

Quantum advantage bounds for a multipartite Gaussian battery

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We demonstrate the possibility of a genuine quantum advantage in the efficiency of quantum batteries by analyzing a model that enables a consistent comparison between quantum and classical regimes. Our system consists of N harmonic oscillator cells coupled to a common thermal reservoir, evolving through Gaussian states. We define the global efficiency as the ratio of extractable work (ergotropy) to stored energy, and derive analytical bounds that distinguish, in order of increasing efficiency, regimes characterized by classical squeezing, quantum squeezing without entanglement, and genuine entanglement. Moreover, numerical simulations support the emergence of a similar hierarchy for the thermodynamic efficiency, defined as the ratio between ergotropy and the total thermodynamic cost of the charging process.

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I. INTRODUCTION

Inspired by the progressive miniaturization of devices devoted to energy storage [1], the concept of Quantum Batteries (QBs) has recently emerged as a hot topic in the panorama of quantum technologies [2–7]. Theoretical proposals and early-stage experimental proofs of principle of devices exploiting quantum features for energy transfer, storage, and manipulation have appeared, based on different platforms. Among them are cavity and circuit quantum electrodynamics [8–21], spin chains and networks [22–29], molecules [30,31], and harmonic oscillators [32–37].

Among the main questions emerging in this domain, as well as in all quantum technologies, arguably the most fundamental is: “Is there a genuine quantum advantage?” As the energy E_B stored into the QB scales with the number N of elementary units composing the quantum battery, similarly to the classical case, research has focused on the charging power $P = E_B/t^*$, where t^* is the charging time. A superextensive scaling, $P \propto N^k$ with $k > 1$, can be achieved via two distinct mechanisms [38].

On the one hand, a *collective advantage* can arise, associated with many-body induced accelerations in the QB evolution from the initial empty state to a charged state, which may also occur in classical models [8,33,39]. On the other hand, a *genuine quantum advantage* occurs in the presence of energy fluctuations scaling more than linearly with N , and

is attributed to entanglement generated among the elementary units composing the QB [40–42]. This latter form of advantage, however, has been demonstrated in models—e.g., when the QB is composed of N spin-1/2 units—that lack a classical analog to compare with. More importantly, in analogy with quantum thermal machines [43–48], a proper thermodynamic quantification of the performance of a QB must take into account the *efficiency* of work extraction. The possibility of a quantum advantage in this context, however, has so far eluded investigation.

The purpose of this paper is to address the possibility of a *quantum advantage in efficiency* by considering a system that can be treated both in a fully quantum regime and in its semiclassical limit, thereby allowing for a fair comparison of the QB performance across different regimes while keeping the same number of cells, stored energy, and temperature. Specifically, we consider a QB composed of N identical harmonic oscillator cells, initially prepared at the same temperature and coupled to a common reservoir. Via an appropriately engineered system-bath interaction, it is possible to generate quantum correlations among the harmonic oscillators composing the QB. These bath-induced correlations are ultimately responsible for an enhancement of the efficiency. By considering the *global efficiency* $\eta_{\text{glob}}(t)$, defined as the ratio between the work extractable via global unitary operations acting on the N cells (*ergotropy* [49–52]) and the stored energy, we derive rigorous bounds establishing a clear quantum advantage: $\eta_{\text{glob}}(t) < \mathcal{B}_{\text{cl}}(t)$ in the case of classical squeezing; $\mathcal{B}_{\text{cl}}(t) < \eta_{\text{glob}}(t) \leq \mathcal{B}_{\text{en}}(t)$ with quantum squeezing without entanglement; and $\mathcal{B}_{\text{en}}(t) < \eta_{\text{glob}}(t) \leq 1$ when entanglement is present. Finally, numerical results support the existence of the same hierarchy of performance for the *thermodynamic efficiency* $\eta_{\text{th}}(t)$, defined as the ratio between the ergotropy and the overall thermodynamic cost of the charging process.

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The paper is organized as follows. In Sec. II, we discuss the model for the QB, namely, a multipartite device composed by N harmonic oscillators coupled with a Caldeira-Leggett thermal bath, and the covariance matrix (CM) encoding the dynamics of the system. Figures of merits such as the stored energy, the ergotropy, and the efficiencies are introduced in Sec. III. Classical and quantum correlations within the device are characterized in Sec. IV. In Sec. V are derived stringent analytical and numerical bounds for the efficiencies, while Sec. VI is devoted to conclusions. Technical details of the calculations and side comments are discussed in few Appendixes.

II. MODEL AND COVARIANCE MATRIX

The total Hamiltonian is given by $H_{\text{tot}} = H_B + H_{\text{int}} + H_R$, with

$$H_B = \sum_{l=1}^N \left(\frac{p_l^2}{2m} + \frac{m\omega_0^2}{2} q_l^2 \right) = \sum_{l=1}^N H_B^{(l)} \quad (1)$$

the Hamiltonian of the QB, composed of N identical harmonic oscillators with equal frequencies and masses. The reservoir is modeled by a set of harmonic oscillators [53–58], $H_R = \sum_{k=1}^{+\infty} \left(\frac{\pi_k^2}{2m_k} + \frac{m_k\omega_k^2}{2} x_k^2 \right)$, interacting with the QB through the coupling [59]

$$H_{\text{int}} = - \sum_{k=1}^{+\infty} c_k x_k \sum_{l=1}^N \alpha_l q_l + \sum_{k=1}^{+\infty} \frac{c_k^2}{2m_k\omega_k^2} \left(\sum_{l=1}^N \alpha_l q_l \right)^2, \quad (2)$$

which is turned on at time $t = 0$. Notice that the coupling with the l th oscillator of the QB is modulated by the factor α_l .

The initial state is described by the density matrix $\rho_{\text{tot}}(0) = \frac{e^{-H_B/k_B T_0}}{Z_B} \otimes \frac{e^{-H_R/k_B T}}{Z_R}$, the tensor product of two initially uncorrelated thermal states, with T_0 the temperature of the QB and T the temperature of the reservoir. This system undergoes a Gaussian dynamics, so that the dynamical properties of the QB are characterized by the time behavior of its covariance matrix [60–64].

Rather than adopting the usual *local* representation, expressed in terms of the canonical quadratures $Q_l = \sqrt{\frac{m\omega_0}{\hbar}} q_l$ and $P_l = \sqrt{\frac{1}{\hbar m\omega_0}} p_l$ (see Appendixes A and B 1 for details), it is useful to switch to a *global* basis, which highlights the *collective* properties of the model. Indeed, Eq. (2) shows that only the combination $\sum_l \alpha_l Q_l(t)$ couples to the reservoir. This observation allows us to introduce a *bright mode* (BM) [65], with quadratures (see Appendix B 2 for details)

$$(Q_{\text{BM}}, P_{\text{BM}}) \equiv \frac{1}{\bar{\alpha}} \sum_{l=1}^N \alpha_l (Q_l, P_l), \quad \bar{\alpha} = \sqrt{\sum_{l=1}^N \alpha_l^2}, \quad (3)$$

the only mode coupling to the reservoir. The remaining $N - 1$ ones are *dark modes* (DMs), completely decoupled from the reservoir. Each DM has energy frozen at the initial value $\frac{\hbar\omega_0}{2} C(T_0)$, with

$$C(T_0) = \coth \left(\frac{\hbar\omega_0}{2k_B T_0} \right). \quad (4)$$

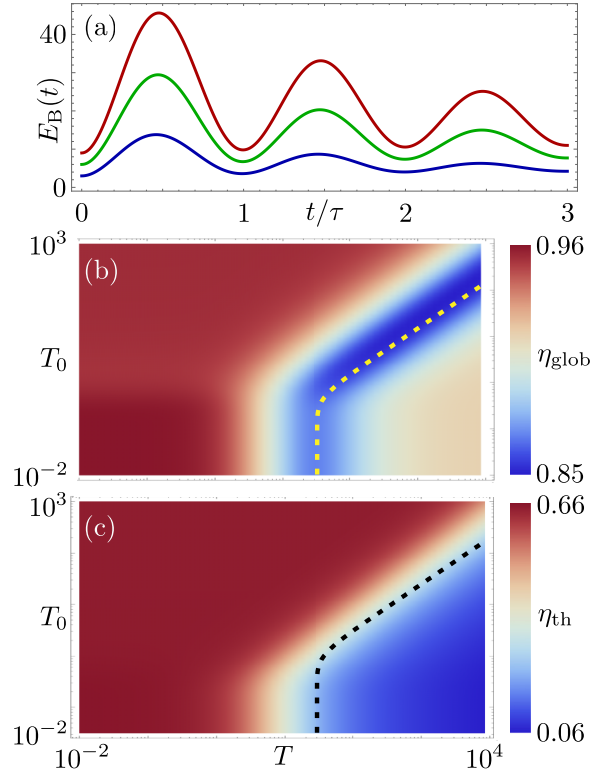


FIG. 1. (a) Plot of $E_B(t)$ (units $\hbar\omega_0$) as a function of t/τ for $N = 6$ (blue), $N = 12$ (green), and $N = 18$ (red) with $k_B T = k_B T_0 = 0.01\hbar\omega_0$. For $N = 6$, density plots of $\eta_{\text{glob}} = \eta_{\text{glob}}(t^*)$ [panel (b)] and $\eta_{\text{th}} = \eta_{\text{th}}(t^*)$ [panel (c)], as a function of T and T_0 (units $\hbar\omega_0/k_B$). The dashed line in panels (b) and (c) represents the boundary $T = T^*$. Here, $\alpha_l = 1$, $\gamma_0 = 5\omega_0$, and $\omega_D = 2\omega_0$.

On this global basis, the covariance matrix of the QB, $\sigma_B^{(\text{glob})}(t)$, takes a block-diagonal form with a single block corresponding to the BM:

$$\Sigma_{\text{BM}}(t) = \begin{pmatrix} \frac{C(T_0)}{2} + \bar{\alpha}^2 a(t) & \bar{\alpha}^2 b(t) \\ \bar{\alpha}^2 b(t) & \frac{C(T_0)}{2} + \bar{\alpha}^2 c(t) \end{pmatrix}, \quad (5)$$

with $a(t)$, $b(t)$, and $c(t)$ given in Appendix B 1. The $N - 1$ DMs are identical and represented by blocks of the form $\Sigma_{\text{DM}} = \frac{1}{2} \text{diag}\{C(T_0), C(T_0)\}$.

III. ENERGY STORAGE AND EXTRACTION

A. Energy and ergotropy

Given the above structure of the covariance matrix, the total energy of the battery at time t , defined as $E_B(t) \equiv \langle H_B(t) \rangle$, can be written as

$$E_B(t) = E_{\text{BM}}(t) + (N - 1) \frac{\hbar\omega_0}{2} C(T_0), \quad (6)$$

where $E_{\text{BM}}(t) = \frac{\hbar\omega_0}{2} \text{Tr}\{\Sigma_{\text{BM}}(t)\}$, the energy associated with the BM, is given by

$$E_{\text{BM}}(t) = \frac{\hbar\omega_0}{2} \{C(T_0) + \bar{\alpha}^2 [a(t) + c(t)]\}. \quad (7)$$

The behavior of $E_B(t)$ is shown in Fig. 1(a) for three different values of N , plotted as a function of the rescaled time t/τ ,

where $\tau = \pi/\Omega_N$. Here, τ sets the typical timescale of the battery, linked to the characteristic frequency $\Omega_N = \bar{\alpha}\sqrt{\gamma_0\omega_D}$, which is determined by the coupling strength to the reservoir γ_0 and the cutoff frequency ω_D of the spectral density (see Appendix A). The stored energy exhibits oscillations with several local maxima, the largest of which occurs at $t = t^* \approx \tau/2$. We note, in particular, that $E_B(t^*) \propto N$ and the charging power $P(t^*) = E_B(t^*)/t^* \propto N^{3/2}$. Such superextensive scaling is analogous to that observed in other systems exhibiting a *collective advantage* [8,10,11,20,33].

Since we are interested in the performances of the battery at maximum stored energy—a standard choice in the context of quantum batteries—we will choose henceforth this particular time t^* as a reference time, especially when in the following we will discuss the efficiency of the charging process.

