones.

APPENDIX B: ADDITIONAL REMARKS AND ELABORATION ON THE ENTANGLEMENT WITNESS

In this section, we provide more details on and/or clarify aspects of some calculations in this paper.

As mentioned earlier, we minimize the value W_{en} ; that is, we solve for a_μ and b_μ such that $\partial_{a_\mu}(W_{\text{en}}) = 0$ and $\partial_{b_\mu}(W_{\text{en}}) = 0$. W_{en} , being quadratic in a_μ and b_μ , is solvable. Note that this gives a higher violation than if were to maximize $W_b - W_{\text{en}}$ by expressing the absolute value as $s(a_y b_z - a_z b_y)$, where s sets the sign (+1 or -1), for reasons we cannot explain currently; the underlying reason must be that optimizing $s(a_y b_z - a_z b_y)$ (and choosing the pairing of s and the coefficients that makes this value positive) is not equivalent to optimizing $\sqrt{(a_y b_z - a_z b_y)^2}$.

1. More details on the noiseless entanglement witness

The nonsimplified versions of the noiseless coefficients are

$$a_{y,0} = -\frac{\sqrt{2m\omega}\lambda N \sin(\omega t)}{\{2 + 3\lambda^2 N + \lambda^2 N[\cos(2\omega t) - 4\cos(\omega t)]\}}, \quad (\text{B1})$$

$$b_{y,0} = \sqrt{\frac{2}{m\omega}} \frac{\lambda N[1 - \cos(\omega t)]}{[2 + \lambda^2 N - \lambda^2 N \cos(2\omega t)]}, \quad (\text{B2})$$

$$a_{z,0} = \frac{\sqrt{2m\omega}\lambda N[1 - \cos(\omega t)]}{\{2 + 3\lambda^2 N + \lambda^2 N[\cos(2\omega t) - 4\cos(\omega t)]\}}, \quad (\text{B3})$$

$$b_{z,0} = \sqrt{\frac{2}{m\omega}} \frac{\lambda N \sin(\omega t)}{[2 + \lambda^2 N - \lambda^2 N \cos(2\omega t)]}, \quad (\text{B4})$$

which, when written to $O(\lambda)$, give Eqs. (25)–(28).

The lowest order of the expectation values $\langle \hat{J}_x \rangle$, $\langle \hat{J}_x^2 \rangle$, $\langle \hat{J}_y \rangle$, $\langle \hat{J}_y^2 \rangle$, $\langle \hat{q}^2 \rangle$, and $\langle \hat{p}^2 \rangle$ is λ^2 . $\langle \hat{q} \hat{J}_x + \hat{J}_x \hat{q} \rangle = \langle \hat{p} \hat{J}_x + \hat{J}_x \hat{p} \rangle = 0$. Very interestingly, to lowest order in λ , $\langle \hat{q} \hat{J}_y + \hat{J}_y \hat{q} \rangle = -a_{y,0,\lambda}$, $\langle \hat{p} \hat{J}_y + \hat{J}_y \hat{p} \rangle = -a_{z,0,\lambda}$, $\langle \hat{q} \hat{J}_z + \hat{J}_z \hat{q} \rangle = -b_{y,0,\lambda}$, and $\langle \hat{p} \hat{J}_z + \hat{J}_z \hat{p} \rangle = -b_{z,0,\lambda}$, where $a_{y,0,\lambda}$, etc. are the expressions in

Eqs. (25)–(28). It must be noted that these coefficients reduced to being $O(\lambda)$ to lowest order in the end, despite the fact that the calculations were conducted to $O(\lambda^2)$.

Another way to understand the coefficients is by investigating how the sum of the spin variances changes as the system evolves. To $O(\lambda^2)$, we find that

$$\sum_{\mu} \text{Var}(\hat{J}_{\mu}) = \frac{N}{2} + \frac{\lambda^2 N^2}{2} [1 - \cos(\omega t)] \quad (\text{B5})$$

for $|\psi_{\text{en}}\rangle$. This shows that there is an overall increase in $[\sum_{\mu} \text{Var}(\hat{J}_{\mu})]$ as the system evolves. However, if we consider the contribution from the rest of the operators in W [see Eq. (15)], we find that the coefficients work with the $\hat{q}$ and $\hat{p}$ operators to bring the overall value of W_{en} to be less than $N/2$; this contribution is given by

$$W_{\text{en},0} - \sum_{\mu} \text{Var}(\hat{J}_{\mu}) = -\lambda^2 N^2 [1 - \cos(\omega t)], \quad (\text{B6})$$

such that the total value of W_{en} is

$$W_{\text{en},0} = \frac{N}{2} - \frac{\lambda^2 N^2}{2} [1 - \cos(\omega t)]. \quad (\text{B7})$$

In essence, the coefficients work in various ways with the terms in Eq. (15) to minimize W_{en} .

2. Elaboration on the entanglement witness with thermal noise

The covariances that contribute to the bath noise are as follows:

$$\langle \langle \Xi(t) \Xi(t) \rangle \rangle = \frac{2\lambda^2 \bar{n}}{Q} [6\omega t - 8\sin(\omega t) + \sin(2\omega t)],$$

$$\langle \langle \Xi(t) \mathcal{Q}(t) \rangle \rangle = \sqrt{\frac{1}{2m\omega}} \frac{16\lambda \bar{n}}{Q} \sin^4\left(\frac{\omega t}{2}\right),$$

