

Probing superfluidity with quantum vortex necklaces and Leggett's bounds

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We theoretically investigate a binary Bose-Einstein condensate in which the majority component hosts a vortex necklace, whose vortex cores act as moving effective potential wells for the minority component. This configuration represents a tunable periodic landscape whose strength is controlled by the intercomponent interaction. As the coupling increases, the minority component undergoes a crossover from a delocalized, nearly perfect superfluid to an array of localized density peaks confined within the vortex cores. Using Gross-Pitaevskii simulations, we compute the moment of inertia of the minority component and show that its superfluid fraction is tightly bracketed by Leggett's bounds constructed from its density profile. Because these bounds depend solely on the spatial density distribution, they may be extracted directly from experimentally accessible measurements. Our results identify vortex necklaces as a controllable tool for tuning and probing superfluidity in two-component condensates and provide a natural framework for exploring superfluid behavior in dynamically evolving and disordered potentials.

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Introduction. Bose-Einstein condensation and superfluidity, while often co-occurring, are fundamentally different phenomena. The former is a static property quantifying the macroscopic occupation of a single quantum state and is represented by a scalar, while the latter is a dynamical quantity measuring the (absence of) response to a weak and constant perturbation and is therefore a tensor [1]. In weakly interacting uniform ultracold bosonic systems, the condensed and superfluid fractions typically coincide, but they can be significantly different in other situations: An ideal Bose-Einstein condensate (BEC), for example, is fully condensed but not superfluid, whereas superfluid liquid helium at zero temperature exhibits complete superfluidity despite having only $\sim 10\%$ condensate fraction [2].

A foundational approach to probing the superfluid fraction originated with the Andronikashvili experiment [3], which measured the frequency shift of a torsional pendulum immersed in liquid helium. As temperature decreases, the emergence of a superfluid component, which decouples from the pendulum's motion, leads to a measurable increase in oscillation frequency, reflecting a reduced effective moment of inertia. This pioneering method inspired a broad family of moment-of-inertia-based techniques [2,4,5] for the determination of the superfluid fraction, including recent applications to toroidal dipolar supersolids [6–11]. Other experiments extracted the superfluid fraction by measuring anisotropy of

sound propagation [12], scissor modes [13], and transport under lattice modulations [14,15].

The idea of probing conduction via rotation, or equivalently through a synthetic gauge field, also arises in Fermi lattice models through the concept of the Drude weight [16], which quantifies the singular part (at zero frequency) of the electrical conductivity. Originally introduced to distinguish metals from insulators [17], it has recently been applied to characterize transport in ringlike multicomponent Fermi systems [18], minimal quantum circuits capable of sustaining persistent currents [19].

In this work, we theoretically investigate a binary BEC mixture in which the majority component (a) hosts a vortex necklace, whose vortex cores act as a moving, angularly modulated potential for the minority component b (see Fig. 1). This setting creates a tunable periodic landscape for the minority component (b), controlled by the intercomponent interaction. As this coupling increases, the minority component evolves from a delocalized, nearly perfect superfluid to a set of localized density peaks confined within the vortex cores. In this system, the intrinsic rotation of the vortex necklace [20–23] may induce angular-momentum transfer to component b , which allows us to evaluate its (nonclassical) moment of inertia and hence its superfluid fraction. We show that the latter is tightly bracketed by Leggett's bounds constructed from its density profile. These bounds depend solely on the spatial density distribution and have been employed experimentally in a variety of ultracold-atom platforms [11–15], highlighting their broad applicability irrespective of the specific microscopic mechanism generating the density modulations. This system represents a clean and controllable platform for exploring how vortex-induced density modulations affect superfluid transport in two-component condensates. The tunability of the intercomponent interaction naturally allows one to investigate the crossover from a