In addition to energy, a key quantity is the *ergotropy* [50–52], which, for a Gaussian system, is directly related to the symplectic eigenvalues of the CM [34,66]. By considering all possible *global unitary* operations acting on the N -oscillator system, we obtain the global ergotropy

$$\mathcal{E}_{\text{glob}}(t) = E_B(t) - \hbar\omega_0 \sum_{j=1}^N \nu_j(t), \quad (8)$$

where $\nu_j(t)$ are the symplectic eigenvalues of $\sigma_B^{(\text{glob})}(t)$, with

$$\nu_1(t) = \frac{1}{2} \sqrt{C^2(T_0) + 2\bar{\alpha}^2[C(T_0)\mathcal{T}(t) + 2\bar{\alpha}^2\Delta(t)]}, \quad (9)$$

stemming from the BM and $\nu_{2 \leq j \leq N} = C(T_0)/2$ from the DMs. Here,

$$\begin{aligned} \mathcal{T}(t) &= a(t) + c(t), \\ \Delta(t) &= a(t)c(t) - b^2(t). \end{aligned} \quad (10)$$

Then,

$$\mathcal{E}_{\text{glob}}(t) = E_{\text{BM}}(t) - \hbar\omega_0 \nu_1(t). \quad (11)$$

B. Global and Thermodynamic efficiency

To assess the performance of the QB, we consider two definitions of efficiency.

The first one is the *global efficiency*, i.e., the ratio between the global ergotropy and the energy stored in the battery [67]:

$$\eta_{\text{glob}}(t) = \frac{\mathcal{E}_{\text{glob}}(t)}{E'_{\text{BM}}(t)} = 1 - \frac{\hbar\omega_0}{E'_{\text{BM}}(t)} [\nu_1(t) - 1/2], \quad (12)$$

where the energy E'_{BM} is obtained from E_{BM} by subtracting the (unextractable) zero-point energy $\hbar\omega_0/2$. The density plot of $\eta_{\text{glob}}(t^*)$ in Fig. 1(b) reveals the emergence of two distinct regions, separated approximately by a characteristic temperature $T^* = \frac{\hbar^2\Omega_N^2}{2k_B^2T} \coth\left(\frac{\hbar\omega_0}{2k_BT_0}\right)$, which results from the interplay between thermal fluctuations in the reservoir and those in the initial state of the QB. When $T > T^*$, the reservoir thermal fluctuations dominate. These fluctuations are not useful for enhancing global work extraction, as they mainly generate classical correlations. Conversely, when $T < T^*$, thermal fluctuations are suppressed, enabling global unitary operations to extract more work than *local* ones (see Appendix C).

The second one is the *thermodynamic efficiency*, a figure of merit that accounts for the *total thermodynamic cost* of the charging process. It is rooted in the concept of entropy production [68–71] $\Sigma_{\text{irr}}(t) = \frac{\Delta E_R(t)}{T} + \Delta S_B(t) \geq 0$ always, due to the second law of thermodynamics. The finite entropy production originates from two contributions: the heat exchanged with the reservoir, $\Delta E_R(t) = \langle H_R(t) \rangle - \langle H_R(0) \rangle$, and the Von Neumann entropy [72] variation of the quantum battery, $\Delta S_B(t) = S_B(t) - S_B(0)$. Since the total system is closed while the battery is coupled to the reservoir, the total energy balance reads $\Delta E_R(t) + \Delta E_B(t) = W(t)$, which links the heat exchanged to the work done to switch on/off the coupling $W(t) = \langle H_{\text{int}}(0) \rangle - \langle H_{\text{int}}(t) \rangle$ (see Appendix D for details) and to the energy accumulated in the battery, $\Delta E_B(t) = \langle H_B(t) \rangle - \langle H_B(0) \rangle$. This allows to rewrite the entropy production as

$$\Sigma_{\text{irr}}(t) = \frac{1}{T} [W(t) - \Delta E_B(t) + T \Delta S_B(t)] \geq 0. \quad (13)$$

To define the efficiency associated to the global ergotropy, we rewrite $\Delta E_B(t)$ in terms of $\mathcal{E}_{\text{glob}}(t)$. Using Eqs. (6) and (11), we have

$$\Delta E_B(t) = \mathcal{E}_{\text{glob}}(t) + \hbar\omega_0 \left[\nu_1(t) - \frac{C(T_0)}{2} \right] \equiv \mathcal{E}_{\text{glob}}(t) - \Delta\epsilon(t).$$

The thermodynamic efficiency of the quantum battery can be now defined using the concept of *exergy* [73–76] $\Phi(t)$, the ratio between the negative contributions to the entropy production (taken with a change of sign), $\Sigma_{\text{irr}}^{(-)}(t)$, and the positive ones, $\Sigma_{\text{irr}}^{(+)}(t)$: $\Phi(t) = -\Sigma_{\text{irr}}^{(-)}(t)/\Sigma_{\text{irr}}^{(+)}(t)$. Since $\Sigma_{\text{irr}}(t) \geq 0$, one has [73–76] $0 \leq \Phi(t) \leq 1$.

Considering our system, we always observe that the switch on/off work is positive: $W(t) > 0$. Under this condition, the exergy $\Phi(t)$ can be written explicitly as

$$\Phi(t) = \frac{\mathcal{E}_{\text{glob}}(t) - \Delta\epsilon(t)\theta[-\Delta\epsilon(t)] - T\theta[-\Delta S_B(t)]\Delta S_B(t)}{W(t) + \theta[\Delta\epsilon(t)]\Delta\epsilon(t) + T\theta[\Delta S_B(t)]\Delta S_B(t)}. \quad (14)$$

It is worth emphasizing that the quantity $\Delta\epsilon$ is not guaranteed to be always negative. Therefore, to keep the formulation as general as possible, it is necessary to include the term $\theta[\Delta\epsilon(t)]\Delta\epsilon(t)$ in the denominator of Eq. (14). Now, in order to obtain a figure of merit associated with the ergotropy, we consider only the first term in the numerator of the exergy. This finally leads to the definition of the *thermodynamic efficiency*

$$\eta_{\text{th}}(t) = \frac{\mathcal{E}_{\text{glob}}(t)}{W(t) + T\theta[\Delta S_B(t)]\Delta S_B(t) + \theta[\Delta\epsilon(t)]\Delta\epsilon(t)}, \quad (15)$$

which always satisfies $0 \leq \eta_{\text{th}}(t) \leq \Phi(t) \leq 1$. Note that for a Gaussian system, the Von Neumann entropy $S_B(t)$ of the quantum battery can be expressed in terms of the symplectic eigenvalues $\nu_l(t)$ of $\sigma_B^{(\text{glob})}(t)$ as $S_B(t) = k_B \sum_{j=1}^N f[\nu_j(t)]$, with [63] $f(\nu) = \sum_{p=\pm 1} p(\nu + \frac{p}{2}) \ln(\nu + \frac{p}{2})$. In our case, using the expression of the symplectic eigenvalues $\nu_l(t)$, we obtain

$$\Delta S_B(t) = S_B(t) - S_B(0) = k_B \left\{ f[\nu_1(t)] - f\left[\frac{C(T_0)}{2}\right] \right\}. \quad (16)$$

IV. CLASSICAL VERSUS QUANTUM CORRELATIONS

From the observations made so far, it is evident that the best overall performances are achieved in the regime $T < T^*$, where the reservoir fluctuations are weak. Interestingly, in this same regime, high values of T_0 —corresponding to a nearly classical battery—can also lead to good performances [see Figs. 1(b) and 1(c)]. This raises two important questions: Are there signatures of *genuine quantum correlations*? And if so, what role do they play in determining the battery's performance? Addressing these questions is the central goal of this work. To do so, we need first of all to characterize the different possible sources of quantum correlations.

A. Classical versus quantum squeezing

The most general nondisplaced, single-mode Gaussian state is known to be a phase-space rotated *squeezed state* [63,64]—where the rotation is due to the fact that, as seen in Eq. (5), $\Sigma_{\text{BM}}(t)$ is in general not diagonal. Diagonalizing it amounts to expressing the state in terms of its principal axes, corresponding to a new set of canonical quadratures, $Q'_{\text{BM}}(t)$ and $P'_{\text{BM}}(t)$. The variances along these axes are given by the eigenvalues of $\Sigma_{\text{BM}}(t)$, namely, $\lambda_{Q'_{\text{BM}}}(t) = \langle Q'^2_{\text{BM}}(t) \rangle$ and $\lambda_{P'_{\text{BM}}}(t) = \langle P'^2_{\text{BM}}(t) \rangle$. The diagonal form of $\Sigma_{\text{BM}}(t)$ is then

$$\Sigma'_{\text{BM}}(t) \equiv \begin{pmatrix} \lambda_{Q'_{\text{BM}}}(t) & 0 \\ 0 & \lambda_{P'_{\text{BM}}}(t) \end{pmatrix} = \begin{pmatrix} e^{-2r(t)} v_1(t) & 0 \\ 0 & e^{2r(t)} v_1(t) \end{pmatrix},$$

with $r(t) = \frac{1}{4} \ln \left[\frac{\lambda_{P'_{\text{BM}}}(t)}{\lambda_{Q'_{\text{BM}}}(t)} \right]$ the squeezing parameter. It is convenient to define $\lambda_{\pm}(t) = \max/\min\{\lambda_{Q'_{\text{BM}}}(t), \lambda_{P'_{\text{BM}}}(t)\}$. Explicitly, we have

$$\lambda_{\pm}(t) = \frac{1}{2} \{C(T_0) + \bar{\alpha}^2 [\mathcal{T}(t) \pm \sqrt{\mathcal{T}^2(t) - 4\Delta(t)}]\}, \quad (17)$$

obtained from Eq. (5), which satisfy

$$\lambda_+(t)\lambda_-(t) = v_1^2(t). \quad (18)$$

If $r(t) \neq 0$, squeezing occurs, and two qualitatively different regimes can be distinguished. When both $\lambda_{\pm}(t) > 1/2$, one refers to *classical squeezing*: both variances are large and dominated by semiclassical fluctuations. In this case, squeezing merely reflects the anisotropy of the two variances. Conversely, when $\lambda_-(t) < \frac{1}{2}$ one of the two variances falls *below the ground state limit*, indicating a regime of *genuinely quantum squeezing*, whose origin cannot be ascribed to any classical effect. Figure 2 displays the regions of classical and quantum squeezing in the (T, T_0) parameter space. They are separated by the yellow contour line corresponding to $\lambda_-(t^*) = 1/2$. The underlying density plot shows the squeezing parameter $r(t^*)$, revealing that the BM is *always squeezed*. As both temperatures decrease, the system transitions into a broad regime of *quantum squeezing*. The high degree of controllability allowed by tuning T and T_0 , which warrants the ability to access distinct squeezing regimes, makes our model an ideal candidate to quantitatively analyze the presence of genuinely quantum advantage. Finally, recalling that the energy of the BM is obtained by tracing $\Sigma_{\text{BM}}(t)$, we obtain

$$E_{\text{BM}}(t) = \frac{\hbar\omega_0}{2} [\lambda_+(t) + \lambda_-(t)]. \quad (19)$$

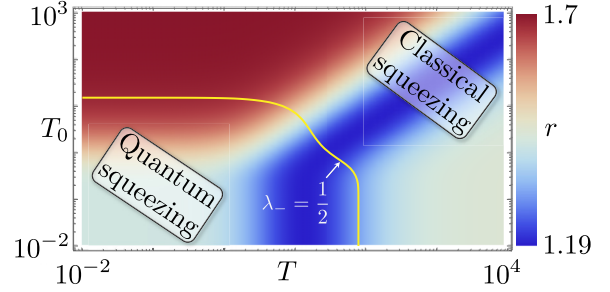


FIG. 2. Density plot of $r = r(t^*)$ as a function of T and T_0 (units $\hbar\omega_0/k_B$). The yellow line corresponds to $\lambda_- = \lambda_-(t^*) = 1/2$. Here, $N = 4$, $\alpha_I = 1$, $\gamma_0 = 5\omega_0$, and $\omega_D = 2\omega_0$.

B. Entanglement

A source of genuinely quantum correlation is the *entanglement* between the different components of the battery, which we assess in this work by means of the logarithmic negativity $\mathcal{N}(t)$, a witness of entanglement [63]. We derive here a closed expression for this quantity.