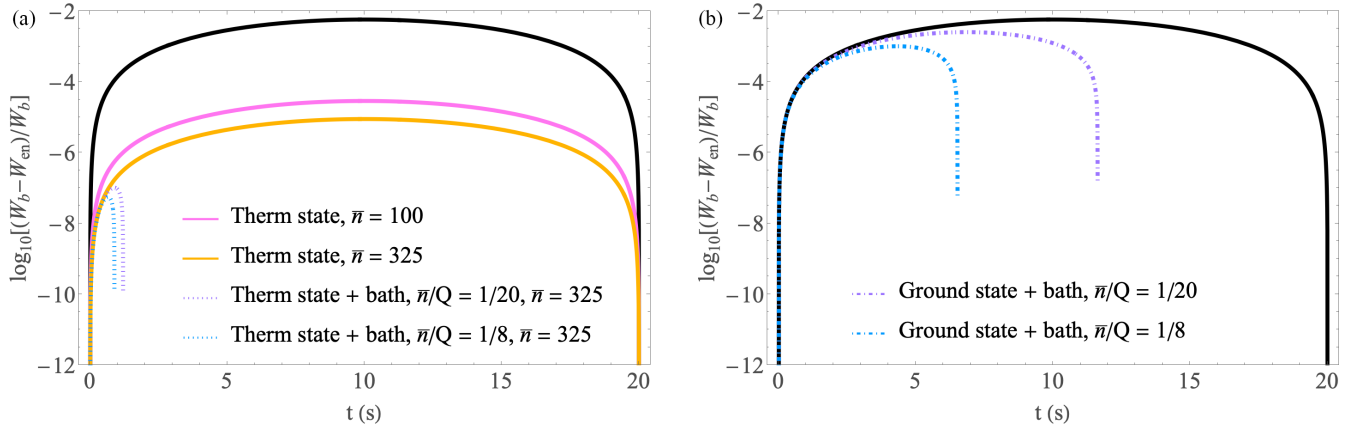


FIG. 8. For potential clarity, we have split the plots in Fig. 2 into two panels. The black line depicts the noiseless case. We have set $\lambda = 10^{-4.5}$, $N = 10^6$, and $\omega = 2\pi/20$ rad/s. White noise depends only on $\bar{n}/Q$. Thus, the scenario where the oscillator starts in a ground state and then remains in contact with a bath can be characterized solely using $\bar{n}/Q$. However, a value for $\bar{n}$ must be set when starting in a thermal state. Since the condition $\bar{n}\lambda^2 \ll 1$ must be satisfied, we consider $\bar{n} = 325$ to be the highest appropriate value. Purple lines correspond to $\bar{n}/Q = 1/20$ and blue lines correspond to $\bar{n}/Q = 1/8$. As can be seen in the plots, the lower the value of $\bar{n}/Q$, the better the violation. Note that introducing a bath results in the violation *not* being seen for the entire oscillator period; where the plots get truncated is when $W_b - W_{\text{en}} < 0$.

$$\begin{aligned}
 \langle\langle \Xi(t) \mathcal{P}(t) \rangle\rangle &= -\sqrt{\frac{m\omega}{2}} \frac{8\lambda\bar{n}}{Q} \left[\frac{\omega t}{2} - \sin(\omega t) + \frac{\sin(2\omega t)}{4} \right], \\
 \langle\langle \mathcal{Q}(t) \mathcal{Q}(t) \rangle\rangle &= \frac{1}{2m\omega} \frac{2\bar{n}}{Q} [2\omega t - \sin(2\omega t)], \\
 \langle\langle \mathcal{P}(t) \mathcal{P}(t) \rangle\rangle &= \frac{m\omega}{2} \frac{2\bar{n}}{Q} [2\omega t + \sin(2\omega t)], \\
 \langle\langle \mathcal{Q}(t) \mathcal{P}(t) \rangle\rangle &= \frac{2\bar{n}}{Q} \sin^2(\omega t).
 \end{aligned} \tag{B8}$$

As examples, $\langle\langle \Xi(t) \Xi(t) \rangle\rangle$ contributes to the noise in the EW term $\text{Var}(\hat{J}_x)$ and $\langle\langle \mathcal{Q}(t) \mathcal{Q}(t) \rangle\rangle$ contributes to the noise in $\langle\langle \hat{q}^2 \rangle\rangle$. Therefore, the expectation values that compose the EW must be modified as was shown in Sec. II C.

Next, we elaborate on the scenario where the oscillator starts in a ground state and remains in contact with a white noise bath. Some additional plots to those in Fig. 2 are shown in Fig. 7, in particular, plots for “reoptimized” coefficients. It can be seen that the noiseless coefficients are still better than the reoptimized ones for the times where the violation is positive.

Finally, we include a split version of the thermal noise plots in Fig. 8, in case this may be preferred by the reader.

APPENDIX C: ELABORATION ON THE MAGNETIC INTERACTION

Here, we present an alternative explanation of Eq. (71). The key physics we focus on are the interactions induced between the mass and the atoms due to the diamagnetic/paramagnetic response of the mass’s material. This produces a field that is roughly dipolar and causes a shift in the magnetic field and in the field gradient near the atoms. Meanwhile, the atoms themselves have a response to the combined external and mass-sourced magnetic fields, which is described by the second-order Zeeman effect, where the ground-state energy shifts proportionally to B^2 [27]. As the atomic separation is small compared to the atom-mass distance, we can focus on the differences in B^2 around the center-of-mass of the atom. As such, we take the derivative $\partial B^2/\partial z$ and multiply by Δz (separation between the atoms) to obtain the potential difference between the two atom locations. We remark that this is a force times a distance, which also corresponds to the differential force the atoms induce on the mass by Newton’s third law. To see the effect of this force, we take a second derivative with respect to $(R_0 - R)\theta$ (distance moved by cylinder) to get $[1/(R_0 - R)](\partial^2 B^2/\partial z \partial \theta)$ and multiply by the oscillator’s zero point motion $q_{\text{zpf}}(c + c^\dagger)$ to convert the force into a Hamiltonian term.

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