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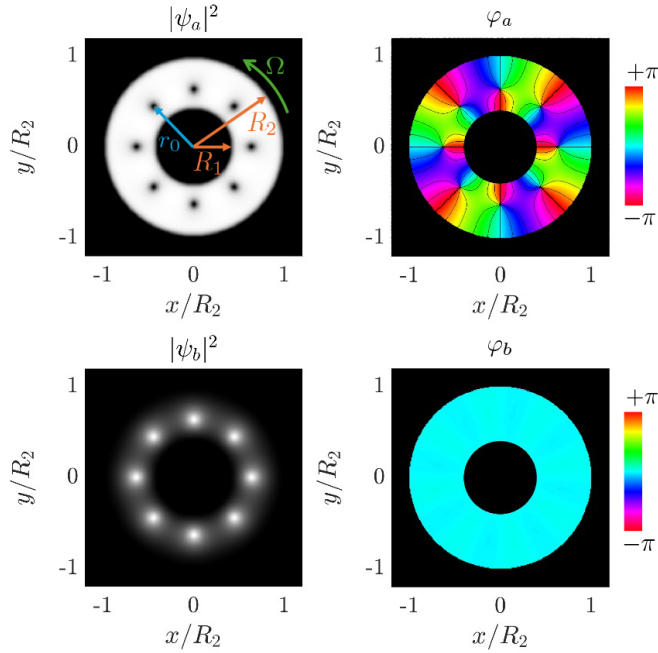


FIG. 1. Upper row: Density (left) and phase (right) of the majority component a , hosting a necklace with $N_v = 8$ vortices, which rotates at frequency Ω . Lower row: Density (left) and phase (right) of the minority component b . Parameters are given in the text; here, $g_{ab}/\sqrt{g_{aa}g_{bb}} = 0.4$.

fully superfluid regime to a localized, Mott-like state [24–26]. Moreover, the same platform inherently lends itself to the study of out-of-equilibrium phenomena, such as disordering or slow ramps across the miscibility threshold, making it ideal for investigating dynamical aspects of superfluidity. The annular geometry we consider is particularly suited to such investigation, in light of recent experimental advances in annular trapping and vortex control [14,27–31].

Rotating vortex necklaces. We consider a binary mixture of atomic Bose-Einstein condensates consisting of N_a atoms of component a and N_b atoms of component b with atomic masses m_a and m_b . We assume $N_a \gg N_b$, so that we can refer to a (b) as the majority (minority) component. The system is subject to a strong harmonic confinement along the z axis, associated with the harmonic-oscillator length d_z , which effectively freezes atomic motion in this direction. In the xy plane, the mixture is trapped in an annular confining potential V_{tr} with hard walls at inner and outer radii R_1 and R_2 (see Fig. 1). At zero temperature, the state of the system is described by two order parameters ψ_a and ψ_b (with normalization $\int |\psi_i|^2 d^2r = N_i$) whose time evolution is governed by the coupled Gross-Pitaevskii equations (GPEs)

$$i\hbar \frac{\partial \psi_i}{\partial t} = \left(-\frac{\hbar^2 \nabla^2}{2m_i} + V_{tr} + \sum_{j=a,b} g_{ij} |\psi_j|^2 \right) \psi_i, \quad i = a, b, \quad (1)$$

where the coupling constants $g_{ij} = \sqrt{2\pi} \hbar^2 a_{ij} / (m_{ij} d_z)$ depend on the intra- and intercomponent s -wave scattering lengths a_{ij} as well as on the reduced masses $m_{ij} = (m_i^{-1} + m_j^{-1})^{-1}$ [32,33].

The GPE for component a admits stationary solutions featuring N_v quantized vortices arranged as a regular polygon. These “vortex necklaces” have been extensively studied in the literature, as they appear in both classical inviscid fluids [20,34,35] and rotating superfluids [21–23,29,36,37]. Similar geometric patterns emerge also in charged plasmas, where they are referred to as “Wigner crystals” (see Ref. [38] and references therein). We stress, however, that vortex necklaces can be dynamically unstable over large regions of their existence domain (i.e., they eventually break up), as thoroughly analyzed in Refs. [23,29,36,39] (see the Supplemental Material [40], which includes Refs. [41–54]). This issue makes the preparation of accurate initial vortex states a crucial requirement, since fluctuations in the initial configuration can strongly affect the subsequent dynamics and the reliable extraction of physical parameters [55].