To begin, we consider a balanced bipartition composed of two subsystems A and B, each with $N/2$ oscillators—assuming N even. Since the dynamics of the QB is entirely captured by the BM, it is sufficient to consider a bipartition of the BM alone. This is done by introducing new quadratures Q_μ, P_μ ($\mu = A, B$), corresponding to the two partitions, defined as

$$\begin{aligned} (Q_A, P_A) &= \frac{\sqrt{2}}{\bar{\alpha}} \sum_{j=1}^{N/2} \alpha_j (Q_j, P_j), \\ (Q_B, P_B) &= \frac{\sqrt{2}}{\bar{\alpha}} \sum_{j=1+N/2}^N \alpha_j (Q_j, P_j). \end{aligned} \quad (20)$$

Given the bipartite state vector $(Q_A(t), P_A(t), Q_B(t), P_B(t))^T$, its covariance matrix can be written in block form as

$$\sigma_{\text{bip}}(t) = \begin{pmatrix} L(t) & K(t) \\ K^t(t) & L(t) \end{pmatrix}, \quad (21)$$

with

$$L(t) = \frac{C(T_0)}{2} \mathbb{1} + K(t), \quad K(t) = \frac{\bar{\alpha}^2}{2} \begin{pmatrix} a(t) & b(t) \\ b(t) & c(t) \end{pmatrix}. \quad (22)$$

The logarithmic negativity is then defined as $\mathcal{N}(t) = \sum_{j=\pm} \max\{0, -\ln[2v_j^{(\text{PT})}(t)]\}$, where $v_-^{(\text{PT})}(t) \leq v_+^{(\text{PT})}(t)$ are the symplectic eigenvalues of the *partially transposed covariance matrix* $\sigma_{\text{bip}}^{(\text{PT})}(t) = \Lambda \sigma_{\text{bip}}(t) \Lambda$, where $\Lambda = \text{diag}\{1, 1, 1, -1\}$ [63]. Entanglement is present if and only if $\mathcal{N}(t) > 0$ [77], which requires at least one symplectic eigenvalue $v_j^{(\text{PT})}(t) < 1/2$. However, since it is always $v_+^{(\text{PT})}(t)v_-^{(\text{PT})}(t) \geq 1/4$, only $v_-^{(\text{PT})}(t)$ can be smaller than $1/2$ which leads to

$$\mathcal{N}(t) = \max\{0, -\ln[2v_-^{(\text{PT})}(t)]\}. \quad (23)$$

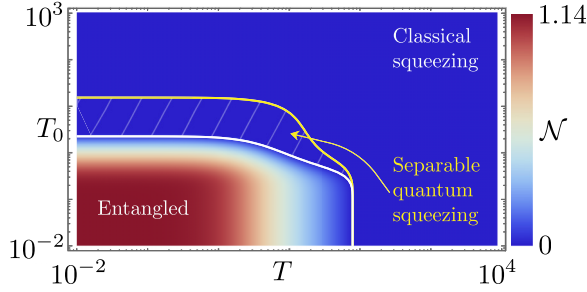


FIG. 3. Density plot of $\mathcal{N} = \mathcal{N}(t^*)$ as a function of T and T_0 (units $\hbar\omega_0/k_B$). The white line, $v_-^{(\text{PT})}(t^*) = 1/2$, marks the boundary between entangled and separable states. The yellow line, $\lambda_-(t^*) = 1/2$, separates the regions of quantum and classical squeezing. Here, $N = 4$, with a balanced partition $N_A = N_B = 2$, $\alpha_l = 1$, $\gamma_0 = 5\omega_0$, and $\omega_D = 2\omega_0$.

To evaluate $v_-^{(\text{PT})}(t)$, we follow standard procedures [63]. After some algebra, we find

$$v_-^{(\text{PT})}(t) = \frac{\sqrt{C(T_0)}}{2} \sqrt{C(T_0) + \bar{\alpha}^2[\mathcal{T}(t) - \sqrt{\mathcal{T}^2(t) - 4\Delta(t)}]}. \quad (24)$$

Comparing Eqs. (24) and (17), we immediately obtain the important relation

$$v_-^{(\text{PT})}(t) = \sqrt{\frac{C(T_0)}{2}} \sqrt{\lambda_-(t)}. \quad (25)$$

This equality allows us to conclude that the presence of entanglement ($v_-^{(\text{PT})}(t) < 1/2$) directly implies $\lambda_-(t) < \frac{1}{2}$. Thus, for a balanced bipartition, the condition “Entanglement $\Rightarrow$ Quantum squeezing” is always fulfilled, while the converse is not true.

Figure 3 shows the typical behavior of $\mathcal{N} = \mathcal{N}(t^*)$ as a function of T and T_0 . As expected, entanglement emerges at low T and T_0 , within the region bounded by the white line ($v_-^{(\text{PT})}(t^*) = 1/2$). This region lies entirely inside the yellow boundary of quantum squeezing ($\lambda_-(t^*) = 1/2$), thus confirming that entanglement implies quantum squeezing.

C. Quantum discord

We conclude this part by briefly commenting about another relevant measure of nonclassical correlations: the quantum discord [78–80]. As reported in the literature [81] and confirmed in Appendix E for our system, this quantity does not exhibit any sharp feature either at the boundary between classical and quantum squeezing (yellow curve in Fig. 3) or at the boundary between separable and nonseparable states (white curve in Fig. 3). Thus, although it can quantify the overall amount of quantum correlations and may be linked to ergotropy [51], the quantum discord is not helpful in our case since it does not clearly distinguish between different types of quantum correlations, in contrast with the other quantities discussed above. Such a clear distinction is, in turns, crucial to derive analytical bounds on the efficiency of the battery and to prove the existence of genuine quantum advantage as we are going to show here below.

V. QUANTUM ADVANTAGE

We compare here the classical versus quantum regimes by choosing two configurations with the same number of oscillators N , equal temperatures T_0 and T , and same energy $E_B(t)$ stored in the battery—and consequently the same BM energy $E_{\text{BM}}(t)$. Our aim is to show that quantum squeezing and entanglement improve the global efficiency.

We begin by deriving a bound $\mathcal{B}_{\text{cl}}(t)$ that $\eta_{\text{glob}}(t)$ satisfies when quantum squeezing occurs. To do so, we start by rewriting the global efficiency $\eta_{\text{glob}}(t)$ in Eq. (12), using the link derived in Eq. (18). We have

$$\eta_{\text{glob}}(t) = 1 - \frac{\hbar\omega_0}{E'_{\text{BM}}(t)} [\sqrt{\lambda_-(t)\lambda_+(t)} - 1/2]. \quad (26)$$

Then, exploiting Eq. (19) we express $\lambda_+(t)$ in terms of $\lambda_-(t)$ as $\lambda_+(t) = \frac{2E_{\text{BM}}(t)}{\hbar\omega_0} - \lambda_-(t)$. Inserting this expression into $\eta_{\text{glob}}(t)$, we can express it in terms of $\lambda_-(t)$ only:

$$\eta_{\text{glob}}(t) = 1 - \frac{1}{\kappa(t)} [\sqrt{[2\kappa(t) + 1 - \lambda_-(t)]\lambda_-(t)} - 1/2], \quad (27)$$

where

$$\kappa(t) = \frac{E'_{\text{BM}}(t)}{\hbar\omega_0} > 0. \quad (28)$$

Notice that since $\lambda_+(t) \geq \lambda_-(t)$, it follows that $\lambda_-(t) \leq \kappa(t) + 1/2$. Therefore, $\eta_{\text{glob}}(t)$ is monotonically decreasing in $\lambda_-(t)$. To obtain the explicit expression for the classical bound $\mathcal{B}_{\text{cl}}(t)$, we impose $\lambda_-(t) = 1/2$, corresponding to the boundary between classical and quantum squeezing, in Eq. (27) obtaining

$$\mathcal{B}_{\text{cl}}(t) = 1 - \frac{1}{2\kappa(t)} [\sqrt{4\kappa(t) + 1} - 1]. \quad (29)$$

We can then conclude that if $\lambda_-(t) > 1/2$ (classical squeezing), one has $0 \leq \eta_{\text{glob}}(t) < \mathcal{B}_{\text{cl}}(t)$, while, on the other hand, if $\lambda_-(t) < 1/2$ (quantum squeezing), it is $\mathcal{B}_{\text{cl}}(t) < \eta_{\text{glob}}(t) \leq 1$. This proves that quantum squeezing increases the performances of the battery with respect to a classically squeezed regime. Moving from the classical to the quantum regime is then possible by varying the value of $\lambda_-(t)$. Notice that this procedure is achievable, even if the energy $E_{\text{BM}}(t)$ is fixed, by varying γ_0 and ω_D .

Figure 4(a) shows an example of the enhancement of both $\eta_{\text{glob}}(t^*)$ and $\eta_{\text{th}}(t^*)$ due to the presence of quantum squeezing. In particular, as long as $\lambda_- > 1/2$ (classical squeezing), the global efficiency $\eta_{\text{glob}}(t^*)$ (blue curve) remains below the bound $\mathcal{B}_{\text{cl}}(t^*)$ (dashed line), in full agreement with the results discussed above. Conversely, for $\lambda_- < 1/2$ (quantum squeezing), the global efficiency exceeds this bound. The thermal efficiency $\eta_{\text{th}}(t^*)$ (red curve) exhibits a qualitatively similar behavior.

We now proceed to obtain, within the regime of quantum squeezing ($\lambda_-(t) < 1/2$), a further bound $\mathcal{B}_{\text{en}}(t)$ that $\eta_{\text{glob}}(t)$ satisfies when entanglement is present ($v_-^{(\text{PT})} \geq 1/2$). To find $\mathcal{B}_{\text{en}}(t)$, we start from Eq. (25), which gives $\lambda_-(t) = 2[v_-^{(\text{PT})}(t)]^2/C(T_0)$. The bound is obtained imposing $v_-^{(\text{PT})}(t) = 1/2$, which corresponds to $\lambda_-(t) = [2C(T_0)]^{-1}$, into Eq. (27). This yields $\mathcal{B}_{\text{en}}(t) = \eta_{\text{glob}}(t)|_{\lambda_-(t)=[2C(T_0)]^{-1}}$ or, in

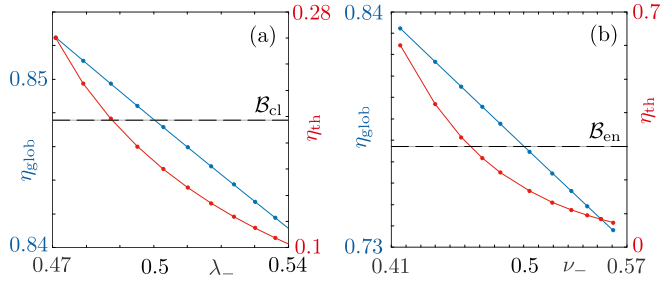


FIG. 4. Efficiencies $\eta_{\text{glob}} = \eta_{\text{glob}}(t^*)$ (blue, left scale) and $\eta_{\text{th}} = \eta_{\text{th}}(t^*)$ (red, right scale) as a function of (a) λ_- , for $k_B T_0 = 6\hbar\omega_0$ and $E_B(t^*) = 67.07\hbar\omega_0$, and (b) ν_- for $k_B T_0 = \hbar\omega_0$ and $E_B(t^*) = 10.41\hbar\omega_0$. The dashed lines represent the bounds $\mathcal{B}_{\text{cl}} = \mathcal{B}_{\text{cl}}(t^*)$ [panel (a)] and $\mathcal{B}_{\text{en}} = \mathcal{B}_{\text{en}}(t^*)$ [panel (b)]. In all panels, $k_B T = 0.01\hbar\omega_0$, $\alpha_l = 1$, and $N = 6$.

explicit form,

$$\mathcal{B}_{\text{en}}(t) = 1 - \frac{1}{2\kappa(t)} \left[\sqrt{\left[4\kappa(t) + 2 - \frac{1}{C(T_0)} \right] \frac{1}{C(T_0)}} - 1 \right], \quad (30)$$

with $\mathcal{B}_{\text{cl}}(t) < \mathcal{B}_{\text{en}}(t) \leq 1$ always. This leads, within the quantum squeezing regime, to

$$\begin{aligned} \mathcal{B}_{\text{cl}}(t) < \eta_{\text{glob}}(t) &\leq \mathcal{B}_{\text{en}}(t) \quad \text{without entanglement} \\ \left(\frac{1}{2C(T_0)} \leq \lambda_-(t) < \frac{1}{2} \right), \\ \mathcal{B}_{\text{en}}(t) < \eta_{\text{glob}}(t) &\leq 1 \quad \text{with entanglement} \\ \left(\lambda_-(t) < \frac{1}{2C(T_0)} \right). \end{aligned} \quad (31)$$

These *additional* new bounds prove that entanglement increases the performances of the QB. As shown in Fig. 4(b), the impact of entanglement on the efficiencies $\eta_{\text{glob}}(t^*)$ and $\eta_{\text{th}}(t^*)$ is qualitatively similar to that discussed above for quantum squeezing. In particular, the global efficiency $\eta_{\text{glob}}(t^*)$ (blue curve) remains below the bound $\mathcal{B}_{\text{en}}(t^*)$ (dashed line) for $\nu_- > 1/2$ (absence of entanglement), and exceeds this bound only for $\nu_- < 1/2$, i.e., when entanglement is present. Note that also in this case the behavior of the thermal efficiency $\eta_{\text{th}}(t^*)$ (red curve) is qualitatively similar.