For component b , in the absence of intercomponent interactions ($g_{ab} = 0$), the density distribution $|\psi_b|^2$ is approximately uniform within the annulus. When a repulsive intercomponent interaction is introduced ($g_{ab} > 0$), the vortex necklace in the majority component acts as an effective optical lattice along the azimuthal direction θ , with vortex cores serving as potential wells for the minority component.

A crucial property of this system, which we will extensively exploit, is that vortex necklaces are not stationary in the laboratory frame, but instead rotate rigidly—at least to a good approximation, before dynamical instabilities set in (see the Supplemental Material [40])—due to mutual interactions between vortices and the presence of boundaries. For vortex necklaces of radius $r_0 = \sqrt{R_1 R_2}$, the precession frequency Ω simply reads

$$\Omega(r_0) = \frac{\hbar}{m_a r_0^2} \left(n_1 + \frac{N_v - 1}{2} \right), \quad (2)$$

with $n_1 \in \mathbb{Z}$ the number of quanta of circulation around the inner boundary of the annulus [23].

Probing the superfluid fraction. From the perspective of component- b atoms, component a acts as a *rotating* lattice potential. That means that the vortex necklace in component a imparts not only a density modulation on ψ_b , but also a nonzero angular momentum. As noted above, a rotating potential provides an ideal framework for measuring the superfluid fraction of a quantum fluid. In this sense, the majority component a serves as a natural probe for assessing the superfluid properties of the minority component b . In what follows, we focus on the diagonal azimuthal component of the superfluid tensor, $f_{\theta\theta}$, which quantifies the azimuthal response of component b to an azimuthal stimulus. This scalar, referred to as the “superfluid fraction,” is defined as

$$f_{\theta\theta} = 1 - \lim_{\Omega \rightarrow 0} \frac{\langle \psi_b | \hat{L}_z | \psi_b \rangle}{\Omega \int m_b |\psi_b|^2 r^2 d^2r}, \quad (3)$$

where $\hat{L}_z = -i\hbar \partial_\theta$ is the z component of the angular-momentum operator. The value of $f_{\theta\theta}$ lies between 0 and 1 and captures the deviation of the system’s actual angular momentum $\langle \psi_b | \hat{L}_z | \psi_b \rangle$ from its classical rigid-body counterpart $\Omega \int m_b |\psi_b|^2 r^2 d^2r$ [4].

To remain within linear response and compute $f_{\theta\theta}$ at the smallest possible Ω , we choose a circulation along the inner

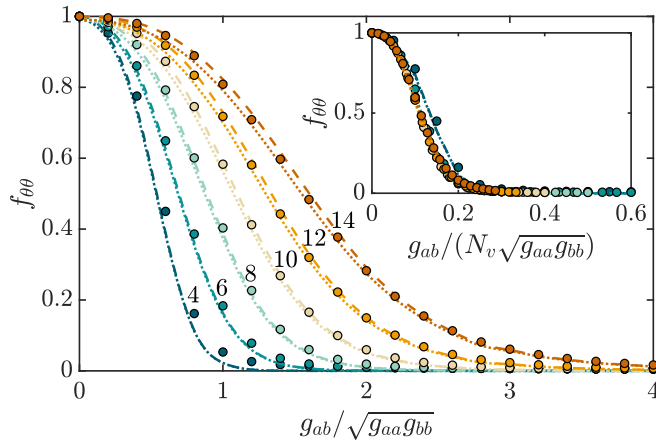


FIG. 2. Superfluid fraction of component b as a function of the intercomponent coupling g_{ab} , for different values of N_v (black labels). Colored dots correspond to the superfluid fraction $f_{\theta\theta}$ [Eq. (3)], while dotted and dashed lines correspond, respectively, to the Leggett's bounds $f_{\theta\theta}^-$ and $f_{\theta\theta}^+$ defined in Eq. (4), interpolated with a cubic spline. Inset: Same data, shown rescaling the horizontal axis by a factor N_v .

ring given by $n_1 = -N_v/2$ and we compute the superfluid fraction defined in Eq. (3) determining (by imaginary-time propagation) the stationary solutions of the system (of the type illustrated in Fig. 1) in a reference frame rotating with frequency Ω given by Eq. (2) [this requires adding an additional term $-\Omega \hat{L}_z \psi_i$ in each of the two equations in Eq. (1)]. For concreteness, we consider a Bose-Bose mixture of ^{23}Na and ^{39}K atoms, representing components a and b , respectively, having intracomponent scattering lengths $a_{aa} = 52 a_0$ and $a_{bb} = 8 a_0$, with a_0 the Bohr radius [56,57]. This mixture is particularly suited to implement the discussed protocol, as the intercomponent scattering length a_{ab} can be widely tuned via a Feshbach resonance across the miscible-immiscible transition [57]. We further choose experimentally relevant values $N_a \approx 2 \times 10^5$, $N_b \approx 5 \times 10^3$, $d_z \approx 0.5 \mu\text{m}$, $R_1 = 20 \mu\text{m}$, and $R_2 = 50 \mu\text{m}$, leading to $r_0 \approx 32 \mu\text{m}$ and $|\Omega| \approx 1.4 \text{ rad/s}$. The resulting superfluid fractions are shown by colored dots in Fig. 2, where they are plotted as a function of the miscibility parameter $g_{ab}/\sqrt{g_{aa}g_{bb}}$ for various vortex numbers N_v . These results correspond to stationary GPE solutions, but we note that this mixture also allows for a dynamical variation of g_{ab} , providing a route to investigate the out-of-equilibrium properties of the superfluid fraction (see the Supplemental Material [40]).

A powerful method for estimating the degree of superfluidity of a given system is provided by the so-called Leggett's bounds [58,59]. Quite remarkably, these provide lower and upper limits for the superfluid fraction based solely on the density distribution $|\psi_b|^2$. The upper bound f^+ is derived variationally and is therefore rigorous, meaning that the actual superfluid fraction cannot exceed this limit. In practice, however, small deviations above f^+ occasionally appear in our numerical data (notably for low N_v). We attribute those to numerical and postprocessing inaccuracies rather than to a violation of the bound itself. In contrast, the lower bound f^- is heuristic and generally holds only for dilute Bose gases,

where the system remains fully condensed [12,60]. A growing number of studies have benchmarked these bounds against established measurements of the superfluid fraction [12–15,60], highlighting their limitations across different configurations [60,61]. When applied to the azimuthal response, these may be computed as

$$f_{\theta\theta}^- = \frac{4\pi^2}{N_b} \int \frac{r dr}{\int \frac{d\theta}{|\psi_b|^2}} \quad \text{and} \quad f_{\theta\theta}^+ = \frac{4\pi^2}{N_b} \frac{1}{\int r dr |\psi_b|^2}. \quad (4)$$

The values of the upper and lower bounds that we extract from the densities of component b computed in imaginary time are illustrated in Fig. 2 (dashed and dotted lines). Notably, the condition $f_{\theta\theta}^- \leq f_{\theta\theta} \leq f_{\theta\theta}^+$ is always satisfied (except at very small values of N_v , as discussed above). In any case, our results support the conclusion that the superfluid fraction of a dilute Bose gas (which is an intrinsically dynamic property) can be reliably estimated, to a very high degree of accuracy, from its density profile only.

Interaction-driven localization and drop of superfluidity. When $g_{ab} = 0$, the two quantum fluids are fully decoupled. In this case, ψ_b remains uniform along the θ direction, carrying zero angular momentum. Consequently, the superfluid fraction computed according to Eq. (3) is exactly one. This result is consistent with the Leggett's bounds, which also yield $f_{\theta\theta}^+ = f_{\theta\theta}^- = 1$ as the density distribution $|\psi_b(r, \theta)|^2$ is independent of θ . As g_{ab} increases, component b becomes more and more confined within the vortex cores of component a (see upper row of Fig. 3). Consequently, the density profile $|\psi_b|^2$ acquires a stronger angular dependence, and the superfluid fraction decreases. This trend is consistently captured both by direct calculations using Eq. (3) and by the Leggett's bounds (see colored dots, as well as dashed and dotted lines, in Fig. 2). In the deeply immiscible regime ($g_{ab} \gg \sqrt{g_{aa}g_{bb}}$), the density modulations become so strong that component b effectively forms a necklace of spatially disconnected density peaks. In this insulating state $f_{\theta\theta}$ vanishes, indicating a complete loss of superfluid transport across the system. This behavior represents the precursor of the celebrated Mott transition [12,24–26]. Notably, in this regime, the vortices in component a effectively bind to massive cores, giving rise to massive quantum vortices, a class of dressed topological excitations currently under intense theoretical scrutiny [22,23,37,62–69]. A related superfluid-to-insulator transition has been recently discussed in the context of dipolar gases, where rotational symmetry becomes spontaneously broken in the supersolid and droplet phases [10,11,70–73].