The above bounds have been shown at $t = t^*$, where the maximum of the battery energy $E_B(t)$ occurs. However, their particular value is very stable with respect to t , as shown in Appendix F.

VI. CONCLUSION

We have demonstrated that multipartite Gaussian QBs can exhibit a genuine quantum advantage in their global efficiency, with analytically derived bounds establishing a clear hierarchy from classical squeezing to quantum squeezing and entanglement. Our numerical results also suggest a similar quantum advantage for the thermodynamic efficiency, which captures the overall thermodynamic cost of a QB, thereby further strengthening our findings.

The interaction between the harmonic oscillators composing the QB and the bath constitutes the key physical resource responsible for the observed quantum advantage. Remarkably, the environment, which is often regarded as detrimental to quantum correlations, plays here an active role in their generation, ultimately enhancing the battery efficiency. Moreover, in our case, the environment-induced entanglement is associated with higher efficiency, which can be understood as follows. Ergotropy is operationally defined via the construction of the passive state corresponding to a given density matrix, obtained by rearranging its eigenvalues so that the largest populations occupy the lowest-energy eigenstates of the system Hamiltonian. Removing the constraint of separability, entangled states can—at least in principle—enable a more efficient global reorganization of populations, resulting in greater ergotropy. Although this mechanism is not universal for arbitrary many-body quantum systems [51], it emerges as a robust and well-defined feature of the Gaussian dissipative quantum battery studied here.

The proposed model can be realized in the context of quantum LC circuits [82], embedded in a dissipative environment with tunable dissipation strength [82–84]. This generalizes the analysis in Ref. [36] from a single LC quantum battery to a series of N coupled elements. Indeed, each single cell composing the Gaussian QB can be thought of as a single LC element, where the capacitance C plays the role of the mass m and the inductance maps as $L \rightarrow 1/(m\omega_0^2)$. The associated circuit energy is the sum of the energies of all individual LC cells and corresponds to the QB Hamiltonian given in Eq. (1). The dissipative environment can be modeled using a lumped-element model [36,85]. In this framework, the dissipation strength is inversely proportional to the environment resistance, i.e., $\gamma_0 \propto R_{\text{env}}^{-1}$.

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DATA AVAILABILITY

All data shown in the manuscript have been produced using a Wolfram Mathematica code. They will be provided upon reasonable request.

APPENDIX A: LANGEVIN EQUATION AND ITS SOLUTION

From the total Hamiltonian of the N -oscillator quantum battery coupled to a reservoir—see Eqs. (1) and (2)—one can follow standard procedures [36,54,58,86] to derive a coupled set of Langevin equations of motion for the battery oscillator

operators $q_l(t)$. They read

$$\ddot{q}_l(t) + \omega_0^2 q_l(t) + \alpha_l \int_0^t ds \gamma(t-s) \sum_{l'=1}^N \alpha_{l'} \dot{q}_{l'}(s) + \alpha_l \gamma(t) \sum_{l'=1}^N \alpha_{l'} q_{l'}(0) = \frac{\alpha_l}{m} \xi(t), \quad (\text{A1})$$

with damping kernel $\gamma(t)$ and thermal noise operator $\xi(t)$ defined as

$$\gamma(t) = \frac{\theta(t)}{m} \sum_{k=1}^{\infty} \frac{c_k^2}{m_k \omega_k^2} \cos(\omega_k t),$$

$$\xi(t) = \sum_{k=1}^{\infty} c_k \left[x_k(0) \cos(\omega_k t) + \frac{\pi_k(0)}{m_k \omega_k} \sin(\omega_k t) \right]. \quad (\text{A2})$$

Notice that the noise depends only on the reservoir operators $x_k(0)$, $\pi_k(0)$ at initial time with the reservoir assumed to be in equilibrium at temperature T . This implies $\langle \xi(t) \rangle = 0$ with reservoir correlator [54]

$$\langle \xi(t) \xi(t') \rangle = \hbar \int_0^{\infty} \frac{d\omega}{\pi} J(\omega) \times \left\{ \coth \left(\frac{\hbar \omega}{2k_B T} \right) \cos[\omega(t-t')] - i \sin[\omega(t-t')] \right\}, \quad (\text{A3})$$

written in terms of the spectral density [54]

$$J(\omega) = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{c_k^2}{m_k \omega_k} \delta(\omega - \omega_k). \quad (\text{A4})$$

Applying the Laplace transform to the Langevin equation, we get

$$[\lambda^2 + \omega_0^2 + \lambda \alpha_l^2 \tilde{\gamma}(\lambda)] \tilde{q}_l(\lambda) + \alpha_l \lambda \tilde{\gamma}(\lambda) \sum_{l' \neq l} \alpha_{l'} \tilde{q}_{l'}(\lambda) = \lambda q_l(0) + \dot{q}_l(0) + \frac{\alpha_l \tilde{\xi}(\lambda)}{m}.$$

The above set of equations can be now written in the compact form $\chi^{-1}(\lambda) \tilde{\mathbf{q}}(\lambda) = \tilde{\mathbf{y}}(\lambda)$, with the N -dimensional vector $\tilde{\mathbf{q}}^t(\lambda) = (\tilde{q}_1(\lambda), \dots, \tilde{q}_N(\lambda))$, the external vector $\tilde{\mathbf{y}}^t(\lambda) = (\lambda q_1(0) + \dot{q}_1(0) + \alpha_1 \tilde{\xi}(\lambda)/m, \dots, \lambda q_N(0) + \dot{q}_N(0) + \alpha_N \tilde{\xi}(\lambda)/m)$, and the $N \times N$ inverse resolvent matrix:

$\chi^{-1}(\lambda) = \begin{pmatrix} \Delta_1(\lambda) & \lambda \alpha_1 \alpha_2 \tilde{\gamma}(\lambda) & \dots & \lambda \alpha_1 \alpha_N \tilde{\gamma}(\lambda) \\ \lambda \alpha_2 \alpha_1 \tilde{\gamma}(\lambda) & \Delta_2(\lambda) & \dots & \lambda \alpha_2 \alpha_N \tilde{\gamma}(\lambda) \\ \dots & \dots & \dots & \dots \\ \lambda \alpha_N \alpha_1 \tilde{\gamma}(\lambda) & \lambda \alpha_N \alpha_2 \tilde{\gamma}(\lambda) & \dots & \Delta_N(\lambda) \end{pmatrix}, \quad (\text{A5})$

with diagonal elements $\Delta_l(\lambda) = \lambda^2 + \omega_0^2 + \lambda \alpha_l^2 \tilde{\gamma}(\lambda)$. Upon inversion, one obtains the formal solution for the oscillator operators $\tilde{\mathbf{q}}(\lambda)$ in the Laplace domain:

$$\tilde{\mathbf{q}}(\lambda) = \chi(\lambda) \tilde{\mathbf{y}}(\lambda), \quad (\text{A6})$$

with the explicit form of the matrix elements of $\chi(\lambda)$ given by

$$\chi_{ll}(\lambda) = \frac{\lambda^2 + \omega_0^2 + \sum_{j \neq l} \alpha_j^2 \lambda \tilde{\gamma}(\lambda)}{[\lambda^2 + \omega_0^2][\lambda^2 + \omega_0^2 + \bar{\alpha}^2 \lambda \tilde{\gamma}(\lambda)]}, \quad (\text{A7})$$

$$\chi_{ll'}(\lambda)|_{l \neq l'} = \frac{-\alpha_l \alpha_{l'} \lambda \tilde{\gamma}(\lambda)}{[\lambda^2 + \omega_0^2][\lambda^2 + \omega_0^2 + \bar{\alpha}^2 \lambda \tilde{\gamma}(\lambda)]}. \quad (\text{A8})$$

Here, we introduce the global interaction weight

$$\bar{\alpha} = \sqrt{\sum_{j=1}^N \alpha_j^2}. \quad (\text{A9})$$

From the solution in Eq. (A6), the behavior of the position operators in the time domain follows immediately. They naturally separate in a homogeneous part, depending on the initial conditions, and in a thermal contribution due to the noise: $q_l(t) = q_{l,h}(t) + q_{l,th}(t)$, with

$$q_{l,h}(t) = \sum_{l'=1}^N [q_{l'}(0) \dot{\chi}_{ll'}(t) + \dot{q}_{l'}(0) \chi_{ll'}(t)],$$

$$q_{l,th}(t) = \int_0^t ds \frac{\xi(s)}{m} \sum_{l'=1}^N \alpha_{l'} \chi_{l,l'}(t-s). \quad (\text{A10})$$

Here, $\chi_{ll'}(t)$ are the components of the matrix elements given in Eq. (A8) transformed back to the time domain, satisfying the boundary conditions $\chi_{l,l'}(0) = 0$, $\dot{\chi}_{l,l'}(0) = \delta_{l,l'}$ and $\ddot{\chi}_{l,l'}(0) = 0$.

We now proceed to write the explicit form of the resolvent matrix reservoir $\chi(\lambda)$ in the case of the Drude damping, given by

$$\gamma(t) = \gamma_0 \omega_D e^{-\omega_D t} \theta(t), \quad \tilde{\gamma}(\lambda) = \frac{\gamma_0 \omega_D}{\lambda + \omega_D}, \quad (\text{A11})$$

where γ_0 is the damping strength and ω_D the cutoff frequency of the reservoir. Inserting this form in Eq. (A8), we obtain

$$\chi_{ll}(\lambda) = \frac{(\lambda^2 + \omega_0^2)(\lambda + \omega_D) + \lambda \Omega_0^2 \sum_{l'=1, l' \neq l}^N \alpha_{l'}^2}{\lambda^2 + \omega_0^2} \prod_{i=3}^5 \frac{1}{(\lambda + \mu_i)}, \quad \chi_{ll'}(\lambda)|_{l \neq l'} = \frac{-\alpha_l \alpha_{l'} \lambda \Omega_0^2}{\lambda^2 + \omega_0^2} \prod_{i=3}^5 \frac{1}{(\lambda + \mu_i)}, \quad (\text{A12})$$

with $\Omega_0 = \sqrt{\gamma_0 \omega_D}$.

Notice that $\chi_{ll'}(\lambda)$ has five poles. Two of them are free, completely decoupled from the reservoir: $\mu_{1,2} = \pm i \omega_0$, while

the other three, μ_i with $i = 3, 4, 5$, satisfy the equation

$$(\omega_0^2 + \mu_i^2)(\omega_D - \mu_i) - \mu_i \bar{\alpha}^2 \Omega_0^2 = 0. \quad (\text{A13})$$

They depend on the coupling with the reservoir via the global interaction $\bar{\alpha}$ defined in Eq. (A9). With the above decomposition, the key expressions of the time-dependent matrix elements $\chi_{ll'}(t)$ are obtained via an inverse Laplace transform:

$$\chi_{ll'}(t) = \frac{\sin(\omega_0 t)}{\omega_0} \delta_{ll'} + \alpha_l \alpha_{l'} \left[A(t) - \frac{\sin(\omega_0 t)}{\bar{\alpha}^2 \omega_0} \right],$$

$$A(t) = \Omega_0^2 \sum_{i=3}^5 \mu_i e^{-\mu_i t} \left[\prod_{r=1; r \neq i}^5 \frac{1}{(\mu_r - \mu_i)} \right]. \quad (\text{A14})$$

Notice that the function $A(t)$ does not explicitly depend on the local coupling α_l , but only on the poles μ_i . In addition, due to the initial conditions on $\chi_{ll'}(t)$, we have $A(0) = 0$, $\dot{A}(0) = \frac{1}{\bar{\alpha}^2}$, and $\ddot{A}(0) = 0$.

$$\sigma_B(t) = \begin{pmatrix} \sigma_{Q_1, Q_1}(t) & \sigma_{Q_1, P_1}(t) & \cdots & \sigma_{Q_1, Q_N}(t) & \sigma_{Q_1, P_N}(t) \\ \sigma_{P_1, Q_1}(t) & \sigma_{P_1, P_1}(t) & \cdots & \sigma_{P_1, Q_N}(t) & \sigma_{P_1, P_N}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{Q_N, Q_1}(t) & \sigma_{Q_N, P_1}(t) & \cdots & \sigma_{Q_N, Q_N}(t) & \sigma_{Q_N, P_N}(t) \\ \sigma_{P_N, Q_1}(t) & \sigma_{P_N, P_1}(t) & \cdots & \sigma_{P_N, Q_N}(t) & \sigma_{P_N, P_N}(t) \end{pmatrix} \quad (\text{B2})$$

and depends on symmetrized correlation functions of the form $\sigma_{A,B}(t) = \frac{1}{2} \langle (A(t)B(t) + B(t)A(t)) \rangle - \langle A(t) \rangle \langle B(t) \rangle$. We recall the equilibrium initial conditions of the N oscillators, at temperature T_0 , given by the density matrix: $e^{-H_B/k_B T_0}/Z_B$. This implies $\langle Q_l(t) \rangle = \langle P_l(t) \rangle = 0$, $[Q_l(t), P_l(t)] = i$, $\langle Q_l(0)P_l(0) + P_l(0)Q_l(0) \rangle = 0$, and

$$\langle Q_l^2(0) \rangle = \langle P_l^2(0) \rangle = \frac{C(T_0)}{2}, \quad (\text{B3})$$

with $C(T_0) = \coth(\frac{\hbar\omega_0}{2k_B T_0})$.

Following Eq. (A10) and taking into account that the averages between homogeneous and thermal contributions are always zero, the CM elements decouple into a homogeneous and a thermal contribution:

$$\sigma_{A,B}(t) = \sigma_{A,B}^{(h)}(t) + \sigma_{A,B}^{(th)}(t). \quad (\text{B4})$$

The calculation of the homogeneous part is straightforward and we quote here the main steps only. We start by inserting, in the averages, the homogeneous behaviors of $q_l(t)$ and $\dot{q}_l(t)$, written in Eq. (A10):

$$\sigma_{Q_l, Q_{l'}}^{(h)}(t) = \frac{C(T_0)}{2} \sum_{j=1}^N [\dot{\chi}_{l,j}(t) \dot{\chi}_{l',j}(t) + \omega_0^2 \chi_{l,j}(t) \chi_{l',j}(t)],$$

$$\sigma_{Q_l, P_{l'}}^{(h)}(t) = \frac{C(T_0)}{2\omega_0} \sum_{j=1}^N [\dot{\chi}_{l,j}(t) \ddot{\chi}_{l',j}(t) + \omega_0^2 \chi_{l,j}(t) \dot{\chi}_{l',j}(t)],$$

$$\sigma_{P_l, P_{l'}}^{(h)}(t) = \frac{C(T_0)}{2\omega_0^2} \sum_{j=1}^N [\ddot{\chi}_{l,j}(t) \ddot{\chi}_{l',j}(t) + \omega_0^2 \dot{\chi}_{l,j}(t) \dot{\chi}_{l',j}(t)]. \quad (\text{B5})$$

To reach a compact form, it is sufficient to use the explicit time behavior of the matrix elements $\chi_{ll'}(t)$ of the resolvent,

APPENDIX B: LOCAL AND GLOBAL COVARIANCE MATRIX

In this part, we introduce the covariance matrix, associated with the QB, using local and global approaches.

1. Local covariance matrix

We start by introducing the dimensionless quadrature operators of the QB:

$$Q_l(t) = \sqrt{\frac{m\omega_0}{\hbar}} q_l(t), \quad P_l(t) = \sqrt{\frac{1}{\hbar m \omega_0}} p_l(t) = \sqrt{\frac{m}{\hbar \omega_0}} \dot{q}_l(t). \quad (\text{B1})$$

The general form of the CM is

quoted in Eq. (A14). The final results are

$$\sigma_{Q_l, Q_{l'}}^{(h)}(t) = \frac{C(T_0)}{2} \delta_{ll'} + \alpha_l \alpha_{l'} a_h(t),$$

$$\sigma_{Q_l, P_{l'}}^{(h)}(t) = \alpha_l \alpha_{l'} b_h(t),$$

$$\sigma_{P_l, P_{l'}}^{(h)}(t) = \frac{C(T_0)}{2} \delta_{ll'} + \alpha_l \alpha_{l'} c_h(t). \quad (\text{B6})$$

Here, we introduced the three functions $a_h(t)$, $b_h(t)$, and $c_h(t)$:

$$a_h(t) = \frac{C(T_0)}{2} \left[\bar{\alpha}^2 (\dot{A}^2(t) + \omega_0^2 A^2(t)) - \frac{1}{\bar{\alpha}^2} \right],$$

$$b_h(t) = \frac{C(T_0)}{2\omega_0} \bar{\alpha}^2 \dot{A}(t) [\ddot{A}(t) + \omega_0^2 A(t)],$$

$$c_h(t) = \frac{C(T_0)}{2\omega_0^2} \left[\bar{\alpha}^2 (\ddot{A}^2(t) + \omega_0^2 \dot{A}^2(t)) - \frac{\omega_0^2}{\bar{\alpha}^2} \right]. \quad (\text{B7})$$

We remind that the global interaction weight $\bar{\alpha}$ and the function $A(t)$ are defined in Eqs. (A9) and (A14), respectively.

Finally, we evaluate the thermal contributions. As a starting point, we write the time-dependent evolutions of $q_{l,th}(t)$ and $\dot{q}_{l,th}(t)$ in Eq. (A10) in the compact form:

$$q_{l,th}(t) = \int_0^t ds \frac{\xi(s)}{m} \bar{\chi}_l(t-s),$$

$$\dot{q}_{l,th}(t) = \int_0^t ds \frac{\xi(s)}{m} \frac{d}{dt} \bar{\chi}_l(t-s), \quad (\text{B8})$$

with

$$\bar{\chi}_l(t) \equiv \sum_{l'=1}^N \alpha_{l'} \chi_{l,l'}(t). \quad (\text{B9})$$

Inserting $\chi_{l,l'}(t)$ given in Eq. (A14), the above resolvent function can be rewritten as

$$\bar{\chi}_l(t) = \alpha_l \bar{\alpha}^2 A(t) = \alpha_l \sum_{i=3}^5 e^{-\mu_i t} \mu_i \Omega_N^2 \times \left[\prod_{r=1; r \neq i}^5 \frac{1}{(\mu_r - \mu_i)} \right] \equiv \alpha_l \sum_{i=3}^5 e^{-\mu_i t} \bar{\chi}^{(i)}, \quad (\text{B10})$$

where, in the next-to-last equality, we introduced the characteristic frequency

$$\Omega_N = \bar{\alpha} \sqrt{\gamma_0 \omega_D} = \bar{\alpha} \Omega_0. \quad (\text{B11})$$

Inserting now Eq. (B8) in the CM elements, we have

$$\begin{aligned} \sigma_{Q_l Q_{l'}}^{(\text{th})}(t) &= \frac{\omega_0}{2m\hbar} \int_0^t ds_1 \\ &\times \int_0^t ds_2 \bar{\chi}_l(t-s_1) \bar{\chi}_{l'}(t-s_2) \langle \{\xi(s_1), \xi(s_2)\} \rangle, \\ \sigma_{Q_l P_{l'}}^{(\text{th})}(t) &= \frac{1}{2m\hbar} \int_0^t ds_1 \int_0^t ds_2 \bar{\chi}_l(t-s_1) \\ &\times \left[\frac{d}{dt} \bar{\chi}_{l'}(t-s_2) \right] \langle \{\xi(s_1), \xi(s_2)\} \rangle, \\ \sigma_{P_l P_{l'}}^{(\text{th})}(t) &= \frac{1}{2\hbar m \omega_0} \int_0^t ds_1 \int_0^t ds_2 \\ &\times \left[\frac{d}{dt} \bar{\chi}_l(t-s_1) \right] \left[\frac{d}{dt} \bar{\chi}_{l'}(t-s_2) \right] \langle \{\xi(s_1), \xi(s_2)\} \rangle, \end{aligned} \quad (\text{B12})$$

with the symmetric reservoir correlator represented using the average in Eq. (A3):

$$\begin{aligned} \frac{1}{2} \langle \{\xi(s_1), \xi(s_2)\} \rangle &= \hbar \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} J(\omega) \\ &\times \left\{ \coth \left(\frac{\hbar\omega}{2k_B T} \right) \cos[\omega(s_1 - s_2)] \right\}. \end{aligned} \quad (\text{B13})$$

With this Fourier representation, the two time integrals can be easily evaluated, due to the exponential time behavior of $\bar{\chi}_l(t)$ in Eq. (B10). The final results are

$$\begin{aligned} \sigma_{Q_l Q_{l'}}^{(\text{th})}(t) &= \alpha_l \alpha_{l'} a_{\text{th}}(t), \\ \sigma_{Q_l P_{l'}}^{(\text{th})}(t) &= \alpha_l \alpha_{l'} b_{\text{th}}(t), \\ \sigma_{P_l P_{l'}}^{(\text{th})}(t) &= \alpha_l \alpha_{l'} c_{\text{th}}(t), \end{aligned} \quad (\text{B14})$$

where

$$\begin{aligned} a_{\text{th}}(t) &= \frac{\omega_0}{m} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[J(\omega) \coth \left(\frac{\hbar\omega}{2k_B T} \right) \right] \sum_{j,r=1}^3 L_{j,r}, \\ b_{\text{th}}(t) &= -\frac{1}{m} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[J(\omega) \coth \left(\frac{\hbar\omega}{2k_B T} \right) \right] \sum_{j,r=1}^3 \mu_j L_{j,r}, \end{aligned}$$

$$c_{\text{th}}(t) = \frac{1}{m\omega_0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[J(\omega) \coth \left(\frac{\hbar\omega}{2k_B T} \right) \right] \sum_{j,r=1}^3 \mu_j \mu_r L_{j,r}, \quad (\text{B15})$$

with

$$J(\omega) = m\gamma_0 \omega_D^2 \frac{\omega}{\omega^2 + \omega_D^2} \quad (\text{B16})$$

the Drude spectral density and

$$\begin{aligned} L_{j,r} &= \frac{\bar{\chi}^{(j)} \bar{\chi}^{(r)}}{(\mu_r + i\omega)(\mu_j - i\omega)} \\ &\times [1 + e^{-(\mu_j + \mu_r)t} - e^{-(\mu_j - i\omega)t} - e^{-(\mu_r + i\omega)t}]. \end{aligned} \quad (\text{B17})$$

We remind that the weights $\bar{\chi}^{(j)}$, associated with the poles μ_j , are written in Eq. (B10).

We conclude by summarizing the total expressions of the CM matrix elements. We have

$$\sigma_{\text{B}}(t) = \frac{C(T_0)}{2} \mathbb{1} + \begin{pmatrix} \Sigma^{(1,1)}(t) & \dots & \Sigma^{(1,N)}(t) \\ \vdots & \ddots & \vdots \\ \Sigma^{(N,1)}(t) & \dots & \Sigma^{(N,N)}(t) \end{pmatrix}, \quad (\text{B18})$$

where

$$\Sigma^{(i,j)}(t) = \alpha_i \alpha_j \begin{pmatrix} a(t) & b(t) \\ b(t) & c(t) \end{pmatrix} \quad (\text{B19})$$

is the block matrix due to the QB/reservoir coupling with elements:

$$\begin{aligned} a(t) &= a_{\text{h}}(t) + a_{\text{th}}(t), \\ b(t) &= b_{\text{h}}(t) + b_{\text{th}}(t), \\ c(t) &= c_{\text{h}}(t) + c_{\text{th}}(t), \end{aligned} \quad (\text{B20})$$

given in Eqs. (B7) and (B15).