The superfluid fraction further depends on the number of vortices in the necklace. As N_v increases, the spacing between adjacent density peaks in $|\psi_b|^2$ decreases, leading to a greater overlap of their tails. This enhanced overlap facilitates phase coherence between neighboring density peaks, effectively strengthening superfluid behavior. As a result, higher- N_v configurations transition to the Mott-like regime at larger values of g_{ab} compared to their smaller- N_v counterparts (see also the inset of Fig. 2).

The scenario described above is further confirmed by the analysis of an alternative indicator: the superfluid velocity of

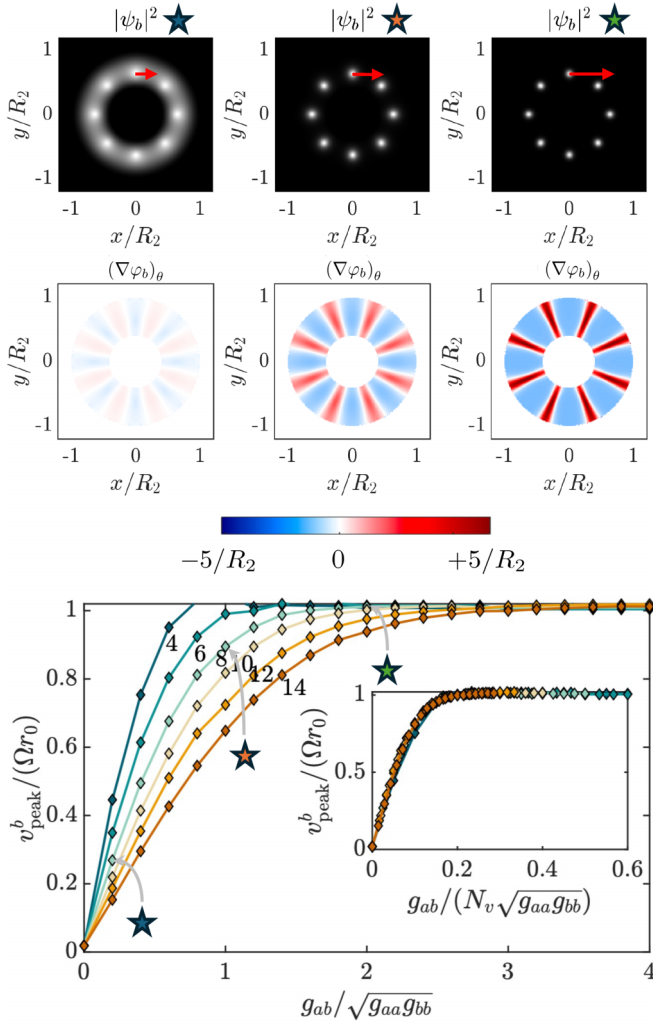


FIG. 3. Upper row: Density of component b for three different values of g_{ab} ; red arrows denote the relevant velocity field evaluated at the density peaks. Second row: Azimuthal gradient of the relevant phase field. Lower panel: Peak velocity (5) as a function of g_{ab} for different values of N_v (black labels). Colored stars serve as a visual guide and correspond to $g_{ab}/\sqrt{g_{aa}g_{bb}} = \{0.2, 1.0, 2.0\}$. Inset: Data collapse on the same curve upon rescaling the horizontal axis by a factor N_v .