2. Global covariance matrix

In the previous Appendix, we described a *local* approach to the problem, with the CM written in terms of the quadratures of each individual oscillator. Here, we outline the *global* approach, discussed in the main part, where the CM is expressed in terms of global coordinates.

All starts observing that the bright mode, described by the quadratures $Q_{\text{BM}}(t) \equiv \frac{1}{\alpha} \sum_{l=1}^N \alpha_l Q_l(t)$ and $P_{\text{BM}}(t) \equiv \frac{1}{\alpha} \sum_{l=1}^N \alpha_l P_l(t)$ [see Eq. (3)], is the only one coupled to the reservoir and thus evolving nontrivially, as shown in Eq. (2). There then exist $N-1$ *freely evolving* dark modes, with canonical quadratures $Q_{\text{DM}}^{(j)}(t), P_{\text{DM}}^{(j)}(t)$ ($2 \leq j \leq N$), spanning a degenerate subspace orthogonal to the BM. A convenient way to generate them is to introduce a unitary transformation U acting on the local quadratures such that $\mathbf{R}' = U\mathbf{R}$, with

$$\begin{aligned} \mathbf{R} &= (Q_1, P_1, \dots, Q_N, P_N)^T, \\ \mathbf{R}' &= (Q_{\text{BM}}, P_{\text{BM}}, Q_{\text{DM}}^{(2)}, P_{\text{DM}}^{(2)}, \dots, Q_{\text{DM}}^{(N)}, P_{\text{DM}}^{(N)})^T, \end{aligned} \quad (\text{B21})$$

where the time argument has been omitted here for compactness. Here,

$$U = \bar{U} \otimes \mathbb{1}_{2 \times 2}, \quad \text{with} \quad \bar{U} = \mathbb{1}_{N \times N} - 2 \frac{\mathbf{v}\mathbf{v}^T}{|\mathbf{v}|^2}, \quad (\text{B22})$$

with $\bar{U}$ is a Householder reflector [87], with $\mathbf{v} = \hat{\mathbf{A}} - \mathbf{e}_1$, $\hat{\mathbf{A}} = \frac{1}{\alpha}(\alpha_1, \dots, \alpha_N)^T$, and $\mathbf{e}_1 = (1, 0, \dots, 0)^T$. Note that since $\frac{\mathbf{v}\mathbf{v}^T}{|\mathbf{v}|^2}$ is a projector, $\bar{U}$ and thus U are orthogonal, hence also unitary. The matrix U is also *symplectic*, satisfying $U\Omega U^T = \Omega$, where $\Omega = \bigoplus_{j=1}^N J$, with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (\text{B23})$$

the standard 2×2 symplectic matrix. The proof follows immediately by exploiting the orthogonality of U and the definition of Ω . The matrix $\bar{U}$ maps $\mathbf{e}_1$ to $\hat{\mathbf{A}}$ (the BM) and provides an orthonormal basis for the DMs. The symplectic lift $U = \bar{U} \otimes \mathbb{1}_{2 \times 2}$ ensures the correct interleaved order of quadratures.

The global operators are canonical, satisfying $[Q_{\text{BM}}(t), P_{\text{BM}}(t)] = [Q_{\text{DM}}^{(j)}(t), P_{\text{DM}}^{(j)}(t)] = i$, while all other

commutators are zero. Noting that H_{B} of Eq. (1) is written in terms of the local quadratures as $H_{\text{B}} = \frac{\hbar\omega_0}{2} \mathbf{R}^T \cdot \mathbf{R}$, it is clear from the above unitary transformation that $H_{\text{B}} = \frac{\hbar\omega_0}{2} \mathbf{R}^T \cdot (UU^T) \mathbf{R}^T \equiv \frac{\hbar\omega_0}{2} \mathbf{R}^T \cdot \mathbf{R}^T$; hence, in the global picture, the battery Hamiltonian can be decomposed as $H_{\text{B}} = H_{\text{BM}} + H_{\text{DM}}$, where

$$H_{\text{BM}} = \frac{\hbar\omega_0}{2} [P_{\text{BM}}^2 + Q_{\text{BM}}^2],$$

$$H_{\text{DM}} = \frac{\hbar\omega_0}{2} \sum_{j=2}^N [P_{\text{DM}}^{(j)2} + Q_{\text{DM}}^{(j)2}], \quad (\text{B24})$$

with $[H_{\text{BM}}, H_{\text{DM}}] = 0$.

We are now in the position to derive the CM in the global basis, $\sigma_{\text{B}}^{(\text{glob})}(t)$. Its general form is

$$\sigma_{\text{B}}^{(\text{glob})}(t) = \begin{pmatrix} \sigma_{Q_{\text{BM}}, Q_{\text{BM}}}(t) & \sigma_{Q_{\text{BM}}, P_{\text{BM}}}(t) & \cdots & \sigma_{Q_{\text{BM}}, Q_{\text{DM}}^{(N)}}(t) & \sigma_{Q_{\text{BM}}, P_{\text{DM}}^{(N)}}(t) \\ \sigma_{P_{\text{BM}}, Q_{\text{BM}}}(t) & \sigma_{P_{\text{BM}}, P_{\text{BM}}}(t) & \cdots & \sigma_{P_{\text{BM}}, Q_{\text{DM}}^{(N)}}(t) & \sigma_{P_{\text{BM}}, P_{\text{DM}}^{(N)}}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{Q_{\text{DM}}^{(N)}, Q_{\text{BM}}}(t) & \sigma_{Q_{\text{DM}}^{(N)}, P_{\text{BM}}}(t) & \cdots & \sigma_{Q_{\text{DM}}^{(N)}, Q_{\text{DM}}^{(N)}}(t) & \sigma_{Q_{\text{DM}}^{(N)}, P_{\text{DM}}^{(N)}}(t) \\ \sigma_{P_{\text{DM}}^{(N)}, Q_{\text{BM}}}(t) & \sigma_{P_{\text{DM}}^{(N)}, P_{\text{BM}}}(t) & \cdots & \sigma_{P_{\text{DM}}^{(N)}, Q_{\text{DM}}^{(N)}}(t) & \sigma_{P_{\text{DM}}^{(N)}, P_{\text{DM}}^{(N)}}(t) \end{pmatrix}. \quad (\text{B25})$$

From Eq. (B24), in the global picture the initial density matrix decomposes as

$$\rho_{\text{tot}}(0) = \frac{e^{-H_{\text{BM}}/k_{\text{B}}T_0}}{Z_{\text{BM}}} \otimes \frac{e^{-H_{\text{DM}}/k_{\text{B}}T_0}}{Z_{\text{DM}}} \otimes \frac{e^{-H_{\text{R}}/k_{\text{B}}T}}{Z_{\text{R}}}. \quad (\text{B26})$$

From this thermal condition at $t = 0$, there are no correlations between the BM and any of the DMs, and the same holds also between two *different* DMs; this implies $\sigma_{\text{B}}^{(\text{glob})}(0) = \frac{C(T_0)}{2} \mathbb{1}$. Since the DMs are decoupled from the reservoir, they evolve freely and therefore their density matrix remains thermal at all times. In addition, they are also decoupled from the BM and this implies that the off-diagonal blocks of $\sigma_{\text{B}}^{(\text{glob})}(t)$ are always null. Therefore, we have

$$\sigma_{\text{B}}^{(\text{glob})}(t) = \Sigma_{\text{BM}}(t) \bigoplus_{j=2}^N \Sigma_{\text{DM}}^{(j)}(t), \quad (\text{B27})$$

with $\Sigma_{\text{DM}}^{(j)}(t) = \Sigma_{\text{DM}} = \text{diag}\{\frac{C(T_0)}{2}, \frac{C(T_0)}{2}\} \forall j$; namely, it is in block-diagonal form. Finally, the explicit form of the BM block,

$$\Sigma_{\text{BM}}(t) = \frac{1}{2} \begin{pmatrix} \langle \{Q_{\text{BM}}(t), Q_{\text{BM}}(t)\} \rangle & \langle \{Q_{\text{BM}}(t), P_{\text{BM}}(t)\} \rangle \\ \langle \{P_{\text{BM}}(t), Q_{\text{BM}}(t)\} \rangle & \langle \{P_{\text{BM}}(t), P_{\text{BM}}(t)\} \rangle \end{pmatrix}, \quad (\text{B28})$$

is promptly obtained from the definition of the bright mode coordinates in terms of the local ones. From Eq. (B18), the BM block is

$$\Sigma_{\text{BM}}(t) = \begin{pmatrix} \frac{C(T_0)}{2} + \bar{\alpha}^2 a(t) & \bar{\alpha}^2 b(t) \\ \bar{\alpha}^2 b(t) & \frac{C(T_0)}{2} + \bar{\alpha}^2 c(t) \end{pmatrix}. \quad (\text{B29})$$

From Eq. (B24), we immediately observe that $E_{\text{B}}(t) = \frac{\hbar\omega_0}{2} \text{Tr}\{\sigma_{\text{B}}^{(\text{glob})}(t)\}$ and exploiting the explicit expressions of

$\Sigma_{\text{BM}}(t)$ and Σ_{DM} , one immediately finds that

$$E_{\text{B}}(t) = E_{\text{BM}}(t) + (N-1) \frac{\hbar\omega_0}{2} C(T_0), \quad (\text{B30})$$

with $E_{\text{BM}}(t) = \frac{\hbar\omega_0}{2} \{C(T_0) + \bar{\alpha}^2 [a(t) + c(t)]\}$, as quoted in Eq. (5).

By symplectic invariance, the symplectic spectrum $\{v_j(t)\}$ ($1 \leq j \leq N$) is easily obtained from $\sigma_{\text{B}}^{(\text{glob})}(t)$. For the BM, we identify $v_1(t) = \sqrt{\det\{\Sigma_{\text{BM}}(t)\}}$ and obtain

$$v_1(t) = \frac{1}{2} \sqrt{C^2(T_0) + 2\bar{\alpha}^2 [C(T_0)\mathcal{T}(t) + 2\bar{\alpha}^2 \Delta(t)]}, \quad (\text{B31})$$

where $\mathcal{T}(t) = a(t) + c(t)$ and $\Delta(t) = a(t)c(t) - b^2(t)$. The DM blocks remain thermal, with

$$v_{j \in [2, N]} = \sqrt{\det\{\Sigma_{\text{DM}}\}} = \frac{C(T_0)}{2}. \quad (\text{B32})$$

APPENDIX C: LOCAL VERSUS GLOBAL EFFICIENCIES

With a multipartite battery, different extraction protocols can be employed. The two extreme cases are a *local* one, with N unitary detectors each addressing a single oscillator, and a *global* one, with a single unitary detector acting on the whole battery. In the *local* case, one has

$$\mathcal{E}_{\text{loc}}(t) = E_{\text{B}}(t) - \hbar\omega_0 \sum_{l=1}^N v_1^{(l)}(t), \quad (\text{C1})$$

where $v_1^{(l)}(t)$ is the symplectic eigenvalue of the matrix $\frac{C(T_0)}{2} \mathbb{1} + \Sigma^{(l,l)}(t)$ [see Eq. (B18)]. It is given by

$$v_1^{(l)}(t) = \frac{1}{2} \sqrt{C^2(T_0) + 2\alpha_l^2 [C(T_0)\mathcal{T}(t) + 2\alpha_l^2 \Delta(t)]}. \quad (\text{C2})$$

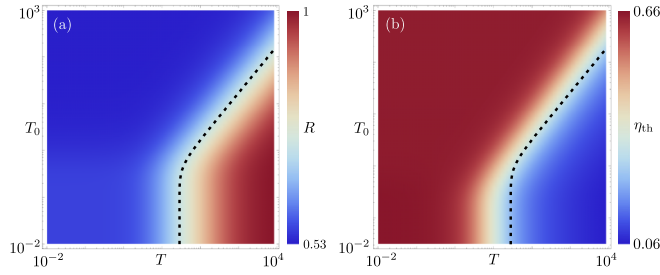


FIG. 5. Density plot as a function of T and T_0 (units $\hbar\omega_0/k_B$) for (a) $R = R(t^*)$ and (b) $\eta_{\text{th}} = \eta_{\text{th}}(t^*)$. The black dashed line in both panels represents the boundary $T = T^* = \frac{\hbar\Omega_N^2}{2k_B T} \coth\left(\frac{\hbar\omega_0}{2k_B T_0}\right)$. The other parameters are $N = 6$, $\alpha_l = 1$, $\gamma_0 = 5\omega_0$, and $\omega_D = 2\omega_0$.

For a global extraction, the ergotropy is instead written in terms of the symplectic eigenvalues $v_j(t)$ of $\sigma_B^{(\text{glob})}(t)$. We have

$$\mathcal{E}_{\text{glob}}(t) = E_B(t) - \hbar\omega_0 \sum_{j=1}^N v_j(t). \quad (\text{C3})$$

From expressions (B31) and (B32), we easily arrive at the compact form:

$$\mathcal{E}_{\text{glob}}(t) = E_{\text{BM}}(t) - \hbar\omega_0 v_1(t). \quad (\text{C4})$$

Notice that the bright mode is brought to a passive (effective thermal) state via a desqueezing Gaussian operation. After this step, no further unitary—either Gaussian or non-Gaussian—can reduce its energy any further.

We are now in the position to assess the most favorable extraction protocol. For this task, we analyze the behavior of the ratio $R(t) = \mathcal{E}_{\text{loc}}(t)/\mathcal{E}_{\text{glob}}(t)$. A plot of $R(t^*)$ is shown in Fig. 5(a) as a function of T and T_0 . One observes two regions with different values, separated by the characteristic temperature $T^* = \frac{\hbar^2\Omega_N^2}{2k_B^2 T} \coth\left(\frac{\hbar\omega_0}{2k_B T_0}\right)$, emerging from the competition between the reservoir thermal fluctuations and those of the initial condition of the QB. When $T > T^*$ (red region), thermal fluctuations are large and they are not useful to enhance the global extraction since they only induce classical correlations, then $R \sim 1$. On the other hand, when $T < T^*$ (blue region), thermal fluctuations are suppressed allowing to extract more work via global operations, $R < 1$. For comparison, we reproduce in panel (b) the thermal efficiency $\eta_{\text{th}}(t^*)$ defined in Eq. (9) and already shown there in Fig. 1(c). As we can see, the behavior of $R(t^*)$ is opposite to that of $\eta_{\text{th}}(t^*)$. Indeed, for $T > T^*$, the efficiency $\eta_{\text{th}}(t^*)$ is strongly reduced since thermal fluctuations impose a significant entropic cost. We then conclude that in order to optimize the thermal cost, the best protocol of extraction is the global one at $T < T^*$.

Another relevant case to mention is the one of a balanced bipartition where the work can be extracted either from the $N/2$ oscillators composing subsystem A or from the $N/2$ oscillators composing subsystem B (with even N). Here, we inspect the purity of each subsystem $\mathcal{P}_A = \mathcal{P}_B$, expressed as

$$\mathcal{P}_A = \mathcal{P}_A(t^*) = \frac{1}{2v_1^{(A)}(t^*)[C(T_0)]^{\frac{N}{2}-1}}, \quad (\text{C5})$$

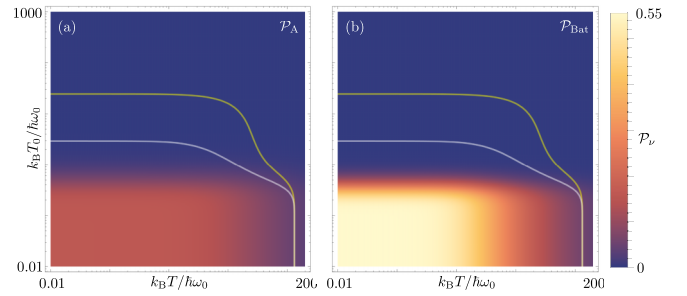


FIG. 6. Density plots of (a) the purity $\mathcal{P}_A = \mathcal{P}_A(t^*)$ of subsystem A with $N/2$ oscillators and (b) the purity $\mathcal{P}_{\text{Bat}} = \mathcal{P}_{\text{Bat}}(t^*)$ of the battery as a function of T and T_0 . The yellow contour line denotes the separation between classical and quantum squeezing ($\lambda_- = 1/2$), and the white contour line is the border between nonentangled and entangled states ($\nu_- = 1/2$). Here, $N = 6$, $\alpha_l = 1$, $\gamma_0 = 5\omega_0$, and $\omega_D = 2\omega_0$.

where

$$v_1^{(A)}(t) = \frac{1}{2} \sqrt{C^2(T_0) + 2\bar{\alpha}_A^2 [C(T_0)\mathcal{T}(t) + 2\bar{\alpha}_A^2 \Delta(t)]}, \quad (\text{C6})$$

with $\bar{\alpha}_A^2 = \sum_{l=1}^{N/2} \alpha_l^2$. This quantity is shown in Fig. 6(a), where one can observe that it remains well below unity, implying a reduced performance for local extraction protocols. However, the purity of the whole battery,

$$\mathcal{P}_{\text{Bat}} = \mathcal{P}_{\text{Bat}}(t^*) = \frac{1}{2v_1(t^*)[C(T_0)]^{N-1}}, \quad (\text{C7})$$

with $v_1(t)$ defined in Eqs. (B31) and (6), is always larger than $\mathcal{P}_A$ [see Fig. 6(b)]: in the words of Schrödinger, for an entangled state, “the best possible knowledge of the whole does not include the best possible knowledge of its parts” [88]. Note that while the mixedness of the subsystems reduces the local ergotropy, this does not preclude—or contradicts—an enhancement of the globally extractable work. On the contrary, correlations activate work that is inaccessible under local operations and make it available through global operations, thereby increasing the battery efficiency. To substantiate this point, in Fig. 7 we consider the ratio

$$\mathcal{R} = \mathcal{R}(t^*) = \frac{\mathcal{E}_A(t^*) + \mathcal{E}_B(t^*)}{\mathcal{E}_{\text{glob}}(t^*)}, \quad (\text{C8})$$

with $\mathcal{E}_{A/B}$ defined in analogy to the global ergotropy in Eq. (C3), but restrained to the proper partition of the system and replacing $v_1(t^*)$ with $v_1^{(A/B)}(t^*)$. This ratio compares the sum of the ergotropies extracted locally from the two partitions A and B with the global ergotropy, plotted as a function of the logarithmic negativity $\mathcal{N} = \mathcal{N}(t^*)$, which characterizes the presence of entanglement. We observe that, as the entanglement between the two partitions increases, $\mathcal{R}$ decreases, indicating that entanglement enhances global work extraction relative to local extraction.

APPENDIX D: ENERGY COST TO SWITCH ON/OFF THE COUPLING

Switching on (at time $t = 0$) and off (at time t) the coupling term H_{int} between the QB and the thermal reservoir requires

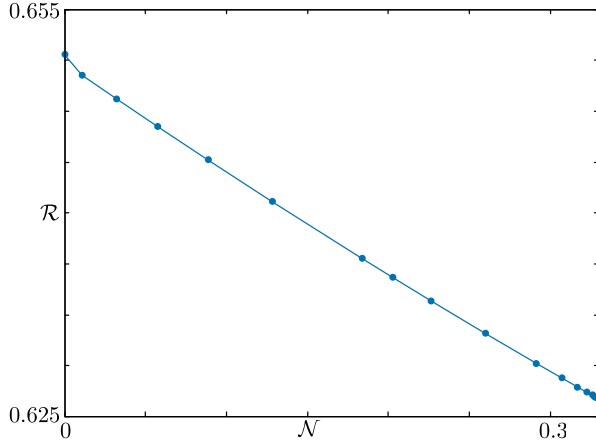


FIG. 7. Plot of the ergotropy ratio $\mathcal{R} = \mathcal{R}(t^*)$ —see text—as a function of the negativity $\mathcal{N} = \mathcal{N}(t^*)$ for $N = 6$, $N_A = N_B = 3$, $\alpha_l = 1$, $k_B T_0 = \hbar\omega_0$, and $k_B T = 0.01\hbar\omega_0$ at fixed energy of the battery $E_B = 10.41\hbar\omega_0$. The negativity $\mathcal{N}$ is varied tuning γ_0 and ω_D , keeping fixed the energy of the battery E_B , N , $N_A = N_B = N/2$, T_0 , and T .

performing work, thereby implying an energy cost. Here, we derive the closed-form expression for this work, defined as

$$W(t) = \langle H_{\text{int}}(0) \rangle - \langle H_{\text{int}}(t) \rangle. \quad (\text{D1})$$

We start by recalling the interaction Hamiltonian

$$H_{\text{int}} = - \sum_{k=1}^{+\infty} c_k x_k \sum_{l=1}^N \alpha_l q_l + \sum_{k=1}^{+\infty} \frac{c_k^2}{2m_k \omega_k^2} \left(\sum_{l=1}^N \alpha_l q_l \right)^2. \quad (\text{D2})$$

The switching on term at $t = 0$ has a simple expression:

$$\langle H_{\text{int}}(0) \rangle = \sum_{k=1}^{+\infty} \frac{c_k^2}{2m_k \omega_k^2} \sum_{l=1}^N \alpha_l^2 \langle q_l^2(0) \rangle = \frac{\hbar \bar{\alpha}^2 \Omega_0^2}{4\omega_0} C(T_0), \quad (\text{D3})$$

where we have exploited the initial thermal average $\langle q_l^2(0) \rangle = \frac{\hbar C(T_0)}{2m\omega_0}$ and we recall the definition

$$\Omega_0^2 = \sum_{k=1}^{+\infty} \frac{c_k^2}{m m_k \omega_k^2} = \gamma_0 \omega_D. \quad (\text{D4})$$

In order to express the switching off work at time t , we use the following relation, valid for any l :

$$\alpha_l \sum_{k=1}^{+\infty} c_k x_k(t) = m \left[\ddot{q}_l(t) + \omega_0^2 q_l(t) + \Omega_0^2 \alpha_l \sum_{l'=1}^N \alpha_{l'} q_{l'}(t) \right], \quad (\text{D5})$$

obtained from the equation of motion for the battery oscillators [36]. Inserting this expression into the general form of Eq. (D2), we arrive at the following average:

$$\begin{aligned} \langle H_{\text{int}}(t) \rangle = & -m \sum_{l=1}^N \left[\frac{1}{2} \langle \{q_l(t), \ddot{q}_l(t)\} \rangle + \omega_0^2 \langle q_l^2(t) \rangle \right. \\ & \left. - \frac{\Omega_0^2}{4} \sum_{l'=1}^N \alpha_l \alpha_{l'} \langle \{q_l(t), q_{l'}(t)\} \rangle \right]. \quad (\text{D6}) \end{aligned}$$

As a final step, we express the above relation in terms of the CM elements, using $\langle \{q_l(t), \ddot{q}_l(t)\} \rangle = \frac{d^2}{dt^2} \langle q_l^2(t) \rangle - 2\langle \dot{q}_l^2(t) \rangle$. The final expression is

$$\begin{aligned} \langle H_{\text{int}}(t) \rangle = & -\hbar \bar{\alpha}^2 \left\{ \frac{\ddot{a}(t)}{2\omega_0} - \omega_0 [c(t) - a(t)] \right. \\ & \left. + \frac{\Omega_0^2}{4\omega_0} [C(T_0) + 2\bar{\alpha}^2 a(t)] \right\}. \quad (\text{D7}) \end{aligned}$$

APPENDIX E: COMMENTS ON QUANTUM DISCORD

Quantum discord captures all nonclassical correlations, whether separable or entangled. In order to characterize its behavior in the context of our model, we have calculated it following the general prescriptions for Gaussian states given in Refs. [78–80]. Considering our standard balanced bipartition with two blocks containing $N/2$ oscillators, we start by recalling that the covariance matrix of the bipartition has the block form

$$\sigma_{\text{bip}}(t) = \begin{pmatrix} L(t) & K(t) \\ K^t(t) & L(t) \end{pmatrix}, \quad (\text{E1})$$

with $L(t) = \frac{C(T_0)}{2} \mathbb{1} + K(t)$ and

$$K(t) = \frac{\bar{\alpha}^2}{2} \begin{pmatrix} a(t) & b(t) \\ b(t) & c(t) \end{pmatrix}. \quad (\text{E2})$$