component b evaluated at the density peaks

$$v_{\text{peak}}^b = \frac{\hbar}{m_b} \left| \frac{1}{r} \frac{\partial \varphi_b}{\partial \theta} \right|_{\mathbf{r}_*}, \quad (5)$$

where φ_b is the phase of the wavefunction ψ_b and $\mathbf{r}_*$ is the position of a peak in $|\psi_b|^2$. As illustrated in the first row of Fig. 3 (see, in particular, the red arrows), v_{peak}^b increases together with the loss of superfluid fraction, i.e., as component- b atoms become increasingly localized within the vortex cores. In the weak-coupling limit ($g_{ab} \rightarrow 0$), v_{peak}^b tends to zero. In this regime, the underlying potential imposed by component a (which rotates at frequency Ω) induces only local compressions and expansions in component b , without generating a significant net tangential flow.

Conversely, in the strongly immiscible regime ($g_{ab} \gg \sqrt{g_{aa}g_{bb}}$) component- b atoms fully localize in the vortex

cores, which behave as pointlike particles. In this regime, they exhibit uniform circular motion with a tangential velocity given by Ωr_0 . Effectively, component b forms a rigid necklace, where density peaks move collectively in a manner dictated by classical rotational dynamics. This crossover from a superfluid state to a state dominated by rigid-body motion is illustrated in the lower panel of Fig. 3. As previously discussed for $f_{\theta\theta}$, vortex necklaces with a larger number of vortices are more resilient to the onset of Mott-like localization and thus retain a significant superfluid fraction even at moderately large values of the miscibility parameter (see inset).

To further characterize this crossover, we compute the azimuthal phase gradient $(\nabla\varphi_b)_\theta = r^{-1}\partial_\theta\varphi_b$. This quantity is illustrated in the middle row of Fig. 3, showing that its magnitude increases as g_{ab} increases. Moreover, one can observe that this phase gradient exhibits opposite signs in different regions: It is negative at the density peaks, reflecting their clockwise motion, while it is positive in the low-density regions between peaks. This structure originates from the irrotational nature of the superfluid velocity field and has been previously reported in different contexts, including rotating supersolids [74].

Conclusions. We have theoretically analyzed an annular quasi-two-dimensional binary Bose-Einstein condensate where the majority component a hosts a rotating vortex necklace that acts as a moving periodic landscape for the minority component b . Within this setting, the intrinsic rotation of the necklace [20–23] induces angular-momentum transfer to component b , allowing us to compute its moment of inertia and hence its superfluid fraction [2,4,10]. We have shown that the latter is tightly bracketed by Leggett’s bounds [58,59], constructed solely from the density distribution of component b . Since these bounds rely only on spatial density modulations, they provide an experimentally accessible estimator. Our analysis reveals a smooth crossover from a nearly perfect superfluid to a strongly localized, Mott-like state in which component b forms an effectively classical array of density peaks, collectively moving as a rigid structure. The proposed framework offers a versatile theoretical setting to explore how vortex-induced density modulations and intercomponent interactions govern superfluid transport in two-component condensates. Beyond its equilibrium applications, this setup exhibits two key features that make it especially well suited for investigating nonequilibrium phenomena, as detailed in the Supplemental Material [40]. First, the intrinsic dynamical instability of vortex necklaces [23,29,75,76] offers a natural avenue to study superfluidity in dynamically evolving, disordered landscapes. Second, the possibility to dynamically tune or quench the intercomponent interaction provides a controlled means to probe the time-dependent behavior of the superfluid fraction, an aspect that remains largely unexplored.

While our analysis focused on the linear-response regime, extending the study to larger rotation frequencies offers a compelling connection to the physics of finite-circulation states in Josephson junction arrays [14]. These systems provide powerful testbeds for quantum circuits, where the interplay of macroscopic phase coherence, nonlinear effects, and dissipation governs key functionalities in quantum

computing, simulation, and metrology. In this context, our platform opens promising avenues for future developments in quantum transport, atomtronics, and superfluid circuits [19,77,78].

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Data availability. The data that support the findings of this Letter are not publicly available. The data are available from the authors upon reasonable request.

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