We are now able to obtain a closed expression for the discord. We introduce the short notation $\bar{L}(t) = \det\{L(t)\}$, $\bar{K}(t) = \det\{K(t)\}$, $\bar{D}(t) = \det\{\sigma_{\text{bip}}(t)\}$,

$$g(t) = [\bar{D}(t) - \bar{L}^2(t)]^2 - \bar{K}^2(t) \left[\frac{1}{4} + \bar{L}(t) \right] [\bar{L}(t) + 4\bar{D}(t)], \quad (\text{E3})$$

and

$$E_{\text{min}}(t) = \begin{cases} \frac{2\bar{K}^2(t) + [\bar{L}(t) - \frac{1}{4}] [4\bar{D}(t) - \bar{L}(t)] + 2|\bar{K}(t)| \sqrt{\bar{K}^2(t) + [\bar{L}(t) - \frac{1}{4}] [4\bar{D}(t) - \bar{L}(t)]}}{[2\bar{L}(t) - \frac{1}{2}]^2}, & \text{if } g(t) \leq 0, \\ \frac{\bar{L}^2(t) - \bar{K}^2(t) + \bar{D}(t) - \sqrt{\bar{K}^4(t) + [\bar{D}(t) - \bar{L}^2(t)]^2 - 2\bar{K}^2(t) [\bar{D}(t) + \bar{L}^2(t)]}}{2\bar{L}(t)}, & \text{if } g(t) > 0. \end{cases} \quad (\text{E4})$$

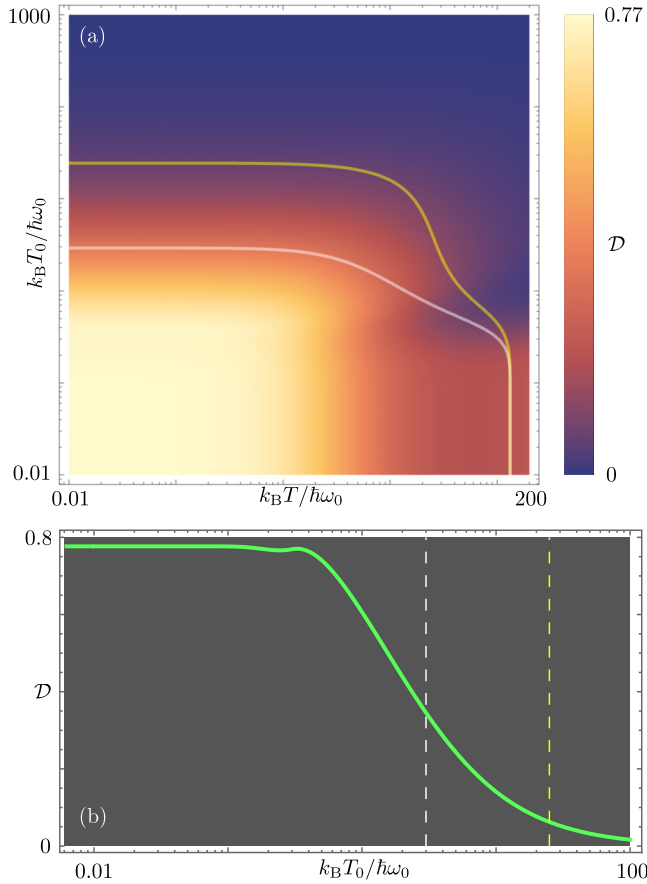


FIG. 8. (a) Density plot of the discord $\mathcal{D} = \mathcal{D}(t^*)$ as a function of T and T_0 (units $\hbar\omega_0/k_B$). (b) Plot of $\mathcal{D}$ as a function of T_0 obtained cutting the density plot at $k_B T = 0.1\hbar\omega_0$ (green line). In the top panel, the yellow contour line denotes the separation between classical and quantum squeezing, and the white contour line is the border between nonentangled and entangled states. In the bottom panel, the yellow and the white dashed lines denote the values of T_0 corresponding to the above lines. Here, $N = 6$, $\alpha_l = 1$, $\gamma_0 = 5\omega_0$, and $\omega_D = 2\omega_0$.

Then, the discord is given by

$$\mathcal{D}(t) = f[\sqrt{\bar{L}(t)}] - f[v_{\text{bip}}^{(+)}(t)] - f[v_{\text{bip}}^{(-)}(t)] + f[\sqrt{E_{\text{min}}(t)}], \quad (\text{E5})$$

with $f(x) = (x + \frac{1}{2}) \ln(x + \frac{1}{2}) - (x - \frac{1}{2}) \ln(x - \frac{1}{2})$ and $v_{\text{bip}}^{(\pm)}(t)$ the symplectic eigenvalues of $\sigma_{\text{bip}}(t)$.

Explicitly, we have $\bar{L}(t) = \frac{1}{4}\{C^2(T_0) + \bar{\alpha}^2[C(T_0)\mathcal{T}(t) + \bar{\alpha}^2\Delta(t)]\}$, $\bar{K}(t) = \frac{\bar{\alpha}^4}{4}\Delta(t)$, $\bar{D}(t) = \frac{C^2(T_0)}{4}v_1^2(t)$, $v_{\text{bip}}^{(+)}(t) = v_1(t)$, and $v_{\text{bip}}^{(-)}(t) = \frac{C(T_0)}{2}$.

The typical behavior of $\mathcal{D}(t^*)$ is reported in Fig. 8. The top panel shows a density plot of $\mathcal{D}(t^*)$ as a function of T and T_0 . This quantity is different from zero over the entire parameter range, signaling the ubiquitous presence of quantum correlations in all the considered regimes: the classically squeezed one, the quantum-squeezed one (separated from the former by the yellow line), and the entangled one (bounded by the white line). While the discord increases upon entering the quantum-squeezed regime and grows further in the

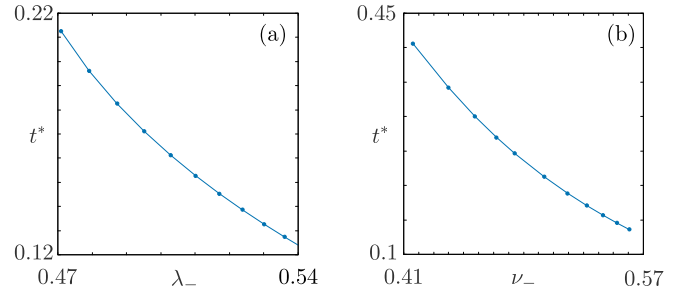


FIG. 9. Plots of t^* (units ω_0^{-1}) (a) as a function of λ_- for $k_B T_0 = 6\hbar\omega_0$ and $E_B(t^*) = 67.07\hbar\omega_0$ and (b) as a function of ν_- for $k_B T_0 = \hbar\omega_0$ and $E_B(t^*) = 10.41\hbar\omega_0$. In both panels, $k_B T = 0.01\hbar\omega_0$, $\alpha_l = 1$, and the quantities are calculated tuning γ_0 and ω_D , keeping fixed the energy of the battery E_B , N , T_0 , and T .

entangled regime, it does not display sharp features. This is in agreement with the fact that, as discussed in Ref. [81], in an open quantum system (as is our case) sharp transitions to states with strictly zero quantum discord cannot occur. To further clarify this point, the bottom panel of the same figure shows a cut of the density plot at fixed $k_B T = 0.01\hbar\omega_0$. It is evident that the discord does not exhibit any distinctive features either at the boundary between classical and quantum squeezing (white vertical dashed line) or at the boundary between separable and entangled states (yellow vertical dashed line). Thus, notwithstanding its usefulness in quantifying the overall amount of quantum correlations, the discord does not allow one to sharply distinguish between their nature; in particular, there is no lower bound on the discord associated with entanglement.

APPENDIX F: STABILITY OF THE BOUNDS

The bounds $\mathcal{B}_{\text{cl}}(t)$ and $\mathcal{B}_{\text{en}}(t)$ discussed in the main text have been derived for an arbitrary value of time t . However, since we are interested in the performance of the battery at maximum stored energy—a standard choice in the context of quantum batteries—we focused on efficiencies at the particular time $t = t^*$ that maximizes $E'_{\text{BM}}(t)$. In general terms, one has $t^* = \delta \frac{\tau}{2}$, where

$$\tau = \frac{\pi}{\Omega_N} = \frac{\pi}{\sqrt{\bar{\alpha}^2 \gamma_0 \omega_D}}, \quad (\text{F1})$$

and the prefactor δ is a function of the various parameters involved in the problem. In all the explored regimes, we have always found that $\delta \sim 0.2 \div 1$; thus t^* is always on the order of $\tau/2$.

Figure 9 shows the behavior of t^* (in units ω_0^{-1}) (a) as a function of λ_- around the transition from classical to quantum squeezing and (b) as a function of ν_- around the transition from a separable to an entangled battery state. In both cases, we clearly see that t^* increases with correlations. While the efficiencies benefit from the presence of quantum correlations or entanglement (see Fig. 3), the increase of t^* with correlations leads to a corresponding reduction in power. This power-efficiency trade-off is analogous to what is observed in thermal machines [43–48], where high efficiency

is typically achieved at the expense of power, with the limiting case of Carnot efficiency corresponding to vanishing power.

Finally, we show that the bounds are robust with respect to deviations from $t = t^*$. This is illustrated in Fig. 10, which shows the relative variations of the bounds

$$\Delta_B^{(j)}(t) = \frac{|\mathcal{B}_j(t) - \mathcal{B}_j(t^*)|}{\mathcal{B}_j(t^*)}, \quad \text{with } j \in \{\text{cl}, \text{en}\}, \quad (\text{F2})$$

as a function of $\frac{t-t^*}{\tau}$, where $\tau = \frac{\pi}{\sqrt{\alpha^2 \gamma_0 \omega_D}}$. This choice allows for a fair comparison among different working points. Considering same relevant working points, the results show that, even for relative time variations exceeding 10%, both $\Delta_B^{(\text{cl})}(t)$ [panel (a)] and $\Delta_B^{(\text{en})}(t)$ [– panel (b)] vary only within a few percent.

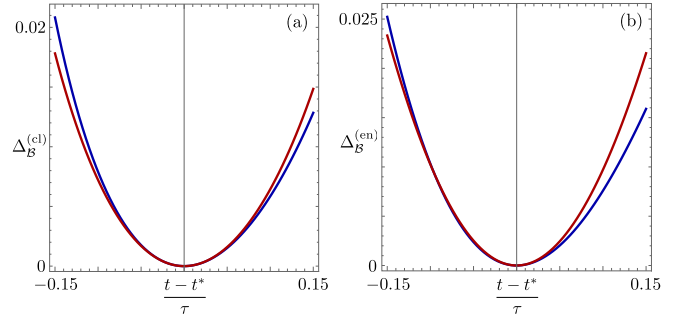


FIG. 10. Plot of (a) $\Delta_B^{(\text{cl})} = \Delta_B^{(\text{cl})}(t)$, the relative variation of $\mathcal{B}_{\text{cl}}(t)$ and (b) $\Delta_B^{(\text{en})} = \Delta_B^{(\text{en})}(t)$, the relative variation of $\mathcal{B}_{\text{en}}(t)$, as a function of $\frac{t-t^*}{\tau}$, where $\tau = \frac{\pi}{\sqrt{\alpha^2 \gamma_0 \omega_D}}$. For panel (a), $k_B T_0 = 9\hbar\omega_0$, with $\gamma_0 = 0.99\omega_0$ and $\omega_D = 12.5\omega_0$ (red curve), and $\gamma_0 = 1.63\omega_0$ and $\omega_D = 3.49\omega_0$ (blue curve). For panel (b), $k_B T_0 = \hbar\omega_0$, with $\gamma_0 = 0.528\omega_0$ and $\omega_D = 7.344\omega_0$ (red curve), and $\gamma_0 = 1.571\omega_0$ and $\omega_D = 0.922\omega_0$ (blue curve). Other common parameters: $N = 6$, $\alpha_l = 1$, and $k_B T = 0.01\hbar\omega_0$.

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